

Decay estimates for Rivière's equation, with applications to regularity and compactness

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Abstract

We derive a selection of energy estimates for a generalisation of a critical equation on the unit disc in \mathbb{R}^2 introduced by Rivière. Applications include sharp regularity results and compactness theorems which generalise a large amount of previous geometric PDE theory, including some of the theory of harmonic and almost-harmonic maps from surfaces.

1 Introduction

Suppose $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution to

$$-\Delta u = \Omega \cdot \nabla u \tag{1}$$

where here and throughout this paper B_1 is the unit disc in \mathbb{R}^2 , $\Omega \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$, and we are using the notation $[\Omega \cdot \nabla u]^i = \langle \Omega_j^i, \nabla u^j \rangle$. This equation, first considered in this generality by Rivière [12], generalises a number of interesting equations appearing naturally in geometry, including the harmonic map equation, the H -surface equation and, more generally, the Euler-Lagrange equation of any conformally invariant elliptic Lagrangian which is quadratic in the gradient. A central issue is the regularity of u implied by virtue of it satisfying the equation (1). *A priori*, the right-hand side of the equation looks like quite a general L^1 function, and standard elliptic regularity theory does not seem to help. However, Rivière [12] showed that any solution must necessarily be continuous and even in $W^{2,1+\epsilon}$ for some $\epsilon > 0$ [13], thus generalising the famous regularity theory of Hélein [8], for example. In most known interesting special cases of this equation, one happens to know that $|\Omega|$ can be estimated linearly in terms of $|\nabla u|$, i.e. we have $|\Omega \cdot \nabla u| \leq C|\nabla u|^2$ and then a standard bootstrapping argument can be applied to improve the regularity of u to $W^{2,p}$ for any $p < \infty$. Moreover typically in such cases Ω can be viewed as a smooth function of u and ∇u , at which point we can conclude

that u is smooth using Schauder theory (and the embedding $W^{2,p}(\mathbb{R}^2) \hookrightarrow C^{1,1-\frac{2}{p}}(\mathbb{R}^2)$ for $p > 2$).

In this paper we investigate what sort of regularity and compactness properties we can deduce for solutions of the general equation (1), and even more general inhomogeneous equations with the same special structure. It is easy to convince oneself that it is unreasonable to expect regularity better than $W^{2,2}$ in general. However, we will show that we *do* have regularity up to this level, or the best possible regularity when there is an inhomogeneity.

Theorem 1.1. *Suppose that $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution on the unit disc in \mathbb{R}^2 to*

$$-\Delta u = \Omega \cdot \nabla u + f, \quad f \in L^p(B_1, \mathbb{R}^m) \quad (2)$$

where $\Omega \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$ and $p \in (1, 2)$. Then $u \in W_{loc}^{2,p}(B_1)$. In particular, if $f \equiv 0$, then $u \in W_{loc}^{2,p}$ for all $p \in [1, 2)$ and $u \in W_{loc}^{1,q}$ for all $q \in [1, \infty)$.

Moreover, for $U \subset\subset B_1$, there exist $\eta_0 = \eta_0(p, m) > 0$ and $C = C(p, m, U) < \infty$ so that if $\|\Omega\|_{L^2(B_1)} \leq \eta_0$ then

$$\|u\|_{W^{2,p}(U)} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}). \quad (3)$$

This theorem omits the borderline case $p = 2$ for good reason; even in the case that $f \equiv 0$, one can find solutions so that u is neither $W^{2,2}$ nor Lipschitz. Moreover, examples with $f \equiv 0$ show that the first derivatives of u need not even lie in BMO , and (consequently) the second derivatives need not even lie in the Lorentz space $L^{2,\infty}$ (see Appendices A.2 and A.3 for definitions if necessary).

As a corollary of our theorem, we see that $f \in L^p$ implies that u lies in $C^{0,2(1-\frac{1}{p})}$, hence recovering a result of Rupflin [14] in the case of two-dimensional domains. Rivière has informed us that our regularity assertion in the particular case $f \equiv 0$ will also be made in the final version of [13], based on a different proof.

We remark that the estimate (3) fails without the smallness of Ω hypothesis. More precisely, there exist a sequence $\Omega_k \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$ uniformly bounded in L^2 , and a sequence of weak solutions $u_k \in W^{1,2}(B_1, \mathbb{R}^m)$ to the equation

$$-\Delta u_k = \Omega_k \cdot \nabla u_k,$$

uniformly bounded in $W^{1,2}$, such that u_k is unbounded in any $W^{2,p}$ space with $p \in (1, 2)$. (A sequence of harmonic maps undergoing bubbling would provide an example, for instance the sequence given in Section 8.)

Estimate (3) implies that for any sequence $\Omega_k \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$ with $\|\Omega_k\|_{L^2(B_1)} \leq \eta_0$, and any sequence of weak solutions $u_k \in W^{1,2}(B_1, \mathbb{R}^m)$ to the equation

$$-\Delta u_k = \Omega_k \cdot \nabla u_k + f_k,$$

with u_k uniformly bounded in $W^{1,2}$ and f_k uniformly bounded in some space L^p for $p \in (1, 2)$, we may deduce that u_k is locally uniformly bounded in $W^{2,p}$. By the theorem of Rellich-Kondrachov, we can deduce that u_k is precompact in $W^{1,t}(B_{1/2})$ for any $t < \frac{2p}{2-p}$.

In this paper, we work somewhat harder to prove a stronger compactness result, extending a recent theorem of Li and Zhu [11], in which we assume merely that the inhomogeneous terms f_k are bounded in $L \ln L$ (a space larger than any of the L^p spaces with $p > 1$, but slightly smaller than L^1 ; see Appendix A.3 for more information on this space, and definitions, if necessary).

Theorem 1.2 (Compactness). *Suppose that we have a sequence $\{u_n\} \subset W^{1,2}(B_1, \mathbb{R}^m)$ of weak solutions to*

$$-\Delta u_n = \Omega_n \cdot \nabla u_n + f_n$$

on the unit disc in \mathbb{R}^2 , where $\{\Omega_n\} \subset L^2(B_1, so(m) \otimes \mathbb{R}^2)$ and $\{f_n\} \subset L \ln L(B_1, \mathbb{R}^m)$. Suppose also that there exists $\Lambda < \infty$ such that

$$\|u_n\|_{L^1(B_1)} + \|f_n\|_{L \ln L(B_1)} \leq \Lambda.$$

Then there exist an $\eta_2 = \eta_2(m) > 0$ and $u \in W_{loc}^{1,2}(B_1, \mathbb{R}^m)$ such that if $\|\Omega_n\|_{L^2(B_1)} \leq \eta_2$ then after passing to a subsequence

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{W^{1,2}(B_{1/2})} = 0.$$

We will show in Section 8 that this result fails if we replace $L \ln L$ by the related local Hardy space h^1 (see Appendix A.2). In the special case that $\{\Omega_n\}$ is a precompact set in L^1 , and u_n is uniformly bounded in $W^{1,2}$, this result was proved recently by Li and Zhu [11].

Remark 1.3. The compactness result is ruling out concentration of energy as is done in [11] - i.e. concentration of $\|\nabla u_n\|_{L^2}^2$. In contrast, we do not rule out concentration of $\|\nabla u_n\|_{L^{2,1}}^2$ or of the corresponding second order quantity $\|\nabla^2 u_n\|_{L^1}$. However, it will follow from our estimates (and in particular, (6) below) that if these latter concentrations occur we must have f_n concentrating in $L \ln L$.

Even in the classical case that $\Omega \equiv 0$ there is a consequence of such compactness which may be worth remarking, although one which would follow from previously known theory.

Corollary 1.4. *On the ball in \mathbb{R}^2 , the embedding*

$$L \ln L(B_1) \hookrightarrow H^{-1}(B_1)$$

is compact, where H^{-1} is the dual space of $W_0^{1,2}$.

In contrast, the example given in Section 8 also serves to show that the embedding $h^1(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$ is *not* compact, which has been pointed out previously by Yuxin Ge [6, Remark 4.4]. (Strictly speaking, the example given in Section 8 and that of [6, Remark 4.4] show that the embedding $\mathcal{H}^1(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$ is not compact, but since $\mathcal{H}^1(\mathbb{R}^2) \hookrightarrow h^1(\mathbb{R}^2)$, this is sufficient - see Appendix A.2.)

At the heart of this paper is a collection of energy/decay estimates which we summarise in the following theorem.

Theorem 1.5 (Main supporting theorem). *Suppose $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution to*

$$-\Delta u = \Omega \cdot \nabla u + f, \quad f \in L \ln L(B_1, \mathbb{R}^m) \quad (4)$$

on the unit disc in \mathbb{R}^2 , where $\Omega \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$. Writing $\bar{u} = \frac{1}{|B_1|} \int_{B_1} u$,

1. *there exist $\eta = \eta(m) > 0$ and $K_1 = K_1(m) < \infty$ such that if $\|\Omega\|_{L^2(B_1)} \leq \eta$, then for all $r \in (0, 1/2]$ we have*

$$\|\nabla u\|_{L^2(B_r)}^2 \leq K_1 \left(\|\Omega\|_{L^2(B_1)}^2 \|\nabla u\|_{L^2(B_1)}^2 + r^2 \|u - \bar{u}\|_{L^1(B_1)}^2 + \|f\|_{L^1(B_1)} \|f\|_{L \ln L(B_1)} \right) \quad (5)$$

and

$$\|\nabla^2 u\|_{L^1(B_r)} \leq K_1 \left(\|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + r^2 \|u - \bar{u}\|_{L^1(B_1)} + \|f\|_{L \ln L(B_1)} \right); \quad (6)$$

2. *for all $\delta > 0$ there exist $\eta = \eta(m, \delta) > 0$ and $K_2 = K_2(m, \delta) < \infty$ such that if $\|\Omega\|_{L^2(B_1)} \leq \eta$, then for all $r \in (0, 1]$ we have*

$$\begin{aligned} \|\nabla u\|_{L^2(B_r)}^2 &\leq (1 + \delta) r^2 \|\nabla u\|_{L^2(B_1)}^2 + \\ &+ K_2 \left(\|\Omega\|_{L^2(B_1)}^2 \|\nabla u\|_{L^2(B_1)}^2 + \|f\|_{L^1(B_1)} \|f\|_{L \ln L(B_1)} \right). \end{aligned} \quad (7)$$

Although we will not need it in this work, we note that the first part of the theorem will also yield estimates for ∇u in the Lorentz space $L^{2,1}$ (by the embedding $W^{1,1} \hookrightarrow L^{2,1}$).

