Commutative Algebra II
5 Koszul complex, regular sequences

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5 Koszul complex and regular local rings

We start with two alternative definitions of the Koszul complex. Recall that a chain complex of R-modules $M_\bullet, d_\bullet^M$ is a sequence of R-modules M_i and homomorphisms

$$\cdots \to M_{m+1} \xrightarrow{d_{m+1}^M} M_m \xrightarrow{d_m^M} M_{m-1} \to \dots$$

such that the composite of any two consecutive maps is zero. Given two chain complexes of R-modules $M_\bullet, d_\bullet^M$ and $N_\bullet, d_\bullet^N$, we can form the double complex $M_\bullet \otimes_A N_\bullet$

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
M_0 \otimes_A N_2 \arrow[d] & M_1 \otimes_A N_2 \arrow[l] \arrow[d] & M_2 \otimes_A N_2 \arrow[l] \arrow[d] & \arrow[l] \\
M_0 \otimes_A N_1 \arrow[d] & M_1 \otimes_A N_1 \arrow[l] \arrow[d] & M_2 \otimes_A N_1 \arrow[l] \arrow[d] & \arrow[l] \\
M_0 \otimes_A N_0 & M_1 \otimes_A N_0 \arrow[l] & M_2 \otimes_A N_0 \arrow[l] & \arrow[l]
\end{tikzcd}$$

where the horizonal differentials are given by $\delta^h := d_m^M \otimes 1 \colon M_m \otimes_A N_n \to M_{m-1} \otimes_A N_n$ and the vertical differentials are given by $\delta^v := (-1)^m 1 \otimes d_n^N \colon M_m \otimes_A N_n \to M_m \otimes_A N_{n-1}$. Because of this we see that each square in the double complex, anti-commutes. This is done on purpose since it allows to define a chain complex from this double complex as follows. We define the *total complex* $\mathrm{Tot}^\oplus(M_\bullet \otimes_A N_\bullet)$ of the above tensor double complex to have degree $d \geq 0$ part

$$\mathrm{Tot}^\oplus(M_\bullet \otimes_A N_\bullet)_d = \bigoplus_{m+n=d} M_m \otimes_A N_n$$

and differential $\delta := \delta^h + \delta^v$. The fact that the squares in the double complex anti-commute means that $\delta^2 = 0$ and so this is indeed a complex. Visually this is just taking diagonal slices in our double complex along lines of slope -1 and the differentials map one diagonal slice to the previous one in the only way possible, that is, in each summand you go down and to the left. Here a useful remark is that the total complex is commutative in the sense that $\mathrm{Tot}^\oplus(M_\bullet \otimes_A N_\bullet)$ and $\mathrm{Tot}^\oplus(N_\bullet \otimes_A M_\bullet)$ are isomorphic as chain complexes. It is also associative in the sense that $\mathrm{Tot}^\oplus(\mathrm{Tot}^\oplus(M_\bullet \otimes_A N_\bullet) \otimes_A P_\bullet)$ and $\mathrm{Tot}^\oplus(M_\bullet \otimes_A \mathrm{Tot}^\oplus(N_\bullet \otimes_A P_\bullet))$ are isomorphic as chain complexes.

Definition 5.1 (Koszul complex 1)**.** Let A be a ring and $x \in A$. We define the *Koszul complex* $K(x)$ to be the complex

$$0 \to A \xrightarrow{x} A \to 0$$

given by multiplication by x. Now for $x_1, \dots, x_n \in A$, we define the Koszul complex $K(x_1, \dots, x_n)$ inductively by $K(x_1, x_2) := \mathrm{Tot}^\oplus(K(x_1) \otimes_A K(x_2))$, and
$K(x_1, \dots, x_n) := \mathrm{Tot}^\oplus(K(x_1, \dots, x_{n-1}) \otimes_A K(x_n))$.

This definition will be extremely useful when proving stuff about *Koszul homology* that we define later.

Definition 5.2 (Koszul complex 2)**.** Given $x_1, \dots, x_n \in A$, we define the Koszul complex

$$\cdots \to K_n \to K_{n-1} \to \cdots \to K_1 \to K_0 \to 0$$

where $K_0 := A$ and for $p \geq 1$ we have $K_p := \bigwedge^p(\bigoplus_{i=1}^n Ae_i)$, the pth exterior algebra of the free A-module of rank n, which is the free A-module of rank $\binom{n}{p}$ with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$

We leave the equivalence of the two definitions as an exercise.

