Modularity and the Fermat Equation over Totally Real Fields

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joint work with Nuno Freitas (Bayreuth/MPIM)

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Motivation

Theorem (Wiles)

The only solutions to the equation

$$a^p + b^p + c^p = 0, \qquad p \geq 5 \text{ prime}$$

satisfy $abc = 0$.

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Theorem (Wiles, Breuil, Conrad, Diamond, Taylor)

All elliptic curves over $\mathbb{Q}$ *are modular.*

More Motivation

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Semistable elliptic curves over $\mathbb{Q}(\sqrt{2})$ *and* $\mathbb{Q}(\sqrt{17})$ *are modular.*

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"... the numerology required to generalise the work of Ribet and Wiles directly continues to hold for $\mathbb{Q}(\sqrt{2})$*... there are no other real quadratic fields for which this is true ..."(Jarvis and Meekin)*

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There are at most finitely many j-invariants of elliptic curves over K that are non-modular.

Theorem (Freitas–Le Hung–S.)

If K is real quadratic, then all elliptic curves over K are modular.

Demystifying the proof of FLT: The Tate Curve

- ℓ prime
- $G_\ell = \mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$
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$$\sigma(\zeta_p) = \zeta_p^a, \qquad \sigma(q^{1/p}) = \zeta_p^b q^{1/p},$$

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Think of ζ_p and $q^{1/p}$ as an $\mathbb{F}_p$-basis for $E[p]$.

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Obtain a representation

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Lemma

- *If $p \mid \upsilon_\ell(q)$ then $\#\overline{\rho}_p(I_\ell) = 1$.*
- *If $p \nmid \upsilon_\ell(q)$ then $\#\overline{\rho}_p(I_\ell) = p$.*

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Global Calculations

- $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
- $E/\mathbb{Q}$ an elliptic curve
- Δ minimal discriminant
- N conductor
- $p \neq 2$ prime
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Better Guess:

$$N(\overline{\rho}_p) = \frac{N}{M_p}, \qquad M_p = \prod_{\substack{\ell || N \\ p | \upsilon_\ell(\Delta)}} \ell.$$

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Then

$$\Delta = 16a^{2p}b^{2p}(a^p + b^p)^2 = 16a^{2p}b^{2p}c^{2p}, \qquad N = 2^? \cdot \prod_{\substack{\ell | abc \\ \ell \neq 2}} \ell.$$

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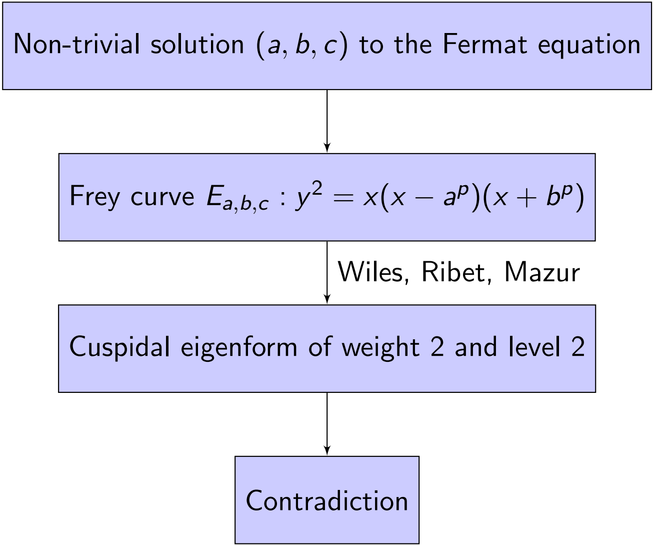
Thus $N(\overline{\rho}_p) = 2^?$. With care, $N(\overline{\rho}_p) = 2$.

Fermat equation $a^p + b^p + c^p = 0$ over $\mathbb{Q}$



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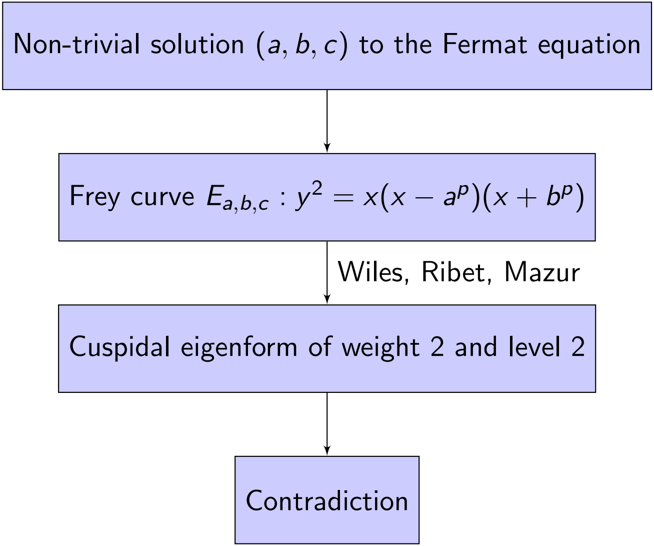

Accident # 1 : there are no newforms of weight 2 **and level** 2.

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Accident # 2: $h(\mathbb{Z}) = 1$.

Fermat $a^p + b^p + c^p = 0$ over a totally real field K

Non-trivial solution (a, b, c) to the Fermat equation

↓

Frey curve $E = E_{a,b,c} : y^2 = x(x - a^p)(x + b^p)$

↓ modulo big theorems and conjectures . . .

Hilbert cuspidal eigenform of weight 2 and one of many levels

Conclusion: $\overline{\rho}_{E,p} \sim \overline{\rho}_{\mathfrak{f},\varpi}$ (where $\varpi \mid p$) for some Hilbert eigenform of parallel weight 2 and at one of these levels.

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$$B(\mathfrak{f},\mathfrak{q}) := (a_{\mathfrak{q}}(\mathfrak{f}) - \mathbb{N}(\mathfrak{q}) - 1)(a_{\mathfrak{q}}(\mathfrak{f}) + \mathbb{N}(\mathfrak{q}) + 1) \prod_{|t| \le 2\sqrt{\mathbb{N}(\mathfrak{q})}} (a_{\mathfrak{q}}(\mathfrak{f}) - t)$$

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Conclusion: $\overline{\rho}_{E,p} \sim \overline{\rho}_{\mathfrak{f},\varpi}$ (where $\varpi \mid p$) for some Hilbert eigenform of parallel weight 2 and at one of these levels.

Let $\mathfrak{q}$ be a prime of K. Then

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So ϖ divides

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CONTRADICTION!
Conclusion: $\mathfrak{f}$ has rational eigenvalues.

Asymptotic Fermat $a^p + b^p + c^p = 0$ over a totally real field K

Non-trivial solution (a, b, c) to the Fermat equation with p large

↓

Frey curve $E_{a,b,c} : y^2 = x(x - a^p)(x + b^p)$

↓ Jarvis, Fujiwarwa, Rajaei, Merel, Momose,.

Hilbert eigenform of weight 2 and level ??, rational eigenvalues

↓ modulo an 'Eichler–Shimura' conjecture

E/K with full 2-torsion, $j(E) \in \mathcal{O}_K[1/2]$, additional properties

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where $s > t > 0$ and

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Therefore, the density of $d_{s,t}$

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Answer: $\delta(d_{s,t}) = 0$.

The Asymptotic FLT

Theorem (Freitas–S.)

If we assume a suitable "Eichler–Shimura" conjecture, then the asymptotic FLT holds for almost all real quadratic fields.

Unconditionally, *the asymptotic FLT holds for* $5/6$ *of real quadratic fields.*

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Thank You!