The estimates of the first part of the theorem are interior estimates which have the weakest norms of u on the right-hand side. By combining them with a standard covering argument, we will also derive the following optimal global estimate:

Theorem 1.6. *With u and f as in Theorem 1.5 and $U \subset\subset B_1$ there exist an $\eta_1 = \eta_1(m) > 0$ and $C = C(m, U) < \infty$ such that if $\|\Omega\|_{L^2(B_1)} \leq \eta_1$, then*

$$\|u\|_{W^{2,1}(U)} \leq C \left(\|u\|_{L^1(B_1)} + \|f\|_{L \ln L(B_1)} \right).$$

The second part of Theorem 1.5 is used to obtain both the regularity result Theorem 1.1 and the compactness result Theorem 1.2.

We remark that Theorems 1.1, 1.2, 1.5 and 1.6 all fail if we drop the antisymmetry hypothesis on Ω .

The paper is laid out as follows. In Section 3 we prove the main supporting Theorem 1.5, which is central to the other results, then we go on to prove Theorem 1.6 in Section 4. This allows us to prove the compactness Theorem 1.2 in Section 5 and its corollary in Section 6. We leave the regularity Theorem 1.1 to Section 7 and finally in Section 8 we give an example to show that the compactness result fails if we replace $L \ln L$ by the local Hardy space h^1 .

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2 Preliminaries

To begin, we describe some properties of the space $L \ln L$, and record the behaviour of the equations (2), (4) and various norms under scaling. In addition, we have collected a number of known results in an appendix.

2.1 Estimates for $L \ln L$

For the definition of $L \ln L$ and f^* , see Appendix A.3.

Lemma 2.1. *Suppose $f \in L \ln L(B_r(x_0))$ and $r \in (0, 1/2]$. Then there exists $C < \infty$ such that*

$$\|f\|_{L^1(B_r(x_0))} \leq C \left[\ln \left(\frac{1}{r} \right) \right]^{-1} \|f\|_{L \ln L(B_r(x_0))}.$$

Proof. Notice that

$$\begin{aligned} 0 &\leq r^2 \int_0^{|B_1|} f^*(r^2 t) \ln \left(2 + \frac{1}{t} \right) dt \\ &= \int_0^{|B_r(x_0)|} f^*(s) \ln \left(2 + \frac{r^2}{s} \right) ds \\ &= \int_0^{|B_r(x_0)|} f^*(s) \ln r^2 ds + \int_0^{|B_r(x_0)|} f^*(s) \ln \left(\frac{2}{r^2} + \frac{1}{s} \right) ds \\ &\leq -2 \ln \left(\frac{1}{r} \right) \|f\|_{L^1(B_r(x_0))} + C \|f\|_{L \ln L(B_r(x_0))}, \end{aligned}$$

where the final inequality is obtained by noticing that $s \leq \pi r^2 < 1$ which implies $\frac{2}{r^2} + \frac{1}{s} \leq \frac{2\pi+1}{s} \leq \left(2 + \frac{1}{s}\right)^C$ for some fixed C . \square

The following lemma indicates that $L \ln L$ norms do not deteriorate under scaling. However we emphasise that they need not improve, unlike L^p norms for $p > 1$.

Lemma 2.2. Suppose $f \in L \ln L(B_r(x_0))$ where $r \in (0, 1/2]$. Defining $\hat{f} := r^2 f(x_0 + rx)$ there exists $C < \infty$ such that

$$\|\hat{f}\|_{L \ln L(B_1)} \leq C \|f\|_{L \ln L(B_r(x_0))}.$$

Proof. First we calculate

$$\begin{aligned} \hat{f}^*(t) &= \inf\{s \geq 0 : |\{x \in B_1 : |\hat{f}(x)| > s\}| \leq t\} \\ &= \inf\{s \geq 0 : |\{x \in B_1 : |r^2 f(x_0 + rx)| > s\}| \leq t\} \\ &= \inf\{s \geq 0 : |\{y \in B_r(x_0) : |f(y)| > \frac{s}{r^2}\}| r^{-2} \leq t\} \\ &= r^2 f^*(r^2 t) \end{aligned}$$

therefore

$$\begin{aligned} \|\hat{f}\|_{L \ln L(B_1)} &\leq C \int_0^{|B_1|} \hat{f}^*(t) \ln \left(2 + \frac{1}{t}\right) dt \\ &= C \int_0^{|B_1|} r^2 f^*(r^2 t) \ln \left(2 + \frac{1}{t}\right) dt \\ &= C \int_0^{|B_r(x_0)|} f^*(s) \ln \left(2 + \frac{r^2}{s}\right) ds \\ &\leq C \|f\|_{L \ln L(B_r(x_0))}. \end{aligned}$$

□

2.2 Scaling

There will be several occasions when we will require estimates on some small ball $B_{R/2}(x_0)$ in terms of quantities on the ball $B_R(x_0)$. In this section, we make a note of what scaling we will be taking, and how each relevant quantity, and the equation itself, behave under this operation.

Let u be a solution to (2) or (4). For $x_0 \in B_1$, let $R > 0$ be such that $B_R(x_0) \subset B_1$. Now we rescale u by defining $\hat{u}(x) := u(x_0 + Rx)$, $\hat{f}(x) := R^2 f(x_0 + Rx)$ and $\hat{\Omega}(x) := R\Omega(x_0 + Rx)$.

We have $\hat{u} \in W^{1,2}(B_1, \mathbb{R}^m)$ and

$$\begin{aligned} -\Delta \hat{u}(x) &= -R^2 \Delta u(x_0 + Rx) \\ &= R\Omega(x_0 + Rx) \cdot R\nabla u(x_0 + Rx) + R^2 f(x_0 + Rx) \\ &= \hat{\Omega}(x) \cdot \nabla \hat{u}(x) + \hat{f}(x), \end{aligned}$$

i.e. the same equation as before. The quantities of which we will need to keep track are:

1. $\|\nabla \hat{u}\|_{L^p(B_r)} = R^{1-\frac{2}{p}} \|\nabla u\|_{L^p(B_{rR}(x_0))}$ for any $r \in [0, 1]$

2. $\|\hat{\Omega}\|_{L^2(B_1)} = \|\Omega\|_{L^2(B_R(x_0))}$
3. $\|\hat{u}\|_{L^p(B_1)} = R^{-\frac{2}{p}}\|u\|_{L^p(B_R(x_0))}$
4. $\|\hat{f}\|_{L^p(B_1)} = R^{2(1-\frac{1}{p})}\|f\|_{L^p(B_R(x_0))}$
5. $\|\hat{f}\|_{L \ln L(B_1)} \leq C\|f\|_{L \ln L(B_R(x_0))}$

where the final estimate is following from Lemma 2.2.

3 Proof of the decay estimates, Theorem 1.5

Most of the work in the proof will be common to both parts of the theorem. We will be referring to the δ of the second part with the understanding that in the case of the first part, we could just set $\delta = 1$.

We start off with $\eta = \epsilon$, taken from Lemma A.4, and will assume throughout that $\|\Omega\|_{L^2(B_1)} \leq \eta$, with the understanding that the upper bound η will be lowered at finitely many points during the proof. For our weak solution u to (4) corresponding to Ω , we will assume, without loss of generality, that $\bar{u} = \frac{1}{|B_1|} \int_{B_1} u = 0$.

To begin with we use Rivière's decomposition of Ω (Lemma A.4) in order to rewrite the equation (4) (equations (10) and (11) below). Lemma A.4 gives us $A \in W^{1,2}(B_1, GL_m(\mathbb{R})) \cap L^\infty(B_1, GL_m(\mathbb{R}))$, $B \in W^{1,2}(B_1, gl_m(\mathbb{R}))$ and $C = C(m) < \infty$ so that

$$\nabla A - A\Omega = \nabla^\perp B \quad (8)$$

and

$$\|\nabla A\|_{L^2(B_1)} + \|\nabla B\|_{L^2(B_1)} + \|dist(A, SO(m))\|_{L^\infty(B_1)} \leq C\|\Omega\|_{L^2(B_1)}. \quad (9)$$

Now by (8) we have

$$\begin{aligned} \operatorname{div}(A\nabla u) &= \nabla A \cdot \nabla u + A\Delta u \\ &= \nabla A \cdot \nabla u - A\Omega \cdot \nabla u - Af \\ &= \nabla^\perp B \cdot \nabla u - Af \end{aligned} \quad (10)$$

and trivially

$$\operatorname{curl}(A\nabla u) = \nabla^\perp A \cdot \nabla u. \quad (11)$$

We note here that the above equations only hold in a weak sense, and more care should be taken in their calculation. We illustrate this for (10): *A priori* $\operatorname{div}(A\nabla u)$ is a distri-

bution, so for $\phi \in C_c^\infty(B_1)$ we have

$$\begin{aligned}
\operatorname{div}(A\nabla u)[\phi] &= - \int_{B_1} A\nabla u \cdot \nabla \phi \\
&= \int_{B_1} (\nabla A \cdot \nabla u)\phi - \nabla(\phi A) \cdot \nabla u \\
&= \int_{B_1} (\nabla A \cdot \nabla u)\phi - (A\Omega \cdot \nabla u)\phi - Af\phi \quad \text{since } u \text{ weakly solves (4)} \\
&= \int_{B_1} (\nabla^\perp B \cdot \nabla u - Af)\phi = (\nabla^\perp B \cdot \nabla u - Af)[\phi].
\end{aligned}$$

We will now essentially carry out a Hodge decomposition of $A\nabla u$ in B_1 using the expressions (10) and (11). We first extend all the quantities arising above to functions on \mathbb{R}^2 .