Definition 5.3 (Koszul homology). Let M be an A-module and $x_1, \dots, x_n \in A$. The *Koszul homology* with coefficients in M is defined by

$$H_p(x_1, \dots, x_n, M) := H_p(M \otimes_A K(x_1, \dots, x_n))$$

For a chain complex $C_\bullet$ of A-modules, we define the Koszul homology with coefficients in $C_\bullet$ to be

$$H_p(x_1, \dots, x_n, C_\bullet) := H_p(\mathrm{Tot}^\oplus(C_\bullet \otimes_A K(x_1, \dots, x_n))).$$

Remark. An easy check shows that we always have

$$H_0(x_1, \dots, x_n, M) = M/(x_1, \dots, x_n)M$$
$$H_n(x_1, \dots, x_n, M) = \{\xi \in M \mid x_1\xi = \dots = x_n\xi = 0\}.$$

Theorem 5.1 (Künneth formula for Koszul homology). *Let $C_\bullet, d_\bullet$ be a chain complex of A-modules and $x \in A$. Then we have a short exact sequence*

$$0 \to H_0(x, H_q(C_\bullet)) \to H_q(x, C_\bullet) \to H_1(x, H_{q-1}(C_\bullet)) \to 0$$

Proof. A calculation involving double complexes shows that the total complex $\mathrm{Tot}^\oplus(C_\bullet \otimes_A K(x))$ is just

$$\mathrm{Tot}^\oplus(C_\bullet \otimes_A K(x))_{q+1} = C_{q+1} \oplus C_q$$

with differential

$$\Delta_{q+1} := \begin{pmatrix} d_{q+1} & (-1)^q x \\ 0 & d_q \end{pmatrix}$$

We end up with the short exact sequence of chain complexes

$$0 \to C_\bullet \to \mathrm{Tot}^\oplus(C_\bullet \otimes_A K(x)) \to C_\bullet[-1] \to 0$$

where $C_\bullet[-1]$ denotes the complex $C_\bullet$ shifted in degree by -1 and same differential (that is, $C_q[-1] = C_{q-1}$). This short exact sequence is given explicitly by

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_{q+1} \arrow[r, hook] \arrow[d, "d_{q+1}"] & C_{q+1} \oplus C_q \arrow[r, two heads] \arrow[d, "\Delta_{q+1}"] & C_q \arrow[r] \arrow[d, "d_q"] & 0 \\
0 \arrow[r] & C_q \arrow[r, hook] \arrow[d] & C_q \oplus C_{q-1} \arrow[r, two heads] \arrow[d] & C_{q-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

One checks that the squares commute and hence this is indeed a short exact sequence of chain complexes. Now taking homology gives us a long exact sequence

$$\dots \longrightarrow H_{q+1}(C_\bullet[-1]) \xrightarrow{\delta_q} H_q(C_\bullet) \longrightarrow H_q(x, C_\bullet) \longrightarrow H_q(C_\bullet[-1]) \xrightarrow{\delta_{q-1}} H_{q-1}(C_\bullet) \longrightarrow \dots$$

which simplifies to

$$\dots \longrightarrow H_q(C_\bullet) \xrightarrow{\delta_q} H_q(C_\bullet) \longrightarrow H_q(x, C_\bullet) \longrightarrow H_{q-1}(C_\bullet) \xrightarrow{\delta_{q-1}} H_{q-1}(C_\bullet) \longrightarrow \dots$$

