MA131 - Analysis 1

Workbook 9
Series III

Autumn 2004

Contents

4.14 Series with Positive and Negative Terms

With the exception of the Null Sequence Test, all the tests for series convergence and divergence that we have considered so far have dealt only with series of non-negative terms. Series with *both* positive and negative terms are harder to deal with.

Exercise 1 Why don't we have to separately consider series which have only negative terms?

4.15 Alternating Series

One very special case is a series whose terms alternate in sign from positive to negative. That is, series of the form $\sum(-1)^{n+1}a_n$ where $a_n \geq 0$.
Example $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is an alternating series.

Assignment 1
Let $s_n = \sum_{r=1}^{n}(-1)^{r+1}/r$. Prove that s_n is convergent using the following steps. Make sure that your proof also proves the inequality $\left|\sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k}\right| \leq \frac{1}{n}$.

1. Prove that $s_{2n+2} - s_{2n}$ is positive. This shows that the sequence (s_{2n}) is strictly increasing.
2. Now find $s_{2n+3} - s_{2n+1}$ and prove that it is negative. This shows that (s_{2n+1}) is strictly decreasing.
3. Using the fact that
$$s_2 < s_{2n} + \frac{1}{2n+1} = s_{2n+1} < s_1,$$
deduce that (s_{2n}) and (s_{2n+1}) are both convergent to the same limit.
4. To show that s_n converges to s, prove that $s_{2n+1} - \frac{1}{2n+1} \leq s \leq s_{2n} + \frac{1}{2n+1}$ and then show that $|s_n - s| < \frac{1}{n}$ for both odd and even $n \in \mathbb{N}$. Conclude that $(s_n) \to s$.

All The Way
Don't stop once you've proved that (s_{2n}) and (s_{2n+1}) converge. You still have to show that the whole sequence of partial sums (s_n) converges.

Assignment 2
Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Find a value of N so that $|\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - s| \leq 10^{-6}$.

Theorem *Alternating Series Test*
Suppose (a_n) is non-negative, decreasing and null. Then the alternating series $\sum(-1)^{n+1}a_n$ is convergent.

Example Since $\left(\frac{1}{n}\right)$ is a decreasing null sequence of positive terms, this test tells us right away that $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is convergent.

Similarly, $\left(\frac{1}{n^2}\right)$ is a decreasing null sequence of positive terms, therefore $\sum \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots$ is convergent.

Assignment 3
Find a sequence (a_n) which is non-negative and decreasing but where $\sum(-1)^{n+1}a_n$ is divergent and a sequence (b_n) which is non-negative and null but where $\sum(-1)^{n+1}b_n$ is divergent.

Assignment 4
Using the steps below (a generalisation of assignment 1) prove the Alternating Series Test. Suppose that (a_n) is non-negative, decreasing and null. Let $s_n = \sum_{r=1}^{n}(-1)^{r+1}a_r$.

1. Show that $s_{2n+2} - s_{2n} > 0$ and that $s_{2n+3} - s_{2n+1} < 0$.
2. Prove that $s_{2n} \le s_{2n} + a_{2n+1} = s_{2n+1} \le s_1$.
3. Show that the sequences (s_{2n}) and (s_{2n+1}) are both convergent to the same limit, s say.
4. Deduce that $\sum(-1)^{n+1}a_n$ is convergent. Make sure your proof includes the inequality $|s - \sum_{k=1}^{n}(-1)^{k+1}a_k| \le a_n$.

Assignment 5
The series $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \dots$ converges to $\sin 1$. Explain how to use the series to calculate $\sin 1$ to within an error of 10^{-10}.

Assignment 6
Using the Alternating Series Test where appropriate, show that each of the following series is convergent.

