THE CANONICAL SHRINKING SOLITON ASSOCIATED TO A RICCI FLOW

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4 February 2010

Abstract

To every Ricci flow on a manifold $\mathcal{M}$ over a time interval $I \subset \mathbb{R}_-$, we associate a shrinking Ricci soliton on the space-time $\mathcal{M} \times I$. We relate properties of the original Ricci flow to properties of the new higher-dimensional Ricci flow equipped with its own time-parameter. This geometric construction was discovered by consideration of the theory of optimal transportation, and in particular the results of the second author [18], and McCann and the second author [12]; we briefly survey the link between these subjects.

1 Introduction

In 1982, Hamilton [7] introduced the study of Ricci flow, which evolves a Riemannian metric g on a manifold $\mathcal{M}$ under the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g(t)), \tag{1.1}$$

for t in some time interval $I \subset \mathbb{R}$. Since then, the subject has developed steadily, and has become established as an effective bridge between analysis, geometry and topology (see for example [13], [14], [15] and the overview in [17]).

The initial progress relevant to the present paper was Hamilton's discovery in 1993 of the so-called Harnack quantities (see [8] for more information) and by 1995, Nolan Wallach [9, §14] had proposed that these quantities should arise as some sort of curvature of some higher-dimensional manifold or bundle associated to the Ricci flow. This idea was developed by Chow and Chu [2] who considered the space-time manifold $\mathcal{M} \times I$, and defined a pair $(\tilde{g}, \widetilde{\nabla})$ of a metric on its cotangent bundle *degenerate* in the time direction and a $\tilde{g}$-compatible torsion-free connection (which is not unique owing to the degeneracy of $\tilde{g}$) so that the derivatives in the time coordinate direction of the components of $(\tilde{g}, \widetilde{\nabla})$ resemble the formulae one can compute for the evolution of the components of the metric and its Levi-Civita connection under Ricci flow. (See [2] for more details.) It turns out that Hamilton's matrix Harnack quadratic is almost the Riemannian curvature of that space-time connection. An improved correspondence is established in the work of Chow and Knopf [4] by considering Ricci flow with a 'cosmological term'. (An example of such a flow would be $\bar{g}(\bar{t}) := \frac{1}{t} g(t)$, for $\bar{t} = \log t$, where $g(t)$ is a Ricci flow.)

In 2002, Perelman [13, §6] made a new breakthrough along these lines involving the construction of an essentially Ricci-flat manifold of dimension unbounded from above, which we now describe. The starting point is a Ricci flow $g(\cdot)$ which once seen with respect to a reverse time parameter $\tau := C - t$ (for some $C \in \mathbb{R}$) is defined for τ lying in some interval $I \subset \mathbb{R}_+$. Let $N \in \mathbb{N}$ be a large natural number, and consider the manifold $\tilde{\mathcal{M}} := \mathcal{M} \times I \times S^N$ equipped with the metric $\tilde{g}$ defined by

$$\tilde{g}_{ij} = g_{ij}; \quad \tilde{g}_{00} = \frac{N}{2\tau} + R; \quad \tilde{g}_{\alpha\beta} = \tau g_{\alpha\beta},$$

with all remaining metric coefficients $\tilde{g}_{0i}$, $\tilde{g}_{0\alpha}$ and $\tilde{g}_{i\alpha}$ equal to zero, where i, j are coordinate indices on the $\mathcal{M}$ factor, α, β are those on the S^N factor, 0 represents the index of the time coordinate $\tau \in I$, the scalar curvature is written R, and $g_{\alpha\beta}$ is the metric on the round S^N of sectional curvature $\frac{1}{2N}$.

The significance of the manifold $(\tilde{\mathcal{M}}, \tilde{g})$ is that it is *Ricci-flat* up to errors of order $\frac{1}{N}$. This allowed Perelman to formally apply the Bishop-Gromov comparison theorem in order to discover his *reduced volume* [13]. By setting $\tau = -t$, Hamilton's Harnack quantities [8] can be recovered from the full curvature tensor, up to errors of order $\frac{1}{N}$, although Perelman's construction works for $\tau > 0$ while Harnack estimates hold only for $t > 0$. Indeed, even starting with a Ricci flow having positive curvature operator, the curvature operator of $\tilde{g}$ need *not* have a sign, even ignoring errors.

More recently, the theory of optimal transportation has been introduced into the study of Ricci flow, with papers by McCann and the second author [12], the second author [18] and then Lott [11]. We give more details in Section 3, but for now we mention that a notion of $\mathcal{L}$-optimal transportation was introduced in [18] which can be used to recover all the important monotonic quantities for Ricci flow that were discovered by Perelman [13] in his analysis of finite-time singularities for Ricci flow (see [18] and [11]).

The starting point for this paper is the proposal of John Lott that one might be able to make a formal justification of the results in [18] by applying optimal transportation theory developed for manifolds of positive Ricci curvature, directly to Perelman's construction $(\tilde{\mathcal{M}}, \tilde{g})$ (see also [11]). This seems to be problematic using existing optimal transportation theory. However, consideration of what alternative construction analogous to Perelman's $(\tilde{\mathcal{M}}, \tilde{g})$ could lie behind the results of [18] turns out to be fruitful; in this paper we are thus led to the following theorem in which we construct the *Canonical Shrinking Soliton* associated to a Ricci flow on $\mathcal{M}$.