Let $Ex : W^{1,2}(B_1) \rightarrow W^{1,2}(\mathbb{R}^2) \cap W_0^{1,2}(B_2)$ be a bounded extension operator. Denote $\tilde{u} = Ex(u) \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^m)$ and note that since we are assuming $\int_{B_1} u = 0$, by the Poincaré inequality and by standard properties of Ex we have

$$\|\tilde{u}\|_{W^{1,2}(\mathbb{R}^2)} \leq C\|u\|_{W^{1,2}(B_1)} \leq C\|\nabla u\|_{L^2(B_1)}$$

and $u = \tilde{u}$ in B_1 .

For A , first let $\hat{A} = A - \frac{1}{|B_1|} \int_{B_1} A$ and $\tilde{A} = Ex(\hat{A}) \in W^{1,2}(\mathbb{R}^2, gl_m(\mathbb{R}))$. Noting that $\int_{B_1} \hat{A} = 0$ and using the same argument as for u we have

$$\|\tilde{A}\|_{W^{1,2}(\mathbb{R}^2)} \leq C\|\nabla A\|_{L^2(B_1)}$$

here we have used that $\nabla \tilde{A} = \nabla \hat{A} = \nabla A$ in B_1 . Notice also that $\tilde{A}\nabla \tilde{u} + \left(\frac{1}{|B_1|} \int_{B_1} A\right) \nabla \tilde{u} = A\nabla u$ in B_1 .

We carry out the same extension for B to get \tilde{B} as above for A . We extend f by zero (without relabelling), so by Appendix A.3, $f \in h^1(\mathbb{R}^2)$ with $\|f\|_{h^1(\mathbb{R}^2)} \leq C\|f\|_{L \ln L(B_1)}$.

Now we define

$$\begin{aligned}
D &:= N[\nabla^\perp \tilde{B} \cdot \nabla \tilde{u}], \\
E &:= N[\nabla^\perp \tilde{A} \cdot \nabla \tilde{u}], \\
F &:= -N[Af],
\end{aligned}$$

where N is the Newtonian potential (see Appendix A.1). Note that the quantity Af is well defined on the whole of \mathbb{R}^2 by the definition of f . Finally let

$$H := \tilde{A}\nabla \tilde{u} + \left(\frac{1}{|B_1|} \int_{B_1} A\right) \nabla \tilde{u} - \nabla D - \nabla F - \nabla^\perp E.$$

The first thing to notice about H is that

$$H = A\nabla u - \nabla D - \nabla F - \nabla^\perp E \quad (12)$$

in B_1 . Hence we have

$$\operatorname{div}(H) = \operatorname{div}(A\nabla u) - \Delta(D + F) = \operatorname{div}(A\nabla u) - \nabla^\perp \tilde{B} \cdot \nabla \tilde{u} + Af = 0$$

weakly in B_1 , and a similar calculation shows $\operatorname{curl}(H) = 0$ weakly in B_1 . (Again care must be taken in checking these.) Therefore H is harmonic in B_1 (i.e. corresponds to a harmonic 1-form).

Suppose $r \in (0, 1]$. (For some estimates later it will need to be less than $\frac{1}{2}$.)

Without loss of generality, we may assume that $\delta \in (0, 1]$. (Recall that when addressing the first part of the theorem, we are just setting $\delta = 1$.) For η small enough, depending on δ , we may assume (by (9)) that A is close to a special-orthogonal matrix in the sense that both A and A^{-1} change the length of any vector by at most a factor of $1 + \delta$. Therefore

$$\begin{aligned} \|\nabla u\|_{L^2(B_r)}^2 &\leq (1 + \delta)^2 \|A\nabla u\|_{L^2(B_r)}^2 \\ &\leq (1 + 3\delta) \|A\nabla u\|_{L^2(B_r)}^2 \\ &\leq (1 + 4\delta) \|H\|_{L^2(B_r)}^2 + C(\|\nabla D\|_{L^2(B_r)}^2 + \|\nabla F\|_{L^2(B_r)}^2 + \|\nabla E\|_{L^2(B_r)}^2), \end{aligned} \quad (13)$$

where C is dependent on δ . In order to obtain the inequalities of Theorem 1.5 we estimate $\|H\|_{L^2(B_r)}$, $\|\nabla D\|_{L^2(B_r)}$, $\|\nabla F\|_{L^2(B_r)}$ and $\|\nabla E\|_{L^2(B_r)}$.

First we consider $\nabla D = \nabla N[\nabla^\perp \tilde{B} \cdot \nabla \tilde{u}]$ and $\nabla E = \nabla N[\nabla^\perp \tilde{A} \cdot \nabla \tilde{u}]$. Notice that by the work of Coifman-Lions-Meyer-Semmes [2] and the fact that $\nabla N : \mathcal{H}^1(\mathbb{R}^2) \rightarrow L^{2,1}(B_1)$ is a bounded linear operator (see Appendix A.4) we have,

$$\begin{aligned} \|\nabla D\|_{L^2(B_1)} + \|\nabla E\|_{L^2(B_1)} &\leq C(\|\nabla D\|_{L^{2,1}(B_1)} + \|\nabla E\|_{L^{2,1}(B_1)}) \\ &= C\left(\|\nabla N[\nabla^\perp \tilde{B} \cdot \nabla \tilde{u}]\|_{L^{2,1}(B_1)} + \|\nabla N[\nabla^\perp \tilde{A} \cdot \nabla \tilde{u}]\|_{L^{2,1}(B_1)}\right) \\ &\leq C\left(\|\nabla^\perp \tilde{B} \cdot \nabla \tilde{u}\|_{\mathcal{H}^1(\mathbb{R}^2)} + \|\nabla^\perp \tilde{A} \cdot \nabla \tilde{u}\|_{\mathcal{H}^1(\mathbb{R}^2)}\right) \\ &\leq C(\|\nabla B\|_{L^2(B_1)} + \|\nabla A\|_{L^2(B_1)}) \|\nabla u\|_{L^2(B_1)} \\ &\leq C\|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} \end{aligned} \quad (14)$$

where we have also used the continuous embedding $L^{2,1} \hookrightarrow L^2$ and the estimate from Lemma A.4.

For $\nabla F = -\nabla N[Af]$ we use (Appendix A.4) that the Riesz potential $\nabla N : L^1(B_1) \rightarrow L^{2,\infty}(B_1)$ is a bounded operator; also $\nabla N : h^1(\mathbb{R}^2) \rightarrow L^{2,1}(B_1)$ is bounded. We will also use the following: $L^{2,\infty}$ is the dual of $L^{2,1}$; if $f \in L \ln L(B_1)$ then for any $g \in L^\infty$, $gf \in L \ln L(B_1)$ and $\|gf\|_{L \ln L(B_1)} \leq \|g\|_{L^\infty} \|f\|_{L \ln L(B_1)}$ and finally we use the continuous

embedding $L \ln L(B_1) \hookrightarrow h^1(\mathbb{R}^2)$ (see Appendix A.3). We have

$$\begin{aligned}
\|\nabla F\|_{L^2(B_1)}^2 &\leq C\|\nabla F\|_{L^{2,\infty}(B_1)}\|\nabla F\|_{L^{2,1}(B_1)} \\
&\leq C\|Af\|_{L^1(B_1)}\|Af\|_{h^1(\mathbb{R}^2)} \\
&\leq C\|f\|_{L^1(B_1)}\|Af\|_{L \ln L(B_1)} \\
&\leq C\|f\|_{L^1(B_1)}\|f\|_{L \ln L(B_1)}.
\end{aligned} \tag{15}$$

Also, using merely the boundedness of $\nabla N : L^1(B_1) \rightarrow L^{2,\infty}(B_1) \hookrightarrow L^1(B_1)$ (Appendix A.4), we have

$$\|\nabla F\|_{L^1(B_1)} \leq C\|f\|_{L^1(B_1)}. \tag{16}$$

From here, we proceed differently in order to prove the two different parts of the theorem. For the first part, we now estimate $\|H\|_{L^1(B_{2/3})}$ and apply standard estimates for harmonic functions in order to estimate $\|H\|_{L^2(B_r)}$: Using Lemma A.2, and estimates (14) and (16), we have

$$\begin{aligned}
\|H\|_{L^1(B_{2/3})} &\leq C\left(\|\nabla u\|_{L^1(B_{2/3})} + \|\nabla D\|_{L^1(B_{2/3})} + \|\nabla E\|_{L^1(B_{2/3})} + \|\nabla F\|_{L^1(B_{2/3})}\right) \\
&\leq C\left(\|u\|_{L^1(B_1)} + \|f\|_{L^1(B_1)} + \|\Omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}\right).
\end{aligned} \tag{17}$$

Since H is harmonic we have pointwise estimates on H and its derivatives on the interior of $B_{2/3}$ in terms of $\|H\|_{L^1(B_{2/3})}$, and in particular

$$\begin{aligned}
\|H\|_{L^\infty(B_{1/2})} + \|\nabla H\|_{L^\infty(B_{1/2})} &\leq C\|H\|_{L^1(B_{2/3})} \\
&\leq C\left(\|u\|_{L^1(B_1)} + \|f\|_{L^1(B_1)} + \|\Omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}\right)
\end{aligned}$$

by (17). Therefore if we consider $r \in (0, \frac{1}{2}]$, then

$$\begin{aligned}
\|H\|_{L^2(B_r)}^2 &\leq \pi r^2\|H\|_{L^\infty(B_r)}^2 \\
&\leq Cr^2\left(\|u\|_{L^1(B_1)}^2 + \|f\|_{L^1(B_1)}^2 + \|\Omega\|_{L^2(B_1)}^2\|\nabla u\|_{L^2(B_1)}^2\right),
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\|\nabla H\|_{L^1(B_r)} &\leq \pi r^2\|\nabla H\|_{L^\infty(B_r)} \\
&\leq Cr^2\left(\|u\|_{L^1(B_1)} + \|f\|_{L^1(B_1)} + \|\Omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}\right).
\end{aligned} \tag{19}$$

Now, looking back at inequality (13) and using (14), (15) and (18) we have

$$\|\nabla u\|_{L^2(B_r)}^2 \leq C\left(\|\Omega\|_{L^2(B_1)}^2\|\nabla u\|_{L^2(B_1)}^2 + r^2\|u\|_{L^1(B_1)}^2 + \|f\|_{L^1(B_1)}\|f\|_{L \ln L(B_1)}\right)$$

which is the first inequality that we seek from the first part of the theorem.