1. $\sum \frac{(-1)^{n+1}n^2}{n^3+1}$
2. $\sum \left(-\frac{1}{2}\right)^n$
3. $\sum \frac{2|\cos\frac{n\pi}{2}|+(-1)^n n}{\sqrt{(n+1)^3}}$
4. $\sum \frac{1}{n}\sin\frac{n\pi}{2}$

4.16 General Series

Series with positive terms are easier because we can attempt to prove that the partial sums (s_n) converge by exploiting the fact that (s_n) is increasing. In the general case, (s_n) is not monotonic. We can still try to apply Cauchy's test for convergence, however, since this applies to any sequence.

Assignment 7
Show that (s_n) is a Cauchy sequence means that for any $\epsilon > 0$ there exists N such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ whenever $n > m \ge N$.

Before exploiting the Cauchy test we shall give one new definition: If $\sum a_n$ is a series with positive and negative terms, we can form the series $\sum |a_n|$, all of whose terms are non-negative.

Definition
The series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Example The alternating series $\sum \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent because $\sum |\frac{(-1)^{n+1}}{n^2}| = \sum \frac{1}{n^2}$ is convergent.

The series $\sum \left(-\frac{1}{2}\right)^n$ is absolutely convergent because $\sum \left(\frac{1}{2}\right)^n$ converges.

Assignment 8
Is the series $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ absolutely convergent?

Exercise 2 For what values of x is the Geometric Series $\sum x^n$ absolutely convergent?

Absolutely convergent series are important for the following reason.

Theorem
Every absolutely convergent series is convergent.

Assignment 9
Let $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} |a_i|$. Prove this result using the following steps:

1. Show that $|s_n - s_m \le |t_n - t_m|$ whenever $n > m$.
2. Show that if t_n is convergent then s_n is Cauchy and hence convergent.

Assignment 10
Is the converse of the theorem true: "Every convergent series is absolutely convergent"?

The Absolute Convergence Theorem breathes new life into all the tests we developed for series with non-negative terms: if we can show that $\sum |a_n|$ is convergent, using one of these tests, then we are guarenteed that $\sum a_n$ converges as well.

Exercise 3 Show that the series $\sum \frac{\sin n}{n^2}$ is convergent.

We see that $0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$. Therefore $\sum \frac{|\sin n|}{n^2}$ is convergent by the Comparison Test. It follows that $\sum \frac{\sin n}{n^2}$ is convergent by the Absolute Convergence Theorem.

The Ratio Test can be modified to cope directly with series of mixed terms.

Theorem *Ratio Test*
Suppose $a_n \neq 0$ and $|\frac{a_{n+1}}{a_n}| \to l$. Then $\sum a_n$ converges absolutely (and hence converges) if $0 \leq l < 1$ and diverges if $l > 1$.

Proof. If $0 \leq l < 1$, then $\sum |a_n|$ converges by the "old" Ratio Test. Therefore $\sum a_n$ converges by the Absolute Convergence Theorem.

If $l > 1$, we are guarenteed that $\sum |a_n|$ diverges, but this does not, in itself, prove that $\sum a_n$ diverges (why not?). We have to go back and modify our original proof.

We know that there exists N such that $||\frac{a_{n+1}}{a_n}| - l| < \frac{1}{2}(l-1)$ when $n > N$. Therefore, $1 < \frac{1}{2}(1+l) < \frac{|a_{n+1}|}{|a_n|}$ when $n > N$. It follows that $0 < |a_{N+1}| < |a_{N+2}|$ and by induction that $0 < |a_{N+1}| < |a_n|$ when $n > N+1$. Clearly the sequence $(|a_n|)$ is not null, hence (a_n) is not null. This being the case, $\sum a_n$ diverges by the Null Sequence Test. ∎

Example Consider the series $\sum \frac{x^n}{n}$. When $x = 0$ the series is convergent. (Notice that we cannot use the Ratio Test in this case.)

Now let $a_n = \frac{x^n}{n}$. When $x \neq 0$ then $|\frac{a_{n+1}}{a_n}| = |\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}| = \frac{n}{n+1}|x| \to |x|$. Therefore $\sum \frac{x^n}{n}$ is convergent when $|x| < 1$ and divergent when $|x| > 1$, by the Ratio Test.

