Existence of Ricci flows of incomplete surfaces

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Abstract

We prove a general existence result for instantaneously complete Ricci flows starting at an arbitrary Riemannian surface which may be incomplete and may have unbounded curvature. We give an explicit formula for the maximal existence time, and describe the asymptotic behaviour in most cases.

1 Introduction

Hamilton's Ricci flow [Ham82] takes a Riemannian metric g_0 on a manifold $\mathcal{M}$ and deforms it under the evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2\mathrm{Ric}\big[g(t)\big] \\ g(0) = g_0 \end{cases} \tag{1.1}$$

There is now a good well-posedness theory for this PDE which we summarise in the following theorem.

Theorem 1.1. (Hamilton [Ham82], DeTurck [DeT03], Shi [Shi89], Chen-Zhu [CZ06].) *Given a complete Riemannian manifold $(\mathcal{M}^n, g_0)$ with bounded curvature $\big|\mathrm{Rm}[g_0]\big| \leq K_0$, there exists $T > 0$ depending only on n and K_0, and a Ricci flow $g(t)$ for $t \in [0, T)$ satisfying (1.1) with bounded curvature and for which $\big(\mathcal{M}, g(t)\big)$ is complete for each $t \in [0, T)$. Moreover, any other complete, bounded curvature Ricci flow $\tilde{g}(t)$ with $\tilde{g}(0) = g_0$ must agree with $g(t)$ while both flows exist.*

We will call the Ricci flow whose existence is asserted by this theorem the *Hamilton-Shi* Ricci flow.

This paper is dedicated to the more general problem of posing Ricci flow in the case that the initial manifold $(\mathcal{M}, g_0)$ may be incomplete, and may have curvature unbounded above and/or below. Without further conditions, this problem is ill-posed, with extreme nonuniqueness. However, in [Top] the second author introduced the restricted class of *instantaneously complete* Ricci flows, proving the following theorem in the case that $\dim \mathcal{M} = 2$.

Theorem 1.2. (Topping [Top].) *Let $(\mathcal{M}^2, g_0)$ be any smooth metric on any surface $\mathcal{M}^2$ (without boundary) with Gaussian curvature bounded above by $K_0 \in \mathbb{R}$. Then for $T > 0$ sufficiently small so that $K_0 < \frac{1}{2T}$, there exists a smooth Ricci flow $g(t)$ on $\mathcal{M}^2$ for $t \in [0, T]$, such that $g(0) = g_0$ and $g(t)$ is complete for all $t \in (0, T]$.*

The curvature of $g(t)$ is uniformly bounded above, and $g(t)$ is ***maximally stretched*** *in the sense that if $\big(\tilde{g}(t)\big)_{t\in[0,\tilde{T}]}$ is any Ricci flow on $\mathcal{M}^2$ with $\tilde{g}(0) \leq g(0)$ (with $\tilde{g}(t)$ not necessarily complete or of bounded curvature) then*

$$\tilde{g}(t) \leq g(t) \quad \text{for every } t \in \big[0, \min\{T, \tilde{T}\}\big].$$

From now on we will call a Ricci flow $\big(\mathcal{M}, g(t)\big)_{t\in I}$ where $I = [0,T)$ or $I = [0,T]$ **instantaneously complete**, if $g(t)$ is complete for all $t \in I$ with $t > 0$.

The significance of this class of instantaneously complete Ricci flows lies in the conjecture, also from [Top], that the solution $g(t)$ should be *unique* within this class. See Conjecture 1.5 below.

In this paper we improve Theorem 1.2 in three ways. First, we drop the need for the upper curvature bound on the initial metric; second, we give a precise formula for the maximal existence time in all cases; third we show that the (rescaled) Ricci flow converges to a hyperbolic metric whenever there exists such a metric to which it can converge.

Theorem 1.3. (Main theorem.) *Let $\big(\mathcal{M}^2, g_0\big)$ be a smooth Riemannian surface which need not be complete, and could have unbounded curvature. Depending on the conformal type, we define $T \in (0,\infty]$ by*

$$T := \begin{cases} \frac{1}{8\pi}\operatorname{vol}_{g_0}\mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathcal{S}^2, \\ \frac{1}{4\pi}\operatorname{vol}_{g_0}\mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathbb{C} \text{ or } (\mathcal{M}, g_0) \cong \mathbb{R}P^2, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a smooth Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ such that

1. *$g(0) = g_0$;*
2. *$g(t)$ is instantaneously complete;*
3. *$g(t)$ is maximally stretched,*

and this flow is unique in the sense that if $\big(\tilde{g}(t)\big)_{t\in[0,\tilde{T})}$ is any other Ricci flow on $\mathcal{M}$ satisfying 1,2 and 3, then $\tilde{T} \leq T$ and $\tilde{g}(t) = g(t)$ for all $t \in [0, \tilde{T})$.

If $T < \infty$, then we have

$$\operatorname{vol}_{g(t)}\mathcal{M} = \left\{ \begin{array}{ll} 8\pi(T-t) & \text{if } (\mathcal{M}, g_0) \cong \mathcal{S}^2, \\ 4\pi(T-t) & \text{otherwise,} \end{array} \right\} \quad \longrightarrow \quad 0 \quad \text{as } t \nearrow T,$$

and in particular, T is the maximal existence time. Alternatively, if $\mathcal{M}$ supports a complete hyperbolic metric H conformally equivalent to g_0 (in which case $T = \infty$) then we have convergence of the rescaled solution*

$$\frac{1}{2t} g(t) \longrightarrow H \quad \text{smoothly locally as } t \to \infty.$$

If additionally there exists a constant $M > 0$ such that $g_0 \leq MH$ then the convergence is global: For any $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\eta \in (0,1)$ there exists a constant $C = C(k, \eta, M) > 0$ such that for all $t \geq 1$

$$\left\| \frac{1}{2t} g(t) - H \right\|_{C^k(\mathcal{M}, H)} \leq \frac{C}{t^{1-\eta}} \quad \overset{t\to\infty}{\longrightarrow} \quad 0.$$

*We call a metric H *hyperbolic* if it has constant Gaussian curvature $K[H] \equiv -1$.

In fact, in this latter case, for all $t>0$ we have $\left\| \frac{1}{2t} g(t) - H \right\|_{C^0(\mathcal{M},H)} \le \frac{C}{t}$, and even

$$0 \le \frac{1}{2t} g(t) - H \le \frac{M}{2t} H.$$

Note, for any smooth Riemannian manifold $(\mathcal{M}, g)$ we denote the C^k norm of a smooth tensor field T on $\mathcal{M}$ by

$$\left\| T \right\|_{C^k(\mathcal{M},g)} = \sum_{j=0}^{k} \sup_{\mathcal{M}} \left| \nabla_g^j T \right|_g .$$

The final part of Theorem 1.3 suggests that the Ricci flow has a uniformising effect in great generality. Note that in the special case of *compact* surfaces, an elegant and complete theory[†] has been developed by Hamilton and Chow to this effect:

Theorem 1.4. (Hamilton [Ham88], Chow [Cho91].) *Let $(\mathcal{M}^2, g_0)$ be a smooth, compact Riemannian surface without boundary. Then there exists a unique Ricci flow $g(t)$ with $g(0) = g_0$ for all $t \in [0, T)$ up to a maximal time*

$$T = \begin{cases} \frac{1}{4\pi\chi(\mathcal{M})} \operatorname{vol}_{g_0} \mathcal{M} & \text{if } \chi(\mathcal{M}) > 0 \\ \infty & \text{otherwise,} \end{cases}$$

where $\chi(\mathcal{M})$ denotes the Euler characteristic of $\mathcal{M}$. Moreover, the rescaled solution

$$\begin{cases} \frac{1}{2(T-t)}\, g(t) & \text{if } \chi(\mathcal{M}) > 0 \\ g(t) & \text{if } \chi(\mathcal{M}) = 0 \\ \frac{1}{2t}\, g(t) & \text{if } \chi(\mathcal{M}) < 0 \end{cases}$$

converges smoothly to a conformal metric of constant Gaussian curvature $1, 0, -1$ resp. as $t \to T$.

Although our result proves that the instantaneously complete Ricci flow will always uniformise a manifold of hyperbolic type, irrespective of whether it is complete or not, and Theorem 1.4 proves that it does the same in the spherical case, the asymptotic behaviour is more involved on the remaining manifolds whose universal cover is conformally $\mathbb{C}$. Nevertheless, the techniques of this paper can be applied to that case, for example to address the conjecture in [IJ09, §1].

Theorem 1.3 asserts not just the existence but also the uniqueness of the given Ricci flow, although only within the class of maximally stretched solutions. In fact, the message of [Top] is that uniqueness should hold in the much more general class in which this maximally stretched condition is dropped:

Conjecture 1.5. (Topping [Top].) The solution of Theorem 1.3 is *unique* within the class of instantaneously complete Ricci flows $g(t)$ with $g(0) = g_0$.

