

# MA131 - Analysis 1

## Workbook 4 Sequences III

Autumn 2004

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## 2.13 Roots

We can use the results we've established in the last workbook to find some interesting limits for sequences involving roots. We will need more technical expertise and low cunning than have been required hitherto. First a simple inequality.

### Bernoulli's Inequality

When  $x > -1$  and  $n$  is a natural number,

$$(1+x)^n \geq 1+nx.$$

**Exercise 1** Sketch a graph of both sides of Bernoulli's inequality in the cases  $n = 2$  and  $n = 3$ .

For *non-negative* values of  $x$  Bernoulli's inequality can be easily proved using the Binomial Theorem, which expands the left-hand side:

$$\begin{aligned} (1+x)^n &= 1+nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 \\ &\quad + \dots + nx^{n-1} + x^n \\ &\geq 1+nx. \end{aligned}$$

What difficulties do we have with this line of argument if  $x < 0$ ?

### Assignment 1

Finish off the following proof of Bernoulli's Inequality for  $x > -1$  using *mathematical induction*. Note down where you use the fact that  $x > -1$ .

**Proof.** We want to show that  $(1+x)^n \geq 1+nx$  where  $x > -1$  and  $n$  is a natural number. This is true for  $n = 1$  since  $(1+x)^1 = 1+x$ .

Now assume  $(1+x)^k \geq 1+kx$ . Then

$$(1+x)^{k+1} = (1+x)^k(1+x).$$

**Exercise 2** Use a calculator to explore the sequences  $(2^{1/n})$ ,  $(10^{1/n})$  and  $(1000^{1/n})$ . Repeated use of the square root button will give a subsequence in each case.

### Proposition

If  $x > 0$  then  $(x^{1/n}) \rightarrow 1$ .

**Example**  $100000000000^{1/n} \rightarrow 1$  and also  $0.000000000001^{1/n} \rightarrow 1$ .

To prove this result you might follow the following fairly cunning steps (although other proofs are very welcome):

### Strictly Speaking

Bernoulli's Inequality is actually *strict* unless  $x = 0$ ,  $n = 0$  or  $n = 1$ .

### Binomial Theorem

For all real values  $x$  and  $y$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .

### The Bernoulli Boys

Bernoulli's Inequality is named after Jacques Bernoulli, a Swiss mathematician who used it in a paper on infinite series in 1689 (though it can be found earlier in a 1670 paper by an Englishman called Isaac Barrow).

### Roots

$x^{1/n} = \sqrt[n]{x}$  is the positive  $n^{\text{th}}$  root of  $x$ .

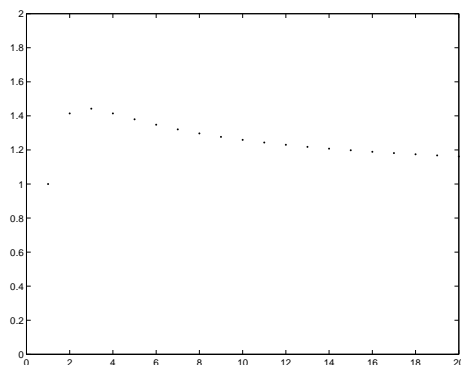


Figure 1: The sequence  $(n^{1/n})$ .

### Assignment 2

1. First assume that  $x \geq 1$  and deduce that  $x^{1/n} \geq 1$ .
2. Let  $a_n = x^{1/n} - 1$  and use Bernoulli's inequality to show that  $x \geq 1 + na_n$ .
3. Use the Sandwich Rule to prove that  $(a_n)$  is a null sequence.
4. Deduce that  $(x^{1/n}) \rightarrow 1$ .
5. Show that  $(x^{1/n}) \rightarrow 1$  when  $0 < x < 1$  by considering  $(1/x^{1/n})$ .

### Assignment 3

Use a calculator to explore the limit of  $(2^n + 3^n)^{1/n}$ . Now find the limit of the sequence  $(x^n + y^n)^{1/n}$  when  $0 \leq x \leq y$ . (Try to find a sandwich for your proof.)

### Proposition

$(n^{1/n}) \rightarrow 1$ .

See figure 1 for a graph of this sequence.

