MA131 - Analysis 1

Workbook 3
Sequences II

Autumn 2004

Contents

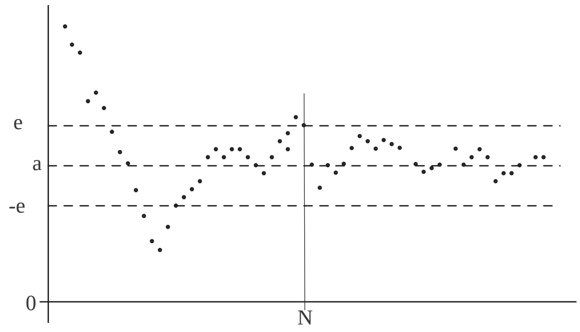

Figure 1: Convergent sequences; first choose ε, then find N.

2.8 Convergent Sequences

Plot a graph of the sequence $(a_n) = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$ To what limit do you think this sequence tends? What can you say about the sequence $(a_n - 1)$? For $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$ find an N such that $|a_n - 1| < \epsilon$ whenever $n > N$.

Definition
Let $a \in \mathbb{R}$. A sequence (a_n) *tends to* a if, for each $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon$ for all $n > N$.

See figure **??** for an illustration of this definition.

We use the notation $(a_n) \to a$, $a_n \to a$, as $n \to \infty$ and $\lim_{n\to\infty} a_n = a$ and say that (a_n) *converges to* a, or the *limit* of the sequence (a_n) as n tends to infinity is a.

Example Prove $(a_n) = \left(\frac{n}{n+1}\right) \to 1$.

Let $\epsilon > 0$. We have to find a natural number N so that

$$|a_n - 1| = \left|\frac{n}{n+1} - 1\right| < \epsilon$$

when $n > N$. We have

$$\left|\frac{n}{n+1} - 1\right| = \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}.$$

Hence it suffices to find N so that $\frac{1}{n} < \epsilon$ whenever $n > N$. But $\frac{1}{n} < \epsilon$ if and only if $\frac{1}{\epsilon} < n$ so it is enough to choose N to be a natural number with $N > \frac{1}{\epsilon}$. Then, if $n > N$ we have

$$|a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Lemma
$(a_n) \to a$ if and only if $(a_n - a) \to 0$.

Good N-ough

Any N that works is good enough - it doesn't have to be the smallest possible N.

Recycle

Have a closer look at figure **??**, what has been changed from figure 6 of workbook 2? It turns out that this definition is very similar to the definition of a null sequence.

Elephants Revisited

A null sequence is a special case of a convergent sequence. So **memorise** this definition and get the other one for free.

Proof. We know that $(a_n - a) \to 0$ means that for each $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon$ when $n > N$. But this is exactly the definition of $(a_n) \to a$. ■

We have spoken of *the* limit of a sequence but can a sequence have more than one limit? The answer had better be "No" or our definition is suspect.

Theorem *Uniqueness of Limits*
A sequence cannot converge to more than one limit.

Assignment 1
Prove the theorem by assuming $(a_n) \to a$, $(a_n) \to b$ with $a < b$ and obtaining a contradiction. [Hint: try drawing a graph of the sequences with a and b marked on]

Theorem
Every convergent sequence is bounded.

Assignment 2
Prove the theorem above.

2.9 "Algebra" of Limits

Theorem
$a, b \in \mathbb{R}$. Suppose $(a_n) \to a$ and $(b_n) \to b$. Then

$$(ca_n + db_n) \to ca + db \qquad \textit{Sum Rule} \text{ for Sequences}$$

$$(a_n b_n) \to ab \qquad \textit{Product Rule} \text{ for Sequences}$$

$$\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}, \text{ if } b \neq 0 \qquad \textit{Quotient Rule} \text{ for Sequences}$$

There is another useful way we can express all these rules: *If* (a_n) *and* (b_n) *are convergent then*

$$\lim_{n\to\infty} (ca_n + db_n) = c \lim_{n\to\infty} a_n + d \lim_{n\to\infty} b_n \qquad \textit{Sum Rule}$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \qquad \textit{Product Rule}$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}, \text{ if } \lim_{n\to\infty}(b_n) \neq 0 \qquad \textit{Quotient Rule}$$

Connection
It won't have escaped your notice that the Sum Rule for *null sequences* is just a special case of the Sum Rule for *sequences.* The same goes for the Product Rule.
Why don't we have a Quotient Rule for null sequences?