This conjecture was partially resolved in [GT10], and in this paper we make further progress. Precisely, we have the following two partial results:

Theorem 1.6. *In the setting of Theorem 1.3, if $(\mathcal{M}, g_0)$ is* not *conformally equivalent to any hyperbolic surface and $\left(\tilde{g}(t)\right)_{t\in[0,\tilde{T})}$ is any instantaneously complete Ricci flow with $\tilde{g}(0) = g_0$, then $\tilde{T} \le T$ and $\tilde{g}(t) = g(t)$ for all $t \in [0, \tilde{T})$.*

[†] For a survey see [CK04, §5] or [Gie07].

We stress that no curvature assumption is made on the competitor $\tilde{g}(t)$ in the theorem above. The theorem can be compared to the results of Chen which would imply this type of strong uniqueness in the case that $(\mathcal{M}, g_0)$ is complete, of bounded curvature and with controlled geometry [Che09].

The following can be viewed as a generalisation of a result from [GT10].

Theorem 1.7. *Let $\big(\mathcal{M}^2, H\big)$ be a complete hyperbolic surface. Suppose $\big(g_1(t)\big)_{t\in[0,T_1]}$ and $\big(g_2(t)\big)_{t\in[0,T_2]}$ are two instantaneously complete Ricci flows on $\mathcal{M}$ which are conformally equivalent to H, with*

(i) $g_1(0) = g_2(0)$;
(ii) *there exists $M > 0$ such that $g_i(0) \leq MH$;*
(iii) *there exists $\varepsilon \in \big(0, \min\{T_1, T_2\}\big]$ such that the curvature of each $g_i(t)$ is bounded above for a short time interval $[0, \varepsilon]$.*

Then $g_1(t) = g_2(t)$ for all $t \in \big[0, \min\{T_1, T_2\}\big]$.

Since the complete uniqueness conjecture has not been fully resolved above in the case that the initial metric is conformally equivalent to some complete hyperbolic metric, it is apriori conceivable that in the case that $(\mathcal{M}, g_0)$ is complete and of bounded curvature, the Ricci flow we construct in Theorem 1.3 could be different from the standard Hamilton-Shi solution of Theorem 1.1. We rule out this possibility in the following theorem.

Theorem 1.8. *Let $\big(\mathcal{M}^2, g_0\big)$ be a complete Riemannian surface with bounded curvature, and let $\big(g(t)\big)_{t\in[0,T)}$ be the corresponding solution constructed in Theorem 1.3. Then $g(t)$ agrees with the Hamilton-Shi Ricci flow of Theorem 1.1 as long as the latter flow exists.*

The combination of Theorem 1.3 and Theorem 1.8 generalises a number of other results which have appeared recently, proved with different techniques:

- In the special case that the initial surface $\big(\mathcal{M}, g_0\big)$ is complete, topologically finite, with negative Euler characteristic $\chi(\mathcal{M}) < 0$, and on each end of $\mathcal{M}$ the initial metric g_0 is asymptotic to a multiple of a hyperbolic cusp metric [JMS09] or of a funnel metric [AAR09], Ji-Mazzeo-Sesum or Albin-Aldana-Rochon resp. show existence and smooth uniform convergence of the normalised Ricci flow to the unique complete metric of constant curvature in the conformal class of g_0. Since the number of cusp or funnel ends is finite and their complement is compact we observe that there exists a conformally equivalent metric H of constant negative curvature with $H \geq g_0$ and we may apply alternatively Theorem 1.3.

- If $\big(\mathcal{M}^2, g_0\big)$ is a complete Riemannian surface with asymptotically conical ends and negative Euler characteristic $\chi(\mathcal{M}) < 0$, then Isenberg-Mazzeo-Sesum show in [IMS10] the existence of a Ricci flow $g(t)$ on $\mathcal{M}$ for all $t \in [0, \infty)$, with $g(0) = g_0$, and smooth local convergence of the rescaled flow $t^{-1}g(t)$ to a complete metric of constant negative curvature and finite area in the conformal class of g_0. This is a special case of Theorem 1.3.

- For an initial metric g_0 on the disc $\mathcal{D}$ which is bounded above and below by positive multiples of the complete hyperbolic metric H, Schnürer-Schulze-Simon show existence and smooth uniform convergence of the normalised Ricci flow to H in [SSS10]. Theorem 1.3 implies that the metrical equivalence can be weakened to $g_0 \leq MH$ for some $M > 0$. Moreover, without this condition we still have smooth local convergence.

The article is organised as follows: After summarising some of the specific properties of the Ricci flow in two dimensions in the following paragraph, we show in Section 2 several apriori estimates for instantaneously complete Ricci flows which have initially a multiple of a hyperbolic metric as an upper barrier. These include sharp barriers (above and below) at later times, the smooth convergence of the rescaled flow, curvature estimates and thus long time existence. The main ingredients here come from Chen's very general apriori estimate for the scalar curvature of a complete Ricci flow [Che09] and Yau's version of the Schwarz Lemma [Yau73]. In Section 3.1 we exploit these properties to improve the existence result from [Top] (Theorem 1.2) on the disc to the case where we might have initially unbounded curvature, by applying the very same Theorem 1.2 locally where we have both bounded curvature and the estimates from Section 2. We also prove Theorem 1.7. In Section 3.2 we prove a lower barrier for Ricci flows on $\mathbb{C}$ only requiring the instantaneous completeness of the flow. This barrier is sufficient to use a comparison principle by Rodriguez-Vazquez-Esteban [RVE97] and gain uniqueness in this class, leading to a proof of Theorem 1.6. Finally, in Section 3.3, we bring all these ingredients together to prove the Main Theorem 1.3.

Ricci flows on surfaces

Since our results address the two-dimensional case, we briefly recall some special features of Ricci flows on surfaces. On a two-dimensional manifold, the Ricci curvature is simply the Gaussian curvature K times the metric: $\mathrm{Ric}[g] = K[g]\, g$. The Ricci flow then moves within a fixed conformal class, and if we pick a local isothermal complex coordinate $z = x + \mathrm{i}y$ and write the metric in terms of a scalar conformal factor $u \in C^\infty(\mathcal{M})$

$$g = \mathrm{e}^{2u}\, |\mathrm{d}z|^2$$

where $|\mathrm{d}z|^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, then the evolution of the metric's conformal factor u under Ricci flow is governed by the nonlinear scalar PDE

$$\frac{\partial}{\partial t} u = \mathrm{e}^{-2u}\, \Delta u = -K[u]. \tag{1.2}$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is defined in terms of the local coordinates and we abuse notation by abbreviating $K[g]$ by $K[u]$.

The definition of a *maximally stretched* Ricci flow can now be viewed as a height-maximality of the conformal factor, i.e. $\mathrm{e}^{2u(t)}\, |\mathrm{d}z|^2$ is maximally stretched if and only if we have $u(t) \geq v(t)$ for any other conformal solution $\mathrm{e}^{2v(t)}\, |\mathrm{d}z|^2$ with $u(0) \geq v(0)$.

There is an independent interest and extensive literature on (1.2) which after the change of variables $v = \mathrm{e}^{2u}$ is called the *logarithmic fast diffusion equation* on $\mathbb{R}^2$:

$$\frac{\partial}{\partial t} v = \Delta \log v. \tag{1.3}$$

In a physical context it models the evolution of the thickness of a thin colloidal film spread over a flat surface if the van der Waals forces are repulsive. For details we refer to [DdP95], [DD96] and [RVE97] plus the references therein. We will appeal to some of these results in the case that the universal cover of the surface is conformally $\mathbb{C}$.

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2 Ricci flows with an upper hyperbolic barrier

In this section we will derive some estimates for Ricci flows mainly under the assumption that they are initially bounded from above by some (possibly large) multiple of a hyperbolic metric. No curvature assumptions will be made on the flow at any time, and yet we derive pointwise and also derivative bounds which will be of fundamental importance in the later proofs. In particular, we apply these estimates even when there is no hyperbolic metric conformally equivalent to the initial metric of the Ricci flow; the trick will be to restrict to smaller compactly contained subdomains where such a metric will exist.

2.1 C^0 bounds

Lemma 2.1. *Let $\big(\mathcal{M}^2, H\big)$ be a complete hyperbolic surface and let $\big(g(t)\big)_{t\in[0,T]}$ be a Ricci flow on $\mathcal{M}$ which is conformally equivalent to H.*

(i) If $g(t)$ is instantaneously complete, then

$$(2t)H \leq g(t) \qquad \text{for all } t \in (0,T]. \tag{2.1}$$

(ii) If there exists a constant $M > 0$ such that $g(0) \leq MH$, then

$$g(t) \leq \big(2t + M\big)H \qquad \text{for all } t \in [0,T]. \tag{2.2}$$

Proof. (i) To establish the lower barrier (2.1) we use Chen's apriori estimate for the scalar curvature (Corollary B.2) to obtain the lower curvature bound $-\frac{1}{2t} \leq K[g(t)]$ for all $t \in (0,T]$. Yau's Schwarz Lemma (Theorem B.3) allows us then to compare $g(t)$ with H, establishing (2.1). (For further details on Yau's result we refer to [GT10, §2].)