In order to get the second estimate (6) of the first part of the theorem, we return to the Hodge decomposition (12) which tells us that

$$\nabla u = A^{-1}(H + \nabla D + \nabla F + \nabla^\perp E)$$

in B_1 , and therefore

$$\nabla^2 u = \nabla A^{-1} \cdot (H + \nabla D + \nabla F + \nabla^\perp E) + A^{-1} (\nabla H + \nabla^2 D + \nabla^2 F + \nabla \nabla^\perp E),$$

with

$$\begin{aligned} \|\nabla^2 u\|_{L^1(B_r)} &\leq \|\nabla A^{-1} \cdot (H + \nabla D + \nabla F + \nabla^\perp E)\|_{L^1(B_r)} \quad (=I) \\ &\quad + \|A^{-1} (\nabla H + \nabla^2 D + \nabla^2 F + \nabla \nabla^\perp E)\|_{L^1(B_r)}. \quad (=II) \end{aligned} \quad (20)$$

Using (9), (14), (15) and (18) we have (assuming also $\eta < 1$)

$$\begin{aligned} I &\leq \|\nabla A^{-1}\|_{L^2(B_r)} \|H + \nabla D + \nabla E + \nabla^\perp F\|_{L^2(B_r)} \\ &\leq C (r \|\Omega\|_{L^2(B_1)} \|u\|_{L^1(B_1)} + \|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + \|f\|_{L \ln L(B_1)}). \end{aligned} \quad (21)$$

Now we use the fact that the operators $\nabla^2 N : h^1(\mathbb{R}^2) \rightarrow L^1(B_1)$ and $\nabla^2 N : \mathcal{H}^1(\mathbb{R}^2) \rightarrow L^1(\mathbb{R}^2)$ are bounded (see Appendix A.2) and the estimates (9) and (19) to conclude

$$\begin{aligned} II &\leq \|A^{-1}\|_{L^\infty(B_r)} \|\nabla H + \nabla^2 D + \nabla^2 F + \nabla \nabla^\perp E\|_{L^1(B_r)} \\ &\leq C (r^2 \|u\|_{L^1(B_1)} + \|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + \|f\|_{L \ln L(B_1)}). \end{aligned} \quad (22)$$

Since we have assumed without loss of generality that $\int_{B_1} u = 0$, by an application of the Poincaré inequality and looking back at (20), (21) and (22) we have

$$\|\nabla^2 u\|_{L^1(B_r)} \leq C (\|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + r^2 \|u\|_{L^1(B_1)} + \|f\|_{L \ln L(B_1)}),$$

as desired.

For the second part of the theorem, we return to a general $r \in (0, 1]$. We will now control H using the standard decay estimate

$$\|H\|_{L^2(B_r)}^2 \leq r^2 \|H\|_{L^2(B_1)}^2$$

which holds since $|H|^2$ is subharmonic.

Then using (13) and (14) and (15) again, we find that

$$\begin{aligned} \|\nabla u\|_{L^2(B_r)}^2 &\leq (1 + 4\delta) \|H\|_{L^2(B_r)}^2 + C (\|\nabla D\|_{L^2(B_r)}^2 + \|\nabla F\|_{L^2(B_r)}^2 + \|\nabla E\|_{L^2(B_r)}^2) \\ &\leq (1 + 4\delta) r^2 \|H\|_{L^2(B_1)}^2 + C (\|\nabla D\|_{L^2(B_1)}^2 + \|\nabla F\|_{L^2(B_1)}^2 + \|\nabla E\|_{L^2(B_1)}^2) \\ &\leq (1 + 5\delta) r^2 \|A \nabla u\|_{L^2(B_1)}^2 + C (\|\nabla D\|_{L^2(B_1)}^2 + \|\nabla F\|_{L^2(B_1)}^2 + \|\nabla E\|_{L^2(B_1)}^2) \\ &\leq (1 + 5\delta) (1 + \delta)^2 r^2 \|\nabla u\|_{L^2(B_1)}^2 + C (\|\nabla D\|_{L^2(B_1)}^2 + \|\nabla F\|_{L^2(B_1)}^2 + \|\nabla E\|_{L^2(B_1)}^2) \\ &\leq (1 + 100\delta) r^2 \|\nabla u\|_{L^2(B_1)}^2 + C \left(\|\Omega\|_{L^2(B_1)}^2 \|\nabla u\|_{L^2(B_1)}^2 + \|f\|_{L^1(B_1)} \|f\|_{L \ln L(B_1)} \right) \end{aligned}$$

Thus, by repeating the argument with δ reduced by a factor of 100, we conclude the proof.

4 Proof of the $W^{2,1}$ estimate, Theorem 1.6

We can say immediately that the η_1 whose existence is claimed in the theorem can be chosen as $\eta_1^2 = \min\left\{\eta^2, \frac{\epsilon_0}{K_1}\right\}$, where ϵ_0 is that given in Lemma A.7 corresponding to $k = 4$, and K_1 and η are from the first part of Theorem 1.5.

We would like to rescale the first estimate (5) of the first part of Theorem 1.5, in the case that $r = \frac{1}{2}$. Indeed, adopting the notation of Section 2.2, we know that

$$\begin{aligned}\|\nabla \hat{u}\|_{L^2(B_{1/2})}^2 &\leq K_1 \left(\|\hat{\Omega}\|_{L^2(B_1)}^2 \|\nabla \hat{u}\|_{L^2(B_1)}^2 + \|\hat{u}\|_{L^1(B_1)}^2 + \|\hat{f}\|_{L^1(B_1)} \|\hat{f}\|_{L \ln L(B_1)} \right) \\ &\leq K_1 \|\hat{\Omega}\|_{L^2(B_1)}^2 \|\nabla \hat{u}\|_{L^2(B_1)}^2 + C \left(\|\hat{u}\|_{L^1(B_1)}^2 + \|\hat{f}\|_{L \ln L(B_1)}^2 \right)\end{aligned}$$

and (again by Section 2.2) this translates to

$$\|\nabla u\|_{L^2(B_{R/2}(x_0))}^2 \leq K_1 \|\Omega\|_{L^2(B_R(x_0))}^2 \|\nabla u\|_{L^2(B_R(x_0))}^2 + C \left(R^{-4} \|u\|_{L^1(B_R(x_0))}^2 + \|f\|_{L \ln L(B_R(x_0))}^2 \right).$$

Using our upper bound for η and the fact that $R \leq 1$ we have, in particular

$$\|\nabla u\|_{L^2(B_{R/2}(x_0))}^2 \leq \epsilon_0 \|\nabla u\|_{L^2(B_R(x_0))}^2 + CR^{-4} \left(\|u\|_{L^1(B_1)}^2 + \|f\|_{L \ln L(B_1)}^2 \right).$$

Letting $\Gamma = C \left(\|u\|_{L^1(B_1)}^2 + \|f\|_{L \ln L(B_1)}^2 \right)$ we are precisely in the set-up of Lemma A.7, since this estimate is true in particular for all $B_{2R}(x_0) \subset B_1$. Therefore

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C \left(\|u\|_{L^1(B_1)} + \|f\|_{L \ln L(B_1)} \right). \quad (23)$$

It remains to improve this estimate to control the second derivatives, and for that we use the second estimate (6) of the first part of Theorem 1.5, in the case $r = \frac{1}{2}$, which we then scale by a factor $\frac{1}{2}$ to give:

$$\|\nabla^2 u\|_{L^1(B_{1/4})} \leq C \left(\|\Omega\|_{L^2(B_{1/2})} \|\nabla u\|_{L^2(B_{1/2})} + \|u\|_{L^1(B_{1/2})} + \|f\|_{L \ln L(B_{1/2})} \right).$$

Combining with (23) then yields

$$\|\nabla^2 u\|_{L^1(B_{1/4})} \leq C \left(\|u\|_{L^1(B_1)} + \|f\|_{L \ln L(B_1)} \right),$$

and a simple rescaling and covering argument gives us that for any compactly contained $U \subset\subset B_1$ there is a $C = C(U, m) < \infty$ such that

$$\|u\|_{W^{2,1}(U)} \leq C \left(\|u\|_{L^1(B_1)} + \|f\|_{L \ln L(B_1)} \right).$$

5 Proof of the compactness, Theorem 1.2

Here we pick $\eta_2 = \min\{\eta_1, \eta, \sqrt{\frac{1}{2K_2}}\}$ where η_1 is from Theorem 1.6, and η and K_2 are from the second part of Theorem 1.5 for the choice $\delta = 1$. We know (by Theorem

1.6) that for all $U \subset\subset B_1$, our sequence $\{u_n\}$ is uniformly bounded in $W^{2,1}(U)$, so by the Sobolev embedding theorem there exists some $u \in W_{loc}^{1,2}(B_1)$ such that (up to a subsequence) $u_n \rightharpoonup u$ weakly in $W^{1,2}(B_{2/3})$. We also know that $\{\nabla u_n\}$ is uniformly bounded in $W^{1,1}(B_{2/3})$, so by Lemma A.6 (with $\nabla u_n = V_n$) if we have

$$\lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(B_r(x))} = 0 \quad (24)$$

for all $x \in B_{2/3}$, then

$$\nabla u_n \rightarrow \nabla u$$

strongly in $L_{loc}^2(B_{2/3})$ which would prove the theorem. Therefore, it remains to prove (24).

Now pick $x_0 \in B_{2/3}$ and $R \in (0, 1/2]$ small enough such that $B_R(x_0) \subset B_{2/3}$. Applying the second part of Theorem 1.5 to the rescaled scenario from Section 2.2 (for each n) yields (for $r \in (0, 1]$)

$$\|\nabla \hat{u}_n\|_{L^2(B_r)}^2 \leq K_2 \|\hat{\Omega}_n\|_{L^2(B_1)}^2 \|\nabla \hat{u}_n\|_{L^2(B_1)}^2 + 2r^2 \|\nabla \hat{u}_n\|_{L^2(B_1)}^2 + K_2 \|\hat{f}_n\|_{L^1(B_1)} \|\hat{f}_n\|_{L \ln L(B_1)}$$

and reversing the scaling leaves us with

$$\begin{aligned} \|\nabla u_n\|_{L^2(B_{rR}(x_0))}^2 &\leq K_2 \|\Omega_n\|_{L^2(B_R(x_0))}^2 \|\nabla u_n\|_{L^2(B_R(x_0))}^2 \\ &\quad + 2r^2 \|\nabla u_n\|_{L^2(B_R(x_0))}^2 + K_2 \|f_n\|_{L^1(B_R(x_0))} \|f_n\|_{L \ln L(B_R(x_0))} \\ &\leq \frac{1}{2} \|\nabla u_n\|_{L^2(B_R(x_0))}^2 \\ &\quad + C \left(r^2 \|\nabla u_n\|_{L^2(B_R(x_0))}^2 + \left[\ln \left(\frac{1}{R} \right) \right]^{-1} \|f_n\|_{L \ln L(B_R(x_0))}^2 \right) \end{aligned}$$

using Lemma 2.1.