We now claim that the connecting homomorphisms δ_q are just multiplication by $(-1)^q x$. To see this we trace the steps of the snake lemma. Let $c \in C_{n+1}[-1] = C_n$ be a cycle (that is, $d_n(c) = 0$). Then c is the image of $(c', c) \in C_{n+1} \oplus C_n$ for some $c' \in C_{n+1}$. Applying Δ_{n+1} to (c', c) we get

$$(d_{n+1}(c') + (-1)^n xc, d_n(c)) = (d_{n+1}(c') + (-1)^n xc, 0) \in C_n \oplus C_{n-1},$$

since c is a cycle. Then by construction of the connecting homomorphism on homology, we have that $\delta_q([c]) = [d_{q+1}(c') + (-1)^q xc] = (-1)^q x[c]$, since $[d_{n+1}(c')] = 0$ as $d_{n+1}(c')$ is a boundary. This proves our claim. Hence from the long exact sequence above, we get a short exact sequence

$$0 \longrightarrow \frac{H_q(C_\bullet)}{xH_q(C_\bullet)} \longrightarrow H_q(x, C_\bullet) \longrightarrow \{[c] \in H_{q-1}(C_\bullet) \mid |x[c] = 0\} \longrightarrow 0$$

But this is just the short exact sequence in question, by the remark before the theorem. ☐

Corollary 5.2. *Let $x \in A$ and $C_\bullet$ a chain complex of A-modules. Then the Koszul homology $H_q(x, C_\bullet)$ is annihilated by x.*

Proof. The leftmost map in the above short exact sequence is given explicitly by

$$\iota^* \colon \frac{H_q(C_\bullet)}{xH_q(C_\bullet)} \longrightarrow H_q(x, C_\bullet) \ , \quad [c_q] + xH_q(C_\bullet) \mapsto [(c_q, 0)]$$

where c_q is a cycle. Let $[(c_q, c_{q-1})] \in H_q(x, C_\bullet)$ for some cycle $(c_q, c_{q-1}) \in C_q \oplus C_{q-1}$. This gives

$$(0, 0) = \Delta(c_q, c_{q-1}) = (d_q(c_q) + (-1)^{q-1} xc_{q-1}, d_{q-1}(c_{q-1})).$$

It follows that $[(c_q, c_{q-1})] = [(c_q, 0)] + [0, c_{q-1})] = \iota^*([c_q] + xH_q(C_\bullet)) + [(0, c_{q-1})]$. Hence

$$x[(c_q, c_{q-1})] = \iota^*(x[c_q] + xH_q(C_\bullet)) + [(0, xc_{q-1})] = 0 + (-1)^{q-1}[(0, d_q(c_q)).$$

But now $\Delta_{q+1}(0, c_q) = ((-1)^q xc_q, d_q(c_q))$ and so $x[(c_q, c_{q-1})] = (-1)^{q-1}[\Delta_{q+1}(0, c_q)] - [(xc_q, 0)]$. The first term is zero since $\Delta_{q+1}(0, c_q)$ is a boundary, and we have already showed that the second term is zero. ☐

Corollary 5.3. *Let $x_1, \ldots, x_n \in A$ and let $C_\bullet$ be a chain complex of A-modules. Then the ideal $(x_1, \ldots, x_n)$ of A annihilates the Koszul homology $H_q(x_1, \ldots, x_n, C_\bullet)$.*

Proof. This follows from Corollary 5.2 together with the commutativity and associativity of the total complex associated to a tensor double complex. ☐

Definition 5.4 (Regular sequence)**.** Let M be an A-module. A sequence $x_1, \ldots, x_n \in A$ is *M-regular* if

(1) $M/(x_1, \ldots, x_n)M \neq 0$;

(2) x_1 is a nonzero divisor of M; and

(3) x_i is a nonzero divisor of $M/(x_1, \ldots, x_{i-1})M$ for $2 \leq i \leq n$.

Theorem 5.4. (1) *Let M be an A-module and $x_1, \ldots, x_n$ an M-regular sequence. Then the Koszul cohomology vanishes: $H_p(x_1, \ldots, x_n, M) = 0$ for all $p > 0$.*

(2) *If $A, \mathfrak{m}$ is a Noetherian local ring, M a finite A-module, $x_1, \ldots, x_n \in \mathfrak{m}$ and $H_1(x_1, \ldots, x_n, M) = 0$, then $x_1, \ldots, x_n$ is an M-regular sequence.*

