MA4J8 Commutative algebra II

Summary of the second part of the course

Contents of Chapter 7 Homological algebra: There is a useful summary of what we need from homological algebra in [Ma], Appendix B from p. 274. The Appendix by Groutides on my website is also useful. I intend that my notes will eventually contain detailed sections on projective modules, and on injective modules. The reason for the proliferation of appendices is that no-one wants to spend several weeks of lectures giving all the necessary background on homological algebra, most of which is elementary but long. Although the homological appendix comes last in the notes, it consists of prerequisites the reader who has not met the material needs to absorb first.

Contents of Chapter 5 This treats complexes and syzygies, the Koszul complex and regular sequences. Free and projective resolutions of finite modules, the first form of the Hilbert syzygies theorem and the Auslander–Buchsbaum refinement.

Contents of Chapter 6 This puts together regular sequences and the applications of Ext groups in homological algebra to treat Cohen–Macaulay rings and modules and Gorenstein rings. The basic point is that conditions on depth are equivalent to the vanishing of Ext groups in some range. The Ext groups work both ways, as contravariant functors in the first variable and as covariant functors in the second, and a main aim is to join the two together to give the best treatment.

5 Introduction to syzygies and complexes

5.1 Introduction

For a nice ring A and a finite A-module M, consider this picture:

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_{n-1} \leftarrow P_n. \tag{5.1}$$

I commonly assume

(1) (5.1) is an exact sequence of A-modules.

(2) Each P_i is a finite free A-module $P_i = b_i A = \bigoplus Ae_{ij}$.

(3) (Sometimes) the sequence has length n, and the complex is also exact at the P_n term so the final map $P_{n-1} \leftarrow P_n \leftarrow 0$ is injective.

(4) (Sometimes) A, the modules P_i and the maps are graded.

The structure (5.1) is called a *free resolution* of M, or a *finite free resolution* if (3) holds. This idea appears frequently in all kinds of arguments.

I spell out (5.1): $P_0 = b_0 A$ is a free A-module of rank b_0 mapping surjectively to M – it specifies b_0 generators of M. Exactness of (5.1) at P_0 means that P_1 maps surjectively to $\ker\{P_0 \to M\}$, so P_1 corresponds to writing generators for the submodule of A-linear relations between the given generators of M.

Now P_2 corresponds to the relations between the relations, that are called *syzygies*. (Greek for "yoke" – the relations are yoked together like a pair of oxen in ploughing, or are subject to linear dependence relations like stars in conjunction.) I give a discussion from scratch.

A free resolution (5.1) is minimal if each P_i provides a minimal set of generators of $\ker\{P_{i-1} \to P_{i-2}\}$. This happens if and only if every entry of the matrix representing $P_i \to P_{i-1}$ is not a unit, so in the maximal ideal.

I prefer to write the maps left-to-right for three reasons:

- The object under study is M, and the surjective map $P_0 = b_0 A \twoheadrightarrow M$ means choosing generators of M. At a basic level, the argument starts here.

- In general, whether the free resolution ends after n steps with an injective map $P_{n-1} \leftarrow P_n$ from a free module P_n is part of the problem: it only holds under special conditions. The Hilbert syzygies theorem gives conditions under which it holds.

- If the free modules have specified bases $P_i = b_i A = \bigoplus Ae_{ij}$, each map $P_{i-1} \leftarrow P_i$ is a $b_{i-1} \times b_i$ matrix M_i, taking $(u_1, \dots u_{b_i}) \in P_i$ (as a column vector) to matrix product $M_i \mathrm{col}(u_1, \dots u_{b_i}) \in P_{i-1})$. Syzygy modules and free resolutions are a standard item of computer algebra, and it is almost always most convenient to write the complex in this order so that composition of maps is written as $M_1 M_2 = 0$, etc.

