MA131 - Analysis 1

Workbook 8
Series II

Autumn 2004

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4.9 Convergence of series - continued

Usually we are doomed to failure if we seek a formula for the sum of a series. Nevertheless we can often tell whether the series converges of diverges without explicitly finding the sum. To do this we shall establish a variety of convergence tests that allow us in many cases to work out from the formula for the terms a_n whether the series converges or not.

Explicit Sums

For most convergent series there is no simple formula for the sum $\sum_{n=1}^{\infty} a_n$ in terms of standard mathematical objects. Only in very lucky cases can we sum the series explicitly, for instance geometric series, telescoping series, various series found by contour integration or by Fourier expansions. But these cases are so useful and so much fun that we mention them often.

4.10 Series with positive terms

Series with positive terms are easier than general series since the partial sums (s_n) form an increasing sequence and we have already seen that monotonic sequences are easier to cope with than general sequences.

All our convergence tests are based on the most useful test - the comparison test - which you have already proven.

Assignment 1
Use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p \in [2, \infty)$ and diverges if $p \in (0, 1]$. [Hint - you already know the answer for $p = 1$ or 2.]

Assignment 2
Use the Comparison Test to determine whether each of the following series converges or diverges.

$$1. \sum \frac{1}{\sqrt[3]{n^2+1}} \qquad 2. \sum \frac{1}{\sqrt[3]{n^7+1}} \qquad 3. \sum \left(\sqrt{n+1} - \sqrt{n}\right)$$

Sometimes the series of which we want to find the sum looks quite complicated. Often the best way to find a series to compare it with is to look at which terms dominate in the original series.

Example Consider the series $\sum \frac{\sqrt{n}+2}{n^{3/2}+1}$. We can rearrange the n^{th} term in this series as follows:

$$\frac{\sqrt{n}+2}{n^{3/2}+1} = \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}}$$

As n gets large then $\frac{1}{\sqrt{n}}$ gets small so the dominant term in the numerator is the 1 and in the denominator is the n. Thus a possible series to compare it with is $\sum \frac{1}{n}$. Since this diverges, we want to show that our series is greater than some multiple of $\sum \frac{1}{n}$:

$$\begin{aligned} \frac{\sqrt{n}+2}{n^{3/2}+1} &= \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}} \\ &> \frac{1+\frac{2}{\sqrt{n}}}{2n} \\ &> \frac{1}{2n} \end{aligned}$$

hence by the comparision test, $\sum \frac{\sqrt{n}+2}{n^{3/2}+1}$ diverges.

Assignment 3
Use this technique with the Comparison Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

1. $\sum \dfrac{1}{n(n+1)(n+2)}$ 2. $\sum \dfrac{5^n + 4^n}{7^n - 2^n}$

4.11 Ratio Test

The previous tests operate by comparing two series. Choosing a Geometric Series for such a comparison gives rise to yet another test which is simple and easy but unsophisticated.

Exercise 1 These questions give you the ideas needed to construct a proof of the upcoming theorem. They rely on previous topics you have already met; the Shift Rule, the Comparison Test and the behaviour of Geometric Series.

1. Let $a_n = n^2/2^n$. Prove that if $n \geq 3$, then $a_{n+1}/a_n \leq 8/9$. By using this inequality for $n = 3, 4, 5, \ldots$ prove that $a_{n+3} \leq (8/9)^n a_3$. Using the Comparison Test and results concerning the convergence of the Geometric Series (from last week) show that $\sum a_{n+3}$ is convergent. Now use the Shift Rule to show that $\sum a_n$ is convergent.

2. We now generalise the results of the previous question. Suppose we have a series $\sum a_n$ of positive terms for which
$$0 < \frac{a_{n+1}}{a_n} < \rho < 1 \quad \text{for all } n.$$
Show that $a_{n+1} < \rho^n \cdot a_1$. Use the Comparison Test and behaviour of the Geometric Series to prove that $\sum a_n$ is convergent.