**Proof.** The proof is similar to that of the previous lemma but we have to be cunning and first show that  $(n^{1/2n}) \rightarrow 1$ . Since  $n \geq 1$  we have  $n^{1/2n} \geq 1$ . Therefore,

$$\begin{aligned} \sqrt{n} &= (n^{1/2n})^n = (1 + (n^{1/2n} - 1))^n \\ &\geq 1 + n(n^{1/2n} - 1) > n(n^{1/2n} - 1) \end{aligned}$$

using Bernoulli's inequality. Rearranging, we see that  $1 \leq n^{1/2n} < \frac{1}{\sqrt{n}} + 1$ . So  $(n^{1/2n}) \rightarrow 1$  by the Sandwich Theorem. Hence  $(n^{1/n}) = (n^{1/2n})^2 \rightarrow 1$  by the Product Rule. ■

### Auto Roots

$n^{1/n} = \sqrt[n]{n}$  is the  $n^{\text{th}}$  root of  $n$ . Natural numbers approach unity by rooting themselves.

What happens if we first write  $n = (n^{1/n})^n = (1 + (n^{1/n} - 1))^n$  and apply Bernoulli's Inequality? Do we get anywhere?

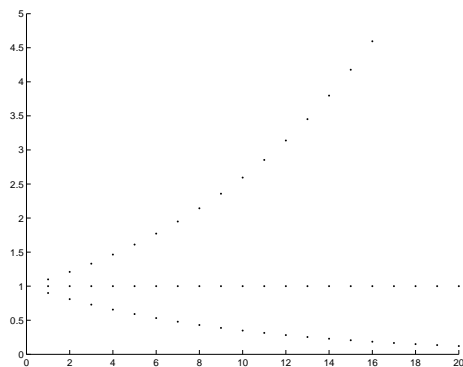


Figure 2: The sequence  $(x^n)$  with three slightly different values of  $x$ .

## 2.14 Powers

### Assignment 4

Explore, with a calculator if necessary, and then write down a conjectured limit for the power sequence  $(x^n)$ . (Warning: you should get four different possible answers depending on the value of  $x$ .)

To prove your conjectures you can use Bernoulli's inequality again. Note that if  $x > 1$  then  $x^n = (1 + (x - 1))^n \geq 1 + n(x - 1)$ . To prove your conjecture for  $0 < x < 1$  look at the sequence  $1/x^n$  and then use lemma 2.11. Then treat all other values of  $x$  such as  $x = 0$ ,  $x = 1$ ,  $-1 < x < 0$ , and  $x \leq -1$ .

Many sequences are not exactly powers but grow or shrink at least as fast as a sequence of powers so that we can compare them with (or sandwich them by) a geometric sequence. A useful idea to formalise this is to consider the ratio of two successive terms:  $a_{n+1}/a_n$ . If this is close to a value  $x$  then the sequence  $(a_n)$  might behave like the sequence  $(x^n)$ . We explore this idea.

### Ratio Lemma, Version 1

Let  $a_0, a_1, a_2, \dots$  be a sequence of positive numbers. Suppose  $0 < l < 1$  and  $\frac{a_{n+1}}{a_n} \leq l$  for all  $n$ . Then  $(a_n) \rightarrow 0$ .

### Assignment 5

Prove this lemma by first using induction to show  $a_n \leq l^n a_0$  and using the fact that  $(l^n)$  is a null sequence.

A small but useful improvement of the above lemma is as follows:

### Ratio Lemma, Version 2

Let  $a_0, a_1, a_2, \dots$  be a sequence of positive numbers. Suppose  $0 < l < 1$  and  $\frac{a_{n+1}}{a_n} \leq l$  eventually. Then  $(a_n) \rightarrow 0$ .

### Assignment 6

Prove this by using version 1 of the Ratio Lemma and using the Shift Rule.

#### Examples

1. Show that  $\frac{n^2}{2^n} \rightarrow 0$
2. Show that  $\frac{n!}{1000^n} \rightarrow \infty$

#### Proof

1. The ratio of successive terms is  $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 \rightarrow \frac{1}{2}$ .  
So, taking  $\epsilon = \frac{1}{4}$  in the definition of convergence, we have  $\frac{1}{4} \leq \frac{a_{n+1}}{a_n} \leq \frac{3}{4}$  for large  $n$ . The Ratio Lemma then implies that  $\frac{n^2}{2^n} \rightarrow 0$ .
2. Let  $a_n = \frac{1000^n}{n!}$ . Then  $\frac{a_{n+1}}{a_n} = \frac{1000^{n+1}/(n+1)!}{1000^n/n!} = \frac{1000}{n+1} \leq \frac{1}{2}$  for all  $n \geq 1999$ .  
The Ratio Lemma says that  $\frac{1000^n}{n!} \rightarrow 0$  so that  $\frac{n!}{1000^n} \rightarrow \infty$ .