What if $|x| = 1$? When $x = 1$ then $\sum \frac{x^n}{n} = \sum \frac{1}{n}$ which is divergent. When $x = -1$ then $\sum \frac{x^n}{n} = \sum -\frac{(-1)^{n+1}}{n}$ which is convergent.

Theorem *Ratio Test Extension*
Suppose $a_n \neq 0$ and $|\frac{a_{n+1}}{a_n}| \to \infty$, then $\sum a_n$ diverges.

Assignment 11
Prove this theorem.

Assignment 12
Determine for which values of x the following series converge and diverge. [Make sure you don't neglect those values for which the Ratio Test doesn't apply.]

1. $\sum \frac{x^n}{n!}$
2. $\sum \frac{n}{x^n}$
3. $\sum n!x^n$
4. $\sum \frac{(2x)^n}{n}$
5. $\sum \frac{(4x)^{3n}}{\sqrt{n+1}}$
6. $\sum (-nx)^n$

4.17 Euler's Constant

Our last aim in this booklet is to find an explicit formula for the sum of the alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

On the way we shall meet Euler's constant, usually denoted by γ, which occurs in several places in mathematics, especially in number theory.

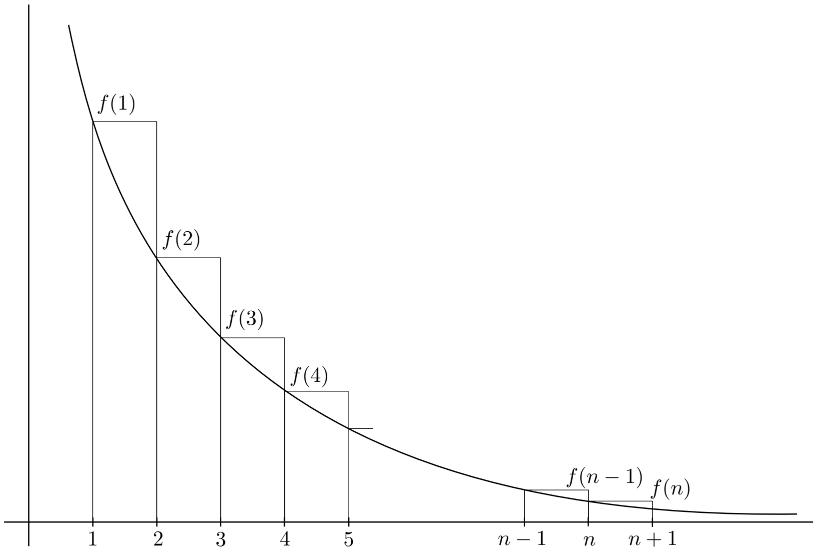

Figure 1: Calculating a lower bound of an integral.

Euler's Constant

The limit of the sequence

$$D_n = \sum_{i=1}^{n} \frac{1}{i} - \log(n+1)$$

is called Euler's Constant and is usually denoted by γ (gamma). Its value has been computed to over 200 decimal places. Its value to 14 decimal places is 0.57721566490153. No-one knows whether γ is rational or irrational.

Assignment 13

Let $D_n = \sum_{i=1}^{n} \frac{1}{i} - \int_1^{n+1} \frac{1}{x} dx = \sum_{i=1}^{n} \frac{1}{i} - \log(n+1)$.

1. Draw a copy of figure 1 and mark in the areas represented by D_n.
2. Show that (D_n) is increasing.
3. Show that (D_n) is bounded - and hence convergent.

The limit of the sequence $D_n = \sum_{i=1}^{n} \frac{1}{i} - \log(n+1)$ is called Euler's Constant and is usually denoted by γ.