Proof. **(1)** By induction on $n \geq 1$. If $n = 1$, $H_p(x_1, M)$ is the homology of the complex

$$0 \to M \xrightarrow{x_1} M \to 0$$

and hence $H_p(x_1, M) = 0$ for $p \geq 2$ and $H_p(x_1, M) = \ker(x_1) = 0$ since x_1 is M-regular by assumption. Now let $n > 1$. Let $C_\bullet$ be the complex $M \otimes_A K(x_1, \ldots, x_{n-1})$. Then $H_p(x_1, \ldots, x_n, M) = H_p(x_n, C_\bullet)$. Hence by Theorem 5.1 we have a short exact sequence

$$0 \to H_0(x_n, H_p(C_\bullet)) \to H_p(x_1, \ldots, x_n, M) \to H_1(x_n, H_{p-1}(C_\bullet)) \to 0$$

By definition, $H_p(C_\bullet) = H_p(x_1, \dots x_{n-1}, M)$ which is zero for $p > 0$ by induction. Thus from the short exact sequence we see that $H_p(x_1, \dots, x_n, M) = 0$ for $p > 1$ since the terms on the left and on the right vanish. Now for $p = 1$, the left most term still vanishes, hence we have

$$H_p(x_1, \dots, x_n, M) \simeq H_1(x_n, H_0(x_n, \dots, x_{n-1}, M)).$$

The latter is equal to the 1st homology of the complex

$$0 \to M/(x_1, \dots, x_{n-1})M \xrightarrow{x_n} M/(x_1, \dots, x_{n-1})M \to 0$$

which is zero, since x_n is $M/(x_1, \dots, x_{n-1})M$-regular.

(2) We again proceed by induction on $n \geq 1$. The result is trivial for $n = 1$. Let $n > 1$ and again let $C_\bullet$ be the complex $M \otimes_A K(x_1, \dots, x_{n-1})$. By the proof of Theorem 5.1, we have an exact sequence

$$H_1(C_\bullet) \xrightarrow{-x_n} H_1(C_\bullet) \to H_1(x_n, C_\bullet) = H_1(x_1, \dots, x_n, M) = 0$$

Hence we have that $H_1(C_\bullet) = x_n H_1(C_\bullet)$. Now since A is Noetherian and M is finite, $H_1(C_\bullet) = H_1(x_1, \dots, x_{n-1}, M)$ is a finite A-module. Since $x_n \in \mathfrak{m}$, we get $H_1(x_1, \dots, x_{n-1}, M) = 0$ from Nakayama's Lemma. Thus by induction, $x_1, \dots, x_{n-1}$ is an M-regular sequence. Now using Theorem 5.1 again, we have a short exact sequence

$$0 \to H_0(x_n, H_1(C_\bullet)) \to H_1(x_1, \dots, x_n, M) \to H_1(x_n, H_0(C_\bullet)) \to 0$$

where the two leftmost terms are zero. Thus $H_1(x_n, H_0(C_\bullet)) = H_1(x_n, M/(x_1, \dots, x_{n-1})M) = 0$, which implies x_n is $M/(x_1, \dots, x_{n-1})M$-regular and thus $(x_1, \dots, x_n)$ is an M-regular sequence. □

Corollary 5.5. *If $x_1, \dots, x_n \in A$ is a regular sequence, the Koszul complex $K(x_1, \dots, x_n)$ is a finite free resolution of $A/(x_1, \dots, x_n)$.*

Proof. This follows from Theorem 5.4, Part 1 and the fact that $H_0(x_1, \dots, x_n, A) = A/(x_1, \dots, x_n)$ □

5.1 Hilbert's syzygy theorem

Let $A = k[x_1, \dots, x_s]$ be the usual graded polynomial ring and M a finite graded A module. Assume M is generated by homogeneous generators $m_1, \dots, m_{r_0}$ of degree $d_{i,0}$. For a nonnegative integer d, we put $A(d)$ for the A-module A but with shifted grading by $-d$. This means that $A(-d)$ is the graded module with jth graded piece $A(-d)_j = A_{j-d}$. (Thus the generator $1 \in A$ has degree d in $A(-d)$). We have a surjective homomorphism of degree 0

$$d_0 \colon \bigoplus_{i=1}^{r_0} A(-d_{i,0}) \longrightarrow M \quad \text{given by } 1_i \mapsto m_i,$$