In both of the examples above we showed that  $\left(\frac{a_{n+1}}{a_n}\right) \rightarrow a$  for  $0 \leq a < 1$ , and then used this to show that  $\frac{a_{n+1}}{a_n} \leq l$  eventually. The following corollary to version 2 of the Ratio Lemma allows us to cut out some of this work.

#### Corollary

Let  $a_0, a_1, a_2, \dots$  be a sequence of positive numbers. If  $\left(\frac{a_{n+1}}{a_n}\right) \rightarrow a$  with  $0 \leq a < 1$  then  $(a_n) \rightarrow 0$ .

**Exercise 3** Prove this corollary by generalising the method used in the worked examples above.

### Assignment 7

State whether the following sequences tend to zero or infinity. Prove your answers:

1.  $\frac{n^{1000}}{2^n}$
2.  $\frac{1.0001^n}{n}$
3.  $\frac{n!}{n^{1000}}$
4.  $\frac{(n!)^2}{(2n)!}$

**Exercise 4** Try using the Ratio Lemma to prove that the sequence  $\frac{1}{n} \rightarrow 0$ . Why does the lemma tell you nothing?

The sequences  $(n^k)$  for  $k = 1, 2, 3, \dots$  and  $(x^n)$  for  $x > 1$  and  $(n!)$  all tend to infinity. Which is quickest? The above examples suggest some general rules which we prove below.

### Assignment 8

Prove that  $\left(\frac{x^n}{n!}\right) \rightarrow 0$  for all values of  $x$ .

#### Powerful Powers

All increasing power sequences grow faster than any polynomial sequence.

#### Powerless Powers

All power sequences are powerless against the factorial sequence  $(n!)$ .

Notice that this result implies that  $(n!)^{1/n} \rightarrow \infty$ , since for any value of  $x$ , eventually we have that  $\left(\frac{x^n}{n!}\right) < 1$  giving that  $x < (n!)^{1/n}$ .

**Assignment 9**

Prove that  $\left(\frac{n!}{n^n}\right) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Assignment 10**

Find the limit of the sequence  $\left(\frac{x^n}{n^k}\right)$  as  $n \rightarrow \infty$  for all values of  $x > 0$  and  $k = 1, 2, \dots$

**Assignment 11**

Find the limits of the following sequences. Give reasons.

1.  $\left(\frac{n^4 11^n + n^9 9^n}{7^{2n} + 1}\right)$
2.  $\left((4^{10} + 2^n)^{1/n}\right)$
3.  $\left(\frac{3n^3 + n \cos^2 n}{n^2 + \sin^2 n}\right)$
4.  $\left((3n^2 + n)^{1/n}\right)$

**Exercise 5** In the box below, make a table of all the general limits you have found, including  $n^k, x^n, \frac{x^n}{n^k}, x^{1/n}, (x^n + y^n)^{1/n}, n^{1/n}, \frac{x^n}{n!}, \frac{n!}{n^n}$ .

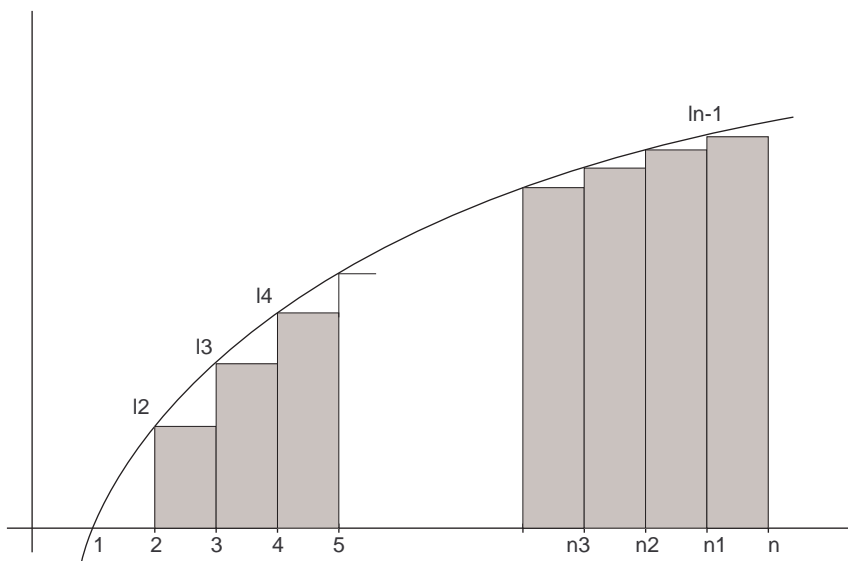


Figure 3: Approximating  $\int_1^n \log x dx$  from below.