(ii) Without loss of generality we may (possibly after lifting to its universal cover) assume $\mathcal{M}^2 = \mathcal{D}$ and write $g(t) = \mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2$. To prove the upper barrier, consider for small $\delta > 0$, $u\big|_{\overline{\mathcal{D}_{1-\delta}}}$ and write the conformal factor of a complete Ricci flow on the disc of radius $1-\delta$ with Gaussian curvature initially $-M^{-1}$ as

$$h_\delta(t,z) := \log\frac{2(1-\delta)}{(1-\delta)^2 - |z|^2} + \frac{1}{2}\log\big(2t + M\big).$$

Note that u is continuous on $[0,T]\times\overline{\mathcal{D}_{1-\delta}}$ and $h_\delta(t,z) \to \infty$ as $z \to \partial\mathcal{D}_{1-\delta}$ for all $t \in [0,T]$. Also, with this choice of M, we have initially $u\big|_{\mathcal{D}_{1-\delta}}(0,\cdot) \leq h_0\big|_{\mathcal{D}_{1-\delta}}(0,\cdot) \leq h_\delta(0,\cdot)$. Therefore the requirements of an elementary comparison principle for the Ricci flow (cf. Theorem A.1) are fulfilled, and we may deduce that $u\big|_{\mathcal{D}_{1-\delta}} \leq h_\delta$ holds throughout $[0,T]\times\mathcal{D}_{1-\delta}$. Since h_δ is continuous in δ, letting $\delta \searrow 0$ yields (2.2). $\square$

2.2 C^k bounds

In this section we bootstrap the estimates of the previous section to obtain estimates for higher derivatives.

Lemma 2.2. *Let $\big(\mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2\big)_{t\in[0,T]}$ be an instantaneously complete Ricci flow on the unit disc $\mathcal{D}$ and for $r \in (0,1]$ let $H_r = \mathrm{e}^{2h_r}\,|\mathrm{d}z|^2$ be the complete hyperbolic metric on the disc $\mathcal{D}_r$ of radius r. If $\mathrm{e}^{2u(0)|_{\mathcal{D}_r}}\,|\mathrm{d}z|^2 \leq MH_r$ for some $M > 0$, then there exists for* *any* $\delta \in \big(0, \min\{r, T\}\big)$ *and* $k \in \mathbb{N}_0$ *a constant* $C_k = C_k(k, r, \delta, M) < \infty$ *such that for all* $t \in [\delta, T]$

$$\Big\| u(t,\cdot) - \frac{1}{2}\log(2t) \Big\|_{C^k(\mathcal{D}_{r-\delta}, |\mathrm{d}z|^2)} \leq C_k. \tag{2.3}$$

Proof. Lemma 2.1 provides the following upper and lower bounds for u,

$$\log \frac{2}{1-|z|^2} + \frac{1}{2}\log(2t) \leq u(t,z) \leq \log \frac{2r}{r^2-|z|^2} + \frac{1}{2}\log\big(2t + M\big) \tag{2.4}$$

for all $t \in (0, T]$ and $z \in \mathcal{D}_r$. Since $\big|u\big|$ cannot be bounded uniformly away from $t = 0$ independently of T, consider the normalised flow $v(t)$ defined by

$$v(t,z) := u(t,z) - s(t) \qquad \text{where} \qquad s(t) := \frac{1}{2}\log(2t),$$

which evolves with the new time scale s according to

$$\begin{aligned} \frac{\partial}{\partial s} v &= \frac{1}{s'(t)} \frac{\partial v}{\partial t} = \frac{1}{s'(t)} \left(\frac{\partial u}{\partial t} - s'(t) \right) \\ &= \frac{1}{s'(t)} \Big(\mathrm{e}^{-2u} \Delta u - s'(t) \Big) = \frac{1}{s'(t)} \Big(\mathrm{e}^{-2s} \mathrm{e}^{-2v} \Delta v - s'(t) \Big) \\ &= \mathrm{e}^{-2v} \Delta v - 1 = \operatorname{div}\big(\mathrm{e}^{-2v} \mathrm{D}v\big) + 2\,\mathrm{e}^{-2v} \big|\mathrm{D}v\big|^2 - 1. \end{aligned} \tag{2.5}$$

From (2.4) we get uniform bounds for $\big|v\big|$ away from $t = 0$, which are independent of T,

$$\log \frac{2}{1-|z|^2} \leq v(t,z) \leq \log \frac{2r}{r^2-|z|^2} + \frac{1}{2}\log \frac{2t+M}{2t}, \tag{2.6}$$

for $z \in \mathcal{D}_r$. Indeed, fixing $\delta \in \big(0, \min\{r, T\}\big]$, there exists a constant $C = C(M, \delta, r) > 0$ such that

$$\sup_{[\delta/2, T] \times \mathcal{D}_{r-\delta/2}} \big|v\big| \leq C < \infty.$$

Thus the evolution equation (2.5) for $v(s)$ is uniformly parabolic on $\big[s(\delta/2), s(T)\big] \times \mathcal{D}_{r-\delta/2}$, which allows us to apply standard parabolic theory (e.g. [LSU68, Theorem V.1.1] to establish Hölder bounds on v and then bootstrap with [LSU68, Theorem IV.10.1]) to obtain, for any $k \in \mathbb{N}_0$, constants $C_k = C_k(k, \delta, C) > 0$ such that

$$\big\| \, v \, \big\|_{C^{k+\frac{\alpha}{2}, 2k+\alpha}\left([\delta, T] \times (\mathcal{D}_{r-\delta}, |\mathrm{d}z|^2)\right)} \leq C_k$$

yielding (2.3). □

Theorem 2.3. *Let* $(\mathcal{M}, H)$ *be a complete hyperbolic surface, and suppose* $\big(g(t)\big)_{t \in [0,T]}$ *is an instantaneously complete Ricci flow on* $\mathcal{M}$ *which is conformally equivalent to* H. *If* $g(0) \leq MH$ *for some constant* $M > 0$, *then for all* $k \in \mathbb{N}_0$ *and any* $\eta \in (0,1)$ *and* $\delta \in (0, T)$ *(however small) there exists a constant* $C = C(k, \eta, \delta, M) < \infty$ *such that for all* $t \in [\delta, T]$ *there holds*

$$\Big\| \frac{1}{2t} g(t) - H \Big\|_{C^k(\mathcal{M}, H)} \leq \frac{C}{t^{1-\eta}}. \tag{2.7}$$

Proof. Fix any point $p \in \mathcal{M}$, and let $\pi : \mathcal{D} \to \mathcal{M}$ be a universal covering of $\mathcal{M}$ with $\pi(0) = p$. Without loss of generality we may then write the pulled back metrics $\pi^* g(t) = \mathrm{e}^{2u(t)} |\mathrm{d}z|^2$ and $\pi^* H = \mathrm{e}^{2h} |\mathrm{d}z|^2$, the latter being the complete hyperbolic metric on the disc. We also have $\mathrm{e}^{2u(0)} \leq M\,\mathrm{e}^{2h}$ by hypothesis.

Using Lemma 2.2 we obtain for every $k \in \mathbb{N}$ constants $C_k' = C_k'(k, \delta, M) > 0$ such that we have uniform C^k-bounds for all $t \in [\delta, T]$

$$\sup_{\mathcal{D}_{1/2}} \left| \mathrm{D}^k \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \right| = \sup_{\mathcal{D}_{1/2}} \left| \mathrm{D}^k \left(\mathrm{e}^{2\left(u(t) - \frac{1}{2} \log 2t\right)} \mathrm{e}^{-2h} \right) \right| \leq C_k'. \tag{2.8}$$

By virtue of Lemma 2.1 there is a much stronger C^0-estimate for all $t \in (0, T]$ on $\mathcal{D}$

$$0 \leq \frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \leq \frac{M}{2t}. \tag{2.9}$$

Now fix $\eta \in (0, 1)$ and combine (2.9) and (2.8) with the interpolation inequality of Lemma B.6 to obtain for every $k \in \mathbb{N}$ constants $C_k'' = C_k''(k, \eta, \delta, M) > 0$ and $l = \lceil k/\eta \rceil$ such that for all $t \in [\delta, T]$

$$\left| \mathrm{D}^k \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \right|_{|\mathrm{d}z|^2} (0) \leq C_k'' t^{-(1-\eta)}. \tag{2.10}$$

Note that the case $k = 0$ of (2.10) is already dealt with by (2.9). Then we estimate using Lemma B.5 (with constant $c = c(k) > 0$) and (2.10) for all $t \in [\delta, T]$

$$\begin{aligned} \left| \nabla_H^k \left(\frac{1}{2t} g(t) - H \right) \right|_H (p) &= \left| \nabla_{\pi^* H}^k \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \pi^* H \right|_{\pi^* H} (0) \\ &\leq c \sum_{j=0}^{k} \left| \mathrm{D}^j \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \right|_{|\mathrm{d}z|^2} (0) \\ &\leq c \sum_{j=0}^{k} C_j'' t^{-(1-\eta)} =: C_k''' t^{-(1-\eta)}. \end{aligned} \tag{2.11}$$

