MA4J8 Commutative algebra II. Worksheet 1

Q1. The compulsory Question 1 of the April 2022 exam The material is either prerequisites or first week's syllabus for this year's module.

1. Prove that a nilpotent element of a ring A is contained in every prime ideal. If $f \in A$ is not nilpotent prove that there is a prime ideal P not containing f.

What does it mean to say that A has Krull dimension 0? If this holds, deduce that the intersection of all maximal ideals of A equals the nilradical of A.

2. If $\varphi\colon A \to B$ is a ring homomorphism and P a prime ideal of B, prove that $\varphi^{-1}(P)$ is a prime ideal of B.

Let S be a multiplicative set in A. Show that the prime ideals of $S^{-1}A$ are in bijection with the prime ideals of A disjoint from S.

3. Let A be an integral domain and $t \in A$ a nonunit. If $x \in A$ is a nonzero multiple of t, say $x = tx_1$, prove that $(x) \subset (x_1)$ is a strict inclusion of ideals.

If A is Noetherian, deduce that $\bigcap_{n=1}^{\infty}(t^n) = 0$.

4. Let A be a ring and M a Noetherian module on which A acts faithfully (that is, no element $a \in A$ acts on M by 0). Prove that A is a Noetherian ring. [Hint: Consider the A-module homomorphism $\varphi\colon A \to \bigoplus_{i=1}^{n} M$ given by $1_A \mapsto (m_1, \dots, m_n)$ where $m_1, \dots, m_n$ generate M.]

5. Describe the equivalence relations on pairs (m, s) that defines the localisation $S^{-1}M$ of an A-module with respect to a multiplicative set S of A. Describe the homomorphism $M \to S^{-1}M$ and say what is its kernel.

In the case $M = A/I$ for I an ideal of A, determine which primes P have $M_P \neq 0$.

6. Given a ring A and prime ideals P_i of A for $i = 1, \dots, n$, suppose that I is an ideal of A not contained in any of the P_i. Prove that I is not contained in the union $\bigcup_{i=1}^{n} P_i$. [Argue by contradiction, and by induction on n.]

7. Define the Zariski topology on the prime spectrum $X = \operatorname{Spec} A$ of a ring A. Introduce the principal open sets X_f for $f \in A$, and prove that they form a basis for the Zariski topology.

Give a necessary and sufficient condition on a set $\{f_\lambda\}_{\lambda \in \Lambda}$ of elements of A for the principal open sets X_{f_λ} to cover X. If it holds, deduce that X is covered by finitely many of them.

Q2 Give the definition of Krull dimension. The *height* of a prime ideal P is the maximum length of a chain $P_0 \subset P_1 \subset \cdots \subset P_h = P$. Prove that $\operatorname{ht}(P) = \dim A_P$.

Q3 An integral domain A has Krull dimension 0 if and only if it is a field. If A is an integral domain and $x \in A$ is nonzero and not a unit, prove that $(x^{i-1}) \supset (x^i)$ is a strict inclusion. Deduce that A is not an Artinian ring. Prove that an Artinian ring A has Krull dimension 0.

Q4 Let $\{m_i\}$ be distinct maximal ideals of a ring A. Show that $m_i + m_j = A$ and $m_i \cap m_j = m_i \cdot m_j$.

Prove that each $\bigcap_{i=1}^{k-1} m_i \supsetneq \bigcap_{i=1}^{k} m_i$ is a strict inclusion. If A is an Artinian ring, deduce that A has only finitely many maximal ideals.

Q5 A basic example in primary decomposition is the ring $A = k[x,y]$, and the ideal $I = (xy, y^2)$. It consists of polynomial functions on the (x,y)-plane that vanish along the x-axis and have a point of multiplicity at least 2 at $(0,0)$. A primary decomposition of I has to take account of the two conditions. There is not choice about (y), the condition that $f(x) \equiv 0$. On the other hand, the second condition can be represented in any number of ways. Show that $I = (y) \cap (x,y)^2$, $I = (y) \cap (x(y + 7x^2))$, and invent infinitely many such expressions.

Determine the associated primes of the A-module A/I. Prime ideals that are not minimal in Ass are called embedded points in algebraic geometry.

Q6 Consider the ring $A = \mathbb{Z}[x]$ and the A-module $M = A/(x^2p, xp^2)$. Show that it has 3 associated primes, and for each $P \in \operatorname{Ass} M$ find an m with $P = \operatorname{ann} m$. [Hint: The ring $A = \mathbb{Z}[x]$ has Krull dimension 2, with arithmetic dimension represented by the (p) for different integer primes and a geometric dimension by the variable x. As motivation, you might like to replace A by $k[x,y]$ with a second variable y of p.)

Q7 Let A be a DVR with local parameter z and $K = \operatorname{Frac} A$, and suppose that p is an integer prime with $1/p \in A$. Consider the field extension of K given by $L = K[t]$ with $t^p = z$, so that L is a K-vector space with basis $\{1, t, t^2, \dots, t^{p-1}$. Prove that $B = A[t]$ is the integral closure of A in L, and is a DVR with local parameter t.