where $e_i := (0, \dots, 1, \dots, 0)$ with 1 in the ith place. On the left, the grading is $\bigoplus_{i=1}^{r_0} A(-d_{i,0}) = \bigoplus_{j \geq 0} (\bigoplus_{i=1}^{r_0} A(-d_{i,0})_j)$, so that 1_i has degree $d_{i,0}$. This makes d_0 a homomorphism of graded A-modules. The kernel $K := \ker(d_0)$ is a homogeneous submodule, and since A is Noetherian, it is generated by finitely many homogeneous elements. Replace M with K and repeat the above process. Iterating this, we get a graded free resolution of M

$$\cdots \to \bigoplus_{i=1}^{r_n} A(-d_{i,n}) \xrightarrow{d_n} \bigoplus_{i=1}^{r_{n-1}} A(-d_{i,n-1}) \to \cdots \to \bigoplus_{i=1}^{r_0} A(-d_{i,0}) \xrightarrow{d_0} M \to 0$$

If we pick a minimal set of generators at each step, the resolution we end up with is called a *minimal graded free resolution* of M. In this case, I claim that $\operatorname{im}(d_n) \subset (x_1, \dots, x_s) \bigoplus_{i=1}^{r_{n-1}} A(-d_{i,n-1})$ for all $n \geq 1$. To see this, suppose not. Since $\operatorname{im}(d_n) = \ker(d_{n-1})$ is a homogeneous ideal, we can find a homogeneous element $(f_1, \dots, f_{r_{n-1}}) \in \operatorname{im}(d_n)$ that is not in $(x_1, \dots, x_s) \bigoplus_{i=1}^{r_{n-1}} A(-d_{i,n-1})$. Because this element is homogeneous, each of the f_i is a homogeneous polynomial and hence it must be the case that f_i is a nonzero constant for some i. Without loss of generality, we may assume that $f_1 = c \in k^\times$. Hence $(1, c^{-1}f_2, \dots, c^{-1}f_{r_{n-1}}) \in \ker(d_{n-1})$. Thus

$$d_{n-1}(e_1) = \sum_{i=2}^{r_{n-1}} c^{-1} f_i d_{n-1}(e_i).$$

However, we chose the $d_{n-1}(e_i)$ to be a minimal set of generators of $\ker(d_{n-2})$, so this is a contradiction.

Theorem 5.6 (Hilbert's syzygy theorem)**.** *Let $A = k[x_1, \dots, x_s]$ be the usual graded polynomial ring and M a finite graded A-module. Then M has a finite free resolution of length at most s.*

Proof. We first take $M = k = A/(x_1, \dots, x_s)$ viewed as an A-module via the trivial action. Clearly, $x_1, \dots, x_s$ is a regular sequence in A and hence by the corollary to Theorem 5.2, the Koszul complex $K(x_1, \dots, x_s)$ is a finite free resolution of k of length $n + 1$.

Now let M be arbitrary. Pick a minimal graded free resolution of M as constructed above

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \to F_0 \to M$$

where F_n is free of rank r_n. Since the resolution is minimal, $\operatorname{im}(d_n) \subset (x_1, \dots, x_s)F_{n-1}$ for all $n \geq 1$. Thus we have a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & F_{n+1} \otimes_A k \arrow[r, "d_{n+1}\otimes 1"] \arrow[d, "\simeq"] & F_n \otimes_A k \arrow[r, "d_n \otimes 1"] \arrow[d, "\simeq"] & F_{n-1} \otimes_A k \arrow[r] \arrow[d, "\simeq"] & \dots \\
\dots \arrow[r] & k^{r_{n+1}} \arrow[r, "0"] & k^{r_n} \arrow[r, "0"] & k^{r_{n-1}} \arrow[r] & \dots
\end{tikzcd}$$

Hence by definition we have that

$$\operatorname{Tor}_n^A(M, k) = k^{r_n},$$

and so $\dim_k \operatorname{Tor}_n^A(M, k) = r_n$. But we can also compute $\operatorname{Tor}_n^A(M, k)$ using a projective resolution of k. In particular, we can use the finite free resolution of k given by the Koszul complex as outlined at the beginning. This resolution has length $s + 1$ and hence $\operatorname{Tor}_n^A(M, k) = 0$ for all $n > s + 1$. In particular we have $r_n = 0$ for all $n > s + 1$ giving us the result. □