Theorem 1.1. *Suppose $g(\tau)$ is a (reverse) Ricci flow – i.e. a solution of $\frac{\partial g}{\partial \tau} = 2\,\mathrm{Ric}(g(\tau))$ – defined for τ within a time interval $(a, b) \subset (0, \infty)$, on a manifold $\mathcal{M}$ of dimension $n \in \mathbb{N}$, and with bounded curvature. Suppose $N \in \mathbb{N}$ is sufficiently large to give a positive definite metric $\hat{g}$ on $\hat{\mathcal{M}} := \mathcal{M} \times (a, b)$ defined by*

$$\hat{g}_{ij} = \frac{g_{ij}}{\tau}; \qquad \hat{g}_{00} = \frac{N}{2\tau^3} + \frac{R}{\tau} - \frac{n}{2\tau^2}; \qquad \hat{g}_{0i} = 0,$$

where i, j are coordinate indices on the $\mathcal{M}$ factor, 0 represents the index of the time coordinate $\tau \in (a, b)$, and the scalar curvature of g is written as R.

Then up to errors of order $\frac{1}{N}$, the metric $\hat{g}$ is a gradient shrinking Ricci soliton on the higher dimensional space $\hat{\mathcal{M}}$:

$$\mathrm{Ric}(\hat{g}) + \mathrm{Hess}_{\hat{g}}\left(\frac{N}{2\tau}\right) \simeq \frac{1}{2}\hat{g}, \tag{1.2}$$

by which we mean that the quantity

$$N\left[\mathrm{Ric}(\hat{g}) + \mathrm{Hess}_{\hat{g}}\left(\frac{N}{2\tau}\right) - \frac{1}{2}\hat{g}\right]$$

is locally bounded independently of N, with respect to any fixed metric on $\hat{\mathcal{M}}$.

It is well known (see for example [17, §1.2.2]) that a Ricci soliton metric on a manifold $\hat{\mathcal{M}}$ induces a self-similar Ricci flow on $\hat{\mathcal{M}}$. In our context this indicates that given a Ricci flow on $\mathcal{M}$ over the time interval (a, b), it is then natural to introduce an additional time parameter and consider Ricci flow using $\mathcal{M} \times (a, b)$ as the underlying manifold, and we adopt this viewpoint in Section 4.

This theorem serves as an example of an application of the theory of optimal transportation to yield a new result in a different field. The route between the theorems of optimal transportation and this construction is described in Sections 3 and 4. It will become apparent, from Section 4 and [18], that our Canonical Shrinking Soliton encodes various monotonic quantities which underpin Perelman's work on Ricci flow [13, 14, 15], including entropies and quantities involving $\mathcal{L}$-length (see Section 3).

Remark 1.2. Our construction encodes Hamilton's Harnack quantities in the same sense as does Perelman's construction. More precisely, the components of the $(4, 0)$ curvature tensor $\tau Rm(\hat{g})$ coincide (up to errors of order $\frac{1}{N}$) with the components of the matrix Harnack expression [8, 13] after setting $t = -\tau$.

One advantage of the notion of Canonical Soliton over the construction of Perelman is that we can use it to find and prove new (and old) Harnack inequalities. Indeed, in [1] we will extend the ideas of this paper to give Canonical *Expanding* Solitons in order to achieve this. A notion of Canonical Steady Soliton, discussed in Section 5, completes the picture.

In Section 6 we investigate a refinement of this construction in which we combine the Ricci flow on $\mathcal{M} \times (a, b)$ suggested by our construction with mean curvature flow of time slices.

Acknowledgements: We thank Robert McCann for discussions relating to an observation of Tom Ilmanen used in Section 3. Part of this work was carried out at the Centre de Recerca Matemàtica, Barcelona, and the Institut Henri Poincaré, Paris, and we thank these institutions for their hospitality. This work was partly supported by the Leverhulme Trust. The first author was partly supported by DGI(Spain) and FEDER Project MTM2007-65852.

2 The calculations

In this section, we give exact formulae for the Christoffel symbols of $\hat{g}$ from Theorem 1.1, and approximate formulae for its Ricci curvatures and for the Hessian of $\frac{N}{2\tau}$ with respect to $\hat{g}$. Theorem 1.1 will then be seen to follow easily.

Proposition 2.1. *In the setting of Theorem 1.1, if Γ^i_{jk} are the Christoffel symbols of $g(\tau)$ at some point $x \in \mathcal{M}$, then the Christoffel symbols of $\hat{g}$ at (x, τ) are given by*

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}; \quad \hat{\Gamma}^i_{j0} = R^i{}_j - \frac{\delta^i{}_j}{2\tau}; \quad \hat{\Gamma}^i_{00} = -\frac{1}{2} g^{ij} \frac{\partial R}{\partial x^j};$$

$$\hat{\Gamma}^0_{jk} = \hat{g}_{00}^{-1} \left(\frac{g_{jk}}{2\tau^2} - \frac{R_{jk}}{\tau} \right); \quad \hat{\Gamma}^0_{i0} = \frac{1}{2\tau} \hat{g}_{00}^{-1} \frac{\partial R}{\partial x^i}; \quad \hat{\Gamma}^0_{00} = \frac{1}{2} \hat{g}_{00}^{-1} \left[-\frac{3N}{2\tau^4} - \frac{R}{\tau^2} + \frac{R_\tau}{\tau} + \frac{n}{\tau^3} \right]$$