Examples Let A be an integral domain, and $x \in A$ a nonzero element. This gives the s.e.s. $0 \to A \xrightarrow{x} A \to A/(x) \to 0$ that we have seen many times. The principal ideal xA is isomorphic to A, that is, it is a free module of rank 1. *This is the only case when an ideal is a free module.*

Suppose $f, g \in A$ are coprime elements of a UFD. If f, g are algebraically independent, you might think that the ideal $I = (f, g)$ would be isomorphic to the direct sum $Af \oplus Ag$.

Of course this never happens. Even in the simplest case $(f, g) = (x, y) \in k[x, y]_{(0,0)})$, the f and g may be algebraically independent (they eliminate different variables), but they are not A-linearly independent as elements of the ideal I. In fact, the map $A \leftarrow 2A$ that takes $(1, 0) \mapsto f$ and $(0, 1) \mapsto g$ does $(a, b) \mapsto af + bg \in A$. This always has $(-g, f)$ in its kernel. Stupid, but true!

If A is a UFD and f, g have no common factors then $af = -bg$ if and only if

$$f = -bc \text{ and } g = ac \text{ for some } c \in A. \tag{5.2}$$

This gives the s.e.s.

$$0 \leftarrow I \leftarrow 2A \leftarrow A \leftarrow 0 \quad \text{with maps } (f, g) \text{ and } \begin{pmatrix} -g \\ f \end{pmatrix} \tag{5.3}$$

as the free resolution of the ideal I. Or we might choose to write

$$0 \leftarrow A/I \leftarrow A \leftarrow 2A \leftarrow A \leftarrow 0 \tag{5.4}$$

as the free resolution of the quotient ring A/I. It is also common to rephrase this as the exact complex

$$A \leftarrow 2A \leftarrow A \leftarrow 0 \quad \text{or} \quad P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \tag{5.5}$$

with 0th homology $H_0(P_\bullet) = A/I$. This is the *Koszul complex* of (f, g), and I elaborate on it later under weaker assumptions.

For $f, g \in k[x_1, \dots, x_n]$ with no common factors, the variety $V(I) = V(f, g) \subset \mathbb{A}^n$ is a codimension 2 complete intersection (assume here that $n \geq 2$ and $V(I) \neq \emptyset$). Its coordinate ring $k[V] = A/I$ (or its local ring $\mathcal{O}_{V,P}$ at a point $P \in V$) has the free resolution of length 2 given by the Koszul complex of (f, g).

These ideas are close to some of the foundations of homological algebra. I don't have weeks to spend on this, but I run through some of it presently, especially the ideas related to the Hom functor and its derived Ext* treated in terms of projective resolutions (usually free resolutions as above), and get some results related to duality.

5.2 The Hilbert syzygies theorem, first proof

I discuss the Hilbert syzygies theorem in more-or-less the original form. Let $S = k[x_1, \dots, x_n]$ be a graded polynomial ring over an infinite field k, and write $m = (x_1, \dots, x_n)$ for the graded maximal ideal.

Theorem 5.1 (Syzygies theorem (1890)) *Suppose that M is a finite graded S-module.*

Then there exist a finite free resolution of the form (5.1)

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_{n-1} \leftarrow P_n \leftarrow 0 \tag{5.6}$$

with $k \leq n$.

Overall shape of the proof The proof is an induction on n, starting at $n = 0$ with the statement that a finite dimensonal vector space has a basis. The inductive step uses two different mechanisms. (I) If a linear form is a nonzerodivisor for M, we can mess around with coordinate changes to ensure that x_n is a nonzerodivisor for M. By induction, we can assume that the result holds for $N = M/x_nM$ as a finite graded module over $\overline{S} = k[x_1, \dots, x_{n-1}]$. Now we can lift a finite free resolution of N to one for M using simple diagram chasing (5.9). Here we use the condition x_n a nonzerodivisor to ensure that the snake lemma gives the required exact sequences.

(II) If all linear forms in the x_i annihilate something in M (which means $m \in \operatorname{Ass} M$), choose generators $m_1, \dots, m_{b_0} \in M$, and write $p\colon P_0 = b_0S \twoheadrightarrow M$ for the standard surjective map. Now switch attention to $\ker p$. This is a submodule of the free S-module P_0, so it is torsion-free: every nonzero element is a nonzerodivisor, so the mechanism of (I) applies to $\ker p$.