Since $p \in \mathcal{M}$ was chosen arbitrarily and the constants C_k''' are independent of p (and of π), (2.11) holds for all $p \in \mathcal{M}$ and we conclude that with $C = C(k, \eta, \delta, M) > 0$,

$$\left\| \frac{1}{2t} g(t) - H \right\|_{C^k(\mathcal{M}, H)} \leq C\, t^{-(1-\eta)}$$

for all $t \in [\delta, T]$. ∎

2.3 Curvature bounds and long time existence

Proposition 2.4. *Let $(\mathcal{M}^2, H)$ be a complete hyperbolic surface and $\big(g(t)\big)_{t\in[0,T]}$ an instantaneously complete Ricci flow on $\mathcal{M}$, conformally equivalent to H. If $g(0) \leq MH$ for some constant $M > 0$, then for all $\delta \in (0, T]$ there exists a constant $B = B(M, \delta) < \infty$ such that for all $t \in [\delta, T]$ there holds*

$$\big| K[g(t)] \big| \leq \frac{B}{t}.$$

Proof. By Theorem 2.3 there exists a constant $C = C(\delta, M) > 0$ such that for all $t \in [\delta, T]$

$$\big| K[g(t)] \big| = \frac{1}{2t} \left| K\left[\frac{1}{2t} g(t) \right] \right| \leq \frac{C}{2t}.$$

∎

Note that from Theorem 2.3 we even have

$$2t\,K[g(t)] = K\left[\frac{1}{2t}g(t)\right] \longrightarrow -1 \quad \text{uniformly as } t \to \infty.$$

By the work of Shi and Chen-Zhu (Theorem 1.1) we can state Hamilton's long time existence result [Ham82, Theorem 14.1] in the more general setting of complete Ricci flows. Although we only need it on surfaces it is also true in higher dimensions.

Lemma 2.5. *For some $T < \infty$ and $\kappa < \infty$ let $\big(g(t)\big)_{t\in[0,T]}$ be a complete Ricci flow on a manifold $\mathcal{M}^n$ with bounded curvature $\big|\mathrm{Rm}[g(t)]\big| \le \kappa$ for all $t \in [0,T]$. Then there exist constants $\tau = \tau(\kappa, n) > 0$, $\tilde\kappa = \tilde\kappa(\kappa, n) < \infty$ and a smooth complete extension $\big(\tilde g(t)\big)_{t\in[0,T+\tau]}$ such that $\tilde g(t) = g(t)$ for all $t \in [0,T]$ and $\big|\mathrm{Rm}[\tilde g(t)]\big| \le \tilde\kappa$ for all $t \in [T, T+\tau]$.*

Moreover, if for some $\varepsilon > 0$ there exists another such complete extension $\big(\bar g(t)\big)_{t\in[0,T+\varepsilon]}$ with bounded curvature and $g(t) = \bar g(t)$ for all $t \in [0,T]$, then $\tilde g(t) = \bar g(t)$ for all $t \in \big[0, T + \min\{\tau, \varepsilon\}\big]$.

Proof. By Theorem 1.1 of Hamilton-Shi there exist a constant $\tau = \tau(\kappa, n) > 0$ and a complete Ricci flow $\big(\tilde g(t)\big)_{t\in[T,T+\tau]}$ starting at $\tilde g(T) = g(T)$ with bounded curvature $\big|\mathrm{Rm}[\tilde g(t)]\big| \le \tilde\kappa < \infty$ for all $t \in [T, T+\tau]$. Combining both solutions we obtain the desired extension $\big(\tilde g(t)\big)_{t\in[0,T+\tau]}$ by setting $\tilde g(t) = g(t)$ for all $t \in [0,T)$.

If $\big(\bar g(t)\big)_{t\in[0,T+\varepsilon]}$ for some $\varepsilon > 0$ is another complete Ricci flow with bounded curvature extending $g(t)$, then $\bar g(t) = \tilde g(t)$ for all $t \in \big[0, T+\min\{\tau,\varepsilon\}\big]$ by Theorem 1.1 of Hamilton or Chen-Zhu. □

Corollary 2.6. *Let $\big(\mathcal{M}^2, H\big)$ be a complete hyperbolic surface, and $\big(g(t)\big)_{t\in[0,T]}$ an instantaneously complete Ricci flow on $\mathcal{M}$, conformally equivalent to H. If $g(0) \le MH$ for some constant $M > 0$, then there exists a unique instantaneously complete extension of $g(t)$ defined for all time $t \in [0, \infty)$.*

Moreover, if for some $\kappa_0 < \infty$ there holds $K[g(t)] \le \kappa_0$ for all $t \in [0,T]$, then there exists a constant $\kappa = \kappa(M, T, \kappa_0) < \infty$ such that

$$K[g(t)] \le \kappa \qquad \text{for all } t \in [0, \infty). \tag{2.12}$$

Proof. The long-time existence is a direct consequence of the preceding Lemma 2.5 and the apriori curvature bounds of Proposition 2.4: We have for all $t \in \big[T/2, T\big]$

$$\Big|K\big[g(t)\big]\Big| \le \frac{B}{t} \le \frac{2B}{T} =: \kappa_1 < \infty. \tag{2.13}$$

By Lemma 2.5 there exist a $\tau > 0$ depending only on κ_1 and a unique extension $\big(\tilde g(t)\big)_{t\in[0,T+\tau]}$ with the very same curvature bound κ_1 in (2.13) on the larger time interval $\big[T/2, T+\tau\big]$. Iterating these arguments we obtain for any $j \in \mathbb{N}$ an extension $\big(\tilde g(t)\big)_{t\in[0,T+j\tau]}$ with bounded curvature $\big|K[\tilde g(t)]\big| \le \kappa_1$ for all $t \in \big[T/2, T+j\tau\big]$, and therefore we can continue $g(t)$ to exist for all time $t \in [0, \infty)$. Since the curvature is bounded away from $t = 0$, by Lemma 2.5 the extension is also unique among other instantaneously complete extensions.

In order to show the uniform upper bound for the curvature in (2.12), note that (2.13) is true for all $t \in \big[T/2, \infty\big)$. Combining that bound with κ_0 for times $t \in [0, T/2]$, we conclude the theorem with $\kappa := \max\{\kappa_0, \kappa_1\}$. □

3 Existence

3.1 Existence of a maximally stretched solution on the disc $\mathcal{D}$

In this section we prove the main existence and asymptotics result in the case that the Ricci flow starts at a Riemannian surface which is conformally the disc.

Theorem 3.1. *Let $(\mathcal{D}, H)$ be the complete hyperbolic disc and let g_0 be a smooth (possibly incomplete) Riemannian metric on $\mathcal{D}$ which is conformally equivalent to H. Then there exists a unique*[‡]*, maximally stretched and instantaneously complete Ricci flow $g(t)$ for all time $t \in [0, \infty)$ with $g(0) = g_0$, and the rescaled flow converges smoothly locally*

$$\frac{1}{2t} g(t) \longrightarrow H \quad \textit{as} \quad t \to \infty.$$

Moreover, if $g_0 \leq MH$ for some (possibly large) $M > 0$ then the convergence above is global: For any $k \in \mathbb{N}_0$ and $\eta \in (0, 1)$ there is a constant $C = C(k, \eta, M) > 0$ such that

$$\left\| \frac{1}{2t} g(t) - H \right\|_{C^k(\mathcal{M}, H)} \leq \frac{C}{t^{1-\eta}}$$

for $t \geq 1$.

Proof. Without loss of generality, we can write $g_0 = \mathrm{e}^{2u_0} |\mathrm{d}z|^2$ for the initial metric and $H = \mathrm{e}^{2h} |\mathrm{d}z|^2$ for the complete hyperbolic metric on the disc $\mathcal{D}$.

Let $\left(D_j\right)_{j \in \mathbb{N}} \subset \mathcal{D}$ be an exhaustion of $\mathcal{D}$, e.g. $D_j = \mathcal{D}_{1 - \frac{1}{j+1}}$, and define $C_j := \sup_{D_j} u_0 < \infty$ and $\kappa_j^0 := \sup_{D_j} K[g_0]$. Furthermore let $H_j = \mathrm{e}^{2h_j} |\mathrm{d}z|^2$ be the complete hyperbolic metric on the smaller disc D_j. Note that there holds

$$g_0\big|_{D_j} \leq \mathrm{e}^{2C_j} H_j. \tag{3.1}$$

Then for each $j \in \mathbb{N}$, by virtue of Theorem 1.2[§] there exist constants $T_j = T_j(\kappa_j^0) > 0$, $\tilde{\kappa}_j = \tilde{\kappa}_j(T_j, \kappa_j^0) < \infty$, and a maximally stretched and instantaneously complete solution $\left(g_j(t)\right)_{t \in [0, T_j]}$ to the Ricci flow on D_j with $g_j(0) = g_0\big|_{D_j}$, and $K[g(t)] \leq \tilde{\kappa}_j$ for all $t \in [0, T_j]$. Since $g_j(0) \leq \mathrm{e}^{2C_j} H_j$, we may apply Corollary 2.6 to show that each $g_j(t)$ can be extended to exist forever and has a uniform (in t) upper curvature bound $K[g_j(t)] \leq \kappa_j = \kappa_j(C_j, \kappa_j^0) < \infty$. Define $u_j(t)$ such that $\mathrm{e}^{2u_j(t)} |\mathrm{d}z|^2 = g_j(t)$.