This is a straightforward computation from the definition of the Christoffel symbols

$$\hat{\Gamma}^a_{bc} := \frac{1}{2} \hat{g}^{ad} \left(\frac{\partial \hat{g}_{cd}}{\partial x^b} + \frac{\partial \hat{g}_{bd}}{\partial x^c} - \frac{\partial \hat{g}_{bc}}{\partial x^d} \right),$$

where a, b, c, d are arbitrary indices, and the equation of Ricci flow. Using the standard formula for the coefficients of the Ricci curvature

$$\hat{R}_{ab} = \frac{\partial \hat{\Gamma}^c_{ab}}{\partial x^c} - \frac{\partial \hat{\Gamma}^c_{ac}}{\partial x^b} + \hat{\Gamma}^c_{ab}\hat{\Gamma}^d_{cd} - \hat{\Gamma}^d_{ac}\hat{\Gamma}^c_{bd},$$

the formula for the coefficients of $\mathrm{Hess}_{\hat{g}}(f)$

$$\hat{\nabla}^2_{ab}(f) = \frac{\partial^2 f}{\partial x^a \partial x^b} - \frac{\partial f}{\partial x^c}\hat{\Gamma}^c_{ab},$$

the equation for the evolution of R

$$R_\tau + \Delta R + 2|\mathrm{Ric}|^2 = 0,$$

(see for example [17, Proposition 2.5.4]) and the contracted Bianchi identity

$$\nabla_i R^i{}_j = \frac{1}{2}\nabla_j R,$$

one readily verifies the following:

Proposition 2.2. *Fixing $\tau > 0$, a time at which the Ricci flow exists, and fixing local coordinates $\{x^i\}$ in a neighbourhood U of some $p \in \mathcal{M}$, then in any neighbourhood $V \subset\subset U \times (a, b)$ of (p, τ), we have*

$$\hat{R}_{ij} \simeq R_{ij}; \qquad \hat{R}_{i0} \simeq -\frac{1}{2}\nabla_i R; \qquad \hat{R}_{00} \simeq -\frac{R_\tau}{2} - \frac{R}{2\tau},$$

where $\simeq$ denotes equality of the coefficients up to an error bounded in magnitude by $\frac{C}{N}$, with $C > 0$ a constant independent of N (but depending on V and the choice of coordinates). Moreover, we have

$$\hat{\nabla}^2_{ij}(\tfrac{N}{2\tau}) \simeq \frac{g_{ij}}{2\tau} - R_{ij}; \qquad \hat{\nabla}^2_{i0}(\tfrac{N}{2\tau}) \simeq \frac{\nabla_i R}{2}; \qquad \hat{\nabla}^2_{00}(\tfrac{N}{2\tau}) \simeq \frac{N}{4\tau^3} + \frac{R}{\tau} - \frac{n}{4\tau^2} + \frac{R_\tau}{2}.$$

By combining the formulae of Proposition 2.2 and the definition of $\hat{g}$, we deduce Theorem 1.1.

Remark 2.3. Although each side of (1.2) evaluated on the pair $(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau})$ will have magnitude of order N, the theorem tells us that their difference does not even have any terms of order 1, only those of order $\frac{1}{N}$. All errors disappear when the original Ricci flow $g(\tau)$ is a homothetically shrinking Einstein manifold, shrinking to nothing at $\tau = 0$.

3 Optimal Transportation on Ricci flows

Suppose that $(\mathcal{M}, g)$ is a closed (compact, no boundary) Riemannian manifold, and ν_1 and ν_2 are two Borel probability measures on $\mathcal{M}$. For $p \in [1, \infty)$, we define the p-Wasserstein distance W_p between ν_1 and ν_2 to be

$$W^g_p(\nu_1, \nu_2) := \left[\inf_{\pi \in \Gamma(\nu_1,\nu_2)} \int_{\mathcal{M}\times\mathcal{M}} d^p(x, y)\, d\pi(x, y)\right]^{\frac{1}{p}}, \tag{3.1}$$

where $d(\cdot, \cdot)$ is the Riemannian distance function induced by g and $\Gamma(\nu_1, \nu_2)$ is the space of Borel probability measures on $\mathcal{M} \times \mathcal{M}$ with marginals ν_1 and ν_2.

A basic principle in the subject – see Sturm and von Renesse [16] and the references therein – is that two probability measures evolving under an appropriate diffusion equation should get closer in the Wasserstein sense provided the manifold satisfies some curvature condition, most famously positive Ricci curvature.

In [12], McCann and the second author showed that this type of contractivity on an *evolving* manifold $(\mathcal{M}, g(\tau))$ *characterises* super-solutions of the Ricci flow (parametrised backwards in time) by which we mean solutions to

$$\frac{\partial g}{\partial \tau} \leq 2\,\mathrm{Ric}(g(\tau)). \tag{3.2}$$

Here the relevant notion of diffusion on an evolving manifold $(\mathcal{M}, g(\tau))$ moves the probability density u (with respect to the evolving Riemannian measure $\mu_{g(\tau)}$) by the parabolic equation

$$\frac{\partial u}{\partial \tau} = \Delta_{g(\tau)} u - \frac{1}{2}\mathrm{tr}\left(\frac{\partial g}{\partial \tau}\right) u,$$

and we refer to the resulting one-parameter families of measures simply as *diffusions*. This way, if we define a local top-dimensional form $\omega(\tau) := u\, dV_{g(\tau)}$, then

$$\frac{\partial \omega}{\partial \tau} = \Delta_{g(\tau)} \omega, \tag{3.3}$$

where Δ is here the connection Laplacian, and thus the evolution of the measures corresponds to Brownian motion. The following characterisation was proved for the W_2 distance in [12]. Tom Ilmanen has pointed out to us (via Robert McCann) that this extends to the case of W_1 distance, giving:

Theorem 3.1. *(cf. [12, Theorem 2]) Suppose that $\mathcal{M}$ is a closed manifold equipped with a smooth family of metrics $g(\tau)$ for $\tau \in [\tau_1, \tau_2] \subset \mathbb{R}$. Then the following are equivalent:*

(A) $g(\tau)$ is a super Ricci flow (i.e. satisfies (3.2));

(B) whenever $\tau_1 < a < b < \tau_2$ and $\nu_1(\tau)$, $\nu_2(\tau)$ are diffusions (as defined above) for $\tau \in (a, b)$, the function $\tau \mapsto W_1^{g(\tau)}(\nu_1(\tau), \nu_2(\tau))$ is weakly decreasing in $\tau \in (a, b)$;

(C) whenever $\tau_1 < a < b < \tau_2$ and $f : \mathcal{M} \times (a, b) \to \mathbb{R}$ is a solution to $-\frac{\partial f}{\partial \tau} = \Delta_{g(\tau)} f$, the Lipschitz constant of $f(\cdot, \tau)$ with respect to $g(\tau)$ is weakly increasing in τ.

Proof. All implications in this theorem are proved exactly as in [12] except for $(A) \implies (B)$ whose proof, pointed out by Ilmanen, we now describe. Suppose that $g(\tau)$ is a super Ricci flow, and that $\nu_1(\tau)$ and $\nu_2(\tau)$ are two diffusions, all defined for τ in a neighbourhood of $\tau_0 \in (a, b)$. By Kantorovich-Rubinstein duality (see e.g. [19, §1.2.1]) we have

$$W_1^{g(\tau)}(\nu_1(\tau), \nu_2(\tau)) := \max\left\{ \int_{\mathcal{M}} \varphi d\nu_1(\tau) - \int_{\mathcal{M}} \varphi d\nu_2(\tau) \;\middle|\; \varphi : \mathcal{M} \to \mathbb{R} \text{ is Lipschitz and } \|\varphi\|_{Lip} \leq 1 \text{ with respect to } g(\tau) \right\}. \tag{3.4}$$

Let $\varphi_0 : \mathcal{M} \to \mathbb{R}$ be a function which achieves the maximum in this variational problem at time τ_0, and extend φ_0 to a function $\varphi : \mathcal{M} \times [\tau_0 - \varepsilon, \tau_0] \to \mathbb{R}$ for some $\varepsilon > 0$ by solving the equation

$$-\frac{\partial \varphi}{\partial \tau} = \Delta_{g(\tau)} \varphi.$$

By the implication $(A) \implies (C)$ of the theorem (proved in [12]) for all $\tau \in [\tau_0 - \varepsilon, \tau_0]$ we have $\|\varphi(\cdot,\tau)\|_{Lip} \leq 1$ and therefore $\varphi(\cdot,\tau)$ can be used as a competitor in the variational problem (3.4) to see that

$$W_1^{g(\tau)}(\nu_1(\tau),\nu_2(\tau)) \geq \int_{\mathcal{M}} \varphi(\cdot,\tau)d\nu_1(\tau) - \int_{\mathcal{M}} \varphi(\cdot,\tau)d\nu_2(\tau).$$

But by an integration by parts formula [17, §6.3] we know that the functions

$$\tau \mapsto \int_{\mathcal{M}} \varphi(\cdot,\tau)d\nu_1(\tau) \qquad \text{and} \qquad \tau \mapsto \int_{\mathcal{M}} \varphi(\cdot,\tau)d\nu_2(\tau)$$

are each independent of τ, so we deduce that

$$W_1^{g(\tau)}(\nu_1(\tau),\nu_2(\tau)) \geq W_1^{g(\tau_0)}(\nu_1(\tau_0),\nu_2(\tau_0)).$$

□

In the next section, we will demonstrate how the manifold $(\hat{\mathcal{M}},\hat{g})$ of Theorem 1.1 arises naturally by trying to reconcile Theorem 3.1 with the main result proved by the second author in [18], which we now rephrase into the most suggestive form for our present purposes. The idea of $\mathcal{L}$-optimal transportation [18] is to transport a probability measure from one time slice of a Ricci flow to another, using a cost function derived from Perelman's $\mathcal{L}$-length. More precisely, given a time interval $[\tau_1,\tau_2] \subset (0,\infty)$ in the domain of definition of the Ricci flow, we consider the cost function $c : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ induced by the Lagrangian $L(x,v,\tau) := \sqrt{\tau}(R(x,\tau) + |v|^2 - \frac{n}{2\tau})$ which gives

$$c(x,y) = \inf_{\gamma} \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(R(\gamma(\tau),\tau) + |\gamma'(\tau)|^2 - \frac{n}{2\tau}\right)d\tau,$$

where the infimum is taken over all C^1 curves $\gamma : [\tau_1,\tau_2] \to \mathcal{M}$ for which $\gamma(\tau_1) = x$ and $\gamma(\tau_2) = y$. Using the definitions from [13, §7] and [18]