Now, using that $\{u_n\}$ is uniformly bounded in $W^{2,1}(B_{2/3})$ and the hypotheses of the theorem, we have

$$\|\nabla u_n\|_{L^2(B_{rR}(x_0))}^2 \leq 1/2 \|\nabla u_n\|_{L^2(B_R(x_0))}^2 + C \left(r^2 + \left[\ln \left(\frac{1}{R} \right) \right]^{-1} \right)$$

Hence

$$\lim_{R \downarrow 0} \lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(B_{rR}(x_0))}^2 \leq 1/2 \lim_{R \downarrow 0} \lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(B_R(x_0))}^2$$

and we have shown that

$$\lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(B_r(x_0))} = 0$$

which proves the theorem.

6 Proof of the compact embedding $L \ln L \hookrightarrow H^{-1}$, Corollary 1.4

In this section we use $\mathcal{H}^1 - BMO$ duality (Appendix A.2 and [5]), the compactness result Theorem 1.2 and the continuous embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$ (Appendix A.4) to prove the compactness of the embedding $L \ln L(B_1) \hookrightarrow H^{-1}(B_1)$.

First we check that the embedding $L \ln L \hookrightarrow H^{-1}$ exists and is continuous. We will realise $f \in L \ln L(B_1)$ as a bounded linear functional on $W_0^{1,2}(B_1)$.

Recall from Appendix A.3 that if $f \in L \ln L(B_1)$ then $f - \bar{f} \in \mathcal{H}^1(\mathbb{R}^2)$ and $\|f - \bar{f}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C\|f\|_{L \ln L(B_1)}$. For $\phi \in W_0^{1,2}(B_1)$ we extend it by zero and calculate

$$\begin{aligned} \int_{B_1} f\phi &= \int (f - \bar{f})\phi + \int \bar{f}\phi \\ &\leq C\|f - \bar{f}\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\phi\|_{BMO(\mathbb{R}^2)} + \frac{1}{\pi}\|f\|_{L^1(B_1)}\|\phi\|_{L^1(B_1)} \\ &\leq C\|f\|_{L \ln L(B_1)}\|\phi\|_{W_0^{1,2}(B_1)}. \end{aligned}$$

Thus $f \in H^{-1}(B_1)$ and $\|f\|_{H^{-1}(B_1)} \leq C\|f\|_{L \ln L(B_1)}$.

Now consider a sequence $\{f_n\} \subset L \ln L(B_1)$ such that $\|f_n\|_{L \ln L(B_1)} \leq \Lambda < \infty$. We can extend each f_n to be zero outside B_1 and consider the sequence of solutions $\{u_n\} \subset W_0^{1,2}(B_2)$ weakly solving

$$-\Delta u_n = f_n \quad \text{on } B_2.$$

By the compactness of Theorem 1.2 for $\Omega_n \equiv 0$ we can conclude that there exists some $u \in W^{1,2}(B_2)$ such that (up to a subsequence) $u_n \rightarrow u$ strongly in $W^{1,2}(B_1)$.

Writing $f = -\Delta u$ (which can clearly be viewed as an element of $H^{-1}(B_1)$) we see that

$$\begin{aligned} \|f_n - f\|_{H^{-1}(B_1)} &= \sup_{\phi \in W_0^{1,2}(B_1) \|\phi\|_{W_0^{1,2}(B_1)}=1} \int (f_n - f)\phi \\ &= \sup_{\phi \in W_0^{1,2}(B_1) \|\phi\|_{W_0^{1,2}(B_1)}=1} \int (\nabla u_n - \nabla u) \cdot \nabla \phi \\ &\leq \|\nabla u_n - \nabla u\|_{L^2(B_1)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

7 Proof of the optimal regularity, Theorem 1.1

The proof will proceed broadly in two steps. First, we will use a type of ‘geometric bootstrapping’ to show that solutions u have almost the optimal regularity claimed.

Lemma 7.1. *Suppose $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution to*

$$-\Delta u = \Omega \cdot \nabla u + f$$

where $\Omega \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$ and $f \in L^p$ for some $p \in (1, 2)$. Then $u \in W_{loc}^{2,\gamma}(B_1)$ for all $\gamma \in [1, p)$.

Once we have got this far, we can quantify the $W^{2,\gamma}$ regularity via the following estimate.

Lemma 7.2. *Let $p \in (1, 2)$ and suppose $\gamma \in (\frac{p+1}{2}, p)$ and $u \in W^{2,\gamma}(B_1, \mathbb{R}^m)$ solves*

$$-\Delta u = \Omega \cdot \nabla u + f$$

where $\Omega \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$ and $f \in L^p$. Then there exists $\eta_3 = \eta_3(p, m) > 0$ and $C = C(p, m) < \infty$ such that if $\|\Omega\|_{L^2(B_1)} \leq \eta_3$ then

$$\|u\|_{W^{2,\gamma}(B_{1/2})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}). \quad (25)$$

Let us assume these two lemmata for the moment and see how Theorem 1.1 follows. First, Lemma 7.1 directly applies, and we deduce $W_{loc}^{2,\gamma}(B_1)$ regularity for $\gamma \in [1, p)$. If we assume that $\|\Omega\|_{L^2(B_1)} \leq \eta_3$, then (25) of Lemma 7.2 holds for $\gamma \in (\frac{p+1}{2}, p)$ (strictly speaking we should make a small rescaling so that we can assume that $u \in W^{2,\gamma}(B_1)$ for $\gamma \in [1, p)$) and by taking the limit $\gamma \nearrow p$, we deduce that

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

By an appropriate covering argument, working on balls small enough so that $\|\Omega\|_{L^2} \leq \eta_3$, we deduce the regularity claimed in the theorem, and the claimed estimate (3).

It remains therefore to prove Lemmata 7.1 and 7.2. We will need, in turn, an additional lemma, which expresses the decay of energy of solutions u .

Lemma 7.3. *Under the hypotheses of Lemma 7.1, set $\alpha = 2(1 - 1/p) \in (0, 1)$. Then there exists $\eta_4 > 0$ depending on p and m such that if $\|\Omega\|_{L^2(B_1)} \leq \eta_4$ then*

$$\sup_{x_0 \in B_{1/2}, r \in (0, 1/2)} r^{-2\alpha} \|\nabla u\|_{L^2(B_r)}^2 < \infty,$$

and in particular, so that

$$\sup_{x_0 \in B_{1/2}, r \in (0, 1/2)} r^{-\alpha} \|\Omega \cdot \nabla u\|_{L^1(B_r)} < \infty.$$

Proof. (Lemma 7.3.) For reasons that will become apparent, choose $\delta \in (0, 1]$ sufficiently small so that

$$\lambda := \frac{(1 + 2\delta)}{4} < 2^{-4(1-1/p)} =: \Lambda \in \left(\frac{1}{4}, 1\right) \quad (26)$$

We can now choose $\eta_4 := \min\{\eta, \sqrt{\frac{\delta}{4K_2}}\}$, where η is from the second part of Theorem 1.5, depending on the δ we have just chosen and therefore on p (as well as m) and where K_2 is also from the second part of Theorem 1.5. Now take an arbitrary point $x_0 \in B_{1/2}$ and any $R \in (0, 1/2)$. Estimate (7) in the case that $r = \frac{1}{2}$, applied to the rescaled quantities defined in Section 2.2 yields

$$\begin{aligned} \|\nabla \hat{u}\|_{L^2(B_{1/2})}^2 &\leq \frac{(1+\delta)}{4} \|\nabla \hat{u}\|_{L^2(B_1)}^2 + K_2 \left(\|\hat{\Omega}\|_{L^2(B_1)}^2 \|\nabla \hat{u}\|_{L^2(B_1)}^2 + \|\hat{f}\|_{L^1(B_1)} \|\hat{f}\|_{L \ln L(B_1)} \right) \\ &\leq \lambda \|\nabla \hat{u}\|_{L^2(B_1)}^2 + C \|\hat{f}\|_{L^p(B_1)}^2. \end{aligned}$$

Reversing the scaling, using Section 2.2, we find that

$$\begin{aligned} \|\nabla u\|_{L^2(B_{R/2}(x_0))}^2 &\leq \lambda \|\nabla u\|_{L^2(B_R(x_0))}^2 + CR^{4(1-1/p)} \|f\|_{L^p(B_R(x_0))}^2 \\ &\leq \lambda \|\nabla u\|_{L^2(B_R(x_0))}^2 + [C\|f\|_{L^p(B_1)}^2] R^{4(1-1/p)}. \end{aligned}$$

Now applying what we have proved for $R = 2^{-k}$, with $k \in \{1, 2, \dots\}$ and using the abbreviation $a_k := \|\nabla u\|_{L^2(B_{2^{-k}}(x_0))}^2$, we find that

$$\begin{aligned} a_{k+1} &\leq \lambda a_k + K_3 2^{-4(1-1/p)k} \\ &= \lambda a_k + K_3 \Lambda^k \end{aligned}$$

where K_3 is independent of x_0 . This recursion relation can be solved to yield

$$a_{k+1} \leq \lambda^k a_1 + K_3 \Lambda \frac{(\Lambda^k - \lambda^k)}{\Lambda - \lambda},$$

and by (26), this simplifies to

$$\|\nabla u\|_{L^2(B_{2^{-k}}(x_0))}^2 =: a_k \leq C \Lambda^k.$$

Thus, for $r \in (0, 1/2]$ we have

$$\|\nabla u\|_{L^2(B_r(x_0))}^2 \leq Cr^{4(1-1/p)},$$

and hence the lemma is proved. \square

Proof. (Lemma 7.1.) Our goal is to prove $W^{2,\gamma}$ regularity, for all $\gamma \in [1, p)$. By applying Calderon-Zygmund theory directly to the equation (2), this will follow if we can show that $\Omega \cdot \nabla u \in L^s$ for all $s \in [1, p)$.