Next observe that owing to Lemma A.2, for all $(t, z) \in [0, \infty) \times \mathcal{D}$ and $j \in \mathbb{N}$ sufficiently large such that $z \in D_j$, the sequence $\left(u_j(t, z)\right)_{j \in \mathbb{N}}$ is (weakly) decreasing. Therefore, for any $\Omega \subset\subset \mathcal{D}$ and $T \in (1, \infty)$, u_j is uniformly bounded above (independently of j) on $[0, T] \times \Omega$ for sufficiently large j. On the other hand by Lemma 2.1(i) we know that the conformal factor of the 'big-bang' Ricci flow $(2t)H$ is a lower barrier for each $g_j(t)$ i.e.

$$h\big|_{D_j} + \frac{1}{2} \log(2t) \leq u_j(t) \tag{3.2}$$

for all $t > 0$. Therefore for any $t_0 \in (0, T)$, u_j is uniformly bounded below (independently of j) on $[t_0, T] \times \Omega$. To obtain a uniform lower bound on u_j near $t = 0$, we will follow the

[‡]In fact, if $\tilde{g}(t)$ is any maximally stretched Ricci flow for $t \in [0, T]$ with $\tilde{g}(0) = g_0$, then $\tilde{g}(t) = g(t)$ for all $t \in [0, T]$.

[§]Combining the techniques of this paper and of [GT10] yields a simpler, more direct proof of this theorem.

ideas of [Top] and appeal to the pseudolocality-type result Theorem B.4 of Chen. Since Ω is compact and g_0 is smooth, we can choose $r_0, v_0 > 0$ sufficiently small such that for all $p \in \Omega$ there holds, for sufficiently large j,

(i) $\mathcal{B}_{g_0}(p; r_0) \subset\subset D_j$, in particular $\mathcal{B}_{g_j(t)}(p; r_0) \subset\subset D_j$ for all $t \in [0, T]$;
(ii) $\big|K[g_0]\big| \leq r_0^{-2}$ on $\mathcal{B}_{g_0}(p; r_0)$;
(iii) $\mathrm{vol}_{g_0} \mathcal{B}_{g_0}(p; r_0) \geq v_0 r_0^2$.

Therefore we may apply Theorem B.4 to each such flow $g_j(t)$ and obtain a constant $\tau = \tau(v_0, r_0) \in (0, T]$ such that for sufficiently large j and $t \in [0, \tau]$

$$\Big|K[g_j(t)]\Big| \leq 2r_0^{-2} \quad \text{on } \Omega. \tag{3.3}$$

By inspection of the Ricci flow equation, this gives us a uniform lower bound on u_j on $[0, \tau] \times \Omega$ for sufficiently large j.

Combining these estimates, we find that we have uniform upper and lower bounds for the decreasing sequence u_j on $[0, T] \times \Omega$ (independent of j, for sufficiently large j) and thus we may apply parabolic regularity to get C^k estimates on the functions u_j (uniform in j, for sufficiently large j) on any compact subset of $[0, \infty) \times \mathcal{D}$. Therefore we may define a smooth function $u : [0, \infty) \times \mathcal{D} \to \mathbb{R}$ by

$$u(t, z) := \lim_{j \to \infty} u_j(t, z),$$

and the corresponding metric flow $g(t) := \mathrm{e}^{2u(t)} |\mathrm{d}z|^2$ must be a smooth Ricci flow, defined for all $t \in [0, \infty)$, with $g(0) = g_0$. By (3.2) we also have $(2t)H \leq g(t)$ on $\mathcal{D}$ for all $t > 0$, so $g(t)$ is instantaneously complete.

To see that $g(t)$ is maximally stretched, let $\tilde{u} : [0, \varepsilon] \times \mathcal{D} \to \mathbb{R}$ be the conformal factor of any other Ricci flow with $\tilde{u}(0, \cdot) \leq u_0$, then the maximality of $u_j(t)$ tells us that $\tilde{u}|_{D_j}(t, z) \leq u_j(t, z)$ for all $z \in D_j$ and $t \in [0, \varepsilon]$, and therefore (taking $j \to \infty$) $\tilde{u}(t, z) \leq u(t, z)$ for all $z \in \mathcal{D}$ and $t \in [0, \varepsilon]$. Obviously $g(t)$ is unique amongst maximally stretched solutions (cf. Remark A.5).

We now conclude the proof by showing the asymptotic convergence: For fixed $\Omega \subset\subset \mathcal{D}$ define $\delta := \mathrm{dist}(\Omega, \partial\mathcal{D}) > 0$, and fix $k \in \mathbb{N}$. Using Lemma B.5 (with constant $C' = C'(\delta, k) > 0$) and Lemma 2.2 we establish uniform C^k-bounds on Ω for all $t \geq 1$

$$\begin{aligned}
\sup_\Omega \left|\nabla_H^k \left(\frac{1}{2t} g(t) - H\right)\right|_H &= \sup_\Omega \left|\nabla_H^k \left(\frac{1}{2t} g(t)\right)\right|_H \leq C' \left\|\frac{1}{2t} g(t)\right\|_{C^k(\mathcal{D}_{1-\delta}, |\mathrm{d}z|^2)} \\
&= \sqrt{2} C' \left\|\mathrm{e}^{2\left(u(t) - \frac{1}{2}\log 2t\right)}\right\|_{C^k(\mathcal{D}_{1-\delta}, |\mathrm{d}z|^2)} \\
&\leq C\Big(k, \delta, \sup_{\mathcal{D}_{1-\delta}} u_0\Big).
\end{aligned} \tag{3.4}$$

For any $r \in (0, 1]$, let H_r be the complete hyperbolic metric on the disc $\mathcal{D}_r$ of radius r. Note that for $0 < s \leq r \leq 1$ we have $H\big|_{\mathcal{D}_s} \leq H_r\big|_{\mathcal{D}_s} \leq H_s$. Using Lemma 2.1 we can estimate for any $r \in (1 - \delta, 1)$ with $M(r) = \inf\big\{M > 0 : g_0\big|_{\mathcal{D}_r} \leq M H_r\big\}$ on $\mathcal{D}_r \supset\supset \Omega$

$$0 \leq \frac{1}{2t} g(t) - H = \left(\frac{1}{2t} g(t) - H_r\right) + \big(H_r - H\big) \leq \frac{M(r)}{2t} H_r + \big(H_r - H\big).$$

Therefore

$$\begin{aligned}
\limsup_{t \to \infty} \left\|\frac{1}{2t} g(t) - H\right\|_{C^0(\Omega, H)} &\leq \limsup_{t \to \infty} \frac{M(r)}{2t} \Big\|H_r\Big\|_{C^0(\Omega, H)} + \Big\|H_r - H\Big\|_{C^0(\Omega, H)} \\
&= \Big\|H_r - H\Big\|_{C^0(\Omega, H)} \quad \xrightarrow{r \nearrow 1} \quad 0.
\end{aligned} \tag{3.5}$$

Combining the local uniform C^k-bounds (3.4) with thc C^0-convergence (3.5) we obtain local convergence in the C^k norm

$$\left\| \frac{1}{2t} g(t) - H \right\|_{C^k(\Omega,H)} \quad \overset{t\to\infty}{\longrightarrow} \quad 0.$$

Finally, if $g_0 \leq MH$ for some constant $M > 0$ then the uniform and smooth convergence of $\frac{1}{2t} g(t)$ to H as $t \to \infty$ is a consequence of Theorem 2.3. □

*Proof of **Theorem 1.7**.* By Corollary B.2 and Corollary 2.6 $g_1(t)$ and $g_2(t)$ safisfy all conditions to compare each flow with the maximally stretched solution of Theorem 3.1 using the same proof as [GT10, Theorem 4.1]. □

3.2 Existence and uniqueness on the complex plane $\mathbb{C}$

Whilst Lemma 2.1 gives a good lower barrier for solutions on the disc $\mathcal{D} \subset \mathbb{C}$, it cannot be used directly on the whole plane $\mathbb{C}$, because the plane does not admit a hyperbolic metric. However, the following Theorem provides such a uniform bound in that case by considering the plane with a disc taken off which does admit a hyperbolic metric.