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R(\gamma(\tau),\tau) + |\gamma'(\tau)|^2)d\tau; \qquad Q(x,\tau_1;y,\tau_2) = \inf_{\gamma} \mathcal{L}(\gamma),$$

where the infimum is again over curves γ as above, we can write

$$c(x,y) = Q(x,\tau_1;y,\tau_2) - n(\sqrt{\tau_2} - \sqrt{\tau_1}).$$

This cost function then induces a distance from one Borel probability measure ν_1 (viewed as existing at time τ_1) to another ν_2 (viewed at time τ_2) via the formula

$$\begin{aligned} \mathcal{D}(\nu_1,\tau_1;\nu_2,\tau_2) &= \inf_{\pi \in \Gamma(\nu_1,\nu_2)} \int_{\mathcal{M}\times\mathcal{M}} Q(x,\tau_1;y,\tau_2)d\pi(x,y) - n(\sqrt{\tau_2} - \sqrt{\tau_1}) \\ &=: V(\nu_1,\tau_1;\nu_2,\tau_2) - n(\sqrt{\tau_2} - \sqrt{\tau_1}), \end{aligned} \tag{3.5}$$

using V as defined in [18]. From the following theorem, one can recover [18] most of Perelman's monotonic quantities (both involving entropies and $\mathcal{L}$-length) which are central in his work on Ricci flow [13, 14, 15].

Theorem 3.2. *(Equivalent to [18, Theorem 1.1].) Suppose that $g(\tau)$ is a Ricci flow on a closed manifold $\mathcal{M}$ over an open time interval containing $[\bar{\tau}_1,\bar{\tau}_2]$, and suppose that $\nu_1(\tau)$ and $\nu_2(\tau)$ are two diffusions (in the same sense as in Theorem 3.1) defined for τ in neighbourhoods of $\bar{\tau}_1$ and $\bar{\tau}_2$ respectively. Then the distance between the diffusions decays in the sense that for $s \geq 1$ sufficiently close to 1,*

$$\mathcal{D}(\nu_1(s\bar{\tau}_1),s\bar{\tau}_1;\nu_2(s\bar{\tau}_2),s\bar{\tau}_2) \leq s^{-\frac{1}{2}}\mathcal{D}(\nu_1(\bar{\tau}_1),\bar{\tau}_1;\nu_2(\bar{\tau}_2),\bar{\tau}_2).$$

This formulation of the theorem indicated to us that we should look for a context in which the result arises as an application of Theorem 3.1. This leads to Theorem 1.1, and we explain the connection in the next section.

4 The relationship between $(\hat{\mathcal{M}}, \hat{g})$ and $\mathcal{L}$-optimal transportation

In this section, as opposed to all others in this paper, we allow ourselves to make purely heuristic arguments. We present a formal argument to recover Theorem 3.2 by applying Theorem 3.1 to a flow starting at the specific manifold $(\hat{\mathcal{M}}, \hat{g})$ of Theorem 1.1. While this is the simplest way to explain the link between our new manifold $(\hat{\mathcal{M}}, \hat{g})$ and optimal transportation, we stress that the main interest here is the reverse flow of ideas: the Ricci soliton $(\hat{\mathcal{M}}, \hat{g})$ was *discovered* by trying to find a context in which Theorem 3.1 formally implied Theorem 3.2.

We begin this section by arguing formally that shortest paths in $(\hat{\mathcal{M}}, \hat{g})$ correspond to $\mathcal{L}$-geodesics for the original Ricci flow. Suppose $x, y \in \mathcal{M}$ and $[\tau_1, \tau_2] \subset (0, \infty)$ lies within the time domain on which the Ricci flow is defined. Consider paths $\Gamma : [\tau_1, \tau_2] \to \hat{\mathcal{M}}$ connecting (x, τ_1) and (y, τ_2) in $\hat{\mathcal{M}}$ of the form $\Gamma(\tau) = (\gamma(\tau), \tau)$, where $\gamma : [\tau_1, \tau_2] \to \mathcal{M}$ satisfies $\gamma(\tau_1) = x$ and $\gamma(\tau_2) = y$. Then

$$\begin{aligned} Length(\Gamma) &= \int_{\tau_1}^{\tau_2} \left| \gamma'(\tau) + \frac{\partial}{\partial \tau} \right|_{\hat{g}} d\tau \\ &= \int_{\tau_1}^{\tau_2} \left[\frac{|\gamma'|^2_{g(\tau)}}{\tau} + \frac{N}{2\tau^3} + \frac{R}{\tau} - \frac{n}{2\tau^2} \right]^{\frac{1}{2}} d\tau \\ &= \int_{\tau_1}^{\tau_2} \left[\frac{N}{2\tau^3} \right]^{\frac{1}{2}} \left(1 + \frac{\tau^2 |\gamma'|^2}{N} + \frac{\tau^2 R}{N} - \frac{\tau n}{2N} + O(\tfrac{1}{N^2}) \right) d\tau \\ &= \sqrt{2N}(\tau_1^{-\frac{1}{2}} - \tau_2^{-\frac{1}{2}}) + \frac{1}{\sqrt{2N}} \left[\mathcal{L}(\gamma) - n(\sqrt{\tau_2} - \sqrt{\tau_1}) \right] + O(\tfrac{1}{N^{3/2}}). \end{aligned} \tag{4.1}$$

Just as in [13, §6], this indicates that minimising paths Γ should essentially arise from $\mathcal{L}$-geodesics γ for large N, and suggests the formula

$$d_{\hat{g}}((x, \tau_1), (y, \tau_2)) = \sqrt{2N}(\tau_1^{-\frac{1}{2}} - \tau_2^{-\frac{1}{2}}) + \frac{1}{\sqrt{2N}} \left[Q(x, \tau_1; y, \tau_2) - n(\sqrt{\tau_2} - \sqrt{\tau_1}) \right] + O(\tfrac{1}{N^{3/2}}). \tag{4.2}$$

