Commutative Algebra II
6 Depth, Cohen–Macaulay and Gorenstein

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6 Depth, Cohen–Macaulay and Gorenstein

Definition 6.1. Let A be a Noetherian ring, I an ideal of A and M a finite A-module. We define the *I-depth* $\operatorname{depth}_I M$ of M as the maximum length of an M-regular sequence contained in I.

If $\operatorname{depth}_I M = 0$, that is, I is contained in the zerodivisors of M, it follows by prime avoidance that $I \subset P$ for some associated prime $P \in \operatorname{Ass} M$.

Before stating and proving results regarding depth and the Ext groups, we recall a lemma[1] that we use in the proof of Theorem 6.2.

Lemma 6.1. *Let A be a ring and N, M modules over A. Assume that N is finitely presented. Then $\operatorname{Hom}_A(N, M)_P \cong \operatorname{Hom}_{A_P}(N_P, M_P)$ for any $P \in \operatorname{Spec} A$. Here finite presentation of N means there exists $A^{\oplus r} \twoheadrightarrow N \to 0$ with finitely generated kernel; this holds when A is Noetherian and N is finite.*

Theorem 6.2. *Fix $n \geq 1$. Let A be a Noetherian ring, I an ideal of A and M a finite A-module with $M/IM \neq 0$. Equivalent conditions:*

(1) $\operatorname{Ext}^i_A(N, M) = 0$ *for all $i < n$ and for every finite A-module N with* $\operatorname{Supp} N \subseteq V(I)$.

(2) $\operatorname{Ext}^i_A(A/I, M) = 0$ *for all $i < n$.*

(3) $\operatorname{Ext}^i_A(N, M) = 0$ *for all $i < n$ and for some finite A-module N with* $\operatorname{Supp} N = V(I)$.

(4) $\operatorname{depth}_I M \geq n$*: there exists an M-regular sequence in I of length n.*

Proof. $1 \Rightarrow 2 \Rightarrow 3$ are trivial given that $\operatorname{Supp}(A/I) = V(I)$.

We prove $3 \Rightarrow 4$ by induction on $n \geq 1$. If $n = 1$, then $\operatorname{Ext}^0_A(N, M) = \operatorname{Hom}_A(N, M) = 0$ for some finite A-module N with $\operatorname{Supp} N = V(I)$. Suppose for a contradiction that there is no M-regular element in I, that is, I only contains zerodivisors of M. Then by Lemma 1.2 (3), I is contained in $\bigcup_{P \in \operatorname{Ass}_A M} P$, and this is a finite union by Lemma 1.2 (4). So by prime avoidance, I is contained in an associated prime P. In other words, $P \in V(I) = \operatorname{Supp} N$ and hence $N_P \neq 0$. So by Nakayama, $N_P/(PA_P)N_P \neq 0$. However, if we write $k(P)$ for the residue field A_P/PA_P, we get

$$\begin{aligned} N_P/(PA_P)N_P &= k(P) \otimes_{A_P} N_P = k(P) \otimes_{A_P} (A_P \otimes_A N) \\ &= (k(P) \otimes_{A_P} A_P) \otimes_A N = k(P) \otimes_A N. \end{aligned}$$

Thus $k(P) \otimes_A N$ is a nonzero $k(P)$-vector space and so by linear algebra, $\operatorname{Hom}_{k(P)}(k(P) \otimes_A N, k(P)) \neq 0$. Pick a nonzero $k(P)$-homomorphism f. Then f is clearly also an A_P-homomorphism. Also, since P is an associated prime, we have an injection, $A/P \hookrightarrow M$ and since localisation is exact, we have an injection $\iota\colon k(P) \hookrightarrow M_P$. Now consider the composite

$$N_P \to N_P/(PA_P)N_P = k(P) \otimes_A N \xrightarrow{f} k(P) \xhookrightarrow{\iota} M_P.$$

This is nonzero and hence $\operatorname{Hom}_{A_P}(N_P, M_P) \neq 0$. So by Lemma 6.1, $\operatorname{Hom}_A(N, M) \neq 0$ which gives a contradiction. This proves the case $n = 1$. Now suppose $n > 1$. By the base case, we must have some $x_1 \in I$ which is M-regular. Put $M_1 := M/x_1M$, then we have a short exact sequence

$$0 \to M \xrightarrow{x} M \to M_1 \to 0,$$

Applying the covariant Ext gives a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_A(N, M_1) \to \operatorname{Ext}^i_A(N, M) \xrightarrow{x} \operatorname{Ext}^i_A(N, M) \to \operatorname{Ext}^i_A(N, M_1) \to \operatorname{Ext}^{i+1}_A(N, M) \to \cdots$$