Theorem 3.2. *Let $\big(\mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2\big)_{t\in[0,T]}$ be a smooth, instantaneously complete Ricci flow on the complex plane $\mathbb{C}$. Then there exists a constant $C = C\big(u|_{[0,T]\times\mathcal{D}_2}, T\big) < \infty$ such that for all $|z| \geq 2$ and $t \in (0,T]$ there holds*

$$u(t,z) \geq -C - \log\big(|z|\log|z|\big) + \frac{1}{2}\log(2t).$$

Proof. Pick any cutoff function $\varphi \in C_c^\infty\big(\mathcal{D}_2,[0,1]\big)$ with $\varphi \equiv 1$ in $\mathcal{D}_{3/2}$, and define

$$\alpha := \sup_{[0,T]\times(\mathcal{D}_2\setminus\mathcal{D}_{3/2})} |u| + \big|\mathrm{D}u\big|_{|\mathrm{d}z|^2} + \big|\mathrm{D}^2 u\big|_{|\mathrm{d}z|^2}.$$

Furthermore, let $\mathrm{e}^{2h}\,|\mathrm{d}z|^2$ be the complete hyperbolic metric on $\mathbb{C}\setminus\overline{\mathcal{D}}$, i.e. $h(z) = -\log\big(|z|\log|z|\big)$. Finally, consider an interpolated metric defined by the conformal factor $v \in C^\infty\big([0,T]\times(\mathbb{C}\setminus\overline{\mathcal{D}})\big)$ given by

$$v(t,z) := \varphi(z)\cdot h(z) + \big(1-\varphi(z)\big)\cdot u(t,z).$$

Note that by Corollary B.2, we have $K[u(t)] \geq -\frac{1}{2t}$ for all $t \in (0,T]$. Thus we can estimate the Gaussian curvature of $\mathrm{e}^{2v(t)}\,|\mathrm{d}z|^2$ for all $t \in (0,T]$, by

$$K\big[v(t,z)\big] \geq \begin{Bmatrix} -1 & \text{for } |z| \leq \frac{3}{2}, \text{ i.e. where } v = h \\ -\beta & \text{for } \frac{3}{2} < |z| < 2 \\ -\frac{1}{2t} & \text{for } |z| \geq 2, \text{ i.e. where } v = u \end{Bmatrix} \geq \min\left\{-\frac{1}{2t}, -\beta\right\}$$

where $\beta = \beta(\alpha,\varphi) \geq 1$. Comparing $\mathrm{e}^{2v(t)}\,|\mathrm{d}z|^2$ with $\mathrm{e}^{2h}\,|\mathrm{d}z|^2$ using Theorem B.3 yields for all $(t,z) \in (0,T]\times(\mathbb{C}\setminus\mathcal{D}_2)$

$$u(t,z) = v(t,z) \geq h(z) + \frac{1}{2}\log\Big(\min\big\{2t,\beta^{-1}\big\}\Big) \geq -C - \log\big(|z|\log|z|\big) + \frac{1}{2}\log(2t),$$

defining $C := \max\Big\{\frac{1}{2}\log(2T), 0\Big\} + \frac{1}{2}\log\beta < \infty$. □

Corollary 3.3. *Let $\big(g_1(t)\big)_{t\in[0,T]}$ and $\big(g_2(t)\big)_{t\in[0,T]}$ be two Ricci flows on $\mathbb{C}$ conformally equivalent to $|\mathrm{d}z|^2$. If $g_2(t)$ is instantaneously complete and $g_1(0) \leq g_2(0)$, then $g_1(t) \leq g_2(t)$ for all $t \in [0,T]$.*

Proof. By Theorem 3.2 the conformal factor $u(t)$ of $g_2(t) = \mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2$ satisfies the decay condition (A.1) of the comparison principle Theorem A.7 by Rodriguez, Vazquez and Esteban, and the Corollary's statement follows. □

It is now easy to prove Theorem 1.6 by dividing into the two cases that the universal cover of $(\mathcal{M}, g_0)$ is conformally $\mathcal{S}^2$ or $\mathbb{C}$.

Theorem 3.4. *Let g_0 be a smooth (possibly incomplete) Riemannian metric on the complex plane $\mathbb{C}$, which is conformally equivalent to $|\mathrm{d}z|^2$. Then there exists a unique instantaneously complete Ricci flow $g(t)$ with $g(0) = g_0$ for all $t \in [0,T)$ up to a maximal time*

$$T = \frac{1}{4\pi}\operatorname{vol}_{g_0}\mathbb{C} \leq \infty.$$

Uniqueness here is in the sense that any other instantaneously complete Ricci flow on $\mathbb{C}$ with initial metric g_0 will agree with $g(t)$ while they both exist. Moreover, the flow $g(t)$ is maximally stretched.

Proof. In [DD96] DiBenedetto and Diller showed that if $\operatorname{vol}_{g_0}\mathbb{C} < \infty$, there exists a maximally stretched and instantaneously complete solution $g(t)$ to the Ricci flow for $t \in [0,T)$ up to a maximal time $T = \frac{1}{4\pi}\operatorname{vol}_{g_0}\mathbb{C}$, with volume decaying linearly to zero as t increases to T.

In the case of infinite volume $\operatorname{vol}_{g_0}\mathbb{C} = \infty$ we are going to approximate the solution by a sequence of finite volume solutions: Define a weakly increasing sequence $\big(g_{0,j}\big)_{j\in\mathbb{N}}$ that converges smoothly locally to g_0, but has finite volume $\operatorname{vol}_{g_{0,j}}\mathbb{C} < \infty$ for all $j \in \mathbb{N}$. Then apply for each $j \in \mathbb{N}$ DiBenedetto and Diller's existence theorem to obtain instantaneously complete solutions $g_j(t)$ with $g_j(0) = g_{0,j}$, defined for all $t \in [0,T_j)$ up to a maximal time $T_j = \frac{1}{4\pi}\operatorname{vol}_{g_{0,j}}\mathbb{C} \to \infty$ as $j \to \infty$. These instantaneously complete solutions allow us to use Corollary 3.3 to see that the sequence $\big(g_j(t)\big)_{j\in\mathbb{N}}$ is also weakly increasing for all $j \geq j_0$ such that $t \leq T_{j_0}$. By Lemma 2.1(ii) there is also a uniform upper barrier on compact subsets of $[0,\infty)\times\mathbb{C}$, hence by parabolic regularity theory, the sequence $\big(g_j(t)\big)_{j\in\mathbb{N}}$ converges locally smoothly on $[0,\infty)\times\mathbb{C}$ to a smooth Ricci flow $g(t)$ with $g(0) = g_0$. Since for all $t \in (0,\infty)$ we have $g(t) \geq g_j(t)$ for all j sufficiently large so that $T_j > t$, we find that $g(t)$ is also complete.

The claimed uniqueness and the maximally stretched property are a direct consequence of Corollary 3.3.

An alternative way of constructing $g(t)$ would be to follow the strategy of Theorem 3.1 but choosing D_j to be an exhaustion of $\mathbb{C}$ rather than $\mathcal{D}$. This gives an instantaneously complete solution which by uniqueness must agree with the solution above. □

3.3 Proof of Theorems 1.3 and 1.8

*Proof of **Theorem 1.3**.* Let $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ be the universal covering of $\mathcal{M}$ with the lifted metric $\tilde{g}_0 = \pi^* g_0$, and let $\Gamma < \operatorname{Isom}(\tilde{\mathcal{M}}, \tilde{g}_0)$ be the discrete subgroup of the isometry group, isomorphic to $\pi_1(\mathcal{M})$, such that $\mathcal{M} \cong \tilde{\mathcal{M}}/\Gamma$. Then as is implicit throughout the paper, by virtue of the uniformisation theorem, $\big(\tilde{\mathcal{M}}, \tilde{g}_0\big)$ is conformally equivalent either to the sphere $\mathcal{S}^2$, to the complex plane $\mathbb{C}$ or to the open unit disc $\mathcal{D}$.

Therefore by Theorems 1.4, 3.4 and 3.1 resp. there exists an instantaneously complete and maximally stretched Ricci flow $\big(\tilde{g}(t)\big)_{t\in[0,T)}$ on $\tilde{\mathcal{M}}$ with $\tilde{g}(0) = \tilde{g}_0$ up to a maximal time

$$T = \begin{cases} \frac{1}{8\pi} \operatorname{vol}_{\tilde{g}_0} \tilde{\mathcal{M}} & \text{if } \tilde{\mathcal{M}} \cong \mathcal{S}^2, \\ \frac{1}{4\pi} \operatorname{vol}_{\tilde{g}_0} \tilde{\mathcal{M}} & \text{if } \tilde{\mathcal{M}} \cong \mathbb{C}, \\ \infty & \text{if } \tilde{\mathcal{M}} \cong \mathcal{D}. \end{cases}$$

By Lemma A.6, Γ acts by isometries on $\big(\tilde{\mathcal{M}}, \tilde{g}(t)\big)$ for every $t \in [0, T)$, so we may quotient $\tilde{g}(t)$ to obtain uniquely a maximally stretched and instantaneously complete solution $g(t) = \pi_* \tilde{g}(t)$ on $\mathcal{M} = \tilde{\mathcal{M}}/\Gamma$ for all $t \in [0, T)$ with $g(0) = g_0$.