5.2 Regular local rings

Let $A, \mathfrak{m}, k$ be a Noetherian local ring. Let $\{\overline{x}_1, \dots, \overline{x}_r\}$ be generators of $\mathfrak{m}/\mathfrak{m}^2$ as an A-module and hence as a k-vector space. Let $E := Ax_1 + \cdots + Ax_r$. Then $\mathfrak{m} = E + \mathfrak{m}^2$. Thus by Nakayama, $E = \mathfrak{m}$. So if we pick $\{\overline{x}_1, \dots, \overline{x}_r\}$ to be a k-basis for $\mathfrak{m}/\mathfrak{m}^2$ we end up with a minimal set of generators of $\mathfrak{m}$ and vice versa.

Definition 5.5. We define the *embedding dimension* of A, denoted by $\operatorname{emb}\dim A$ to be the minimal number of generators of $\mathfrak{m}$; equivalently $\operatorname{emb}\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$.

We always have $\dim A = \operatorname{ht}\mathfrak{m} \leq \operatorname{emb}\dim A$ by Krull's height theorem.

Definition 5.6. We say that a Noetherian local ring A is a *regular local ring* if $\dim A = \operatorname{emb}\dim A$.

Lemma 5.7. *Let $A, \mathfrak{m}$ be a Noetherian local ring and let $x_1, \dots, x_n \in \mathfrak{m}$. Then we have*

$$\dim(A/(x_1, \dots, x_n)) \geq \dim A - n$$

Equality holds if $x_1, \dots, x_n$ is a regular sequence.

Proof. Let $d := \dim(A/(x_1, \dots, x_n)) = \delta(A/(x_1, \dots, x_n))$ by the fundamental theorem of dimension theory. Thus there exists a $\mathfrak{m}/(x_1, \dots, x_n)$-primary ideal $(\overline{y}_1, \dots, \overline{y}_d)$ of $A/(x_1, \dots, x_n)$ that is generated by d elements. Then $\mathfrak{m}/(x_1, \dots, x_n)$ is minimal over $(\overline{y}_1, \dots, \overline{y}_d)$ and hence $\mathfrak{m}$ is minimal over $(x_1, \dots, x_n, y_1, \dots, y_d)$. So $\dim A = \operatorname{ht} \mathfrak{m} \leq n + d$ by Krull's height theorem. Rearranging, we get $d \geq \dim A - n$.

If $x_1, \dots, x_n$ is a regular sequence, we prove by induction on n that $\dim(A/(x_1, \dots, x_n)) \leq \dim A - n$. Recall that for any ring A and any ideal I of A, we have $\dim(A/I) \leq \dim A - \operatorname{ht} I$. Now for $n = 1$, $\dim(A/xA) \leq \dim A - \operatorname{ht}(x)$. But since x is regular, it is a nonzerodivisor of A and so $\operatorname{ht}(x) = 1$ by Krull's Hauptidealsatz. Now notice that $A/(x_1, \dots, x_n) = \frac{A/(x_1, \dots, x_{n-1})}{\overline{x}_n A/(x_1, \dots, x_{n-1})}$, where $\overline{x}_n$ denotes the image of x_n in $A/(x_1, \dots, x_{n-1})$. Since $x_1, \dots, x_n$ is a regular sequence, $\overline{x}_n$ is a nonzerodivisor of $A/(x_1, \dots, x_{n-1})$ and thus again by Krull's Hauptidealsatz, we have $\dim(A/(x_1, \dots, x_n)) \leq \dim(A/(x_1, \dots, x_{n-1})) - 1$ and we are done by induction. $\square$