Roughly speaking, the first step works assuming that $\operatorname{depth} M > 0$, and decreases the dimension by passing to the hyperplane section $x_n = 0$. The lifting argument is called the *hyperplane section principle*. The second step increases the depth if necessary, thus making the first step applicable. I treat this first in a naive way, as if we were still in the 1890s, but we can soup up the result by turning on more recent technology, as I sketch later.

Theorem 5.2 (Hilbert syzygies + Auslander–Buchsbaum) *Let S, m be a regular local ring of dimension n, and M a finite graded S-module of* m-depth $\geq d$. *Then M has a finite free resolution of length $\leq n - d$.* Proof currently omitted. See [Ma] and [Ei] for the modern form.

The argument for (I) in detail. Suppose $n \geq 1$. Assume that x_n is a nonzerodivisor for M, providing the standard short exact sequence

$$0 \to M \xrightarrow{x_n} M \xrightarrow{\pi} N \to 0. \tag{5.7}$$

Now N is a finite module over $\overline{S} = k[x_1, \dots, x_{n-1}]$, so by induction, it has a finite free resolution by graded free $\overline{S}$-modules:

$$0 \leftarrow N \leftarrow Q_0 \leftarrow Q_1 \leftarrow \cdots \leftarrow Q_{n-1} \leftarrow Q_n \leftarrow 0 \tag{5.8}$$

Each Q_i is a finite free graded module. Set[1] $Q_i = b_i\overline{S} = \bigoplus_{j=1}^{b_i} \overline{S}(-a_{ij})$. Write $p\colon Q_0 \twoheadrightarrow N$ – its image is generated by the images $p(e_{0j}) \in N_{0j}$ of the basis elements of Q_0.

5.3 Hyperplane section principle

The lemma below is not confined to the graded polynomial case. It does not aim directly for a whole finite free resolution. Instead, it works with a finite A-module M and its quotient $N = M/xM$ by an element $x \in A$. It concerns generators of M, relations between given generators, and syzygies between the relations.

Lemma 5.3 (Hyperplane section principle) *Let A be a ring, $x \in A$, and let M be a finite A-module. Assume that x is a nonzerodivisor of M and $xM \subsetneq M$. We also work under the alternative assumptions:* either *all of A, M and x are graded with* $\deg x > 0$, or *A, m is local and $x \in m$. Write $\overline{A} = A/(x)$ and $N = M/xM$ as in* (5.7)

(1) **Generators:** *Suppose that $n_j \in N$ are generators of N. Then there exist $m_j \in M$ such that $m_j \mapsto n_j$, and these m_j generate M.*

(2) **Relations:** *Now write $P_0 = b_0S$ for the free S-module corresponding to the generators m_j, and $K_0 = \ker\{P_0 \twoheadrightarrow M\}$ for the relations holding between them. The same construction for the generators n_j of N over $\overline{A}$ is the free module $Q_0 = b_0\overline{A} \to N$, and the submodule and $L_0 = \ker\{Q_0 \to N\}$ of relations between them. Then $K_0 \to L_0$ is surjective.*

    In particular, we can lift every relation $\sum \overline{a}_j n_j \in L_0$ between the generators n_j of N to a relation $\sum a_j m_j \in K_0$, and these lifted relations generate K_0.

(3) **Syzygies:** *A free resolution $Q_\bullet \twoheadrightarrow N$ can be lifted to a resolution $P_\bullet \twoheadrightarrow M$ of the same shape. This means that it has the same Betti numbers, and in the homogeneous case its graded pieces have the same degrees.*

[1] The notation $S(-a)$ means the module S with basis $1 \in S$ an element degree $-a$. The point of this is to keep track of the grading – my resolution complexes have morphisms with entries $S(-a) \to S(-b)$ given by polynomials of degree $a - b \geq 0$, and I view the morphisms as having degree 0.