Polly Want a Cracker?
If you have a parrot, teach it to say:
"The limit of the sum is the sum of the limits."
"The limit of the product is the product of the limits."
"The limit of the quotient is the quotient of the limits."

Example In full detail

$$\begin{aligned}
\lim_{n\to\infty}\frac{(n^2+1)(6n-1)}{2n^3+5} &= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)}{2+\frac{5}{n^3}} \\
&\quad \text{using the Quotient Rule} \\
&= \frac{\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)\right]}{\lim_{n\to\infty}\left(2+\frac{5}{n^3}\right)} \\
&\quad \text{using the Product and Sum Rules} \\
&= \frac{\left(1+\lim_{n\to\infty}\left(\frac{1}{n^2}\right)\right)\left(6-\lim_{n\to\infty}\left(\frac{1}{n}\right)\right)}{2+5\lim_{n\to\infty}\left(\frac{1}{n^3}\right)} \\
&= \frac{(1+0)(6-0)}{2+0} \\
&= 3
\end{aligned}$$

Unless you are asked to show where you use each of the rules you can keep your solutions simpler. Either of the following is fine:

$$\lim_{n\to\infty}\frac{(n^2+1)(6n-1)}{2n^3+5} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)}{2+\frac{5}{n^3}} = \frac{(1+0)(6-0)}{2+0} = 3$$

or

$$\frac{(n^2+1)(6n-1)}{2n^3+5} = \frac{\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)}{2+\frac{5}{n^3}} \to \frac{(1+0)(6-0)}{2+0} = 3$$

Bigger and Better

By induction, the Sum and Product Rules can be extended to cope with any *finite* number of convergent sequences. For example, for three sequences:

$$\lim_{n\to\infty}(a_n b_n c_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \cdot \lim_{n\to\infty} c_n$$

Assignment 3
Use the Sum Rule for *null* sequences to prove the Sum Rule for sequences.

Exercise 1 Show that

$$(a_n-a)(b_n-b)+a(b_n-b)+b(a_n-a) = a_n b_n - ab$$

Assignment 4
Use the identity in Exercise 1 and the rules for *null* sequences to prove the Product Rule for sequences.

Assignment 5
Write a proof of the Quotient Rule. You might like to structure your proof as follows.

1. Note that $(bb_n) \to b^2$ and show that $bb_n > \frac{b^2}{2}$ for sufficiently large n.
2. Then show that eventually $0 \le \left|\frac{1}{b_n}-\frac{1}{b}\right| \le \frac{2}{b^2}|b-b_n|$ and therefore $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$.
3. Now tackle $\frac{a_n}{b_n} = a_n\frac{1}{b_n}$.

Don't Worry

You can still use the Quotient Rule if some of the b_ns are zero. The fact that $b \neq 0$ ensures that there can only be a finite number of these.
Can you see why?
And "eventually", the sequence leaves them behind.

Assignment 6
Find the limit of each of the sequences defined below.

1. $\frac{7n^2+8}{4n^2-3n}$
2. $\frac{2^n+1}{2^n-1}$
3. $\frac{(\sqrt{n}+3)(\sqrt{n}-2)}{4\sqrt{n}-5n}$
4. $\frac{1+2+\cdots+n}{n^2}$

Cunning Required

Do you know a cunning way to rewrite

$$1+2+3+\cdots+n\,?$$

2.10 Further Useful Results

Theorem *Sandwich Theorem for Sequences*
Suppose $(a_n) \to l$ and $(b_n) \to l$. If $a_n \leq c_n \leq b_n$ then $(c_n) \to l$.

Connection

The Sandwich Rule for *null sequences* represents the case when $l = 0$.

This improved Sandwich theorem can be tackled by rewriting the hypothesis as $0 \leq c_n - a_n \leq b_n - a_n$ and applying the earlier Sandwich theorem.

Assignment 7
Prove the Sandwich Theorem for sequences.