By the nature of what we are trying to prove, we may also assume that $\|\Omega\|_{L^2(B_1)} \leq \eta_4$, where η_4 is from Lemma 7.3. If this is not true, would could apply the result, after rescaling, on appropriate small balls where it is true.

By Lemma 7.3, and the first part of Lemma A.3 with h chosen to be $\Omega \cdot \nabla u$, we can deduce that for any $q \in (1, \frac{2-\alpha}{1-\alpha}) = (1, \frac{2}{2-p})$, we have

$$I_1 h \in L^q(B_{1/4}),$$

and in particular, this implies that $u \in W^{1,q}(B_{1/4})$ for some $q > 2$. If we apply the same result over appropriate smaller balls, we have in fact that $u \in W_{loc}^{1,q}(B_1)$ for some $q > 2$. At that point, we know that $\Omega \cdot \nabla u \in L_{loc}^s(B_1)$ with $s = \frac{2q}{2+q} \in (1, \frac{2}{3-p}) \subset (1, p)$.

We will now show that from here it is possible to carry out a geometric bootstrapping argument by using the second part of Lemma A.3. As an aside, we note that a bootstrapping argument using only classical Calderon-Zygmund methods does not work. The bootstrapping claim is that

$$\Omega \cdot \nabla u \in L_{loc}^s(B_1) \text{ with } s \in (1, p) \implies \Omega \cdot \nabla u \in L_{loc}^{s(\frac{2}{2+s-p})}(B_1). \quad (27)$$

This is true because whenever we know that $\Omega \cdot \nabla u \in L_{loc}^s(B_1)$ with $s \in (1, p)$, then we can deduce from Lemma 7.3, and the second part of Lemma A.3 with h chosen to be $\Omega \cdot \nabla u$, that

$$I_1 h \in L^{s\frac{2-\alpha}{1-\alpha}}(B_{1/4}),$$

and recalling that $\frac{2-\alpha}{1-\alpha} = \frac{2}{2-p}$, this is enough to establish this time that $u \in W^{1, \frac{2s}{2-p}}(B_{1/4})$ and hence $\Omega \cdot \nabla u \in L^{s(\frac{2}{2+s-p})}(B_{1/4})$. Again, by a simple covering argument, we deduce that $\Omega \cdot \nabla u \in L_{loc}^{s(\frac{2}{2+s-p})}(B_1)$ as desired.

By iterating the bootstrapping claim (27), we find that $\Omega \cdot \nabla u \in L_{loc}^s(B_1)$ for all $s \in [1, p)$, and the proof is complete. \square

It remains to prove Lemma 7.2.

Proof. (Lemma 7.2.) We begin by applying the Calderon-Zygmund estimate from Lemma A.1, giving some C independent of γ such that

$$\|u\|_{W^{2,\gamma}(B_{1/2})} \leq \frac{C}{\gamma-1} (\|\Delta u\|_{L^\gamma(B_{2/3})} + \|u\|_{L^\gamma(B_{2/3})}),$$

valid for $\gamma \in (1, 2]$ and $u \in W^{2,\gamma}(B_1)$.

For the specific u of the lemma, and $\gamma \in (\frac{1+p}{2}, p)$, we may then compute (also using the inequalities of Sobolev and Hölder, and Lemma A.2)

$$\begin{aligned} \|u\|_{W^{2,\gamma}(B_{1/2})} &\leq C(\|\Omega \cdot \nabla u\|_{L^\gamma(B_{2/3})} + \|f\|_{L^\gamma(B_{2/3})} + \|u\|_{L^2(B_{2/3})}) \\ &\leq C(\|\Omega \cdot \nabla u\|_{L^\gamma(B_{2/3})} + \|f\|_{L^\gamma(B_{2/3})} + \|u\|_{W^{1,1}(B_{2/3})}) \\ &\leq C(\|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_1)} + \|f\|_{L^\gamma(B_1)} + \|u\|_{L^1(B_1)}). \end{aligned} \quad (28)$$

where C depends only on p and m . Now consider any $v \in W^{2,t}(B_{1/2})$ for $t \in [1, 2)$. We can find a t -independent extension operator $Ex : W^{2,t}(B_{1/2}) \rightarrow W^{2,t}(\mathbb{R}^2)$ whose images have compact support in B_1 , so that there exists some $C < \infty$ independent of $t \in [1, 2)$ such that (denoting $Ex(v) = \tilde{v}$)

$$\|\tilde{v}\|_{W^{2,t}(\mathbb{R}^2)} \leq C\|v\|_{W^{2,t}(B_{1/2})} \quad (29)$$

with $v = \tilde{v}$ in $B_{1/2}$. (See for instance [7, Theorem 7.25].)

From here we apply the standard Sobolev embedding for $\nabla \tilde{v} \in W^{1,t} \hookrightarrow L^{\frac{2t}{2-t}}$ to obtain (see [7, Theorem 7.10])

$$\|\nabla \tilde{v}\|_{L^{\frac{2t}{2-t}}} \leq \frac{t}{2-t} \|\nabla^2 \tilde{v}\|_{L^t}.$$

This, coupled with (29) gives us

$$\|\nabla v\|_{L^{\frac{2t}{2-t}}(B_{1/2})} \leq \frac{C}{2-t} \|v\|_{W^{2,t}(B_{1/2})} \quad (30)$$

for all $v \in W^{2,t}(B_{1/2})$, $t \in [1, 2)$ and where C is independent of t .

Using (28) and (30) we have that there exists some $C < \infty$ depending only on p and m such that

$$\|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_{1/2})} \leq C(\|\Omega\|_{L^2(B_1)} \|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_1)} + \|f\|_{L^\gamma(B_1)} + \|u\|_{L^1(B_1)}) \quad (31)$$

for all $\gamma \in (\frac{1+p}{2}, p)$.

We now choose $B_R(x_0) \subset B_1$; using the scaling of Section 2.2, the estimate (31) yields

$$\|\nabla \hat{u}\|_{L^{\frac{2\gamma}{2-\gamma}}(B_{1/2})} \leq C(\|\hat{\Omega}\|_{L^2(B_1)} \|\nabla \hat{u}\|_{L^{\frac{2\gamma}{2-\gamma}}(B_1)} + \|\hat{f}\|_{L^\gamma(B_1)} + \|\hat{u}\|_{L^1(B_1)})$$

which translates (using $R \leq 1$ and Section 2.2) to

$$\|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_{R/2}(x_0))} \leq C(\|\Omega\|_{L^2(B_R(x_0))} \|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_R(x_0))} + R^{-3}(\|f\|_{L^\gamma(B_R(x_0))} + \|u\|_{L^1(B_R(x_0))})).$$

Now let $\beta = \frac{2\gamma}{2-\gamma}$ and raise this inequality to the power β to obtain (noticing $\beta < \frac{2p}{2-p}$)

$$\begin{aligned} \|\nabla u\|_{L^\beta(B_{R/2}(x_0))}^\beta &\leq K \|\Omega\|_{L^2(B_R(x_0))}^\beta \|\nabla u\|_{L^\beta(B_R(x_0))}^\beta \\ &\quad + R^{-\frac{6p}{2-p}} K (\|f\|_{L^\gamma(B_R(x_0))} + \|u\|_{L^1(B_R(x_0))})^\beta \end{aligned}$$

for some specific $K < \infty$ depending only on p and m .

We now wish to apply Lemma A.7: We are able to choose $\eta_3 = (\frac{\epsilon_0}{K})^{\frac{1}{\beta}}$ where ϵ_0 is that of Lemma A.7 corresponding to the choice $k = \frac{6p}{2-p}$. Let $\Gamma = K (\|f\|_{L^\gamma(B_1)} + \|u\|_{L^1(B_1)})^\beta$.

The above estimate holds in particular for any $B_R(x_0)$ such that $B_{2R}(x_0) \subset B_1$, so we have

$$\|\nabla u\|_{L^\beta(B_{R/2}(x_0))}^\beta \leq \epsilon_0 \|\nabla u\|_{L^\beta(B_R(x_0))}^\beta + R^{-\frac{6p}{2-p}} \Gamma.$$

Hence a direct application of Lemma A.7 gives us

$$\|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_{1/2})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

Therefore a simple covering argument yields that there is a $C = C(p, m) < \infty$ such that

$$\|\nabla u\|_{L^{\frac{2\gamma}{2-\gamma}}(B_{2/3})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

and from here another application of Lemmata A.1 and A.2 tells us that there is a $C = C(p, m) < \infty$ such that

$$\|u\|_{W^{2,\gamma}(B_{1/2})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

□

8 $L \ln L$ cannot be replaced by h^1

Here we present a counterexample to the compactness Theorem 1.2 when we allow $f_n \in h^1(B_1)$. Our example will have $\Omega_n \equiv 0$ for all n and $u_n : B_1 \rightarrow \mathbb{S}^2$ will be a sequence of harmonic maps with bounded energy that undergoes bubbling.

Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ be the (inverse of) stereographic projection and take $u_n(x, y) = \pi(nx, ny)$. Since u_n is harmonic for all n we know it solves (see [9])

$$-\Delta u_n = \nabla^\perp B_n \cdot \nabla u_n$$

where $\nabla^\perp(B_n)_j^i = (u_n)^i \nabla(u_n)^j - (u_n)^j \nabla(u_n)^i$, therefore $\|\nabla^\perp B_n\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u_n\|_{L^2(\mathbb{R}^2)}$. Letting $f_n = \nabla^\perp B_n \cdot \nabla u_n$ we have

$$\|f_n\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C\|\nabla u_n\|_{L^2(\mathbb{R}^2)}^2 \leq C\|\nabla \pi\|_{L^2(\mathbb{R}^2)}^2 = \Lambda < \infty.$$

This tells us two things: first that $\|f_n\|_{h^1(B_1)} \leq \Lambda < \infty$ and second that $\|u_n\|_{W^{1,2}(B_1)} \leq \Lambda < \infty$. At this point we have all the hypotheses of the theorem (except that we allow $f_n \in h^1$), but it is easy to see that there can be no subsequence converging locally strongly in $W^{1,2}$ because this sequence forms a bubble at the origin.