Let A be a UFD with $K = \operatorname{Frac} A$. Let $a, b \in A$ be square free and coprime and consider the field extension $L = K(x)$ with $x = \sqrt[3]{a^2 * b}$ the cube root of a^2b. Show that x is integral over A, and also $y = x^2/a$. Show that the relations between x and y are generated by

$$x^2 = ay, \quad xy = ab, \quad y^2 = bx, \quad \text{that is,} \quad \bigwedge^2 \begin{pmatrix} a & y & x \\ x & b & y \end{pmatrix}$$

Q8 Now assume also that A is a Dedekind domain and $1/3 \in A$ (for example, $A = \mathbb{Z}[1/3]$). Prove that the integral closure of A in L is generated by the x, y discussed above. [The Dedekind domain assumption and localisation at each prime reduces to the easy result of Q5 for DVRs. Doing this by direct computation is a lot harder.]

Q9 Let $n = 9k + 1$ be an integer congruent to 1 mod 9, and consider the number field $L = \mathbb{Q}[t]/(t^3 - n)$. Direct calculation gives that $X = 1 + t + t^2$ satisfies

$$X^3 - 3X^2 - 27kX - 81k^2 = (t^3 - n)((1 + t)^3 + n) = 0 \in L.$$

Use this to deduce that $s = (1 + t + t^2)/3$ is integral over $\mathbb{Z}$.

A favourite result from number theory is that the ring of integers of $\mathbb{Q}[\sqrt{n}]$ (the normalisation of $\mathbb{Z}[\sqrt{n}]$) is $\mathbb{Z}[(1 + \sqrt{n})/2]$ if $d = 1 \bmod 2^2$.

The same result for other primes is less well known: the normalisation of $\mathbb{Z}[\sqrt[p]{d}]$ away from p is obtained by taking pth powers out of each power d^i then taking its pth root. Over the prime p, one further step is needed to make the integral closure: in fact if $d \equiv 1 \bmod p^2$ then $s = (\sum_{i=0}^{p-1} d^i)/p$ satisfies a relation similar to that in the case $p = 2, 3$, so that s is integral over $\mathbb{Z}$. Compare [Daniel A. Marcus, Number Fields, Springer, 2018, Chapter 3]. (I learned this from the URSS project of Lucie Gatzmaga.)

Q10 The formal power series ring over a field $k[\![x]\!]$ and the ring of p-adic integers are analogous. Prove that

$$\mathbb{Z}_p = \mathbb{Z}[\![T]\!]/(T - p).$$

[Method: Consider the ideals $(T^i) \subset \mathbb{Z}[\![T]\!]/(T - p)$. Determine the quotient ring by (T^i) and how the quotient by (T^{i+1}) relates to it.]

Philosophical question Where is $\mathbb{F}_p\mathbb{Z}[\![T]\!]$? Does it make sense to ask for a deformation between $\mathbb{Z}_p$ and $\mathbb{F}_p\mathbb{Z}[\![T]\!]$? For example, consider $\mathbb{Z}[\![T]\!]/(\lambda * T - p)$ with λ a parameter, and set $\lambda = 1$ for one and $\lambda = 0$ for the other.

Q11 The point of $\mathbb{Z}_p$ is that it allows to solve equations of various kinds by successive approximation, like formal power series. For roots of polynomials, this the point of Hensel's Lemma (in coming lectures).

For example, we know how to solve $2x = 1$ in $\mathbb{F}_5$ to get $x = 3$. Now can we solve $2x = 1$ in $\mathbb{Z}/(5^2)$ of $\mathbb{Z}/5^3$? For $2x = 1 \bmod 25$, if we try $x = 3 + 5x_1$ with $x_1 = [0, 1, 2, 3, 4]$ we eventually stumble on $x = 3 + 2 \cdot 5 = 13$. In the same way, by trying $x = 13 + 25x_2$, we find that $x = 3 + 2 \cdot 5 + 2 \cdot 5^2 = 63$ has $2x \equiv 1 \bmod 125$.

As a final effort, prove that $h = 3 + 2*p + 2*p^2 + \cdots + 2*p^n + \cdots = 1/2 \in \mathbb{Z}_5$.

Q12 More generally, prove that a p-adic number is rational if and only if its expansion (as a power series in p) is eventually recurrent.

Show that the terms in the p-adic expansion of $1/3$ in $\mathbb{Z}_p$ recur with period 1 if $p \equiv 1 \bmod 3$, and with period 2 if $p \equiv 2 \bmod 3$.

Show that the p-adic expansion of $1/a$ in $\mathbb{Z}_p$ has recurrent terms with period r where $p^r = 1 \bmod a$. Compare with the familiar result for the expansion of $1/a$ as a decimal.