Corollary 5.8. *Let $A, \mathfrak{m}$ be a Noetherian local and M a finite A-module. If $x_1, \dots, x_n \in \mathfrak{m}$ is an M-regular sequence then*

$$\dim(M/(x_1, \dots, x_n)M) = \dim M - r$$

Proof. By definition, we have $\dim M = \dim(A/\operatorname{Ann} M)$ and $\dim(M/(x_1, \dots, x_n)M) = \dim(\frac{A/\operatorname{Ann} M}{(\overline{x}_1, \dots, \overline{x}_n)})$ where $\overline{x}_i$ denotes the image of x_i in $A/\operatorname{Ann} M$. Then by Lemma 5.7, it suffices to show that $\overline{x}_1, \dots, \overline{x}_n$ is a regular sequence in $A/\operatorname{Ann} M$. We do this by induction on $n \geq 1$. Let x be M-regular. Suppose that $\overline{x}a = 0$ in $A/\operatorname{Ann} M$ hence $xa \in \operatorname{Ann} M$. That is for all $m \in M$, $xam = 0$. But x is M-regular, so $am = 0$ for all $m \in M$ and so $\overline{a} = 0$ in $A/\operatorname{Ann} M$, showing that $\overline{x}$ is regular in $A/\operatorname{Ann} M$. Now let $x_1, \dots, x_n$ be an M-regular sequence, then $x_2, \dots, x_n$ is an M/x_1M-regular sequence, and so by induction, $\overline{x}_2, \dots, \overline{x}_n$ is a regular sequence in $\frac{A/\operatorname{Ann} M}{\overline{x}_1}$. But by the case $n = 1$, $\overline{x}_1$ is regular in $A/\operatorname{Ann} M$ and so we are done. $\square$

Note that the above shows that for $A, \mathfrak{m}$ local Noetherian and M finite, the dimension of M is always greater or equal to the maximum length of an M-regular sequence contained in $\mathfrak{m}$. This is a notion that we will encounter again soon.

Lemma 5.9. *Let $A, \mathfrak{m}, k$ be a regular local ring of dimension n. Let $x_1, \dots, x_r \in \mathfrak{m}$ be linearly independent as elements of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ (hence $r \leq n$), Then $A/(x_1, \dots, x_r)$ is a regular local ring and*

$$\dim(A/(x_1, \dots, x_r)) = n - r.$$

Proof. We know by Lemma 5.4 that $\dim(A/(x_1, \dots, x_r)) \geq n - r$. Now since $\overline{x}_1, \dots, \overline{x}_r$ are linearly independent, we can extend to a basis $\{\overline{x}_1, \dots, \overline{x}_r, \overline{x}_{r+1}, \dots, \overline{x}_n\}$ of $\mathfrak{m}/\mathfrak{m}^2$. Then $x_1, \dots, x_n$ is a minimal generating set of $\mathfrak{m}$ and hence

$$\operatorname{emb}\dim(A/(x_1, \dots, x_r)) = n - r \quad \text{and} \quad \dim(A/(x_1, \dots, x_r)) = \operatorname{ht} \mathfrak{m}/(x_1, \dots, x_r) \leq n - r$$

by Krull, as $\mathfrak{m}/(x_1, \dots, x_r)$ is generated by the images of $x_{r+1}, \dots, x_n$. $\square$

Theorem 5.10. *Let A be a regular local ring. Then A is an integral domain.*

Proof. We argue by induction on $n = \dim A = \operatorname{emb}\dim A \geq 0$. If $n = 0$, then $\mathfrak{m} = 0$ and hence A is a field. If $n = 1$, then $\mathfrak{m} = Ax$ and since $\dim A = 1$, we can find a prime $\mathfrak{P} \subsetneq \mathfrak{m}$. Then for $y \in \mathfrak{P}$, $y = ax$ for some $a \in A$. But $ax \in \mathfrak{P}$ and clearly $x \notin \mathfrak{P}$. Hence $\mathfrak{P} = x\mathfrak{P}$ and so by Nakayama, $\mathfrak{P} = 0$ so A is a domain. Now let $n > 1$. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ be the minimal primes of A. Then since $\dim A > 1$, $\mathfrak{m}$ is not contained in any of $\mathfrak{m}^2, \mathfrak{P}_1, \mathfrak{P}_r$. So by prime avoidance, we can find $x \in \mathfrak{m}$, that i snot contained in any of $\mathfrak{m}^2, \mathfrak{P}_1, \mathfrak{P}_r$. Then $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2$ is nonzero and so by Lemma 5.5, $A_1 := A/xA$ is a regular local ring of dimension $n - 1$. By induction we have that A_1 is a domain and hence Ax is a prime ideal of A and so must contain a minimal prime $\mathfrak{P}_i$ for some i. Since by construction $x \notin \mathfrak{P}_i$, the same argument as in the case $n = 1$, gives that $\mathfrak{P}_i = 0$ and so A is a domain. $\square$