A Background and supporting results

A.1 Singular Integrals

We recall here the basics of Calderon-Zygmund theory on the unit ball $B_1 \subset \mathbb{R}^2$, following [7]. Define the Newtonian potential operator N on functions $f \in L^1(B_1)$ (implicitly extended to be zero to $\mathbb{R}^2 \setminus B_1$) by

$$N[f](x) := (\Gamma * f)(x) = \int \Gamma(x - y)f(y) dy$$

where $\Gamma(x) = \frac{1}{2\pi} \ln|x|$. If $f \in C_c^\infty$ then $\Delta N[f] = f$. Writing $w = N[f]$ we have

1. $N : L^p(B_1) \rightarrow L^p(B_1)$ is a bounded operator for all $1 \leq p \leq \infty$.
2. $\nabla w = \nabla N[f] = (\nabla \Gamma) * f$. We will frequently view ∇N as an operator in its own right. It is easy to see that $|\nabla N[f]| \leq \frac{1}{2\pi} I_1[|f|]$ where I_1 is the standard notation for this Riesz potential (defined by convolution with $\frac{1}{|x|}$) and $I_1 : L^p(B_1) \rightarrow L^{\frac{2p}{2-p}}(B_1)$ is a bounded operator for all $1 < p < 2$.
3. (Calderon-Zygmund) Let $f \in L^p(B_1)$, $1 < p < \infty$ and $w = N[f]$. Then $\nabla^2 w = \nabla^2 N[f] = (\nabla^2 \Gamma) * f$ and (as above we see $\nabla^2 N$ as an operator) $\nabla^2 N : L^p(B_1) \rightarrow L^p(B_1)$ is bounded for p in this range. More explicitly we have $w \in W^{2,p}(B_1)$, $\Delta w = f$ almost everywhere, and

$$\|\nabla^2 w\|_{L^p(B_1)} \leq C(p) \|f\|_{L^p(B_1)}. \quad (32)$$

In fact, revisiting a second time the proof of the Calderon-Zygmund estimates (e.g. [7, §9.4], but interpolating between $q = 1$ and $r = 3$) we find that the dependency of C in (32) can be weakened, and one can prove:

Lemma A.1. *Let $\gamma \in (1, 2]$ and suppose $u \in W^{2,\gamma}(B_1)$. Then there exists some $C < \infty$ independent of γ such that*

$$\|u\|_{W^{2,\gamma}(B_{1/2})} \leq \frac{C}{\gamma - 1} (\|\Delta u\|_{L^\gamma(B_{2/3})} + \|u\|_{L^\gamma(B_{2/3})})$$

Of course, even in the L^1 case, we have the following sub-optimal estimate which one can prove using standard estimates for harmonic functions and because the operator $\nabla N : L^1(B_1) \rightarrow L^{2,\infty}(B_1) \hookrightarrow L^1(B_1)$ is bounded (see Appendix A.4).

Lemma A.2. *If $u \in L^1(B_1)$ is a weak solution to*

$$-\Delta u = f \in L^1(B_1),$$

then $u \in W^{1,1}(B_{2/3})$ and there exists $C < \infty$ such that

$$\|\nabla u\|_{L^1(B_{2/3})} \leq C(\|f\|_{L^1(B_1)} + \|u\|_{L^1(B_1)}).$$

We now present a theorem of Adams giving improved estimates on the Riesz potential I_1 if, in addition we have a decay estimate on our function. Given $h \in L^1(B_1, \mathbb{R}^m)$ define a new function $I_1 h$ to be the convolution of $|h|$, extended to be zero on $\mathbb{R}^2 \setminus B_1$, with $|x|^{-1}$, i.e.

$$I_1 h(x) = \int_{B_1} \frac{|h(y)|}{|x-y|} dy.$$

Lemma A.3. *(Adams [1, Propositions 3.1 and 3.2].) Suppose $h \in L^1(B_1)$ and*

$$\sup_{x \in B_{1/2}, r \in (0, 1/2)} r^{-\alpha} \int_{B_r(x)} |h| < \infty,$$

for some $\alpha \in (0, 1)$. Then

1. $I_1 h \in L^q(B_{1/4})$ for any $q \in [1, \frac{2-\alpha}{1-\alpha})$;
2. if in addition $h \in L_{loc}^s(B_1)$ for $s \in (1, \frac{2}{2-\alpha})$, then $I_1 h \in L^{s(\frac{2-\alpha}{1-\alpha})}(B_{1/4})$.

A.2 Hardy Spaces

Pick $\phi \in C_c^\infty(B_1)$ such that $\int \phi = 1$ and let $\phi_t(x) = t^{-2}\phi(\frac{x}{t})$. For a distribution f we say f lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ if $f_* \in L^1(\mathbb{R}^2)$ where

$$f_*(x) = \sup_{t>0} |(\phi_t * f)(x)|$$

with norm $\|f\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|f_*\|_{L^1(\mathbb{R}^2)}$. Clearly we have the continuous embedding $\mathcal{H}^1(\mathbb{R}^2) \hookrightarrow L^1(\mathbb{R}^2)$. The dual space of $\mathcal{H}^1(\mathbb{R}^2)$ is $BMO(\mathbb{R}^2)$ where $BMO := \{g \in L_{loc}^1(\mathbb{R}^2) : \sup_{B \subset \mathbb{R}^2} \frac{1}{|B|} \int_B |g - \bar{g}| < \infty\}$ (see [5]).

Related to \mathcal{H}^1 is the so-called local Hardy space h^1 defined to be those functions for which

$$f_*(x) = \sup_{0<t<1} |(\phi_t * f)(x)| \in L^1(\mathbb{R}^2)$$

with corresponding norm. Again we clearly have the continuous embedding $h^1(\mathbb{R}^2) \hookrightarrow L^1(\mathbb{R}^2)$. For a function f defined in B_1 we say that $f \in h^1(B_1)$ if

$$f_*(x) = \sup_{0<t<1-|x|} |(\phi_t * f)(x)| \in L^1(B_1).$$

By [15, Theorem 1.92] we know that $f \in h^1(B_1)$ if and only if for any $\varphi \in C_c^\infty(B_1)$ with $\int \varphi \neq 0$ there is a constant λ such that $\varphi(f - \lambda) \in \mathcal{H}^1(\mathbb{R}^2)$, with

$$\|\varphi(f - \lambda)\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C\|f\|_{h^1(B_1)},$$

where $C = C(\varphi)$ and λ is chosen such that $\int \varphi(f - \lambda) = 0$.

The Hardy spaces act as replacements to L^1 in Calderon-Zygmund estimates. In particular for $f \in \mathcal{H}^1(\mathbb{R}^2)$, writing $w = N[f]$, we have the estimate (see [9])

$$\|\nabla^2 w\|_{L^1(\mathbb{R}^2)} \leq C\|f\|_{\mathcal{H}^1(\mathbb{R}^2)}, \tag{33}$$

and if $f \in h^1(\mathbb{R}^2)$ then $\nabla^2 w \in L_{loc}^1(\mathbb{R}^2)$ with

$$\|\nabla^2 w\|_{L^1(B_1)} \leq C\|f\|_{h^1(\mathbb{R}^2)}.$$

Moreover $f \in h^1(B_1)$ implies $\nabla^2 w \in L_{loc}^1(B_1)$. The final two assertions follow from (33), [15, Theorem 1.92] and standard elliptic estimates.

A.3 Lorentz Spaces and $L \ln L$

For measurable f define, for $s \geq 0$, the distribution function $\lambda(s) = |\{x : |f|(x) > s\}|$. Assuming $\lim_{s \rightarrow \infty} \lambda(s) = 0$, define the nonincreasing rearrangement $f^* : (0, \infty) \rightarrow [0, \infty)$ by

$$f^*(t) := \inf\{s \geq 0 : \lambda(s) \leq t\}.$$

Here we consider the spaces defined by:

1. $L^{2,1} := \{f : \int t^{-1/2} f^*(t) dt < \infty\}$
2. $L^{2,\infty} := \{f : \sup_{t>0} t^{1/2} f^*(t) < \infty\}$
3. $L \ln L := \{f : \int f^*(t) \ln(2 + \frac{1}{t}) dt < \infty\}$

The quantities above are not norms, but the spaces are all Banach spaces whose norms are equivalent to these quantities respectively. The spaces $L^{2,1}$ and $L^{2,\infty}$ are two examples of Lorentz spaces, which can be thought of as perturbations of the usual L^p spaces. For example the following are all continuous embeddings (see [18])

$$L^p(B_1) \hookrightarrow L^{2,1}(B_1) \hookrightarrow L^2(B_1) = L^{2,2}(B_1) \hookrightarrow L^{2,\infty}(B_1) \hookrightarrow L^q(B_1)$$

for all $q < 2 < p$. The dual space of $L^{2,1}$ is $L^{2,\infty}$ [10].

For the space $L \ln L$ we have the continuous embeddings

$$L^p(B_1) \hookrightarrow L \ln L(B_1) \hookrightarrow L^1(B_1)$$

for all $p > 1$. It is well known [17] that $f \in L \ln L$ if and only if its corresponding maximal function is locally integrable, where the maximal function $M_0(f)(x) = \sup_{t>0} \frac{1}{|B_t(x)|} \int_{B_t(x)} |f|$. By extension of f by zero and comparison of the functions $M_0(f)$ and $f_{\#}$ we see that the following embedding is continuous

$$L \ln L(B_1) \hookrightarrow h^1(\mathbb{R}^2).$$

In fact, if $f \in L \ln L(B_1)$ then $f - \bar{f} \in \mathcal{H}^1(\mathbb{R}^2)$ with $\|f - \bar{f}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|f\|_{h^1(\mathbb{R}^2)} \leq C \|f\|_{L \ln L(B_1)}$, where

$$f - \bar{f} := \begin{cases} f - \frac{1}{|B_1|} \int_{B_1} f & \text{in } B_1; \\ 0 & \text{in } \mathbb{R}^2 \setminus B_1. \end{cases}$$

A.4 Embeddings and Estimates

Listed below are some important miscellaneous results involving Sobolev spaces and the spaces mentioned above.