But by assumption, $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < n$, and hence $\operatorname{Ext}^i_A(N, M_1) = 0$ for all $i < n - 1$. So by induction, there exists an M_1-regular sequence $x_2, \ldots, x_n$ in I and hence $x_1, \ldots, x_n$ is an M-regular sequence in I so we are done.

[1] This belongs earlier with the discussion on localisation $S^{-1} \otimes_A S^{-1}$. Localisation is exact, and compatible with Hom modules. I used it in the proof that locally free implies projective.

To prove $4 \Rightarrow 1$, we again argue by induction on $n \geq 1$. If $n = 1$, then we have an M-regular element $x \in I$. Let N be finite such that $\operatorname{Supp} N \subseteq V(I)$. Since N is finite, $V(\operatorname{Ann} N) = \operatorname{Supp} N$ and so $I \subseteq \operatorname{rad}(\operatorname{Ann} N)$. Thus $x^m \in \operatorname{Ann} N$ for some $m \geq 1$. Let $\varphi \in \operatorname{Ext}^0_A(N, M) = \operatorname{Hom}_A(N, M)$. Then $x^m\varphi(n) = \varphi(x^m n) = \varphi(0) = 0$. But recall that x is M-regular, hence it must be the case that $\varphi(n) = 0$ for all $n \in N$, which proves the base case. Now let $n > 1$ and let $x_1, \ldots, x_n$ be an M-regular sequence in I. Put $M_1 := M/x_1M$, then we have a short exact sequence

$$0 \to M \xrightarrow{x_1} M \to M_1 \to 0$$

Again we get a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_A(N, M_1) \to \operatorname{Ext}^i_A(N, M) \xrightarrow{x_1} \operatorname{Ext}^i_A(N, M) \to \operatorname{Ext}^i_A(N, M_1) \to \operatorname{Ext}^{i+1}_A(N, M) \to \cdots.$$

Now $x_2, \ldots, x_n$ is an M_1-regular sequence in I and so by induction, $\operatorname{Ext}^i_A(N, M_1) = 0$ for all $i < n-1$. Hence from the long exact sequence, we get an injection

$$0 \to \operatorname{Ext}^i_A(N, M) \xrightarrow{x_1} \operatorname{Ext}^i_A(N, M)$$

for all $i < n$. But $\operatorname{Ext}^i_A(N, M)$ is a subquotient of $\operatorname{Hom}_A(N, I^i)$, where $I^\bullet$ is an injective resolution of M. The same argument as in the base case shows that for some $m \geq 1$ multiplication by x_1^m kills the A-module $\operatorname{Hom}_A(N, I^i)$ and so it also kills $\operatorname{Ext}^i_A(N, M)$. So multiplication by x_1^m as an endomorphism of $\operatorname{Ext}^i_A(N, M)$ is both injective (as the composition of injective maps) and the zero map, hence $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < n$. Notice that the assumptions A Noetherian and M finite are not necessary for the proof of $4 \Rightarrow 1$. □

Corollary 6.3. *Let A be a Noetherian ring, I an ideal of A and M a finite A-module. Then*

$$\operatorname{depth}_I M = \inf\{i \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}$$

Proof. Let $d := \operatorname{depth}_I M$. Then we have a regular sequence of maximal length $x_1, \ldots, x_d$ in I. Put $M_i := M/(x_1, \ldots, x_i)M$ for $i = 1, \ldots, d$. We start with the usual short exact sequence

$$0 \to M \xrightarrow{x_1} M \to M_1 \to 0$$

Applying $\operatorname{Ext}^\bullet_A(A/I, -)$ gives a long exact sequence,

$$\cdots \to \operatorname{Ext}^{i-1}_A(A/I, M) \to \operatorname{Ext}^{i-1}_A(A/I, M_1) \to \operatorname{Ext}^i_A(A/I, M) \xrightarrow{x_1} \operatorname{Ext}^i_A(A/I, M) \to \cdots.$$