There are going to be many occasions when we are interested in the behaviour of a sequence "after a certain point", regardless of what went on before that. This can be done by "chopping off" the first N terms of a sequence (a_n) to get a shifted sequence (b_n) given by $b_n = a_{N+n}$. We often write this as (a_{N+n}), so that

$$(a_{N+n}) = a_{N+1},\ a_{N+2},\ a_{N+3},\ a_{N+4},\ \ldots$$

which starts at the term a_{N+1}. We use it in the definition below.

Definition
A sequence (a_n) satisfies a certain property *eventually* if there is a natural number N such that the sequence (a_{N+n}) satisfies that property.

For instance, a sequence (a_n) is *eventually bounded* if there exists N such that the sequence (a_{N+n}) is bounded.

Max and Min

In your proof you may well use the fact that each finite set has a maximum and a minimum. *Is this true of infinite sets?*

Lemma
If a sequence is eventually bounded then it is bounded.

Assignment 8
Prove this lemma.

The next result, called the Shift Rule, tells you that a sequence converges if and only if it converges eventually. So you can chop off or add on any *finite* number of terms at the beginning of a sequence without affecting the convergent behaviour of its infinite "tail".

Theorem *Shift Rule*
Let N be a natural number. Let (a_n) be a sequence. Then $a_n \to a$ if and only if the "shifted" sequence $a_{N+n} \to a$.

Proof. Fix $\epsilon > 0$. If $(a_n) \to a$ we know there exists N_1 such that $|a_n - a| < \epsilon$ whenever $n > N_1$. When $n > N_1$, we see that $N+n > N_1$, therefore $|a_{N+n} - a| < \epsilon$. Hence $(a_{N+n}) \to a$. Conversely, suppose that $(a_{N+n}) \to a$. Then there exists N_2 such that $|a_{N+n} - a| < \epsilon$ whenever $n > N_2$. When $n > N + N_2$ then $n - N > N_2$ so $|a_n - a| = |a_{N+(n-N)} - a| < \epsilon$. Hence $(a_n) \to a$. ■

Corollary *Sandwich Theorem with Shift Rule*
Suppose $(a_n) \to l$ and $(b_n) \to l$. If eventually $a_n \leq c_n \leq b_n$ then $(c_n) \to l$.

Example We know $1/n \to 0$ therefore $1/(n+5) \to 0$.

Exercise 2 Show that the Shift Rule also works for sequences which tend to infinity: $(a_n) \to \infty$ if and only if $(a_{N+n}) \to \infty$.

If all the terms of a convergent sequence sit within a certain interval, does its limit lie in that interval, or can it "escape"? For instance, if the terms of a convergent sequence are all positive, is its limit positive too?

Lemma
Suppose $(a_n) \to a$. If $a_n \geq 0$ for all n then $a \geq 0$.

Assignment 9
Prove this result. [Hint: Assume that $a < 0$ and let $\epsilon = -a > 0$. Then use the definition of convergence to arrive at a contradiction.]

Assignment 10
Prove or disprove the following statement:

"Suppose $(a_n) \to a$. If $a_n > 0$ for all n then $a > 0$."

Theorem *Inequality Rule*
Suppose $(a_n) \to a$ and $(b_n) \to b$. If (eventually) $a_n \leq b_n$ then $a \leq b$.

Assignment 11
Prove this result using the previous Lemma. [Hint: Consider $(b_n - a_n)$.]

Corollary *Closed Interval Rule*
Suppose $(a_n) \to a$. If (eventually) $A \le a_n \le B$ then $A \le a \le B$.

If $A < a_n < B$ it is *not* the case that $A < a < B$. For example $0 < \frac{n}{n+1} < 1$ but $\frac{n}{n+1} \to 1$.

Limits on Limits

Limits cannot escape from closed intervals. They can escape from open intervals - but only as far as the end points.

2.11 Subsequences

A subsequence of (a_n) is a sequence consisting of some (or all) of its terms in their original order. For instance, we can pick out the terms with even index to get the subsequence

$$a_2, a_4, a_6, a_8, a_{10}, \ldots$$

or we can choose all those whose index is a perfect square

$$a_1, a_4, a_9, a_{16}, a_{25}, \ldots$$

In the first case we chose the terms in positions 2,4,6,8,... and in the second those in positions 1,4,9,16,25,...