1. The embedding $W^{1,1}(\mathbb{R}^2) \hookrightarrow L^{2,1}(\mathbb{R}^2)$ is continuous [9].
2. Given $u, v \in W^{1,2}(\mathbb{R}^2)$ then $\nabla u \cdot \nabla^\perp v \in \mathcal{H}^1(\mathbb{R}^2)$ with $\|\nabla u \cdot \nabla^\perp v\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^2)} \|\nabla v\|_{L^2(\mathbb{R}^2)}$. (See [2].)
3. By 1. and the estimates from Appendices A.1 and A.2 on the Newtonian potential we have that the operators $\nabla N : \mathcal{H}^1(\mathbb{R}^2) \rightarrow L_{loc}^{2,1}(\mathbb{R}^2)$ and $\nabla N : \mathcal{H}^1(\mathbb{R}^2) \rightarrow L_{loc}^{2,1}(\mathbb{R}^2)$ are bounded.
4. $\nabla N : L^1(B_1) \rightarrow L^{2,\infty}(B_1)$ is a bounded operator; this follows by standard estimates on convolutions and the fact that $\nabla \Gamma \in L^{2,\infty}$.
5. The embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$ is continuous by Poincaré's inequality.

A.5 Riviere's gauge

The key result from Rivière's work that we will need is the existence of the following perturbation of Coulomb's gauge.

Lemma A.4 (Rivière). *Suppose $\Omega \in L^2(B_1, so(m) \otimes \mathbb{R}^2)$. Then there exists $\epsilon = \epsilon(m) > 0$ such that if $\|\Omega\|_{L^2(B_1)} \leq \epsilon$ we can find $A \in W^{1,2}(B_1, GL_m(\mathbb{R})) \cap L^\infty(B_1, GL_m(\mathbb{R}))$, $B \in W^{1,2}(B_1, gl_m(\mathbb{R}))$ and $C = C(m) < \infty$ where*

$$\nabla A - A\Omega = \nabla^\perp B$$

and

$$\|\nabla A\|_{L^2(B_1)} + \|\nabla B\|_{L^2(B_1)} + \|dist(A, SO(m))\|_{L^\infty(B_1)} \leq C \|\Omega\|_{L^2(B_1)}.$$

A.6 Weak Convergence of Measures and Functions of Bounded Variation

We consider the space of functions of bounded variation $BV(B)$ for any ball $B \subset \mathbb{R}^2$. BV is defined by $BV(B) = \{V \in L^1(B) : \int_B |\nabla V| := \sup_{\phi \in C_0^1(B, \mathbb{R}^2)} \|\phi\|_{L^\infty} \int_B V \operatorname{div} \phi < \infty\}$. In other words it is the space of functions whose distributional derivatives are signed Radon measures with finite total mass. This is a Banach space with norm $\|V\|_{BV(B)} = \|V\|_{L^1(B)} + \int_B |\nabla V|$. It is easy to see that we have the continuous embedding $W^{1,1} \hookrightarrow BV$, moreover in two dimensions we have the continuous embedding $BV(B) \hookrightarrow L^2(B)$ and the compact embeddings $BV(B) \hookrightarrow L^p(B)$ for any $p < 2$ (see for instance [18]).

We also use the standard weak-* compactness available in the space of signed Radon measures with finite total mass, denoted M .

The proof of the next lemma is essentially taken from [3, Theorem 9] and is similar to that stated in [11]. For an integrable function k we implicitly view it as both a function and a measure, i.e. $k = k dx$.

Lemma A.5. *Suppose $\{V_n\} \subset BV(B)$ is a bounded sequence and $B \subset \mathbb{R}^2$ is an open ball. Then there exist at most countable $\{x_j\} \subset B$ and $\{a_j > 0\}$ (where $\sum_j a_j < \infty$) and $V \in BV(B)$ such that (up to a subsequence)*

$$V_n^2 \rightharpoonup V^2 + \sum_j a_j \delta_{x_j}$$

weakly in $M(B)$.

Proof. Since $\{V_n\} \subset BV(B)$ is a bounded sequence, there exists $V \in L^2$ such that (up to a subsequence) $V_n \rightarrow V$ strongly in L^p for all $p < 2$ and $V_n \rightharpoonup V$ weakly in L^2 . Also $\{\nabla V_n\} \subset M(B)$ is bounded so (again up to a subsequence) $\nabla V_n \rightharpoonup \lambda$ (a vector-valued measure) $\in M(B)$. In particular, for all $\phi \in C_c^1(B, \mathbb{R}^2)$

$$\begin{aligned} \int \phi \cdot d\lambda &= \lim_{n \rightarrow \infty} \int \phi \cdot \nabla V_n \, dx \\ &= - \lim_{n \rightarrow \infty} \int \operatorname{div}(\phi) V_n \, dx \\ &= - \int \operatorname{div}(\phi) V \, dx. \end{aligned}$$

In other words $V \in BV(B)$ and $\nabla V = \lambda$.

Now set $g_n := V_n - V$. Note that $|\nabla g_n| \in M(B)$ is bounded so for a subsequence $|\nabla g_n| \rightharpoonup \mu \in M(B)$ where μ is non-negative. Similarly (up to a subsequence) $g_n^2 \rightharpoonup \nu \in M(B)$ where ν is also non-negative. We have that for all $\phi \in C_c^1(B)$, $\phi g_n \in BV(B)$ and by the continuous embedding $BV(B) \hookrightarrow L^2(B)$ we have

$$\left(\int (\phi g_n)^2 \, dx \right)^{1/2} \leq C \int |\nabla(\phi g_n)| \, dx$$

and since $g_n \rightarrow 0$ in L^1 , taking limits gives

$$\left(\int \phi^2 \, d\nu \right)^{1/2} \leq C \int |\phi| \, d\mu.$$

Taking ϕ to be an approximation to the characteristic function on $B_r(x) \subset B$ we get

$$\nu(B_r(x)) \leq C(\mu(B_r(x)))^2$$

for all $B_r(x) \subset B$, and in particular $\nu \ll \mu$.

By standard results for differentiation of measures (see e.g. [4, §1.6 Theorem 2]), for any Borel set $E \subset B$

$$\nu(E) = \int_E D_\mu \nu \, d\mu$$

where $D_\mu \nu = \lim_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$ is a μ -integrable function (this limit exists μ -almost everywhere).

Since μ is a finite, positive Radon measure, there are at most countable points $\{x_j\}$ such that $\mu(\{x_j\}) > 0$, and if $\mu(\{x\}) = 0$ then

$$D_\mu \nu(x) = \lim_{r \downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \leq C \lim_{r \downarrow 0} \mu(B_r(x)) = 0.$$

Letting $X := \cup_j \{x_j\}$ we have $D_\mu \nu = 0$ μ -almost everywhere on $B \setminus X$. Hence $D_\mu \nu$ is a simple function, therefore for Borel $E \subset B$

$$\nu(E) = \int_E D_\mu \nu d\mu = \sum_{\{j: x_j \in E\}} D_\mu \nu(x_j) \mu(\{x_j\}).$$

Setting $a_j := D_\mu \nu(x_j) \mu(\{x_j\})$ we have $\nu = \sum_j a_j \delta_{x_j}$. Now, for $\phi \in C_c^0(B)$

$$\begin{aligned} \sum_j a_j \phi(x_j) &= \lim_{n \rightarrow \infty} \int g_n^2 \phi dx \\ &= \lim_{n \rightarrow \infty} \int (V_n - V)^2 \phi dx \\ &= \lim_{n \rightarrow \infty} \left(\int (V_n^2 - V^2) \phi dx + 2 \int V(V - V_n) \phi dx \right) \end{aligned}$$

where the last term vanishes in the limit since $V_n \rightharpoonup V$ weakly in L^2 . \square

Lemma A.6 (Corollary of Lemma A.5). *Suppose $\{V_n\}$ is as in Lemma A.5. If*

$$\lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} \|V_n\|_{L^2(B_r(x))} = 0$$

for all $x \in B$, then

$$V_n \rightarrow V$$

strongly in $L_{loc}^2(B)$ (same V as in Lemma A.5).

Proof. First we apply Lemma A.5 and viewing $|V_n|^2 dx$ as a sequence in $M(B)$ we notice that the condition $\lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} \|V_n\|_{L^2(B_r(x))} = 0$ simply says that $V_n^2 \rightharpoonup V^2$ weakly in $M(B)$. Therefore, given any open ball $B_r(x) \subset\subset B$ we can apply standard results for Radon measures ([4, §1.9 Theorem 1]) to conclude that (since $\int_{\partial B_r(x)} |V|^2 dx = 0$) $\|V_n\|_{L^2(B_r(x))} \rightarrow \|V\|_{L^2(B_r(x))}$ for all $B_r(x) \subset\subset B$. Hence

$$\begin{aligned} \int_{B_r(x)} (V - V_n)^2 dx &= \int_{B_r(x)} (V_n^2 - V^2) dx + 2 \int_{B_r(x)} V(V - V_n) dx \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

since $V_n \rightharpoonup V$ weakly in L^2 . Therefore $V_n \rightarrow V$ strongly in $L_{loc}^2(B)$. \square

A.7 Absorption lemma

Special cases of the following lemma are widely used in regularity theory.

Lemma A.7. (*Leon Simon [16, §2.8, Lemma 2].*) Let $B_\rho(y) \subset \mathbb{R}^2$ be any ball, $k \in \mathbb{R}$, $\Gamma > 0$, and let φ be any $[0, \infty)$ -valued convex subadditive function on the collection of convex subsets of $B_\rho(y)$; thus $\varphi(A) \leq \sum_{j=1}^N \varphi(A_j)$ whenever A, A_1, A_2, \dots, A_N are convex subsets of $B_\rho(y)$ with $A \subset \bigcup_{j=1}^N A_j$. There is $\epsilon_0 = \epsilon_0(k)$ such that if

$$\sigma^k \varphi(B_{\sigma/2}(z)) \leq \epsilon_0 \sigma^k \varphi(B_\sigma(z)) + \Gamma$$

whenever $B_{2\sigma}(z) \subset B_\rho(y)$, then there exists some $C = C(k) < \infty$ such that

$$\rho^k \varphi(B_{\rho/2}(y)) \leq C\Gamma.$$

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