Algebraic Number Theory
Example Sheet 4

Hand in the answers to questions 3, 6, 7. Deadline 2pm Thursday, Week 10.

(1) Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of $\mathcal{O}_K$ with $\mathfrak{a} \subseteq \mathfrak{b}$.
  (i) Show that $\mathrm{Norm}(\mathfrak{a}) \geq \mathrm{Norm}(\mathfrak{b})$.
  (ii) Show that $\mathrm{Norm}(\mathfrak{a}) = \mathrm{Norm}(\mathfrak{b})$ if and only if $\mathfrak{a} = \mathfrak{b}$.

(2) Let K be a number field. Show that $\mathcal{O}_K$ is a PID if and only if it is a UFD.

(3) Let $K = \mathbb{Q}(\sqrt{-2})$. Show that $\mathcal{O}_K$ is a principal ideal domain. Deduce that every prime $p \equiv 1, 3 \pmod 8$ can be written as $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$.

(4) Compute the class groups of the following quadratic fields

$$\mathbb{Q}(\sqrt{5}), \qquad \mathbb{Q}(\sqrt{-6}), \qquad \mathbb{Q}(\sqrt{-30}).$$

(5) (i) Let α, β be non-zero elements of $\mathcal{O}_K$. Suppose $\langle \alpha \rangle = \langle \beta \rangle$. Show that $\alpha = \beta\varepsilon$ for some $\varepsilon \in U(K)$.
  (ii) Let $\mathfrak{a}$, $\mathfrak{b}$ be non-zero ideals with $\mathfrak{a} + \mathfrak{b} = \langle 1 \rangle$ (we say $\mathfrak{a}$, $\mathfrak{b}$ are coprime). Show that $\mathfrak{a}$, $\mathfrak{b}$ are coprime in the following sense: if $\mathfrak{p}$ is a prime ideal then $\mathfrak{p}$ divides at most one of $\mathfrak{a}$, $\mathfrak{b}$.
  (iii) Let $\mathfrak{a}$, $\mathfrak{b}$ be coprime non-zero ideals. Suppose $\mathfrak{a}\mathfrak{b} = \mathfrak{c}^n$ where $\mathfrak{c}$ is an ideal and n is a positive integer. Show that there are ideals $\mathfrak{c}_1$, $\mathfrak{c}_2$ such that

$$\mathfrak{a} = \mathfrak{c}_1^n, \qquad \mathfrak{b} = \mathfrak{c}_2^n, \qquad \mathfrak{c} = \mathfrak{c}_1\mathfrak{c}_2.$$

  (iv) Give a counterexample, with $K = \mathbb{Q}$, to show that (iii) fails if $\mathfrak{a}$, $\mathfrak{b}$ are not coprime.
  (v) Let $x, y \in \mathbb{Z}$ and satisfy $x^2 + 2 = y^3$. Show that x, y are odd, and deduce that the ideals $\mathfrak{a} = \langle x + \sqrt{-2} \rangle$, $\mathfrak{b} = \langle x - \sqrt{-2} \rangle$ are coprime.
  (vi) Continuing from (v), show carefully that $x + \sqrt{-2} = (u + v\sqrt{-2})^3$ for some $u, v \in \mathbb{Z}$. Hence determine the solutions to $x^2 + 2 = y^3$ with $x, y \in \mathbb{Z}$.

(6) Let $K = \mathbb{Q}(\sqrt{-5})$.
  (a) Show that $\mathrm{Cl}(K) \cong C_2$.
  (b) Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ and suppose $\mathfrak{a}^3$ is principal. Show that $\mathfrak{a}$ is principal.
  (c) Solve $x^2 + 5 = y^3$ with $x, y \in \mathbb{Z}$.

(7) Let $K = \mathbb{Q}(\sqrt[3]{2})$. You may suppose that $1, \sqrt[3]{2}, \sqrt[3]{2}^2$ is an integral basis for $\mathcal{O}_K$. Show that

$$U(K) = \{\pm(\sqrt[3]{2} - 1)^n \ : \ n \in \mathbb{Z}\}.$$

You may need to use `WolframAlpha`, `MATLAB` or a similar package to compute approximations to the embeddings of some algebraic numbers.