We now wish to apply Theorem 3.1 to certain diffusions on a reverse Ricci flow $G(s)$ on *space-time* starting at time $s = 1$ with the metric $G(1) = \hat{g}$ on $\hat{\mathcal{M}} = \mathcal{M} \times (a, b)$. Since $\hat{g}$ satisfies the (approximate) Ricci soliton equation (1.2), the theory of Ricci solitons (see [17, §1.2.2]) tells us that (modulo errors) we can take $G(s)$ to be

$$G(s) := s\, \psi_s^*(\hat{g}) \tag{4.3}$$

where $\psi_1 : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ is the identity, and ψ_s is the family of maps obtained by integrating the vector field $X_s := -\frac{1}{s} \hat{\nabla}(\frac{N}{2\tau})$ with $\hat{\nabla}$ representing the gradient with respect to $\hat{g}$. We may compute

$$X_s = \frac{N}{2s\tau^2} \hat{g}_{00}^{-1} \frac{\partial}{\partial \tau} = \frac{\tau}{s} \frac{\partial}{\partial \tau} + O(\tfrac{1}{N}), \tag{4.4}$$

and by neglecting the error of order $\frac{1}{N}$ (but see also Section 6) this integrates to

$$\psi_s(x, \tau) \simeq (x, s\tau).$$

In particular, the (approximate, reverse) Ricci flow $G(s)$ operates by pulling back $\hat{g}$ in time, and scaling appropriately.

Applying Theorem 3.1, we find that two diffusions on the evolving manifold with metric $G(s)$ should get closer in the W_1 sense as s increases. The Ricci flow $g(\tau)$ we use to generate $(\hat{\mathcal{M}}, \hat{g})$ will be that in the hypotheses of Theorem 3.2. The measures we wish to put into Theorem 3.1 will be derived from the measures $\nu_1(\tau)$ and $\nu_2(\tau)$ appearing in the hypotheses of Theorem 3.2, together with the corresponding $\bar{\tau}_1$ and $\bar{\tau}_2$. Let $F_\tau : \mathcal{M} \to \hat{\mathcal{M}}$ be defined to be the embedding $F_\tau(x) = (x, \tau)$; then the initial measures we wish to put into Theorem 3.1 are $(F_{\bar{\tau}_k})_\# \nu_k(\bar{\tau}_k)$, for $k = 1, 2$, which are each supported on a time-slice in $\hat{\mathcal{M}}$. Because of the extreme stretching of the τ direction in the metric $\hat{g}$ (recall that $\hat{g}_{00}$ is of order N) there is essentially no diffusion of the measures in the τ direction, and we view them as remaining supported in the time slices $\mathcal{M} \times \{\bar{\tau}_k\}$. They then evolve mainly under diffusion in the $\mathcal{M}$ factor, under the Laplacian induced by $G_{ij}(s)$. By (4.3) and the definition of $\hat{g}$ from Theorem 1.1, on the time-slice $\mathcal{M} \times \{\bar{\tau}_k\}$ we have (approximately)

$$G_{ij}(s) = s[\psi_s^*(\hat{g})]_{ij} = s[\hat{g}|_{\mathcal{M} \times \{s\bar{\tau}_k\}}]_{ij} = \frac{g_{ij}(s\bar{\tau}_k)}{\bar{\tau}_k}.$$

Therefore, the measures evolve by diffusion in the $\mathcal{M}$ factor under

$$\Delta_{\frac{g(s\bar{\tau}_k)}{\bar{\tau}_k}} = \bar{\tau}_k \Delta_{g(s\bar{\tau}_k)},$$

which is also the evolution of $s \mapsto \nu_k(s\bar{\tau}_k)$. In other words, we have (formally) deduced that

$$s \mapsto W_1^{G(s)}((F_{\bar{\tau}_1})_\# \nu_1(s\bar{\tau}_1), (F_{\bar{\tau}_2})_\# \nu_2(s\bar{\tau}_2))$$

is weakly decreasing.

If we now push forward this whole construction under the maps ψ_s, and adopt the abbreviation

$$\hat{\nu}_k(\tau) := (F_\tau)_\# \nu_k(\tau),$$

then we find that

$$s \mapsto W_1^{s\hat{g}}(\hat{\nu}_1(s\bar{\tau}_1), \hat{\nu}_2(s\bar{\tau}_2)) = s^{\frac{1}{2}} W_1^{\hat{g}}(\hat{\nu}_1(s\bar{\tau}_1), \hat{\nu}_2(s\bar{\tau}_2)) \tag{4.5}$$

is weakly decreasing.

Now the expansion (4.2) of the Riemannian distance on $(\hat{\mathcal{M}}, \hat{g})$ suggests that

$$W_1^{\hat{g}}(\hat{\nu}_1(\tau_1), \hat{\nu}_2(\tau_2)) \simeq \sqrt{2N}(\tau_1^{-\frac{1}{2}} - \tau_2^{-\frac{1}{2}}) + \frac{1}{\sqrt{2N}} \mathcal{D}(\nu_1(\tau_1), \tau_1; \nu_2(\tau_2), \tau_2).$$

Therefore, the monotonicity in (4.5) implies that

$$s \mapsto \sqrt{2N}(\bar{\tau}_1^{-\frac{1}{2}} - \bar{\tau}_2^{-\frac{1}{2}}) + \frac{s^{\frac{1}{2}}}{\sqrt{2N}} \mathcal{D}(\nu_1(s\bar{\tau}_1), s\bar{\tau}_1; \nu_2(s\bar{\tau}_2), s\bar{\tau}_2)$$

is monotonically decreasing, and hence so is

$$s \mapsto s^{\frac{1}{2}} \mathcal{D}(\nu_1(s\bar{\tau}_1), s\bar{\tau}_1; \nu_2(s\bar{\tau}_2), s\bar{\tau}_2),$$

which is the content of Theorem 3.2 as desired.