Finally, using the relation $|\Gamma| \cdot \operatorname{vol}_{g_0} \mathcal{M} = \operatorname{vol}_{\tilde{g}_0} \tilde{\mathcal{M}}$ we are going to phrase the maximal time T in terms of $\big(\mathcal{M}, g_0\big)$ by distinguishing the only cases:

$$T = \begin{cases} \frac{1}{8\pi} \operatorname{vol}_{g_0} \mathcal{M} < \infty & \text{if } \tilde{\mathcal{M}} \cong \mathcal{S}^2 \text{ and } |\Gamma| = 1 \quad \Longrightarrow \quad \mathcal{M} \cong \mathcal{S}^2 \\ \frac{1}{4\pi} \operatorname{vol}_{g_0} \mathcal{M} < \infty & \text{if } \tilde{\mathcal{M}} \cong \mathcal{S}^2 \text{ and } |\Gamma| = 2 \quad \Longrightarrow \quad \mathcal{M} \cong \mathbb{R}P^2 \\ \frac{1}{4\pi} \operatorname{vol}_{g_0} \mathcal{M} \leq \infty & \text{if } \tilde{\mathcal{M}} \cong \mathbb{C} \text{ and } |\Gamma| = 1 \quad \Longrightarrow \quad \mathcal{M} \cong \mathbb{C} \\ \infty & \text{if } \tilde{\mathcal{M}} \cong \mathbb{C} \text{ and } |\Gamma| = \infty \\ \infty & \text{if } \tilde{\mathcal{M}} \cong \mathcal{D}. \end{cases}$$

The local convergence in the hyperbolic case follows also from Theorem 3.1: For convergence on an arbitrary ball $\mathcal{B}_H(p; r)$ in $\big(\mathcal{M}, H\big)$ choose a point $\tilde{p} \in \pi^{-1}(p)$ and consider the ball $\mathcal{B}_{\pi^* H}(\tilde{p}; r) \subset \mathcal{D}$. Then the local smooth convergence of $\frac{1}{2t} g(t)$ to H on $\mathcal{B}_H(p; r)$ is a consequence of Theorem 3.1 which we can apply to show smooth convergence of $\frac{1}{2t}\tilde{g}(t)$ to $\pi^* H$ on $\mathcal{B}_{\pi^* H}(\tilde{p}; r)$. The global convergence in the case that $g_0 \leq MH$ for some $M > 0$ is the statement of Theorem 2.3. □

Last we provide the proof that in the case of a complete initial surface with bounded curvature our solution does not differ from the Hamilton-Shi Ricci flow.

*Proof of **Theorem 1.8**.* Since $g(t)$ is the unique maximally stretched solution, it agrees with the solution from Theorem 1.2 which has a uniform upper bound to the curvature. On the other hand, Chen's apriori estimate Theorem B.1 provides also a uniform lower bound to the curvature. From [Shi89] or [Ham82] we know that $\tilde{g}(t)$ is complete and of bounded curvature. Therefore by Theorem A.4 $g(t)$ and $\tilde{g}(t)$ must coincide. □

A Comparison principles

In this appendix we state different comparison principles and some direct consequences. We start with an elementary one whose proof can be found in [GT10].

Theorem A.1. [GT10, Theorem A.1] *Let $\Omega \subset \mathbb{C}$ be an open, bounded domain and for some $T > 0$ let $u \in C^{1,2}\big((0,T)\times\Omega\big) \cap C\big([0,T]\times\bar{\Omega}\big)$ and $v \in C^{1,2}\big((0,T)\times\Omega\big) \cap C\big([0,T]\times\Omega\big)$ both be solutions of the Ricci flow equation (1.2) for the conformal factor of the metric. Furthermore, suppose that for each $t \in [0, T]$ we have $v(t, z) \to \infty$ as $z \to \partial\Omega$. If $v(0, z) \geq u(0, z)$ for all $z \in \Omega$, then $v \geq u$ on $[0, T] \times \Omega$.*

As a direct consequence we can show that Ricci flows on discs of different sizes stay ordered.

Lemma A.2. *For $0 < r < R < \infty$ let $\big(g_1(t)\big)_{t\in[0,T]}$ on $\mathcal{D}_R$ and $\big(g_2(t)\big)_{t\in[0,T]}$ on $\mathcal{D}_r$ be two solutions to the Ricci flow which are conformally equivalent on $\mathcal{D}_r$ and satisfy*

(i) $g_1(0)\big|_{\mathcal{D}_r} \leq g_2(0)$,
(ii) $g_2(t)$ *is complete for all* $t \in (0,T]$,
(iii) there exists a constant $\kappa \in (0,\infty)$ *such that* $K[g_2(t)] \leq \kappa$ *for all* $t \in [0,T]$.

Then $g_1(t)\big|_{\mathcal{D}_r} \leq g_2(t)$ *for all* $t \in [0,T)$.

Proof. Without loss of generality write $g_1(t) = \mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2$ and $g_2(t) = \mathrm{e}^{2v(t)}\,|\mathrm{d}z|^2$ with $u(t) \in C^\infty(\mathcal{D}_R)$ and $v(t) \in C^\infty(\mathcal{D}_r)$. For any $\delta \in (0,T)$ define

$$v_\delta(t,z) := v\big(\mathrm{e}^{-2\kappa\delta}(t+\delta), z\big) + \kappa\delta \qquad \text{for } (t,z) \in [0, T-\delta] \times \mathcal{D}_r,$$

which is a slight adjustment of v, again a solution to the Ricci flow (1.2)

$$\left(\frac{\partial}{\partial t} v_\delta - \mathrm{e}^{-2v_\delta}\,\Delta v_\delta\right)(t,z) = \mathrm{e}^{-2\kappa\delta}\left(\frac{\partial}{\partial t} v - \mathrm{e}^{-2v}\,\Delta v\right)\big(\mathrm{e}^{-2\kappa\delta}(t+\delta), z\big) = 0.$$

In order to compare $u\big|_{\overline{\mathcal{D}_r}}$ and v_δ, we are going to check the requirements of Theorem A.1: For the conformal factor of the restricted metric holds $u\big|_{\overline{\mathcal{D}_r}} \in C\big([0,T] \times \overline{\mathcal{D}_r}\big)$. From (ii) follows with Lemma 2.1(i) for all $t \in [0, T-\delta]$

$$v_\delta(t,z) \geq \log\frac{2r}{r^2 - |z|^2} + \frac{1}{2}\log 2(t+\delta) \to \infty \quad \text{as} \quad z \to \partial\mathcal{D}_r.$$

Finally to check the initial condition use the uniform upper bound κ for the curvature of v from (iii) to integrate (1.2), and we may estimate

$$v_\delta(0,z) = v\big(\mathrm{e}^{-2\kappa\delta}\,\delta, z\big) + \kappa\delta \geq v(0,z) - \kappa\,\mathrm{e}^{-2\kappa\delta}\,\delta + \kappa\delta \overset{(i)}{\geq} u(0,z)$$

for all $z \in \mathcal{D}_r$. Thus by Theorem A.1 there holds $v_\delta(t,z) \geq u(t,z)$ for all $(t,z) \in [0, T-\delta] \times \mathcal{D}_r$ and all $\delta \in (0,T)$. Given any $(t,z) \in [0,T) \times \mathcal{D}_r$, we may conclude

$$u(t,z) \leq \lim_{\delta \searrow 0} v_\delta(t,z) = v(t,z).$$

□

The following more geometrical comparison principle from [GT10] will us allow to give a simple proof of the uniqueness of complete Ricci flows with bounded curvature on surfaces.

Theorem A.3. [GT10, Theorem 4.2] *For some $T > 0$ let $\big(g_1(t)\big)_{t\in[0,T]}$ and $\big(g_2(t)\big)_{t\in[0,T]}$ be two conformally equivalent Ricci flows on a surface $\mathcal{M}^2$, and define $Q : [0,T] \times \mathcal{M} \to \mathbb{R}$ to be the function for which $g_1(t) = \mathrm{e}^{2Q(t)}\,g_2(t)$. Suppose further that $g_2(t)$ is complete for each $t \in [0,T]$ and that for some constant $C \geq 0$ we have*

(i) $\big|K[g_2]\big| \leq C$, *(ii)* $K[g_1] \leq C$, *(iii)* $Q \leq C$

on $[0,T] \times \mathcal{D}$. If $g_1(0) \leq g_2(0)$, then $g_1(t) \leq g_2(t)$ for all $t \in [0,T]$.

The contribution of Chen-Zhu [CZ06] to Theorem 1.1 was the uniqueness in the complete case. In our situation where the underlying manifold is two-dimensional, this is far simpler to prove than in the general case; the statement and proof are as follows.

Theorem A.4. *Let $\big(g_1(t)\big)_{t\in[0,T]}$ and $\big(g_2(t)\big)_{t\in[0,T]}$ be two complete Ricci flows on a surface $\mathcal{M}^2$, with uniformly bounded curvature. If $g_1(0) = g_2(0)$, then $g_1(t) = g_2(t)$ for all $t \in [0, T]$.*

Proof. With respect to a local complex coordinate z, let us write $g_1(t) = \mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2$ and $g_2(t) = \mathrm{e}^{2v(t)}\,|\mathrm{d}z|^2$ for some locally defined functions $u(t)$ and $v(t)$, and define $Q := u - v$ globally. Observe that Q is uniformly bounded since $Q = 0$ at $t = 0$ and the curvature of $g_1(t)$ and $g_2(t)$ is uniformly bounded

$$\left|\frac{\partial}{\partial t}Q\right| = \Big|K[u_2] - K[u_1]\Big| \le C < \infty \qquad \Longrightarrow \qquad \Big|Q\Big| \le CT.$$

Therefore we may apply Theorem A.3 twice to obtain $g_1(t) \le g_2(t)$ and $g_2(t) \le g_1(t)$ for all $t \in [0, T]$. □

We will also require an obvious uniqueness property of maximally stretched solutions:

Remark A.5. Let $\big(g(t)\big)_{t\in[0,T]}$ and $\big(\tilde g(t)\big)_{t\in[0,\tilde T]}$ be two conformally equivalent and maximally stretched Ricci flows on $\mathcal{M}^2$ with $g(0) = \tilde g(0)$. Then $g(t) = \tilde g(t)$ for all $t \in \big[0, \min\{T, \tilde T\}\big]$.