Assignment 14

Show that $\sum_{i=1}^{2n-1} \frac{(-1)^{i+1}}{i} = \log 2 + D_{2n-1} - D_{n-1}$. Hence evaluate $\sum \frac{(-1)^{n+1}}{n}$.
Hint: Use the following identity:

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} \\ = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots \frac{1}{2n-2}\right) \end{aligned}$$

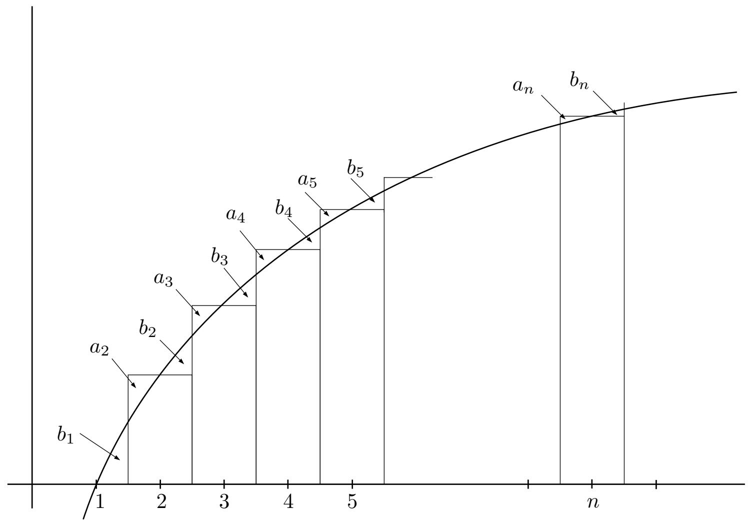

Figure 2: Approximating the integral by the mid point.

4.18 * Application - Stirling's Formula *

Using the alternating series test we can improve the approximations to $n!$ that we stated in workbook 4. Take a look at what we did there: we obtained upper and lower bounds to $\log(n!)$ by using block approximations to the integral of $\int_1^n \log x dx$. To get a better approximation we use the approximation in figure 2.

Now the area of the blocks approximates $\int_1^n \log x dx$ except that there are small triangular errors below the graph (marked as $b_1, b_2, b_3, \dots$) and small triangular errors above the graph (marked as $a_2, a_3, a_4, \dots$).

Note that $\log n! = \log 2 + \log 3 + \cdots + \log n =$ area of the blocks.

Assignment 15
Use the above diagram to show:

$$\log n! - (n + \frac{1}{2}) \log n + n = 1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots - b_{n-1} + a_n$$

[Hint: $\int_1^n \log x dx = n \log n - n + 1$]

The curve $\log n$ is concave down and it seems reasonable, and can be easily proved (try for yourselves), that $a_n \geq b_n \geq a_{n+1}$ and $\lim a_n = 0$.

Assignment 16
Assuming that these claims are true, explain why $1 - b_1 + a_2 - b_2 + \cdots \to C$ as $n \to \infty$.

This proves that $\log n! = (n + \frac{1}{2}) \log n - n + \Sigma_n$ where Σ_n tends to a constant as $n \to \infty$. Taking exponentials we obtain:

$$n! \simeq \text{constant} \cdot n^n e^{-n} \sqrt{n}$$

What is the constant? This was identified with only a little more work by the mathematician James Stirling. Indeed, he proved that:

$$\frac{n!}{n^n e^{-n} \sqrt{2\pi n}} \to 1 \quad \text{as } n \to \infty$$

a result known as Stirling's formula.

Check Your Progress
By the end of this Workbook you should be able to:

- Use and justify the Alternating Series Test: If (a_n) is a decreasing, null sequence of non-negative terms then $\sum (-1)^{n+1} a_n$ is convergent.
- Use the proof of Alternating Series Test to establish error bounds.
- Say what it means for a series to be *absolutely convergent* and give examples of such series.
- Prove that an absolutely convergent series is convergent, but that the converse is not true.
- Use the modified Ratio Test to determine the convergence or divergence of series with positiven and negative terms.
- Prove that $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$.

Name: ....

Supervisor: ....

Class Teacher: ....

MA131 - Analysis 1
Workbook 9 Assignments

Due in 29th Oct at 10am

Detach this sheet and staple it to the front of your solutions