Theorem 5.11. *Let $A, \mathfrak{m}, k$ be a d-dimensional Noetherian local ring. Then equivalent conditions:*

(1) *A is regular*

(2) $\mathrm{gr}_{\mathfrak{m}}(A)$ *is isomorphic as a graded ring to the polynomial ring $k[x_1, \ldots, x_d]$ graded as usual.*

Proof. For $1 \Rightarrow 2$, note that since A is regular of dimension d, $\mathfrak{m}$ is generated by d elements, say $\xi_1, \ldots, \xi_d$. Let $\overline{\xi}_i$ denote the image of ξ_i in $\mathfrak{m}/\mathfrak{m}^2$. Then $\mathrm{gr}_{\mathfrak{m}}(A) = k[\overline{\xi}_1, \ldots, \overline{\xi}_d]$ and we have a surjective graded ring homomorphism

$$\varphi\colon k[x_1, \ldots, x_d] \to k[\overline{\xi}_1, \ldots, \overline{\xi}_d]$$

Let $I := \ker \varphi$. Then I is a homogeneous ideal, and hence φ induces an isomorphism of graded rings. Suppose for contradiction that I is nonzero. Then we can find a nonzero homogeneous element $f \in I_r = I \cap k[x_1, \ldots, x_d]_r$. Now

$$\begin{aligned} \mathrm{length}_A(\mathrm{gr}_{\mathfrak{m}}(A)) &= \mathrm{length}_A((k[x_1, \ldots, x_d]/I)_n) \\ = \dim_k(k[x_1, \ldots, x_d]_n) - \dim_k(I_n) &= \binom{n+d-1}{d-1} - \dim_k(I_n) \end{aligned}$$

For $n > r$, $\dim_k((f)_n) = \dim_k(k[x_1, \ldots, x_d]_{n-r}) = \binom{n+d-r-1}{d-1}$. And since $f \in I$, $\dim_k((f)_n) \leq \dim_k(I_n)$, so

$$\mathrm{length}_A(\mathrm{gr}_{\mathfrak{m}}(A)) \leq \binom{n+d-1}{d-1} - \binom{n+d-r-1}{d-1}$$

which is a polynomial in n of degree $d-2$ for large enough n. In other words, the Hilbert polynomial of $\mathrm{gr}_{\mathfrak{m}}(A)$ has degree at most $d-2$ and so (recall sections 4.1 and 4.2), the Samuel function of A has degree at most $d-1$. But by the fundamental theorem, the degree of the Samuel function must equal the dimension of A which is d. This gives a contradiction.

For $2 \Rightarrow 1$, if $\mathrm{gr}_{\mathfrak{m}}(A)$ is isomorphic to $k[x_1, \ldots, x_d]$ as graded rings, then $\mathfrak{m}/\mathfrak{m}^2$ is generated as a k-vector space by d elements. Hence

$$d = \dim A \leq \operatorname{emb}\dim A \leq d$$

and so we must equality. □

Theorem 5.11 gives an alternative proof of Theorem 5.10: If $A, \mathfrak{m}, k$ is regular local, then by 5.7, the associated graded $\mathrm{gr}_{\mathfrak{m}}(A)$ is a polynomial ring and hence an integral domain. Let $a, b \in A$ be nonzero. Then by Krull's intersection theorem (Theorem 3.5), since a and b are nonzero, we can find integers $n, m \geq 1$ such that $a \in \mathfrak{m}^{n-1} - \mathfrak{m}^n$ and $b \in \mathfrak{m}^{m-1} - \mathfrak{m}^m$. Then the elements $\overline{a} \in \mathfrak{m}^{n-1}/\mathfrak{m}^n$ and $\overline{b} \in \mathfrak{m}^{m-1}/\mathfrak{m}^m$ are nonzero in $\mathrm{gr}_{\mathfrak{m}}(A)$. Hence $\overline{ab} \in \mathfrak{m}^{n+m-2}/\mathfrak{m}^{n+m-1}$ is nonzero. Hence $ab \notin \mathfrak{m}^{n+m-1}$ and thus ab cannot be zero in A.