## 2.15 \* Application - Factorials \*

Factorials  $n!$  occur throughout mathematics and especially where counting arguments are used. The last section showed that the factorial sequence  $(n!)$  is more powerful than any power sequence  $(x^n)$ , but earlier you showed that  $\frac{n!}{n^n} \rightarrow 0$ . In this section we will get excellent approximations to  $n!$  using two clever but very useful tricks. The first is to change the product  $n! = 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$  into a sum by taking logarithms:  $\log n! = \log 2 + \log 3 + \cdots + \log(n-1) + \log n$ .

The second trick, which we shall use repeatedly in future sections, is to approximate the sum by an integral, see figure 3. Here is a graph of the function  $\log(x)$  with a series of blocks, each of width one, lying underneath the graph. The sum of the areas of all the blocks is  $\log 2 + \log 3 + \cdots + \log(n-1)$ . But the area of the blocks is less than the area under the curve between  $x = 1$  and  $x = n$ . So we have:

$$\begin{aligned}
 \log n! &= \log 2 + \log 3 + \cdots + \log(n-1) + \log n \\
 &\leq \int_1^n \log x dx + \log n \\
 &= [x \log x - x]_1^n + \log n \\
 &= (n+1) \log n - n + 1.
 \end{aligned}$$

Taking exponentials of both sides we get the wonderful upper bound

$$n! \leq n^{n+1} e^{-n+1}.$$

### Assignment 12

Use figure 4 to obtain a lower bound on  $n!$  In this case the area of the blocks is greater than the area under the graph.

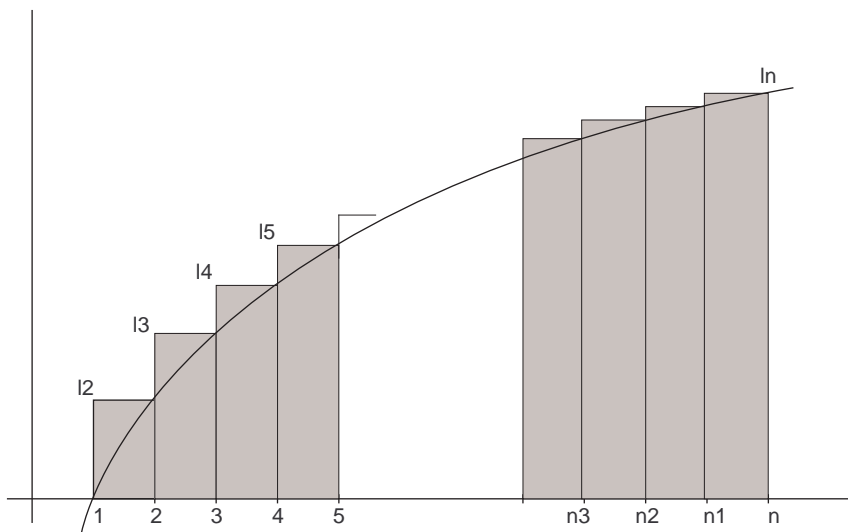


Figure 4: Approximating  $\int_1^n \log x dx$  from above.

### Assignment 13

Use your upper and lower bounds on  $n!$  to find the following limits: (i)  $\left(\frac{n!2^n}{n^n}\right)$   
(ii)  $\left(\frac{n!4^n}{n^n}\right)$

## 2.16 \* Application - Sequences and Beyond \*

In workbook 2 we defined a sequence as an infinite list of numbers. However, the concept of an infinite list of other objects is also useful in mathematics. With that in mind, we define a *sequence of objects* to be an infinite list of those objects.

For example, let  $P_n$  be the regular  $n$ -sided polygon of area 1. Then  $(P_n)$  is a sequence of shapes.