3. Suppose that $\sum a_n$ is a series of positive terms and the sequence of ratios $(a_{n+1}/a_n) \to k < 1$. For some suitable choice of $\varepsilon > 0$ show that there exists an n_0 such that
$$\frac{a_{n+1}}{a_n} < \frac{1}{2}(k+1) < 1 \quad \text{for } n > n_0$$
Let $\rho = \frac{1}{2}(k+1)$ and the Shift Rule and the previous question to prove that $\sum a_n$ is convergent.

4. Suppose that a series of positive terms $\sum a_n$ satisfies $1 \leq a_{n+1}/a_n$ for all n. Deduce that(a_N) is not a null sequence and so $\sum a_n$ is divergent.

5. Suppose that a series of positive terms $\sum a_n$ satisfies $(a_{n+1}/a_n) \to k > 1$. For some suitable choice of $\varepsilon > 0$ show that there exists an n_0 such that
$$1 < \frac{1}{2}(1+k) < \frac{a_{n+1}}{a_n} \quad \text{for } n > n_0.$$
Using the previous question and the Shift Rule show that $\sum a_n$ is divergent.

Theorem *Ratio Test*
Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to l$. Then $\sum a_n$ converges if $0 \leq l < 1$ and diverges if $l > 1$.

Examples

1. Consider the series $\sum \frac{1}{n!}$. Letting $a_n = \frac{1}{n!}$ we obtain $\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$. Therefore $\sum \frac{1}{n!}$ converges.

2. Consider the series $\sum \frac{n^2}{2^n}$. Letting $a_n = \frac{n^2}{2^n}$ we obtain $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2}$. Therefore $\sum \frac{n^2}{2^n}$ converges.

Assignment 4
Tie together your answers to parts 3 and 5 of exercise 1 and write a proof of the Ratio Test.

Assignment 5
Write down an example of a convergent series and a divergent series both of which satisfy the condition $l = 1$. [This shows why the Ratio Test cannot be used in this case.]

Assignment 6
Use the Ratio Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

$$1.\ \sum \frac{2^n}{n!} \qquad 2.\ \sum \frac{3^n}{n} \qquad 3.\ \sum \frac{n!}{n^n}$$

The Missing Case
The case $l = 1$ is omitted from the statement of the Ratio Test. This is because there exist both convergent *and* divergent series that satisfy this condition.

A.K.A.
This test is also called D'Alembert's Ratio Test, after the French mathematician Jean Le Rond D'Alembert (1717 - 1783). He developed it in a 1768 publication in which he established the convergence of the Binomial Series by comparing it with the Geometric Series.

4.12 Integral Test

We can use our integration skills to get hugely useful approximations to sums. Consider a real-valued function $f(x)$ which is non-negative and decreasing for $x \geq 1$. We have sketched such a function in figure 1 (actually we sketched $f(x) = 1/x$).

The shaded blocks lie under the graph of the function so that the total area of all the blocks is less than the area under the graph between $x = 1$ and $x = n$. So:

$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx$$

Forward and Back
In later Analysis courses you will formally define both the integral and the logarithm function. What you learnt at school is fine for now. Using them now gives us more interesting examples.