By the previous theorem, $\operatorname{Ext}^i_A(A/I, M) = 0$ for all $i < d$, thus from the exact sequence we see that $\operatorname{Ext}^i_A(A/I, M_1) = 0$ for all $i < d-1$. Now if $i = d-1$, we have an exact sequence

$$0 = \operatorname{Ext}^{d-1}_A(A/I, M) \to \operatorname{Ext}^{d-1}_A(A/I, M_1) \to \operatorname{Ext}^d_A(A/I, M) \xrightarrow{x_1} \operatorname{Ext}^d_A(A/I, M)$$

Since $x_1 \in I$, multiplication by x_1 kills $\operatorname{Ext}^d_A(A/I, M)$ and so $\operatorname{Ext}^{d-1}_A(A/I, M_1) \cong \operatorname{Ext}^d_A(A/I, M)$We next consider the short exact sequence

$$0 \to M_1 \xrightarrow{x_2} M_1 \to M_2 \to 0$$

Repeating the process above using the information we just obtained about the A-modules $\operatorname{Ext}^i_A(A/I, M_1)$ for $i < d-1$ gives that $\operatorname{Ext}^i_A(A/I, M_2) = 0$ for all $i < d-2$ and $\operatorname{Ext}^{d-2}_A(A/I, M_2) \cong \operatorname{Ext}^{d-1}_A(A/I, M_1)$. Iterating this, we get a chain of isomorphisms

$$\operatorname{Ext}^d_A(A/I, M) \cong \operatorname{Ext}^{d-1}_A(A/I, M_1) \cong \cdots \cong \operatorname{Ext}^1_A(A/I, M_{d-1}) \cong \operatorname{Ext}^0_A(A/I, M_d)$$

Thus if $\operatorname{Ext}^d_A(A/I, M)$ is zero, $\operatorname{Ext}^0_A(A/I, M_d)$ is also zero. Thus by Theorem 6.2, there exists an M_d-regular element $x_{d+1} \in I$. But then $x_1, \ldots, x_d, x_{d+1}$ would be an M-regular sequence contained in I, which contradicts $\operatorname{depth}_I M = d$. □

Remark. If $A, \mathfrak{m}, k$ is local Noetherian and M finite, we simply write $\operatorname{depth} M$ to mean $\operatorname{depth}_{\mathfrak{m}} M = \inf\{i \mid \operatorname{Ext}^i_A(k, M) \neq 0\}$

Corollary 6.4. *Let $A, \mathfrak{m}$ be a local Noetherian ring and M a finite A-module with $\operatorname{depth} M = n$. Then we can extend any M-regular sequence $x_1, \ldots, x_r \in \mathfrak{m}$ to a maximal regular sequence $x_1, \ldots, x_n$ (necessarily $r \leq n$).*

Proof. If $r = n$ we are done so assume $r < n$. We again put $M_i := M/(x_1, \ldots, x_i)M$ for $i = 1, \ldots, r$. Then from the proof of the previous corollary, we have that $\operatorname{Ext}^r_A(A/I, M) \cong \operatorname{Ext}^0_A(A/I, M_r)$. Since $r < n$, we know by Theorem 6.2 that $\operatorname{Ext}^r_A(A/I, M) = 0$ and hence $\operatorname{Ext}^0_A(A/I, M_r) = 0$. But again by Theorem 6.2, this gives us an M_r-regular element $x_{r+1} \in \mathfrak{m}$, and thus $x_1, \ldots, x_r, x_{r+1}$ is an M-regular sequence. This process can be iterated until we reach maximal length. □

Corollary 6.5. *Let $A, \mathfrak{m}$ be a local Noetherian ring and M a finite A-module. If $x_1, \ldots, x_r$ is an M-regular sequence in $\mathfrak{m}$, then*

$$\operatorname{depth}(M/(x_1, \ldots, x_r)M) = \operatorname{depth} M - r.$$

Proof. Clearly, $\operatorname{depth} M \geq \operatorname{depth}(M/(x_1, \ldots, x_r)M) + r$, that is, $\operatorname{depth}(M/(x_1, \ldots, x_r)M) \leq \operatorname{depth} M - r$. Conversely, let $d := \operatorname{depth} M \geq r$. By Corollary 6.4, $x_1, \ldots, x_r$ can be extended to a maximal sequence $x_1, \ldots, x_r, x_{r+1}, \ldots, x_d$. Then $x_{r+1}, \ldots, x_d$ is a regular sequence for $M/(x_1, \ldots, x_r)M$. Thus $\operatorname{depth}(M/(x_1, \ldots, x_r)M) \geq \operatorname{depth} M - r$. □

