MA131 - Analysis 1

Workbook 7
Series I

Autumn 2004

Contents

4 Series

4.1 Definitions

We saw in the last booklet that decimal expansions could be defined in terms of sequences of sums. Thus a decimal expansion is like an infinite sum. This is what we shall be looking at for the rest of the course.

Our first aim is to find a good definition for summing infinitely many numbers. Then we will investigate whether the rules for finite sums apply to infinite sums.

Exercise 1 What has gone wrong with the following argument? Try putting $x = 2$.

If $S = 1 + x + x^2 + \dots$,
then $xS = x + x^2 + x^3 + \dots$,
so $S - xS = 1$,
and therefore $S = \frac{1}{1-x}$.

If the argument were correct then we could put $x = -1$ to obtain the sum of the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$$

as 1/2. But the same series could also be be thought of as

$$(1 - 1) + (1 - 1) + (1 - 1) + \dots$$

with a sum of 0, or as

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

with a sum of 1. This shows us that great care must be exercised when dealing with infinite sums.

We shall repeatedly use the following convenient notation for finite sums: given finite integers $0 \leq m \leq n$ and numbers $(a_n : n = 0, 1, \dots)$ we define

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \dots + a_n$$

Example $1 + 4 + 9 + \dots + 100 = \displaystyle\sum_{k=1}^{10} k^2$

Exercise 2 Express the following sums using the $\sum$ notation:

1. $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dots + \dfrac{1}{3628800}$ 2. $\dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \dots \dfrac{7}{128}$

Exercise 3 Show that $\sum_{k=1}^{n} a_{k-1} = \sum_{k=0}^{n-1} a_k$.

Exercise 4

1. By decomposing $1/r(r+1)$ into partial fractions, or by induction, prove that
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}.$$
Write this result using $\sum$ notation.

2. If
$$s_n=\sum_{r=1}^{n}\frac{1}{r(r+1)},$$
prove that $(s_n)\to 1$ as $n\to\infty$. This result could also be written as
$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=1.$$

A *series* is an expression of the form $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+a_4+\ldots$. As yet, we have not defined what we mean by such an infinite sum. To get the ball rolling, we consider the "partial sums" of the series:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1+a_2 \\ s_3 &= a_1+a_2+a_3 \\ &\vdots \\ s_n &= a_1+a_2+a_3+\cdots+a_n \end{aligned}$$

To have any hope of computing the infinite sum $a_1+a_2+a_3+\ldots$, then the partial sums s_n should represent closer and closer approximations as n is chosen larger and larger. This is just an informal way of saying that the infinite sum $a_1+a_2+a_3+\ldots$ ought to be the limit of the sequence of partial sums.

Serious Sums

The problem of how to deal with infinite sums vexed the analysis of the early 19th century. Some said there wasn't a problem, some pretended there wasn't until inconsistencies in their own work began to unnerve them, and some said there was a terrible problem and why wouldn't anyone listen? Eventually, everyone did.

Definition

Consider the series $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\ldots$ with partial sums (s_n), where

$$s_n=a_1+a_2+a_3+\cdots+a_n=\sum_{i=1}^{n}a_i$$

We say:

1. $\sum_{n=1}^{\infty} a_n$ *converges* if (s_n) converges. If $s_n\to S$ then we call S the sum of the series and we write $\sum_{n=1}^{\infty} a_n = S$.
2. $\sum_{n=1}^{\infty} a_n$ *diverges* if (s_n) does not converge.
3. $\sum_{n=1}^{\infty} a_n$ *diverges to infinity* if (s_n) tends to infinity.
4. $\sum_{n=1}^{\infty} a_n$ *diverges to minus infinity* if (s_n) tends to minus infinity.

Double Trouble

There are two sequences associated with every series $\sum_{n=1}^{\infty} a_n$: the sequence (a_n) and the sequence of partial sums $(s_n) = \left(\sum_{i=1}^{n} a_i\right)$. Do not get these sequences confused!