In general, if we take any strictly increasing sequence of natural numbers $(n_i) = n_1, n_2, n_3, n_4, \ldots$ we can define a subsequence of (a_n) by

$$(a_{n_i}) = a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots$$

Caution

Note that the subsequence (a_{n_i}) is indexed by i not n. In all cases $n_i \ge i$. (Why is this?) *Remember these facts when subsequences appear!*

Definition
A *subsequence* of (a_n) is a sequence of the form (a_{n_i}), where (n_i) is a strictly increasing sequence of natural numbers.

Effectively, the sequence (n_i) "picks out" which terms of (a_n) get to belong to the subsequence. Think back to the definition of convergence of a sequence. Why is it immediate from the definition that if a sequence (a_n) converges to a then all subsequence (a_{n_i}) converge to a? This is a fact which we will be using constantly in the rest of the course.

Notice that the shifted sequence (a_{N+n}) is a subsequence of (a_n).

Prove the obvious

It may seem obvious that every subsequence of a convergent sequence converges, but you should still check that you know how to prove it!

Assignment 12
Let $(a_n) = (n^2)$. Write down the first four terms of the three subsequences (a_{n+4}), (a_{3n-1}) and (a_{2^n}).

Here is another result which we will need in later workbooks.

Assignment 13
Suppose we have a sequence (a_n) and are trying to prove that it converges. Assume that we have have shown that the subsequences (a_{2n}) and (a_{2n+1}) both converge to the same limit a. Prove that $(a_n) \to a$ converges.

Exercise 3 Answer "Yes" or "No" to the following questions, but be sure that you know why and that you aren't just guessing.

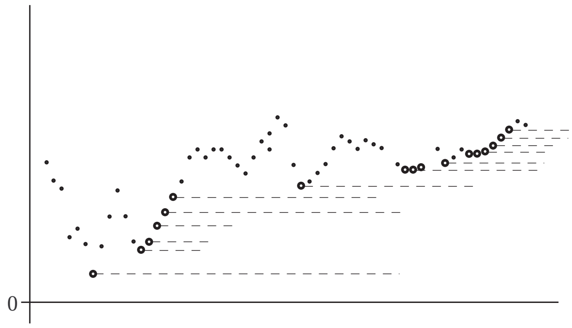

Figure 2: Floor terms are lower bounds for the rest of the sequence.

1. A sequence (a_n) is known to be increasing, but not strictly increasing.
   (a) Might there be a strictly increasing subsequence of (a_n)?
   (b) Must there be a strictly increasing subsequence of (a_n)?
2. If a sequence is bounded, must every subsequence be bounded?
3. If the subsequence $a_2,\ a_3,\ \ldots,\ a_{n+1},\ \ldots$ is bounded, does it follow that the sequence (a_n) is bounded?
4. If the subsequence $a_3,\ a_4,\ \ldots,\ a_{n+2},\ \ldots$ is bounded does it follow that the sequence (a_n) is bounded?
5. If the subsequence $a_{N+1},\ a_{N+2},\ \ldots,\ a_{N+n},\ \ldots$ is bounded does it follow that the sequence (a_n) is bounded?

Lemma
Every subsequence of a bounded sequence is bounded.

Proof. Let (a_n) be a bounded sequence. Then there exist L and U such that $L \leq a_n \leq U$ for all n. It follows that if (a_{n_i}) is a subsequence of (a_n) then $L \leq a_{n_i} \leq U$ *for all* i. Hence (a_{n_i}) is bounded. ■

You might be surprised to learn that every sequence, no matter how bouncy and ill-behaved, contains an increasing or decreasing subsequence.

Theorem
Every sequence has a monotonic subsequence.

We say a_f is a **floor term** of the (a_n) if $a_n \geq a_f$ for all $n \geq f$. So each floor term is "eventually" a lower bound.

Exercise 4 Write down the floor terms of the sequences:

1. $((-1)^n)$ 2. $0, 1, \frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{4}, \frac{1}{7}, \frac{1}{6}, \ldots$ 3. $\left(\frac{1}{n}\right)$

Identify a monotonic subsequence of each.