Theorem 6.6 (Ischebeck's Theorem)**.** *Let $A, \mathfrak{m}$ be a local Noetherian ring, and M, N nonzero finite A-modules. Put $\operatorname{depth} M = k$ and $\dim N = n$. Then*

$$\operatorname{Ext}^i_A(N, M) = 0 \quad \textit{for all } i < k - n.$$

Proof. We argue by induction on $n \geq 0$. If $n = 0$, then $A/\operatorname{Ann} N$ is zero dimensional Noetherian and hence Artinian, with unique prime ideal $\mathfrak{m}/\operatorname{Ann} N$. So $V(\operatorname{Ann} N) = V(\mathfrak{m})$ and hence by Theorem 6.2 $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < k$. Now let $n > 0$. By Corollary 1.3, we have a chain $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$, with $N_i/N_{i-1} \cong A/P_i$, with $P_i \in \operatorname{Spec} A$. Suppose we had the result for $N = A/P$, $P \in \operatorname{Spec} A$. Consider the short exact sequence

$$0 \to A/P_1 \to N_2 \to A/P_2 \to 0$$

Applying the contravariant $\operatorname{Ext}^\bullet_A(-, M)$, we get a long exact sequence

$$\cdots \to \operatorname{Ext}^i_A(A/P_2, M) \to \operatorname{Ext}^i_A(N_2, M) \to \operatorname{Ext}^i_A(A/P_1, M) \to \cdots$$

Then $\operatorname{Ext}^i_A(A/P_1, M) = \operatorname{Ext}^i_A(A/P_2, M) = 0$ for $i < k - \max\{\dim(A/P_1), \dim(A/P_2)\}$ and so

$\operatorname{Ext}^i_A(N_2, M) = 0$ for $i < k - \max\{\dim(A/P_1), \dim(A/P_2)\}$.

Next we consider the short exact sequence

$$0 \to N_2 \to N_3 \to A/P_3 \to 0$$

and repeating the same argument, we get that $\operatorname{Ext}^i_A(N_3, M) = 0$
for all $i < k - \max\{\dim(A/P_1), \dim(A/P_2), \dim(A/P_3)\}$.

Continuing in the same way we get that $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < k - \max_{i=1}^r\{\dim(A/P_i)\}$. But recall that by exactness o localisation, $\operatorname{Supp} N = \bigcup_{i=1}^r \operatorname{Supp}(A/P_i)$, where $\operatorname{Supp}(A/P_i) = V(P_i)$ is closed for each i. Hence $n = \dim N = \max_{i=1}^r\{\dim(A/P_i)\}$, which gives the result. Thus it suffices to prove the induction step for $N = A/P$ with $P \in \operatorname{Spec} A$. Since $n = \dim(A/P) > 0$, P is not maximal. Thus we can find $x \in \mathfrak{m}$ that is not in P (In other words, $x \in \mathfrak{m}$ is $A/P-regular$). Consider the short exact sequence

$$0 \to A/P \xrightarrow{x} A/P \to A/(P + xA) \to 0$$

Applying the contravariant Ext, we get a long exact sequence

$$\cdots \to \operatorname{Ext}_A^{i-1}(A/(P+xA), M) \to \operatorname{Ext}_A^i(A/P, M) \xrightarrow{x} \operatorname{Ext}_A^i(A/P, M) \to \operatorname{Ext}_A^i(A/(P+xA), M) \to \cdots$$

Now since x is A/P-regular and $A/(P+xA) = (A/P)/x(A/P)$, we have by Corollary 5.8, that $\dim(A/(P+xA)) = n-1$. So by induction, $\operatorname{Ext}_A^i(A/(P+xA), M) = 0$ for all $i < k-(n-1) = k-n+1$. Hence we have an isomorphism

$$0 \to \operatorname{Ext}_A^i(A/P, M) \xrightarrow{x} \operatorname{Ext}_A^i(A/P, M) \to 0$$

for all $i < k-n$. But $x \in \mathfrak{m}$ and $\operatorname{Ext}_A^i(A/P, M)$ is a finite A-module, thus by Nakayama, $\operatorname{Ext}_A^i(A/P, M) = 0$ for all $i < k-n$. □