The main question we asked about sequences was whether they converged or not. To examine convergence in general, we need to be able to say when two objects are close to each other. This is not always an easy thing to do. However, given a sequence of objects, we may be able to derive sequences of numbers and examine those sequences to learn something about the original sequence. For example, given a sequence of tables, we could look at the number of legs of each table. This gives us a new sequence of numbers which is related to the original sequence of tables. However, different related sequences can behave in wildly different ways.

We shall demonstrate this by considering the set of solid shapes in  $\mathbb{R}^2$ . Given a sequence of shapes,  $(S_n)$ , two obvious related sequences are the sequence of perimeters,  $(p(S_n))$ , and the sequence of areas,  $(a(S_n))$ .



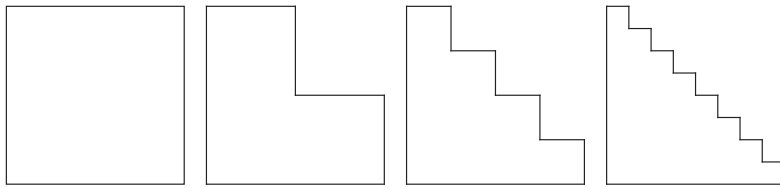


Figure 5:

**Example** Let  $P_n$  be the regular  $n$ -sided polygon centred at the origin which fits exactly inside the unit circle. This sequence starts with  $P_3$ , which is the equilateral triangle.

Let  $a_n = a(P_n)$  be the sequence of areas of the polygons. In workbook 1 you showed that  $a_n = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$  and in workbook 2 you showed that this converges to  $\pi$  as  $n \rightarrow \infty$  which is the area of the unit circle.

**Exercise 6** Let  $p_n = p(P_n)$  be the perimeter of  $P_n$ . Show that  $p_n = 2n \sin\left(\frac{\pi}{n}\right) = 2n \sin\left(\frac{2\pi}{2n}\right) = 2a_{2n}$ .

**Assignment 14**

Show that  $\lim p_n = 2\pi$ .

We see that in both cases we get what we would expect, namely that as the shape looks more and more like the circle, so also the area and perimeter tend to those of the circle.

**Assignment 15**

Consider the sequence of shapes in figure 5. Each is produced from the former by replacing each large step by two half-sized ones. Draw the “limiting” shape. Given that the original shape is a square of area 1, what is the perimeter and area of the  $n^{\text{th}}$  shape? Compare the limits of these sequences with the perimeter and area of the limiting shape.

A famous example of this type of behaviour is the Koch curve, see figure 6. The initial figure is an equilateral triangle of area  $A_1$  and perimeter  $p_1$ . To each side of the triangle is attached another equilateral triangle at the trisection points of the triangle. This process is then applied to each side of the resulting figure and so on.

Let  $p_n$  be the perimeter of the shape at the  $n^{\text{th}}$  stage and  $A_n$  the area.

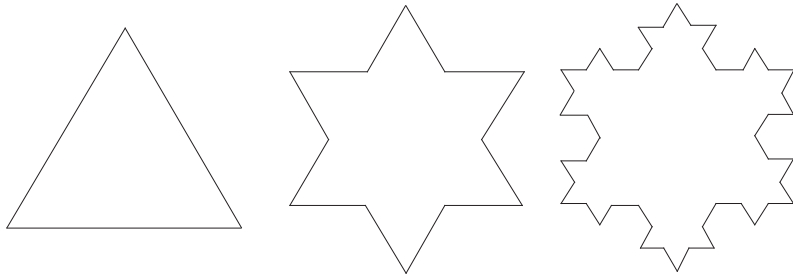


Figure 6:

### Assignment 16

1. What is the number of sides of the shape at the  $n^{\text{th}}$  stage (for  $n = 1$  the answer is 3).
2. Show that  $p_{n+1} = \frac{4}{3}p_n$ .
3. Prove that  $(p_n) \rightarrow \infty$ .
4. Show that, in making the  $(n + 1)^{\text{th}}$  shape, each little triangle being added has area  $\frac{1}{9^n}A_1$ .
5. Show that  $A_{n+1} = A_n + \frac{3}{9} \left(\frac{4}{9}\right)^{n-1} A_1$ .
6. Prove that  $(A_n) \rightarrow \frac{8}{5}A_1$ . (Recall the sum of a geometric series.)

### Check Your Progress

By the end of this Workbook you should be able to:

- Understand, memorise, prove, and use a selection of standard limits involving roots, powers and factorials.