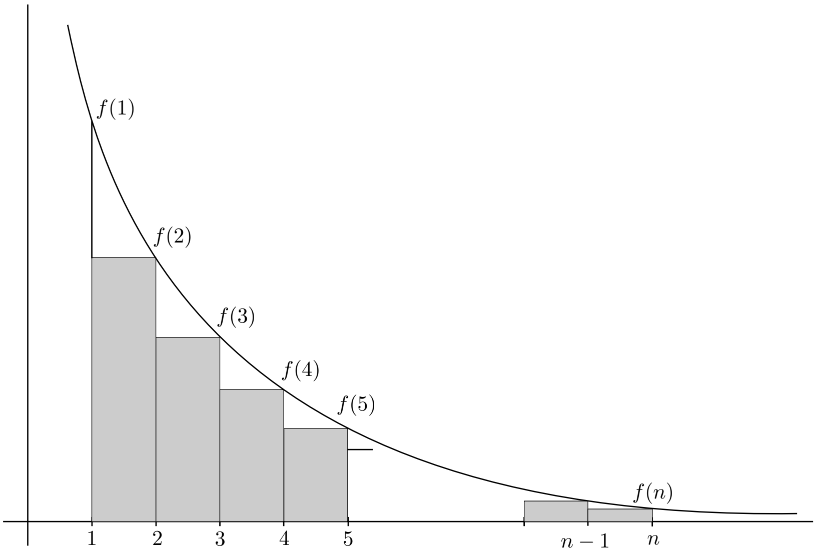

Figure 1: Calculating a lower bound of an integral.

Assignment 7
Draw a similar diagram and use a similar argument to prove the following improvement. If $f(x)$ is a non-negative and decreasing function for all $m \leq x \leq n$ for integers $m < n$ then $\sum_{k=m+1}^{n} f(k) \leq \int_m^n f(x)dx$.

We can use this bound to help us with error estimates. For example, we use a true formula (usually established via complex variable methods or Fourier analysis methods in the second year):

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

If we sum only the first N terms of this series we will reach a total less than $\frac{\pi^2}{6}$. Can we estimate the size of the error?

The error is precisely $\sum_{k=N+1}^{\infty} \frac{1}{k^2}$. If we use assignment 7 we obtain the bound:

$$\sum_{k=N+1}^{n} \frac{1}{k^2} \leq \int_N^n \frac{1}{x^2}dx = -\frac{1}{x}\bigg|_N^n = \frac{1}{N} - \frac{1}{n} \leq \frac{1}{N}$$

Since this is true for any value of n we see that $\sum_{k=N+1}^{\infty} \frac{1}{k^2} = \lim_{n\to\infty} \sum_{k=N+1}^{n} \frac{1}{k^2} \leq \frac{1}{N}$.

So if we sum the first 1,000,000 terms we will reach a total that is within 10^{-6} of $\pi^2/6$.

Assignment 8
Fourier analysis methods also lead to the formula:

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots$$

Find a value of N so that $\sum_{k=1}^{N} \frac{1}{k^4}$ is within 10^{-6} of $\pi^4/90$.

Assignment 9
Draw a diagram where the blocks lie above the graph of the function and use it to prove the following inequality: if $f(x)$ is a non-negative and decreasing function for all $m \leq x \leq n+1$ then $\sum_{k=m}^{n} f(k) \geq \int_{m}^{n+1} f(x)dx$.

Assignment 10
Use your upper and lower bounds in assignments 7 and 9 to show:

$$\sum_{k=101}^{200} \frac{1}{k} = \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{200} \in [0.688, 0.694]$$

We now use these upper and lower bounds to establish a beautiful test for convergence.

Theorem *Integral Test, convergence part*
Suppose the function $f(x)$ is non-negative and decreasing for $x \geq 1$. Then $\sum_{n=1}^{\infty} f(n)$ converges if the increasing sequence $\left(\int_1^n f(x)dx\right)$ is bounded.

Assignment 11
Prove this result by using the upper bound on $\sum_{k=2}^{n} f(k)$ found in exercise 7 and the boundedness condition for convergence of positive series.

Theorem *Integral Test, divergence part*
Suppose the function $f(x)$ is non-negative and decreasing for $x \geq 1$. Then $\sum_{n=1}^{\infty} f(n)$ diverges if the increasing sequence $\left(\int_1^n f(x)dx\right)$ is unbounded.

Assignment 12
Prove this result by using the lower bounds in exercise 9.

