Christmas Homework Present

MA136 Introduction to Abstract Algebra

(1) Show that any subgroup of a cyclic group is cyclic.

(2) Let G be an abelian group. Show that if σ, $\tau \in G$ have orders r, s respectively, then $\sigma\tau$ has order dividing $\operatorname{lcm}(r, s)$. Give a **counterexample** to show that this does not necessarily hold for a non-abelian group.

(3) Write $\mathbb{Z}[2i] = \{a + 2bi : a, b \in \mathbb{Z}\}$. Show that $\mathbb{Z}[2i]$ is a subring of $\mathbb{C}$. Compute its unit group.

(4) Is $\{2a + 2bi : a, b \in \mathbb{Z}\}$ as subring of $\mathbb{C}$?

(5) Which of the following are subrings of $M_{2\times 2}(\mathbb{R})$? If so, are they commutative?

(i) $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$.

(ii) $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

(iii) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{Z} \right\}$.

(6) Let

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \in \mathbb{Z}, b \in \mathbb{R} \right\}.$$

Show that S is a ring under the usual addition and multiplication of matrices. Compute S^*.

(7) Show that $(\mathbb{Z}/7\mathbb{Z})^*$ is cyclic but $(\mathbb{Z}/8\mathbb{Z})^*$ is not.

(8) Show that the only subring of $\mathbb{Z}$ is $\mathbb{Z}$. Show that the only subring of $\mathbb{Z}[i]$ containing i is $\mathbb{Z}[i]$.

(9) Let

$$S = \left\{ \frac{a}{2^r} : a,\ r \in \mathbb{Z},\ r \geq 0 \right\}.$$

Show that S is a ring and find its unit group.

(10) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Show that $\mathbb{Z}[\sqrt{2}]$ is a ring and that $1 + \sqrt{2}$ is unit. What is its order?

(11) Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that $\mathbb{Q}[\sqrt{2}]$ is a field.

(12) Let

$$F = \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) : a, b \in \mathbb{R} \right\}.$$

(a) Show that F is a field (under the usual addition and multiplication of matrices). (**Hint:** Begin by showing that F is a subring of $M_{2\times 2}(\mathbb{R})$. You need to also show that F is commutative and that every non-zero element has an inverse in F.)

(b) Let $\phi : F \to \mathbb{C}$ be given by $\phi \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) = a + bi$. Show that ϕ is a bijection that satisfies $\phi(A + B) = \phi(A) + \phi(B)$ and $\phi(AB) = \phi(A)\phi(B)$.

(c) Show that

$$F' = \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) : a, b \in \mathbb{C} \right\}$$

is not a field.

(13) Let $\zeta = e^{2\pi i/3}$ (this is a cube root of unity). Check that $\overline{\zeta} = \zeta^2$. Let $\mathbb{Z}[\zeta] = \{a + b\zeta : a, b \in \mathbb{Z}\}$.

(a) Show that $\zeta^2 \in \mathbb{Z}[\zeta]$ (**Hint:** the sum of the cube roots of unity is $\ldots$).

(b) Show that $\mathbb{Z}[\zeta]$ is a ring.

(c) Show that ± 1, $\pm\zeta$ and $\pm\zeta^2$ are units in $\mathbb{Z}[\zeta]$.

(d) (Hard) Show that $\mathbb{Z}[\zeta]^* = \{\pm 1, \pm\zeta, \pm\zeta^2\}$. Show that this group is cyclic.

(14) A commutative ring R is an *integral domain* if it satisfies the following property: for all x, $y \in R$, if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(a) Show that every field is an integral domain.

(b) Show that $\mathbb{Z}/m\mathbb{Z}$ is an integral domain if and only if m is prime.

(c) In Question (5) you showed that

$$\{ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix} \right) : a \in \mathbb{R},\ b \in \mathbb{Z} \}$$

is a commutative ring. Is it an integral domain?

(d) Let R be an integral domain, and x a non-zero element of R. Let $f_x : R \to R$ be given by $f_x(y) = xy$.

(i) Show that f_x is injective.

(ii) Suppose R is finite. Show that x is a unit (**Hint:** apply the pigeon-hole principle to f_x.)

(iii) Deduce that a finite integral domain is a field.