MA4J8 Commutative Algebra II

This year I want to condense some of the key points of what I taught in previous years into brief outlines. It is more important to know what the results mean, and how to apply them, than to work line-by-line through formal proofs. These topics include dimension theory (Atiyah-Macdonald, Chap. 11 or Matsumura, Chap. 5), I-adic completion and the Artin-Rees lemma.

Instead, I want to spend more time on free resolutions and the characterisation in terms of regular sequences, leading to an understanding of Cohen-Macaulay and Gorenstein rings.

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Lecture 1 Statement of intent and preview of later stuff

I assume you have on board the definition and basic properties of local rings (A,m) or (A,m,k), e.g. Nakayama's lemma:

> if A,m is a local ring and M a finite A-module then $mM = M$ implies that $M = 0$.

Also assume known the material on Noetherian rings and modules

> e.g. A Noetherian, M finite A-module then M has a finite presentation $M \leftarrow P_0 \leftarrow P_1$, where

$P_0 = n_0 * A$ is a free module of rank n_0, corresponding to generators of M
$P_1 = n_1 * A$ is a free module of rank n_1, corresponding to A-linear relations between the generators that hold in M.

Dimension theory of Noetherian local rings is a whole song and dance: Krull's theorem states that three numbers coincide:

- Krull dimension = maximal length d of chain of prime ideals
  $$P_0 < P_1 <.. P_d$$
- minimal number of generators of an m-primary ideal n
  $n = (x_1,.. x_d)$, where m-primary means rad $n = m$
- order of growth of $m^i/m^{i+1} \sim (d+i \text{ choose } d) \sim 1/d! * k^i$

As a sanity check, consider $k[x_1,..x_d]$

-> it has chain of prime ideals of length d
-> a monomial ideal such as $n = (x_1^{a_1}, x_2^{a_2},.. x_d^{a_d})$ with all $a_i \geq 1$ is m-primary
-> m^i/m^{i+1} is the vector space of homogeneous polynomials in $x_1,.. x_d$ of degree i, which has dimension $(d+i-1 \text{ choose } d-1)$.

For the moment, take this for granted. In what follows, I need the fact that $m^i/m^{(i+1)}$ is a vector space over $k = A/m$ (because multiplication by m take m^i into $m^{(i+1)}$). A minimal set of generators of m map down to a basis of m/m^2 as k-vector space.

Recall that the dual (m/m^2)^dual is the _Zariski tangent space_
of the local ring A,m.

Definition: Regular local ring A,m is a Noetherian local ring
such that m = (x1,.. xd) where d = dim A elements. This
means the maximal ideal is generated by the smallest possible
number of generators.

Consequence: m^i/m^{i+1} is a vector space over k = A/m of dimension
(d+i-1 choose d-1). Just like homogeneous polynomials of deg d.

Model examples are k[x1,.. xd]_m and k[[x1,.. xd]].
That is, the polynomial ring over a field, localised at the maximal
ideal of the origin m = (x1,.. xd); and the formal power series ring.

However, consider the rings
ZZ[x] and its localisations ZZ_(p)[x], ZZ[x]_(p,x),
and its formal completion ZZ_p[[x]].
These are not k-algebras! In fact their additive groups are not
vector spaces over any field. They have _mixed characteristic_:
each of them is an integral domain containing ZZ, so has a field
of fractions containing QQ. We must view it as characteristic
zero. But I can pass to the quotient modulo (p), getting FFp[x]
or FFp[x]_(x) or FF_p[[x]], that are FFp-algebras.

The definition of regular local ring includes ZZ[x]_(p,x) and
ZZ_p[[x]]. The mixed characteristic flavour may be unfamiliar,
but in either case, the maximal ideal is generated by 2 elements
m = (p,x). The generators p and x are objects of a completely
different nature. However, m = (p,x), m^i = (p,x)^i = (p^i,
p^(i-1)*x,.. x^i). Clearly m^i/m^{i+1} is a vector space over k =
A/m.