One application of this uniqueness property is the preservation of the isometry group under a maximally stretched Ricci flow:

Lemma A.6. *Let $\big(g(t)\big)_{t\in[0,T]}$ be a maximally stretched Ricci flow on $\mathcal{M}^n$. Then the isometry group does not shrink under the flow:* $\mathrm{Isom}\big(\mathcal{M}, g(0)\big) \subset \mathrm{Isom}\big(\mathcal{M}, g(t)\big)$ *for all $t \in [0, T]$.*

Proof. Pick any $\phi \in \mathrm{Isom}\big(\mathcal{M}, g(0)\big)$. Since the Ricci flow is invariant under diffeomorphisms, $\phi^* g(t)$ is again a solution to the Ricci flow with $\phi^* g(0) = g(0)$ for all $t \in [0, T]$. Because $g(t)$ is maximally stretched, we have $\phi^* g(t) \le g(t)$ and also $(\phi^{-1})^* g(t) \le g(t)$ for all $t \in [0, T]$. Pulling back the latter inequality by ϕ yields $g(t) \le \phi^* g(t)$. Therefore $g(t) = \phi^* g(t)$ and $\phi \in \mathrm{Isom}\big(\mathcal{M}, g(t)\big)$ for all $t \in [0, T]$. □

The last comparison principle is a result from the theory of the logarithmic fast diffusion equation on $\mathbb{C}$.

Theorem A.7. (Variant of Rodriguez-Vazquez-Esteban [RVE97, Corollary 2.3].) *Let $\big(\mathrm{e}^{2u(t)}\,|\mathrm{d}z|^2\big)_{t\in[0,T]}$ and $\big(\mathrm{e}^{2v(t)}\,|\mathrm{d}z|^2\big)_{t\in[0,T]}$ be two Ricci flows on the plane $\mathbb{C}$. If there exists $C < \infty$ such that $u(t)$ satisfies the decay condition*

$$u(t, z) \ge -C - \log\big(|z|\log|z|\big) + \frac{1}{2}\log(2t) \quad \text{for all } (t, z) \in (0, T) \times \mathbb{C} \tag{A.1}$$

and $u(0) \ge v(0)$, then $u(t) \ge v(t)$ for all $t \in [0, T]$.

B Further supporting results

In [Che09] Chen proves a very general apriori estimate for the scalar curvature of a Ricci flow without requiring anything but the completeness of the solution.

Theorem B.1. (Chen [Che09, Corollary 2.3(i)].) *Let $\big(g(t)\big)_{t\in[0,T]}$ be a smooth complete Ricci flow on a manifold $\mathcal{M}^n$. If $R[g(0)] \geq -\kappa$ for some $\kappa \in [0,\infty]$, then*

$$R\big[g(t)\big] \geq -\frac{n}{2t + \frac{n}{\kappa}} \qquad \text{for all } t \in [0,T].$$

A direct transfer to our situation where we have an instantaneously complete solution on surface is:

Corollary B.2. *Let $\big(g(t)\big)_{t\in[0,T]}$ be a smooth instantaneously complete Ricci flow on a surface $\mathcal{M}^2$. Then*

$$K\big[g(t)\big] \geq -\frac{1}{2t} \qquad \text{for all } t \in (0,T].$$

For further applications and a simple proof of the following special case of the Schwarz lemma of Yau, see [GT10].

Theorem B.3. [Yau73] *Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian surfaces without boundary. If*

(i) $(\mathcal{M}_1, g_1)$ *is complete,*
(ii) $K[g_1] \geq -a_1$ *for some number* $a_1 \geq 0$, *and*
(iii) $K[g_2] \leq -a_2 < 0$,

then any conformal map $f : \mathcal{M}_1 \to \mathcal{M}_2$ satisfies

$$f^*(g_2) \leq \frac{a_1}{a_2} g_1.$$

A more elaborate argument of Chen leads to the following pseudolocality-type result giving 2-sided estimates on the curvature.

Theorem B.4. (Chen [Che09, Proposition 2.9].) *Let $\big(g(t)\big)_{t\in[0,T]}$ be a smooth Ricci flow on a surface $\mathcal{M}^2$. If we have for some $p \in \mathcal{M}$, $r_0 > 0$ and $v_0 > 0$*

(i) $\mathcal{B}_{g(t)}(p; r_0) \subset\subset \mathcal{M}$ *for all* $t \in [0,T]$;
(ii) $\Big|K[g(0)]\Big| \leq r_0^{-2}$ *on* $\mathcal{B}_{g(0)}(p; r_0)$;
(iii) $\mathrm{vol}_{g(0)}\, \mathcal{B}_{g(0)}(p; r_0) \geq v_0 r_0^2$,

then there exists a constant $C = C(v_0) > 0$ such that for all $t \in \big[0, \min\{T, \frac{1}{C} r_0^2\}\big]$

$$\Big|K[g(t)]\Big| \leq 2r_0^{-2} \qquad \text{on } \mathcal{B}_{g(t)}\Big(p; \frac{r_0}{2}\Big).$$

Occasionally we will have to switch between equivalent metrics in arguments, and will use the following elementary fact:

Lemma B.5. *Let $\big(\mathcal{D}, H\big)$ be the complete hyperbolic disc and $T \in \Gamma\big(\mathsf{T}^{(r,s)}\mathcal{D}\big)$ any (r,s) tensor field. Then for every $k \in \mathbb{N}_0$ and $\varrho \in (0,1)$ there exists a constant $C = C(k,\varrho,r,s) > 0$ such that*

$$\frac{1}{C}\big\|T\big\|_{C^k(\mathcal{D}_\varrho, |\mathrm{d}z|^2)} \leq \big\|T\big\|_{C^k(\mathcal{D}_\varrho, H)} \leq C\big\|T\big\|_{C^k(\mathcal{D}_\varrho, |\mathrm{d}z|^2)}.$$

In particular, we have

$$\frac{1}{C}\sum_{j=0}^{k}\left|\nabla^j_{|\mathrm{d}z|^2}T\right|_{|\mathrm{d}z|^2}(0)\le\sum_{j=0}^{k}\left|\nabla^j_{H}T\right|_{H}(0)\le C\sum_{j=0}^{k}\left|\nabla^j_{|\mathrm{d}z|^2}T\right|_{|\mathrm{d}z|^2}(0).$$

In order to bootstrap C^0 convergence into C^k convergence (using C^l bounds) we will need to be able to interpolate:

Lemma B.6. *Let $u:\mathbb{B}^n\to[-1,1]$ be a smooth function such that for all $k\in\mathbb{N}$*

$$\left\|\mathrm{D}^k u\right\|_{L^\infty(\mathbb{B})}<\infty.$$

Then for all $k\in\mathbb{N}$ and $\eta\in(0,1)$ there exist constants $C=C(k,\eta)>0$ and $l:=\lceil k/\eta\rceil$ such that

$$\left|\mathrm{D}^k u\right|(0)\le C\left(1+\left\|\mathrm{D}^l u\right\|_{L^\infty(\mathbb{B})}\right)\|u\|_{L^\infty(\mathbb{B})}^{1-\eta}. \tag{B.1}$$

Proof. By [GT01, Theorem 7.28] (for example) for arbitrary $0<k<l$ there is a constant $C=C(l)>0$ such that for any smooth function $v\in C^\infty(\mathbb{B})\cap L^\infty(\mathbb{B})$ with bounded derivatives

$$\left\|\mathrm{D}^k v\right\|_{L^\infty(\mathbb{B})}\le C\left(\|v\|_{L^\infty(\mathbb{B})}+\left\|\mathrm{D}^l v\right\|_{L^\infty(\mathbb{B})}\right). \tag{B.2}$$

Now define $v(x):=u(\varepsilon x)$ for all $x\in\mathbb{B}$ with $\varepsilon=\|u\|_{L^\infty(\mathbb{B})}^{\frac{1}{l}}\in(0,1]$ and choosing $l=\lceil k/\eta\rceil$ we estimate with (B.2)

$$\begin{aligned}\left\|\mathrm{D}^k u\right\|_{L^\infty(\mathbb{B}_\varepsilon)}=\varepsilon^{-k}\left\|\mathrm{D}^k v\right\|_{L^\infty(\mathbb{B})}&\le C\varepsilon^{-k}\left(\|v\|_{L^\infty(\mathbb{B})}+\left\|\mathrm{D}^l v\right\|_{L^\infty(\mathbb{B})}\right)\\&\le C\left(\varepsilon^{-k}\|u\|_{L^\infty(\mathbb{B}_\varepsilon)}+\varepsilon^{l-k}\left\|\mathrm{D}^l u\right\|_{L^\infty(\mathbb{B}_\varepsilon)}\right)\\&\le C\left(1+\left\|\mathrm{D}^l u\right\|_{L^\infty(\mathbb{B})}\right)\|u\|_{L^\infty(\mathbb{B})}^{1-\frac{k}{l}}\\&\le C\left(1+\left\|\mathrm{D}^l u\right\|_{L^\infty(\mathbb{B})}\right)\|u\|_{L^\infty(\mathbb{B})}^{1-\eta}.\end{aligned}$$

□

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