Commutative Algebra II
7 Appendix on Homological Algebra

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7 Appendix on Homological Algebra

Throughout this short introduction to the basics of homological algebra, we work with an arbitrary ring R (not necessarily commutative) and an R-module M denotes a left R-module unless stated otherwise.

Definition 7.1. A *chain complex* $C_\bullet$ of R-modules is a collection of R-modules and R-module homomorphisms

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots$$

for all $i \in \mathbb{Z}$ such that the composite of any two consecutive maps is zero. The homology of a chain complex $C_\bullet$ is defined as

$$H_i(C_\bullet) := \ker(d_i)/\operatorname{im}(d_{i+1}).$$

Dually, a cochain complex of R-modules $C^\bullet$ is a collection of R-modules and R-module homomorphisms

$$\cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \to \cdots$$

for all $i \in \mathbb{Z}$, such that the composition of any two consecutive maps is zero. The cohomology of a cochain complex $C^\bullet$ is defined as

$$H^i(C^\bullet) := \ker(d^i)/\operatorname{im}(d^{i+1})$$

A (co)chain map between two (co)chain complexes $C_\bullet, D_\bullet$ consists of morphisms $f_n : C_n \to D_n$ that commute with the differentials of $C_\bullet$ and $D_\bullet$ in the natural way. One checks that a (co)chain map $f : C_\bullet \to D_\bullet$ induces a natural map on (co)homology

$$f^* : H_n(C_\bullet) \to H_n(D_\bullet). \quad \text{given by } [c] \mapsto [f(c)].$$

If $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to E_\bullet$ are chain maps then clearly $(gf)^* = g^* f^*$ and $(\mathrm{id}_{C_\bullet})^* = \mathrm{id}_{H_n(C_\bullet)}$. In fancier words, (co)homology defines a covariant functor from the category of chain complexes of R-modules, to the category of R-modules.

Definition 7.2. Two chain maps $f, g : C_\bullet \to D_\bullet$ are *chain homotopic*, if there exists a family of morphisms $h_n : C_n \to D_{n+1}$ that satisfy

$$f_n - g_n = h_{n-1} d_n^C + d_{n+1}^D h_n$$

Two chain complexes are *homotopy equivalent* if there are chain maps $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to C_\bullet$ such that gf is chain homotopic to $\mathrm{id}_{C_\bullet}$ and fg is chain homotopic to $\mathrm{id}_{D_\bullet}$. We have the following two elementary results that we state without proof.

Proposition 7.1. *If $f, g : C_\bullet \to D_\bullet$ are chain homotopic chain maps, then they induce the same map on homology.*

Proposition 7.2. *If $C_\bullet$ and $D_\bullet$ are homotopy equivalent chain complexes, they have isomorphic homology.*

Definition 7.3. An R-module P is *projective* if it satisfies one of the following equivalent conditions

(1) The covariant functor $\mathrm{Hom}_R(P, -)$ sends epimorphism to epimorphisms

(2) The covariant functor $\mathrm{Hom}_R(P, -)$ is exact

As an exercise, check that a free module is projective.

An R-module I is *injective* if it satisfies one of the following equivalent conditions

(1) The contravariant functor $\mathrm{Hom}_R(-, I)$ sends monomorphisms to epimorphisms

(2) The contravariant functor $\mathrm{Hom}_R(-, I)$ is exact

The category of R-modules has *enough projectives*, in the sense that every R-module M has a surjection $P \to M$ from a projective module P (just take a free presentation of M). It can be shown that it also has *enough injectives*, in the sense that every R-module M has an injection $M \to I$ into an injective module I. The injective case is trickier and we omit it here.

Proposition 7.3. *An R-module P is projective if and only if it is a direct summand of a free module.*

Lemma 7.4. *(Baer's criterion) An R-module I is injective if and only if every R-module homomorphism*

$$\mathfrak{a} \to I$$

from an ideal $\mathfrak{a}$ of R extends to a homomorphism $R \to I$.