Corollary 6.7. *Let $A, \mathfrak{m}$ be local Noetherian and M finite. Then for any $P \in \operatorname{Ass}_A M$*

$$\dim(A/P) \geq \operatorname{depth} M.$$

Proof. Suppose for a contradiction that $\operatorname{depth} M > \dim(A/P)$. Then by Theorem 6.4, we have that $\operatorname{Ext}_A^0(A/P, M) = \operatorname{Hom}_A(A/P, M) = 0$. But P is an associated prime, hence we have an injection $A/P \hookrightarrow M$. so we get a contradiction. □

Definition 6.2. Let $A, \mathfrak{m}$ be local Noetherian and M finite. Recall that we always have $\dim M \geq \operatorname{depth} M$ by Corollary 5.8. We say that M is *Cohen–Macaulay* (CM for short) if $M \neq 0$ and $\dim M = \operatorname{depth} M$, that is if the depth of M is as large as possible. The zero module is also Cohen–Macaulay by convention. A local Noetherian ring is Cohen–Macaulay if it is Cohen–Macaulay as a module over itself.

Lemma 6.8. *Let $A, \mathfrak{m}$ be a local Noetherian ring and M a finite A-module. If M is Cohen–Macaulay, then $\dim(A/P) = \dim M = \operatorname{depth} M$ for every associated prime $P \in \operatorname{Ass}_A M$.*

Proof. One can show that $\operatorname{rad}(\operatorname{Ann} M) = \bigcap_{P \in \operatorname{Ass}_A M} P$ and hence $V(\operatorname{Ann} M) = \bigcup_{P \in \operatorname{Ass}_A M} P$ (in this case $\operatorname{Ass}_A M$ is finite). Hence

$$\begin{aligned}\dim M = \dim(V(\operatorname{Ann} M)) &= \dim \bigcup_{P \in \operatorname{Ass}_A M} P \\ &= \max_{P \in \operatorname{Ass}_A M} \dim(V(P)) = \max_{P \in \operatorname{Ass}_A M} \dim(A/P) \\ &\geq \min_{P \in \operatorname{Ass}_A M} \dim(A/P) \geq \dim M.\end{aligned}$$

Here the last inequality follows from Corollary 6.5. Since M is CM, $\dim M = \operatorname{depth} M$ and so the result follows. □

Lemma 6.9. *Let $A, \mathfrak{m}$ be local Noetherian, M finite and $x_1, \ldots, x_r$ an M-regular sequence in $\mathfrak{m}$. Then M is Cohen–Macaulay if and only if $M/(x_1, \ldots, x_r)M$ is Cohen–Macaulay.*

Proof. This follows at once by Corollary 5.8 and Corollary 6.5. □

Lemma 6.10. *Let $A, \mathfrak{m}$ be a local Noetherian ring and M a finite A-module with $\dim M = \delta(M) = n$. Equivalent conditions:*

(1) *M is Cohen–Macaulay (that is, $\operatorname{depth} M = \dim M$).*

(2) *Every system of parameters $x_1, \ldots, x_n$ of M is an M-regular sequence.*

Proof. $2 \Rightarrow 1$ is clear by definition of depth and the fact that $\dim M \geq \operatorname{depth} M$ always holds.

For $1 \Rightarrow 2$, we argue by induction on $n \geq 0$. If $n = 0$, there is nothing to prove. Let $n = 1$. Let $x \in \mathfrak{m}$, with M/xM finite length. Then $\dim(A/(\operatorname{Ann} M + xA)) = \dim(M/xM) = 0$. We now claim that $x \notin \mathfrak{P}$ for any $\mathfrak{P} \in \operatorname{Ass}_A M$. If $x \in \mathfrak{P} \in \operatorname{Ass}_A M$, then $\operatorname{Ann} M + xA \subseteq \mathfrak{P}$ and hence $A/\mathfrak{P}$ is a quotient of $A/(\operatorname{Ann} M + xA)$. So