5 Construction of a space-time steady soliton

In this section we make a steady soliton construction analogous to the shrinking soliton construction of Theorem 1.1. There is also an expanding soliton construction similar to that of Theorem 1.1 which we will use in [1] to prove Harnack inequalities. Only the shrinking case is adapted to Perelman's $\mathcal{L}$-length and his monotonic quantities which are so important in the study of finite-time singularities [13, 14, 15]. However, the steady case is the simplest construction of all and has its own potential applications.

Theorem 5.1. *Suppose $g(\tau)$ is a (reverse) Ricci flow – i.e. a solution of $\frac{\partial g}{\partial \tau} = 2\,\mathrm{Ric}(g(\tau))$ – defined for τ within a time interval $(a,b) \subset \mathbb{R}$, on a manifold $\mathcal{M}$ of dimension $n \in \mathbb{N}$, and with bounded curvature. Suppose $N \in \mathbb{N}$ is sufficiently large to give a positive definite metric $\bar{g}$ on $\hat{\mathcal{M}} := \mathcal{M} \times (a,b)$ defined by*

$$\bar{g}_{ij} = g_{ij}; \qquad \bar{g}_{00} = N + R; \qquad \bar{g}_{0i} = 0,$$

where i, j are coordinate indices on the $\mathcal{M}$ factor and 0 represents the index of the time coordinate $\tau \in (a,b)$.

Then up to errors of order $\frac{1}{N}$, the metric $\bar{g}$ is a gradient steady Ricci soliton on $\hat{\mathcal{M}}$:

$$\mathrm{Ric}(\bar{g}) + \mathrm{Hess}_{\bar{g}}(-N\tau) \simeq 0, \tag{5.1}$$

in the same sense as in Theorem 1.1.

For this metric, the Christoffel symbols can be computed to be

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}; \quad \bar{\Gamma}^i_{j0} = R^i{}_j; \quad \bar{\Gamma}^i_{00} = -\frac{1}{2} g^{ik} \frac{\partial R}{\partial x^k};$$

$$\bar{\Gamma}^0_{jk} = -\frac{1}{N+R} R_{jk}; \quad \bar{\Gamma}^0_{j0} = \frac{1}{2}\frac{\partial}{\partial x^j}\ln(N+R); \quad \bar{\Gamma}^0_{00} = \frac{1}{2}\frac{\partial}{\partial \tau}\ln(N+R).$$

The Ricci coefficients are, up to errors of order $\frac{1}{N}$,

$$\bar{R}_{ij} \simeq R_{ij}; \qquad \bar{R}_{i0} \simeq -\frac{\nabla_i R}{2}; \qquad \bar{R}_{00} \simeq -\frac{R_\tau}{2},$$

and the coefficients of $\mathrm{Hess}_{\bar{g}}(-N\tau)$ are

$$\bar{\nabla}^2_{ij}(-N\tau) \simeq -R_{ij}; \qquad \bar{\nabla}^2_{i0}(-N\tau) \simeq \frac{\nabla_i R}{2}; \qquad \bar{\nabla}^2_{00}(-N\tau) \simeq \frac{R_\tau}{2},$$

which yields Theorem 5.1.

Just as the $(\hat{\mathcal{M}}, \hat{g})$ of Theorem 1.1 encodes Perelman's $\mathcal{L}$-length in its geodesic distance, in the sense of (4.1), the manifold $(\hat{\mathcal{M}}, \bar{g})$ of Theorem 5.1 encodes Li-Yau's length [10]

$$\mathcal{L}_0(\gamma) := \int_{\tau_1}^{\tau_2} (R(\gamma(\tau), \tau) + |\gamma'(\tau)|^2) d\tau$$

which predates $\mathcal{L}(\gamma)$, via the formula (using the same notation as in (4.1))

$$Length(\Gamma) = \sqrt{N}(\tau_2 - \tau_1) + \frac{1}{2\sqrt{N}}\mathcal{L}_0(\gamma) + O(\tfrac{1}{N^{3/2}}).$$

If one considers optimal transportation on this alternative $(\hat{\mathcal{M}}, \bar{g})$, then one recovers the variant of $\mathcal{L}$-optimal transportation using the $\mathcal{L}_0$-length of Li-Yau, as studied in [11].

6 Mean curvature of time-slices

In the framework of Theorem 1.1, the underlying manifold $\mathcal{M}$ of our original Ricci flow can be regarded as a hypersurface $\mathcal{M} \times \{\tau\}$ of the ambient $(\hat{\mathcal{M}}, \hat{g})$. Its mean curvature is $H = H^{\hat{g}}_{\mathcal{M}\times\{\tau\}} = \hat{g}_{00}^{-\frac{1}{2}}(R - \frac{n}{2\tau})$; in fact, the mean curvature vector $\vec{H}^{\hat{g}}_{\mathcal{M}\times\{\tau\}} := -H\nu =$

$-H\hat{g}_{00}^{-\frac{1}{2}}\frac{\partial}{\partial\tau}$ gives precisely the term we neglected in the approximate computation of X_s from (4.4). More precisely, we have the exact formula

$$-\hat{\nabla}\left(\frac{N}{2\tau}\right) = \tau\frac{\partial}{\partial\tau} + \vec{H}^{\hat{g}}_{\mathcal{M}\times\{\tau\}}. \tag{6.1}$$