Lemma 7.5. *Let $A, \mathfrak{m}$ local and P a finite projective A-module. Then P is free.*

Proof. Let $m_1, \dots, m_r$ be a minimal set of generators of P. We have a short exact sequence

$$0 \to K \to \bigoplus_{i=1}^{r} R \xrightarrow{e_i \mapsto m_i} P \to 0$$

which is split since P is projective. Hence $\bigoplus_{i=1}^{r} R = K \oplus P$. We claim that $K = 0$. Let $\sum_{i=1}^{r} \lambda_i e_i \in K$. Then $\sum_{i=1}^{r} \lambda_i m_1 = 0$ in P. Since our set of generators is minimal, all of the λ_i must be nonunits in A and thus must lie inside $\mathfrak{m}$. Thus $K \subset \mathfrak{m} \bigoplus_{i=1}^{r} R \subset \mathfrak{m}K \oplus \mathfrak{m}P$ where we identify K and P with their images in $\bigoplus_{i=1}^{r} R$. But $K \cap P = 0$, hence $K \subset \mathfrak{m}K$ and so $K = \mathfrak{m}K$. Since K is also finitely generated, it must be zero by Nakayama. Hence P is free. ☐

Corollary 7.6. *Let P be a finite projective A-module. Then $P_{\mathfrak{P}}$ is a free $A_{\mathfrak{P}}$-module for all $\mathfrak{P} \in \operatorname{Spec} A$,*

Proof. Since P is projective, we have a finite, free, split presentation

$$0 \to K \to F \to P \to 0$$

with F a finite free A-module. Now since localisation is an additive functor, it preserves split exact sequences, thus we have a split exact sequence

$$0 \to K_{\mathfrak{P}} \to F_{\mathfrak{P}} \to P_{\mathfrak{P}} \to 0$$

where $F_{\mathfrak{P}}$ is a free $A_{\mathfrak{P}}$-module. Thus $P_{\mathfrak{P}}$ is a direct summand of a free $A_{\mathfrak{P}}$-module and so it's projective by Proposition 7.3. But $A_{\mathfrak{P}}$ is local and so by lemma 7.5 $P_{\mathfrak{P}}$ is free. ☐

Remark. The converse of the above corolary is also true if one further assumes that P is finitely presented. In particular, it is true for A Noetherian.

Definition 7.4. Let M be an R-module. A projective resolution of M is a chain complex $P_\bullet, d_\bullet$ and an augmentation map $\epsilon\colon P_0 \to M$ such that

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. Dually, an injective resolution of M, is a cochain complex $I^\bullet, d^\bullet$ together with an augmentation map $\eta\colon M \to I^0$ such that

$$0 \to M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots \to I^n \xrightarrow{d^n} I^{n+1} \to \cdots$$

is exact.

Using that the category of R-modules has enough projectives and injectives one shows by induction that every R-module admits a projective and an injective resolution.

Definition 7.5. Let N be an R-module. We define $\mathrm{Ext}^{\bullet}_R(N, -)$ to be the right derived functor of the left exact functor $\mathrm{Hom}_R(N, -)$. That is for any R-module M,

$$\mathrm{Ext}^i_R(N, M) := H^i(\mathrm{Hom}_R(N, I^{\bullet}))$$

where $I^{\bullet}$ is an injective resolution of M.

Let N be a right R-module. For commutative R, which is always the case in this course, this distinction is not needed. We define $\mathrm{Tor}^R_{\bullet}(N, -)$ to be the left derived functor of the right exact functor $N \otimes_R -$. That is, for any R-module M,

$$\mathrm{Tor}^R_i(N, M) := H_i(N \otimes_R P_{\bullet})$$

where $P_{\bullet}$ is a projective resolution of M.

By right exactness of $N \otimes_R -$ and by left exactness of $\mathrm{Hom}_R(N, -)$, one sees that $\mathrm{Tor}^R_0(N, M) \cong N \otimes_R M$ and $\mathrm{Ext}^1_R(N, M) \cong \mathrm{Hom}_R(N, M)$, independendlty of the choices of projective and injective resolution.