Series Need Sequences

Notice that series convergence is defined entirely in terms of sequence convergence. We haven't spent six weeks working on sequences for nothing!

We sometimes write the series $\sum_{n=1}^{\infty} a_n$ simply as $\sum a_n$.

Example Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. The sequence of partial sums is given by

$$s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{1}{2}\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = 1 - \left(\frac{1}{2}\right)^n$$

Clearly $s_n \to 1$. It follows from the definition that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

We could express the argument more succinctly by writing

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} \left(1 - \left(\frac{1}{2}\right)^n\right) = 1$$

Dummy Variables

Make careful note of the way the variables k and n appear in this example. They are dummies - they can be replaced by any letter you like.

Example Consider the series $\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+\ldots$. Here we have the partial sums:

$$\begin{aligned} s_1 &= a_1 = -1 \\ s_2 &= a_1 + a_2 = 0 \\ s_3 &= a_1 + a_2 + a_3 = -1 \\ s_4 &= a_1 + a_2 + a_3 + a_4 = 0 \\ &\quad \cdots\cdots\cdots \end{aligned}$$

and we can see at once that the sequence $(s_n) = -1, 0, -1, 0, \ldots$ does not converge.

Assignment 1
Look again at the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Plot on two small separate graphs both the sequences $(a_n) = \left(\frac{1}{2^n}\right)$ and $(s_n) = \left(\sum_{k=1}^{n} \frac{1}{2^k}\right)$.

Frog Hopping

Heard about that frog who hops halfway across his pond, and then half the rest of the way, and the half that, and half that, and half that ...?
Is he ever going to make it to the other side?

Assignment 2
Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{1}{10^n}\right)$.

Assignment 3
Reread your answer to exercise 4 and then write out a full proof that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = 1$$

Assignment 4
Show that the series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \ldots$ diverges.
[Hint: Calculate the partial sums $s_1, s_3, s_6, s_{10}, \ldots$.]

4.2 Geometric Series

Theorem *Geometric Series*
The series $\sum_{n=0}^{\infty} x^n$ is convergent if $|x| < 1$ and the sum is $\frac{1}{1-x}$. It is divergent if $|x| \geq 1$.

Exercise 5 $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$ converges.
$\sum_{n=0}^{\infty}(2.1)^n$, $\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty}(-3)^n = 1 - 3 + 9 - 27 + 81 - \dots$ all diverge.

Assignment 5
Prove the theorem [Hint: Use the GP formula to get a formula for s_n].

GP Consultation
How could you ever forget that $a + ax + ax^2 + \cdots + ax^{n-1} = a\left(\frac{x^n-1}{x-1}\right)$ when $x \neq 1$?

4.3 The Harmonic Series

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is called the Harmonic Series. The following grouping of its terms is rather cunning:

$$1 + \underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq 1/2} + \dots$$

Assignment 6
Prove that the Harmonic Series diverges. Structure your proof as follows:

1. Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ be the partial sum. Show that $s_{2n} \geq s_n + \frac{1}{2}$ for all n. (Use the idea in the cunning grouping above).
2. Show by induction that $s_{2^n} \geq 1 + \frac{n}{2}$ for all n.
3. Conclude that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Harmonic History
There are other proofs that the Harmonic Series is divergent, but this is the original. It was contributed by the English mediaeval mathematician Nicholas Oresme (1323-1382) who also gave us the laws of exponents: $x^m.x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$.

Assignment 7
Give, with reasons, a value of N for which $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \geq 10$.

Conflicting Convergence
You can see from this example that the convergence of (a_n) does not imply the convergence of $\sum_{n=1}^{\infty} a_n$.