If we suppose that $\tau_0 \in U \subset\subset (a,b)$, we can pick $\varepsilon > 0$ sufficiently small so that for $s \in (1-\varepsilon, 1+\varepsilon)$ the maps ψ_s arising from integrating the vector field

$$X_s := -\frac{1}{s}\hat{\nabla}(\frac{N}{2\tau})$$

starting with $\psi_1 : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ the identity, restrict to well-defined maps $\mathcal{M}\times U \mapsto \hat{\mathcal{M}}$, diffeomorphic onto their images. It then makes sense at a rigorous level to define $G(s) := s\,\psi_s^*(\hat{g})$ as a flow on $\mathcal{M}\times U$ for $s \in (1-\varepsilon, 1+\varepsilon)$, which is then approximately a reverse Ricci flow as in Section 4. By reducing $\varepsilon > 0$ further, we can also be sure that $\mathcal{M}\times\{s\tau_0\} \subset \psi_s(\mathcal{M}\times U)$ for $s \in (1-\varepsilon, 1+\varepsilon)$.

We then have the following exact statement (i.e. involving no errors at all) relating reverse mean curvature flow to certain one-parameter families of time-slices $\mathcal{M}\times\{\tau\}$, which is in a similar spirit to [3, Lemma 3.2].

Theorem 6.1. *Given U, τ_0, ε, ψ_s and the flow $G(s)$ as above, if we define $\mathcal{F} : \mathcal{M}\times(1-\varepsilon, 1+\varepsilon) \to \mathcal{M}\times U$ by $\mathcal{F}(x,s) = \psi_s^{-1}(x, s\tau_0)$, then the one-parameter family of submanifolds*

$$\mathcal{N}_s := \mathcal{F}(\mathcal{M}\times\{s\}) = \psi_s^{-1}(\mathcal{M}\times\{s\tau_0\})$$

is a reverse mean curvature flow within the flow $G(s)$ in the sense that

$$-\frac{\partial\mathcal{F}(x,s)}{\partial s} = \vec{H}^{G(s)}_{\mathcal{N}_s}(\mathcal{F}(x,s)).$$

Note that here both the (approximate) Ricci flow $G(s)$ and the mean curvature flow $\mathcal{N}_s$ evolve in the same direction – backwards with respect to s.

Proof. By differentiating the expression $\psi_s\circ\psi_s^{-1} = \textit{identity}$ with respect to s, we find that

$$\begin{aligned}(\psi_s)_*\frac{\partial\psi_s^{-1}}{\partial s}(x,s\tau_0) &= -\frac{\partial\psi_s}{\partial s}(\psi_s^{-1}(x,s\tau_0)) = -X_s(x,s\tau_0) = \frac{1}{s}\hat{\nabla}\left(\frac{N}{2\tau}\right)(x,s\tau_0)\\ &= -\tau_0\frac{\partial}{\partial\tau} - \frac{1}{s}\vec{H}^{\hat{g}}_{\mathcal{M}\times\{s\tau_0\}}(x,s\tau_0),\end{aligned} \tag{6.2}$$

where we have used (6.1). Therefore, we may compute

$$\begin{aligned}\frac{\partial}{\partial s}\left(\psi_s^{-1}(x,s\tau_0)\right) &= \frac{\partial\psi_s^{-1}}{\partial s}(x,s\tau_0) + (\psi_s^{-1})_*\left(\tau_0\frac{\partial}{\partial\tau}\right) = -(\psi_s^{-1})_*\left[\frac{1}{s}\vec{H}^{\hat{g}}_{\mathcal{M}\times\{s\tau_0\}}(x,s\tau_0)\right]\\ &= -\frac{1}{s}\vec{H}^{\psi_s^*(\hat{g})}_{\mathcal{N}_s}(\psi_s^{-1}(x,s\tau_0)) = -\vec{H}^{G(s)}_{\mathcal{N}_s}(\mathcal{F}(x,s)).\end{aligned}$$

□

We conclude with an observation about mean curvature flow of one dimension less. Suppose that $(\mathcal{M}, g(\tau))$ is a reverse Ricci flow for $\tau \in (a,b)$ and that for some $(n-1)$-dimensional manifold P, the map $\sigma : P\times(a,b) \to \mathcal{M}$ is a reverse mean curvature flow in the sense that $P_\tau := \sigma(P\times\{\tau\})$ is a family of smooth hypersurfaces of $\mathcal{M}$ and

$$-\frac{\partial\sigma}{\partial\tau}(x,\tau) = \vec{H}^{g(\tau)}_{P_\tau}(\sigma(x,\tau)).$$

Then the space-time track, which is the image of $\Sigma : P \times (a, b) \to \hat{\mathcal{M}}$ defined by

$$\Sigma(x, \tau) := (\sigma(x, \tau), \tau),$$

is ϕ-minimal in $(\hat{\mathcal{M}}, \hat{g})$ for $\phi = \frac{N}{2\tau}$, up to errors of order $\frac{1}{N}$, in the sense that

$$H_\phi := H^{\hat{g}}_{\Sigma(P\times(a,b))} - \langle \hat{\nabla}\phi, \nu_\Sigma \rangle = O(N^{-1}),$$

where ν_Σ is the unit outward normal to $\Sigma(P \times (a, b))$ and we have used the natural generalization of mean curvature introduced in [6]. (See also Chapter 11 §3.10 of [5] for the corresponding property of Perelman's metric $\tilde{g}$ described in Section 1.)

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