$$\dim(A/\mathfrak{P}) \leq \dim(A/(\operatorname{Ann} M + xA)) = 0$$

However, since we assume that M is CM, we have $\dim(A/\mathfrak{P}) = \dim M = 1$ by Lemma 6.8, which gives a contradiction. Thus x is not contained in any associated prime and so it must be M-regular, proving the base case. Now suppose $n > 1$, and let $x_1, \dots, x_n$ be a system of parameters of M. Let $M_1 := M/x_1M$. Then $\dim(M_1) = \delta(M_1) = n-1$ and hence the same argument as in the base case shows that x_1 is M-regular. Thus by Lemma 6.9, M_1 is CM of dimension $n-1$ and so by induction, any system of parameters of M_1, is an M_1-regular sequence. In particular, $x_2, \dots, x_n$ must be an M_1-regular sequence and so $x_1, \dots, x_n$ is an M-regular sequence. □

So far we have defined Cohen–Macaulay local rings (and modules). In our quest for various equivalent ways to define Gorenstein local rings, we first state and prove a result that characterises the injective dimension of a module based on the vanishing of certain Ext groups.

Theorem 6.11. *Let A be a Noetherian ring and N an A-module. Then*

$$\operatorname{inj\,dim} N \leq n \quad \textit{if and only if} \quad \operatorname{Ext}_A^{n+1}(A/P, N) = 0 \quad \textit{for every } P \in \operatorname{Spec} A.$$

Proof. The forwards direction is trivial. Conversely, suppose $\operatorname{Ext}_A^{n+1}(A/P, N) = 0$ for every $P \in \operatorname{Spec} A$, and let M be a finite A-module. We have a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with each $M_i/M_{i-1} \cong A/P_i$ for some $P_i \in \operatorname{Spec} A$. By repeatedly taking short exact sequences induced from the filtration (as in previous proofs), using the long exact sequence of Ext, we deduce that $\operatorname{Ext}_A^{n+1}(M, N) = 0$. Now let

$$0 \to N \to I^0 \xrightarrow{d^0} I^1 \to \cdots \to I^{n-1} \xrightarrow{d^{n-1}} I^n \xrightarrow{d^n} I^{n+1} \to \cdots$$

be an injective resolution of N. Set $C := I^{n-1}/\ker(d^{n-1})$. I claim that C is injective. We have

$$0 \to C \cong \operatorname{im}(d^{n-1}) \to I^n \xrightarrow{d^n} I^{n+1} \to \cdots$$

is exact. That is, $I^n \xrightarrow{d^n} I^{n+1} \xrightarrow{d^{n+1}} I^{n+2} \to \cdots$ is an injective resolution of C. Hence

$$\operatorname{Ext}_A^1(M, C) = \operatorname{Ext}_A^{n+1}(M, N) = 0$$

Now the choice of finite M was arbitrary. In particular this holds for $M = A/I$ where I is any ideal of A. Consider the short exact sequence

$$0 \to I \to A \to A/I \to 0$$

Applying the contravariant $\operatorname{Ext}_A^\bullet(-, C)$ gives the exact sequence

$$0 \to \operatorname{Hom}_A(A/I, C) \to \operatorname{Hom}_A(A, C) \to \operatorname{Hom}_A(I, C) \to \operatorname{Ext}_A^1(A/I, C) = 0$$

Thus the induced map $\operatorname{Hom}_A(A, C) \to \operatorname{Hom}_A(I, C)$ is surjective, that is every A-homomorphism $I \to C$, extends to an A-homomorphism $A \to C$. Hence C is injective by Baer's criterion and thus N has an injective resolution of length n. □

Definition 6.3. Let $A, \mathfrak{m}$ be a local ring and N an A-module. We define the *socle* of N to be the submodule

$$\operatorname{Socle}(N) := \{m \in N \mid \operatorname{Ann}(m) \supset \mathfrak{m}\}$$

This is naturally isomorphic to $\operatorname{Hom}_A(k, N)$ via the map sending an element $f \in \operatorname{Hom}_A(k, N)$ to $f(1)$. We can view $\operatorname{Socle}(N)$ as a k-vector space in a natural way.