4.4 Basic Properties of Convergent Series

Some properties of finite sums are easy to prove for infinite sums:

Theorem *Sum Rule for Series*
Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series. Then, for all real numbers c and d, $\sum_{n=1}^{\infty}(ca_n + db_n)$ is a convergent series and

$$\sum_{n=1}^{\infty}(ca_n + db_n) = c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$$

Proof.

$$\sum_{i=1}^{n}(ca_n + db_n) = c\left(\sum_{i=1}^{n} a_i\right) + d\left(\sum_{i=1}^{n} b_i\right) \to c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$$

■

Theorem *Shift Rule for Series*
Let N be a natural number. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_{N+n}$ converges.

Example We showed that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. It follows that $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent.

Assignment 8
Prove the shift rule.

4.5 Boundedness Condition

If the terms of a series are all non-negative, then we shall show that the boundedness of its partial sums is enough to ensure convergence.

Theorem *Boundedness Condition*
Suppose $a_n \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n) = \left(\sum_{j=1}^{n} a_j\right)$ is bounded.

Assignment 9
Prove this result. Your proof must use the axiom of completeness or one of its consequences - make sure you indicate where this occurs.

4.6 Null Sequence Test

Assignment 10

1. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then the sequence (a_n) tends to zero. (Hint: Notice that $a_{n+1} = s_{n+1} - s_n$ and use the Shift Rule for sequences.)
2. Is the converse true: If $(a_n) \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges?

We have proved that if the series $\sum_{n=1}^{\infty} a_n$ converges then it must be the case that (a_n) tends to zero. The contrapositive of this statement gives us a test for *divergence*:

Theorem *Null Sequence Test*
If (a_n) does not tend to zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

Red Alert
The Null Sequence Test is a test for *divergence* only. You can't use it to prove series convergence.

Example The sequence (n^2) does not converge to zero, therefore the series $\sum_{n=1}^{\infty} n^2$ diverges.

4.7 Comparison Test

The next test allows you to test the convergence of a series by comparing its terms with those of a series whose behaviour you already know.

Theorem *Comparison Test*
Suppose $0 \leq a_n \leq b_n$ for all natural numbers n. If $\sum b_n$ converges then $\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Example You showed in assignment 3 that $\sum \frac{1}{n(n+1)}$ converges. Now $0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$. It follows from the Comparison Test that $\sum \frac{1}{(n+1)^2}$ also converges and via the Shift Rule that the series $\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

Like the Boundedness Condition, you can only apply the Comparison Test (and the other tests in this section) if the terms of the series are non-negative.

Exercise 6 Give an example to show that the test fails if we allow the terms of the series to be negative, i.e. if we only demand that $a_n \leq b_n$.

Assignment 11
Prove the Comparison Test [Hint: Consider the partial sums of both $\sum b_n$ and $\sum a_n$ and show that the latter is increasing and bounded].

Exercise 7 Check that the *contrapositive* of the statement: *"If $\sum b_n$ converges then $\sum a_n$ converges."* gives you the following additional comparison test:

Corollary *Comparison Test extension*
Suppose $0 \leq a_n \leq b_n$. If $\sum a_n$ diverges then $\sum b_n$ diverges.

Examples

1. Note $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$. We know $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{\sqrt{n}}$ diverges too.
2. To show that $\sum \frac{n+1}{n^2+1}$ diverges, notice that $\frac{n+1}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{1}{2n}$. We know that $\sum \frac{1}{2n}$ diverges, therefore $\sum \frac{n+1}{n^2+1}$ diverges.

Assignment 12
Use the Comparison Test to determine whether each of the following series converges or diverges. In each case you will have to think of a suitable series with which to compare it.

$$\text{(i)} \quad \sum \frac{2n^2+15n}{n^3+7} \qquad \text{(ii)} \quad \sum \frac{\sin^2 nx}{n^2} \qquad \text{(iii)} \quad \sum \frac{3^n+7^n}{3^n+8^n}$$

Way To Go

Stare deeply at each series and try to find a simpler series whose terms are very close for large n. This gives you a good idea which series you might hope to compare it with, and whether it is likely to be convergent or divergent. For instance the terms of the series $\sum \frac{n+1}{n^2+1}$ are like those of the series $\sum \frac{1}{n}$ for large values of n, so we would expect it to diverge.