We have some stuff to check before we can sensibly use the above definitions. Namely, we need to verify that Ext and Tor are well defined, in the sense that they are independent of choices. That is, $\mathrm{Ext}^{\bullet}_R(N, M)$ is independent of the choice of injective resolution of M and $\mathrm{Tor}^R_{\bullet}(N, M)$ is independent of the choice of projective resolution of M. We state the following key lemma without proof.

Lemma 7.7 (Comparison theorem)**.** *Let N, M be R-modules, $P_{\bullet} \to M \to 0$ be a projective resolution of M, and $f : M \to N$ a homomorphism. Let $Q_{\bullet} \to N \to 0$ be a resolution of N (not necessarily projective). Then f lifts to a chain map $P_{\bullet} \to Q_{\bullet}$ that is unique up to homotopy.*

For injective resolutions, working in the opposite category gives the dual result.

Now let $P_{\bullet}$ and $Q_{\bullet}$ be two projective resolutions of M. Then by the comparison theorem, we get chain maps $f : P_{\bullet} \to Q_{\bullet}$ and $g : Q_{\bullet} \to P_{\bullet}$, both extending $\mathrm{id} : M \to M$. By the uniqueness in the comparison theorem, gf is chain homotopic to $\mathrm{id}_{P_{\bullet}}$ and fg is chain homotopic to $\mathrm{id}_{Q_{\bullet}}$. Applying the additive functor $N \otimes_R -$ gives us chain homotopies between the induced maps and hence the chain complexes $N \otimes_R P_{\bullet} \to N \otimes_R M$ and $N \otimes_R Q_{\bullet} \to N \otimes_R M$ are chain homotopic and have isomorphic homology by Proposition 7.2. That is,

$$\mathrm{Tor}^R_i(N, M) = H_i(N \otimes_R P_{\bullet}) \cong H_i(N \otimes_R Q_{\bullet}) \quad \text{for } i \geq 1,$$

and for $i = 0$ we already know the result. Hence Tor is well defined. One shows Ext is also well defined similarly.

Now that we have shown that Tor and Ext are well defined, we can talk about *balancing* them. The following is a crucial result that we often use.

Theorem 7.8. *(Balancing Tor and Ext)*

(1) *Let $P_{\bullet} \to N$ and $Q_{\bullet} \to M$ be projective resolutions. Then*

$$H_i(P_{\bullet} \otimes_R M) \cong \mathrm{Tor}^R_i(N, M) = H_i(N \otimes_R Q_{\bullet})$$

That is we can compute the Tor groups by using either a projective resolution of N or a projective resolution of M and indeed we could have defined them either way.

(2) *Let $P_{\bullet} \to N$ be a projective resolution and $M \to I^{\bullet}$ be an injective resolution. Then*

$$H^i(\mathrm{Hom}_R(P_{\bullet}, M)) \cong \mathrm{Ext}^i_R(N, M) = H^i(\mathrm{Hom}_R(N, I^{\bullet}))$$

That is we can compute the Ext groups either by taking a projective resolution of N and then applying the contravariant $\mathrm{Hom}_R(-, M)$ *and taking cohomology, or by first taking an injective resolution of M, applying the covariant* $\mathrm{Hom}_R(N, -)$ *and then taking cohomology.*

Proof. The proof of this result is not hard but the details are rather involved and the full proof is quite lengthy. Consider the total complexes $\mathrm{Tot}^{\oplus}(P_\bullet \otimes_R Q_\bullet)$ and $\mathrm{Tot}^{\otimes}(\mathrm{Hom}_R(P_\bullet, I^\bullet))$ and showing that there are quasi-isomorphisms

$$P_\bullet \otimes_R M \leftarrow \mathrm{Tot}^{\oplus}(P_\bullet \otimes_R Q_\bullet) \to N \otimes_R Q_\bullet \quad \text{and} \quad \mathrm{Hom}_R(P_\bullet, M) \to \mathrm{Tot}^{\oplus}(\mathrm{Hom}_R(P_\bullet, I^\bullet)) \leftarrow \mathrm{Hom}_R(N, I^\bullet)$$

One can prove this by exhibiting these maps in a natural way, and showing their cones are acyclic (possibly using the acyclic assembly lemma). Alternatively, assuming familiarity with spectral sequences, we can prove it using the two convergent spectral sequences arising from the two natural filtrations on the above total complexes. □

Perhaps the most useful result for this course is the following.