Theorem 6.12. *Let $A, \mathfrak{m}, k$ be a local Artinian ring. Equivalent conditions:*

(1) $\operatorname{Socle}(A) \cong k$. *That is the socle of A, is* 1*-dimensional as a k-vector space.*

(2) *A is injective as a module over itself.*

Proof. We first prove $1 \Rightarrow 2$. To say that A is self-injective is the statement that A has zero injective dimension. Thus from Theorem 6.11, it suffices to show that $\mathrm{Ext}^1_A(A/P, A) = 0$ for all $P \in \mathrm{Spec}\, A$. But A is local Artinian and hence $\mathrm{Spec}\, A = \{\mathfrak{m}\}$. Thus we are left with showing that $\mathrm{Ext}^1_A(k, A) = 0$. Since A is Artinian, it admits a Jordan–Hölder series

$$0 = A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset A_n$$

$A_i/A_{i-1} \cong k$ for all $i = 1, \ldots, n$. Here $A_n = A$, and necessarily $A_{n-1} = \mathfrak{m}$ and $A_1 \cong k$. First consider the short exact sequence

$$0 \to k \to A_2 \to k \to 0$$

Applying the contravariant $\mathrm{Ext}^{\bullet}_A(-, A)$ gives a long exact seqeunce

$$0 \to \mathrm{Hom}_A(k, A) \to \mathrm{Hom}_A(A_2, A) \to \mathrm{Hom}_A(k, A) \xrightarrow{\delta_2} \mathrm{Ext}^1_A(k, A) \to \cdots$$

Thus we have

$$\begin{aligned} \ell_A(\mathrm{Hom}_A(A_2, A)) &= 2\,\ell_A(\mathrm{Hom}_A(k, A)) - \ell_A(\mathrm{im}(\delta_2)) \\ &= 2\dim_k(\mathrm{Hom}_A(k, A)) - \ell_A(\mathrm{im}(\delta_2)) \end{aligned}$$

Now consider the short exact sequence

$$0 \to A_2 \to A_3 \to k \to 0$$

Playing the same game gives

$$\ell_A(\mathrm{Hom}_A(A_3, A)) = \ell_A(\mathrm{Hom}_A(A_2, A)) + \ell_A(\mathrm{Hom}_A(k, A)) - \ell_A(\mathrm{im}(\delta_3))$$

$$= 3\dim_k(\mathrm{Hom}_A(k, A)) - \ell_A(\mathrm{im}(\delta_2)) - \ell_A(\mathrm{im}(\delta_3))$$

Continuing in the same way gives

$$\ell_A(\mathrm{Hom}_A(A_n, A)) = n\dim_k \mathrm{Hom}_A(k, A)) - \sum_{i=2}^{n} \ell_A(\mathrm{im}(\delta_i)).$$

But we are assuming that $\mathrm{Socle}(A) = \mathrm{Hom}_A(k, A)$ is 1-dimensional as a k-vector space. Therefore $\dim_k(\mathrm{Hom}_A(k, A)) = 1$. Also,

$$\ell_A(\mathrm{Hom}_A(A_n, A)) = \ell_A(\mathrm{Hom}_A(A, A)) = \ell_A(A) = n$$

(the length of our Jordan–Hölder series). Thus the above equality becomes

$$n = n - \sum_{i=2}^{n} \ell_A(\mathrm{im}(\delta_i))$$

So every term in the sum must be zero and hence $\delta_i = 0$ for all i. In particular, we have an exact sequence

$$0 \to \mathrm{Hom}_A(k, A) \to \mathrm{Hom}_A(A, A) \to \mathrm{Hom}_A(\mathfrak{m}, A) \xrightarrow{\delta_n = 0} \mathrm{Ext}^1_A(k, A) \to \mathrm{Ext}^1_A(A, A) = 0$$

We deduce that $\mathrm{Ext}^1_A(k, A) = 0$ and we are done.

The proof of $2 \Rightarrow 1$ is similar. We use the same Jordan–Hölder series for A, and consider the same short exact sequences arising from this series. However, we assume that A is injective as an A-module, so the functor $\mathrm{Hom}_A(-, A)$ is exact. So at each step $\ell_A(\mathrm{Hom}_A(A_i, A)) = \ell_A(A_{i-1}, A)) + \dim_k(\mathrm{Hom}_A(k, A))$. Thus we end up with

$$n = \ell_A(\mathrm{Hom}_A(A_n, A)) = n\dim_k(\mathrm{Hom}_A(k, A))$$

from which we conclude that the socle of A is 1-dimensional as required. □

We are now finally in a position to define Gorenstein local rings in 4 equivalent ways

Definition 6.4. Let $A, \mathfrak{m}$ be local Noetherian. We say that A is *Gorenstein* if it satisfies any of the four conditions of the next theorem.