Exercise 5

1. If there is an infinite number of floor terms, show that they form a monotonic increasing subsequence.
2. If there is a finite number of floor terms and the last one is a_F, construct a monotonic decreasing subsequence with a_{F+1} as its first term.
3. If there are no floor terms, construct a monotonic decreasing subsequence with a_1 as its first term.

Assignment 14
Turn your answers to Exercise 5 into a proof of the previous theorem.

Sine Time Again
The fact that a sequence has a guaranteed monotonic subsequence doesn't mean that the subsequence is easy to find. Try identifying an increasing or decreasing subsequence of $\sin n$ and you'll see what I mean.

2.12 * Application - Speed of Convergence *

Often sequences are defined *recursively*, that is, later terms are defined in terms of earlier ones. Consider a sequence (a_n) where $a_0 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$, so the sequence begins $a_0 = 1, a_1 = \sqrt{3}, a_2 = \sqrt{\sqrt{3} + 2}$.

Exercise 6 Use induction to show that $1 \leq a_n \leq 2$ for all n.

Now assume that (a_n) converges to a limit, say, a. Then:

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left((a_{n+1})^2 - 2\right) = \left(\lim_{n\to\infty} a_{n+1}\right)^2 - 2 = a^2 - 2$$

So to find a we have to solve the quadratic equation $a^2 - a - 2 = 0$. We can rewrite this as $(a + 1)(a - 2) = 0$, so either $a = -1$ or $a = 2$. But which one is it? The Inequality Theorem comes to our rescue here. Since $a_n \geq 1$ for all n it follows that $a \geq 1$, therefore $a = 2$. We will now investigate the speed that a_n approaches 2.

Assignment 15
Show that $2 - a_{n+1} = \frac{2-a_n}{2+\sqrt{2+a_n}}$. Use this identity and induction to show that $(2 - a_n) \leq \frac{1}{(2+\sqrt{3})^n}$ for all n. How many iterations are needed so that a_n is within 10^{-100} is its limit 2?

An excellent method for calculating square roots is the Newton-Raphson method which you may have met at A-level. When applied to the problem of calculating $\sqrt{2}$ this leads to the sequence given by: $a_0 = 2$ and $a_{n+1} = \frac{1}{a_n} + \frac{a_n}{2}$.

Exercise 7 Use a calculator to calculate a_1, a_2, a_3, a_4. Compare them with $\sqrt{2}$.

Assignment 16
Use induction to show that $1 \leq a_n \leq 2$ for all n. Assuming that (a_n) converges, show that the limit must be $\sqrt{2}$.

We will now show that the sequence converges to $\sqrt{2}$ like a bat out of hell.

Assignment 17
Show that $(a_{n+1} - \sqrt{2}) = \frac{(a_n - \sqrt{2})^2}{2a_n}$. Using this identity show by induction that $|a_n - \sqrt{2}| \leq \frac{1}{2^{2^n}}$. How many iterations do you need before you can guarentee to calculate $\sqrt{2}$ to within an error of 10^{-100} (approximately 100 decimal places)?

Sequences as in Assignment 15 are said to converge *exponentially* and those as in Assignment 17 are said to converge *quadratically* since the error is squared at each iteration. The standard methods for calculating π were exponential (just as is the Archimedes method) until the mid 1970s when a quadratically convergent approximation was discovered.

Check Your Progress
By the end of this Workbook you should be able to:

- Define what it means for a sequence to "converge to a limit".
- Prove that every convergent sequence is bounded.
- State, prove and use the following results about convergent sequences: If $(a_n) \to a$ and $(b_n) \to b$ then:

  Sum Rule: $(ca_n + db_n) \to ca + db$

  Product Rule: $(a_n b_n) \to ab$

  Quotient Rule: $(a_n/b_n) \to a/b$ if $b \neq 0$

  Sandwich Theorem: if $a = b$ and $a_n \leq c_n \leq b_n$ then $(c_n) \to a$

  Closed Interval Rule: if $A \leq a_n \leq B$ then $A \leq a \leq B$
- Explain the term "subsequence" and give a range of examples.