Theorem 7.9. (1) *Let N, M be an R-modules and $0 \to X \to Y \to Z \to 0$ a short exact sequence of R-modules. Then there exist long exact sequences*

$$0 \to \mathrm{Ext}^0_R(N,X) \to \mathrm{Ext}^0_R(N,Y) \to \mathrm{Ext}^0_R(N,Z) \xrightarrow{\delta} \mathrm{Ext}^1_R(N,X) \to \mathrm{Ext}^1_R(N,Y) \to \cdots$$
$$0 \to \mathrm{Ext}^0_R(Z,M) \to \mathrm{Ext}^1_R(Y,M) \to \mathrm{Ext}^0_R(X,M) \xrightarrow{\delta} \mathrm{Ext}^1_R(Z,M) \to \mathrm{Ext}^1_R(Y,M) \to \cdots$$
$$0 \to \mathrm{Tor}_0^R(N,X) \to \mathrm{Tor}_0^R(N,Y) \to \mathrm{Tor}_0^R(N,Z) \xrightarrow{\delta} \mathrm{Tor}_1^R(N,X) \to \mathrm{Tor}_1^R(N,Y) \to \cdots$$
$$0 \to \mathrm{Tor}_0^R(X,M) \to \mathrm{Tor}_0^R(Y,M) \to \mathrm{Tor}_0^R(Z,M) \xrightarrow{\delta} \mathrm{Tor}_1^R(X,M) \to \mathrm{Tor}_1^R(Y,M) \to \cdots$$

Proof. We'll show the first; the others are similar. Let $P_\bullet \to N$ be a projective resolution of N, with differentials d_i. Then we have a short exact sequence of cochain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(P_0,X) & \longrightarrow & \mathrm{Hom}_R(P_0,Y) & \longrightarrow & \mathrm{Hom}_R(P_0,Z) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle -\circ d_1} & & \big\downarrow{\scriptstyle -\circ d_1} & & \big\downarrow{\scriptstyle -\circ d_1} & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_1,X) & \longrightarrow & \mathrm{Hom}_R(P_1,Y) & \longrightarrow & Hom_R(P_1,Z) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle -\circ d_2} & & \big\downarrow{\scriptstyle -\circ d_2} & & \big\downarrow{\scriptstyle -\circ d_2} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the rows are exact since the P_i are projective and hence $\mathrm{Hom}_R(P_i, -)$ is exact. Taking cohomology gives induced maps, and the snake lemma gives us the connecting homomorphisms

$$\mathrm{Ext}^i_R(N,Z) \to \mathrm{Ext}^{i+1}_R(N,X).$$

□

Corollary 7.10. *The following are equivalent:*

(1) *P is projective*

(2) $\mathrm{Ext}^i_R(P,M) = 0$ *for all* $i \geq 1$ *and any* M

(3) $\mathrm{Ext}^1_R(P,M) = 0$ *for any* M

The following are equivalent:

(1) *I is injective*

(2) $\mathrm{Ext}^i_R(N,I) = 0$ *for all* $i \geq 1$ *and any* N

(3) $\mathrm{Ext}^1_R(N,I) = 0$ *for any* N

The following are equivalent:

(1) Q *is flat*

(2) $\operatorname{Tor}_i^R(N, Q) = 0$ *for all* $i \geq 1$ *and any* N

(3) $\operatorname{Tor}_1^R(N, Q) = 0$ *for any* N

Proof. This follows from the above long exact sequences. □