Theorem 6.13. *Let $A, \mathfrak{m}$ be local Noetherian and $\dim A = n$. Let $x_1, \ldots, x_n$ be a system of parameters of A. Equivalent conditions:*

(1) $\mathrm{Ext}^i_A(k, A) = 0$ *for all* $i \neq n$ *and* $\mathrm{Ext}^n_A(k, A) \cong k$.

(2) *A is Cohen–Macaulay and* $\mathrm{Ext}^n_A(k, A) \cong k$.

(3) *A is Cohen–Macaulay and the Artinian quotient $A/(x_1, \ldots, x_n)$ has 1-dimensional socle.*

(4) *A is Cohen–Macaulay and the Artinian quotient $A/(x_1, \ldots, x_n)$ is self-injective.*

The theorem says that Gorenstein is essentially Cohen–Macaulay plus a bit extra. Characterisations 3 and 4 tell us that the extra condition is that we can cut A by a system of parameters (or equivalently by Lemma 6.9, by a regular sequence!) to dimension 0 (that is, Artinian), and the resulting quotient satisfies some nice properties. In other words, n-dimensional Gorenstein is being able to find a regular sequence of length n such that the resulting Artinian quotient satisfies one of the equivalent properties in 3 or 4.

Proof. $1 \Rightarrow 2$ is trivial by using the characterisation of depth in terms of the nonvanishing of Ext groups. For $2 \Leftrightarrow 3$, recall that since A is n-dimensional CM, in particular $\operatorname{depth} A = n$ and hence we've seen before that $\mathrm{Ext}^n_A(k, A) \cong \mathrm{Ext}^0_A(k, A/(x_1, \ldots, x_n)) = \mathrm{Hom}_A(k, A/(x_1, \ldots, x_n))$ form which the result follows. Note that this also shows that 3 and 4 do not depend on the choice of such system of parameters.

Now for $3 \Leftrightarrow$ if 4, we can simply invoke Theorem 6.12, after observing that $\mathrm{Socle}(A/(x_1, \ldots, x_n)) = \mathrm{Hom}_A(k, A/(x_1, \ldots, x_n))$ is isomorphic to $\mathrm{Hom}_{A/(x_1,\ldots,x_n)}(k, A/(x_1, \ldots, x_n))$ as $A/(x_1, \ldots, x_n)$-modules.

Thus we only have $2 \Rightarrow 1$ left to prove. For this we argue by induction on n. If $n = 0$, then $\mathrm{Hom}_A(k, A) \cong k$ and by Theorem 6.12, A is self injective and thus computing the Ext groups, using an injective resolution of A, we get that $\mathrm{Ext}^i_A(k, A) = 0$ for all $i > 0$. Now suppose $n > 0$. Since A is CM, $\operatorname{depth} A = n > 1$ and hence we have some regular element $x \in \mathfrak{m}$. Put $A_1 := A/xA$. Then by Corollary 5.8 and Lemma 6.9, A_1 is $n-1$-dimensional and CM. Thus by induction, $\mathrm{Ext}^i_{A_1}(k, A_1) = 0$ for all $i \neq n-1$ and $\mathrm{Ext}^{n-1}_A(k, A) \cong k$. We also have, for all $i \geq 1$, $\mathrm{Ext}^i_A(k, A) \cong \mathrm{Ext}^{i-1}_{A_1}(k, A_1)$. Hence $\mathrm{Ext}^i_A(k, A) = 0$ for all $i > 0$ and not equal to n, and $\mathrm{Ext}^n_A(k, A) \cong k$. For $i = 0$, let $\varphi \in \mathrm{Ext}^0_A(k, A) = \mathrm{Hom}_A(k, A)$. Then since $x \in \mathfrak{m}$, it annihilates k and we have $x\varphi(1) = \varphi(x) = \varphi(0) = 0$. But recall by assumption that x is A-regular thus we must have that $\varphi(1) = 0$ and thus $\mathrm{Ext}^0_A(k, A) = 0$ which concludes the proof.

□