4.8 * Application - What is e? *

Over the years you have no doubt formed a working relationship with the number e, and you can say with confidence (and the aid of your calculator) that $e \approx 2.718$. But that is not the end of the story.

Just what is this e number?

To answer this question, we start by investigating the mysterious series $\sum_{n=0}^{\infty} \frac{1}{n!}$. Note that we adopt the convention that $0! = 1$.

Assignment 13
Consider the series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ and its partial sums $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!}$.

1. Show that the sequence (s_n) is increasing.
2. Prove by induction that $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for $n > 0$, and.
3. Use the comparison test to conclude that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Now here comes the Big Definition we've all been waiting for...!!!

Definition
$\mathrm{e} := \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

Recall that in the last workbook we showed that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists. We can now show, with some rather delicate work, that this limit equals e.

Assignment 14
Use the Binomial Theorem to show that

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{(k-1)}{n}\right) \leq \sum_{k=0}^{n} \frac{1}{k!}$$

Conclude that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \leq e$.

Binomial Theorem

For all real values x and y and integer $n = 1, 2, \ldots$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Note here we use $0! = 1$.

We now aim to show $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq e$. The first step is to show that for all m and n

$$\left(1+\frac{1}{n}\right)^{m+n} \geq \sum_{k=0}^{m}\frac{1}{k!} \tag{1}$$

By the Binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{m+n} &= \sum_{k=0}^{n+m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k}\\ &\geq \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k}\end{aligned}$$

where we have thrown away the last n terms of the sum. So

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{m+n} &\geq \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)(n+m-1)\dots(n+m-k+1)}{n^k}\\ &\geq \sum_{k=0}^{m}\frac{1}{k!}\end{aligned}$$

which proves equation (1).

Assignment 15
Consider the inequality in equation (1). Fix m and let $n \to \infty$ to conclude that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq \sum_{k=0}^{m}\frac{1}{k!}$. Then let $m \to \infty$ and show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq e$.

You have proved:

Theorem
$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Assignment 16

1. Show that $\left(1-\frac{1}{n+1}\right) = \frac{1}{(1+1/n)}$ and hence find $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$.
2. Use the shift rule to find $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$.

The last exercise in this booklet is the proof that e is an irrational number. The proof uses the fact that the series for e converges very rapidly and this same idea can be used to show that many other series also converge to irrational numbers.

Theorem
e is irrational.

Assignment 17

Prove this result by contradiction. Structure your proof as follows:

1. Suppose $e = \frac{p}{q}$ and show that $e - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{p}{q} - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{k}{q!}$ for some positive integer k.
2. Show that $e - \sum_{i=1}^{q+1} \frac{1}{(i-1)!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots < \frac{1}{q!}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$ and derive a contradiction to part 1.

This exercise will *not* be marked for credit.

Check Your Progress

By the end of this Workbook you should be able to:

- Understand that a series converges if and only if its partial sums converge, in which case $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i\right)$.
- Write down a list of examples of convergent and divergent series and justify your choice.
- Prove that the *Harmonic Series* is divergent.
- State, prove, and use the *Sum* and *Shift Rules* for series.
- State, prove, and use the *Boundedness Condition*.
- Use and justify the *Null Sequence Test*.
- Describe the behaviour of the *Geometric Series* $\sum_{n=1}^{\infty} x^n$.
- State, prove and use the *Comparison Theorem* for series.
- Justify the limit $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ starting from the definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
- Prove that e is irrational.

Name: ....

Supervisor: ....

Class Teacher: ....

MA131 - Analysis 1
Workbook 7 Assignments

Due in 15th Nov at 10am

Detach this sheet and staple it to the front of your solutions