Example The Integral Test gives us another proof of the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Let $f(x) = \frac{1}{x^2}$. We know that this function is non-negative and decreasing when $x \geq 1$. Observe that $\int_1^n f(x)dx = \int_1^n \frac{1}{x^2} = -\frac{1}{x}\Big|_1^n = 1 - \frac{1}{n} \to 1$. Since $f(n) = \frac{1}{n^2}$, the Integral Test assures us that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example If you are familiar with the behaviour of the log function, the Integral Test gives you a neat proof that the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Suppose $f(x) = \frac{1}{x}$. Again, this function is non-negative and decreasing when $x \geq 1$. Observe that $\int_1^n f(x)dx = \int_1^n \frac{1}{x}dx = \log x\Big|_1^n = \log n \to \infty$. Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity.

Assignment 13
Use the Integral Test to investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for values of $p \in (1, 2)$.

Combining this with the result of assignment 1, you have shown:

Corollary
The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

We now examine some series right on the borderline of convergence.

Assignment 14
Show that $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$ is divergent and that $\sum_{n=1}^{\infty} \frac{1}{(n+1)(\log(n+1))^2}$ is convergent.

4.13 * Application - Error Bounds *

If we have established that a series $\sum a_n$ converges then the next question is to calculate the total sum $\sum_{n=1}^{\infty} a_n$. Usually there is no explicit formula for the sum and we must be content with an approximate answer - for example, correct to 10 decimal places.

The obvious solution is to calculate $\sum_{n=1}^{N} a_n$ for a really large N. But how large must N be to ensure the error is small - say less than 10^{-10}? The error is the sum of all the terms we have ignored $\sum_{n=N+1}^{\infty} a_n$ and again there is usually no explicit answer. But by a comparison with a series for which we *can* calculate the sum (i.e. geometric or telescoping series) we can get a useful upper bound on the error.

Example Show how to calculate the value of e to within an error of 10^{-100}.

Solution We shall sum the series $\sum_{n=0}^{N} \frac{1}{n!}$ for a large value of N. Then the error is:

$$
\begin{aligned}
e - \sum_{n=0}^{N} \frac{1}{n!} &= \sum_{n=N+1}^{\infty} \frac{1}{n!} \\
&= \frac{1}{(N+1)!}\left(1 + \frac{1}{N+2} + \frac{1}{(N+2)(N+3)} \cdots\right) \\
&\leq \frac{1}{(N+1)!}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) \\
&= \frac{2}{(N+1)!}
\end{aligned}
$$

Then the error is less than 10^{-100} provided that $\frac{2}{(N+1)!} \leq 10^{-100}$ which occurs when $N \geq 70$.

Assignment 15
The following formula for $\sqrt{e}$ is true, although it will not be proved in this course.

$$\sqrt{e} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \dots$$

Show that the error $\sqrt{e} - \sum_{n=0}^{N} \frac{1}{2^n n!}$ is less than $\frac{1}{2^N (N+1)!}$. Hence find a value of N that makes the error less than 10^{-100}.

Exercise 2 The above examples suggest two ways of calculating e. Either one can use the series $\sum_{n=0}^{N} \frac{1}{n!}$ for a large value of N or use the series $\sum_{n=0}^{N} \frac{1}{2^n n!}$ for a large value of N to approximate $\sqrt{e}$ and then square the answer to approximate e. Explain which method you prefer and why.

Check Your Progress
By the end of this Workbook you should be able to:

- Use and justify the following tests for sequence convergence:
- *Comparison Test:* If $0 \leq a_n \leq b_n$ and $\sum b_n$ is convergent then $\sum a_n$ is convergent.
- *Ratio Test:* If $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to l$ then $\sum a_n$ converges if $0 \leq l < 1$ and diverges if $l > 1$.
- *Integral Test:* If $f(x)$ is non-negative and decreasing for $x \geq 1$ then $\sum f(n)$ converges if and only if $\int_1^n f(x)dx$ converges, and $\sum f(n)$ tends to infinity if and only if $\int_1^n f(x)dx = \infty$.
- You should also be able to use comparisons to establish error bounds when evalutating infinite sums.

Name: ....

Supervisor: ....

Class Teacher: ....

MA131 - Analysis 1
Workbook 8 Assignments

Due in 22th Oct at 10am

Detach this sheet and staple it to the front of your solutions