Proof of (1) In the local case, the m_j generate M modulo mM, so the result follows by Nakayama's lemma. Finite graded modules offer a different (and older) trick: induction on the degree of homogeneous elements. In fact, for $c \in M$, write $\pi(c) \in N$ as the combination $\pi(c) = \sum \alpha_j n_j$, and pick $a_j \in A$ with $\pi(a) = \alpha_j$. Then $c - \sum a_j m_j$ is in $\ker \pi$, so is divisible by x. That is, $c - \sum a_j m_j = xc'$ with $\deg c' = \deg c - 1$. Now by induction on the degree we can assume that $c' \in \sum A m_j$, which proves the lemma.

Proof of (2) We have seen that we can get generators of M (giving P_0 with the surjective map $P_0 \twoheadrightarrow M$) by lifting generators of N. We want to control the kernel K_0 of $P_0 \twoheadrightarrow M$ in the same way, in terms of the kernel L_0 of $Q_0 \twoheadrightarrow N$. (2) asserts that the surjective map $P_0 \to Q_0$ induces a surjective map $K_0 \to L_0$. This comes from the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & K_0 & \to & P_0 & \to & M & \to & 0 \\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \\
0 & \to & K_0 & \to & P_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_0 & \to & Q_0 & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{5.9}
$$

In (5.9) the horizontal rows $0 \to K_0 \to P_0 \to M \to 0$ and $0 \to L_0 \to Q_0 \to N \to 0$ are exact, and the maps $P_0 \twoheadrightarrow Q_0$ and $M \twoheadrightarrow N$ are surjective by construction.

Here I use the assumption that x is M-regular: the top right vertical map $M \xrightarrow{x} M$ is injective. The snake lemma applied to the second and third row gives the long exact sequence

$$0 \to \ker x \to \ker x \to \ker x \xrightarrow{\delta} \operatorname{coker}\{K_0 \to L_0\} \to \operatorname{coker} \to \operatorname{coker} \to 0.$$

Since at the top right $P_0 \to M$ is surjective, it follows that at the bottom left $\operatorname{coker}\{K_0 \to L_0\} = 0$, that is, $K_0 \twoheadrightarrow L_0$ is surjective.

Proof of (3) This follows from (2) by applying it with $M \twoheadrightarrow N$ replaced by $P_0 \twoheadrightarrow Q_0$ and then successively by $P_i \twoheadrightarrow Q_i$. $\square$

Proof of Theorem 5.1 If some linear form in the x_i is M-regular, we can change coordinates so that it is x_n, and the Lemma allows us to decrease the dimension of S. If we can't do that, choose generators of M and the corresponding surjection $P_0 \twoheadrightarrow M$ from a free module P_0. The kernel $K_0 = \ker\{P_0 \twoheadrightarrow M\}$ is a submodule of a free module, so is torsion free. In this case, every nonzero element of S is M-regular, and in particular x_n. Then we can decrease n by passing to the quotient by x_n. The initial step of passing from M to K_0 added 1 to the length of the resolution chain, but the next step cuts the dimension down by 1, so by induction, we get a free graded resolution of length $\leq n$. $\square$

The top score for the length of a free resolution is achieved at $M = S/m = k$, with length n given by the Koszul complex $K(x_1, \dots, x_n)$. We set $P_0 = S$, and the kernel $K_0 = \ker S \to k$ is the maximal ideal m itself. This is torsion free, but has depth only 1 for the reason described in Section 5.5: in this case K_0/x_nK_0 as an $\overline{S}$ module is isomorphic to the quotient field k as the module $k[x_0, \dots, x_{n-1}]/(x_0, \dots, x_{n-1})$.

5.4 Regular sequences and the Koszul complex

I go back to the Koszul complex. Let A be a ring and I an ideal, and let M be an A-module (the case $M = A$ is often the most useful).

Definition 5.4 An element $s \in I$ is *M-regular* or is *regular for M* if it is a nonzerodivisor but not a unit, that is, $s \colon M \to M$ is injective but not surjective.

A sequence of elements $s_1, \dots, s_n \in I$ is a *regular sequence* for M if s_i is regular for $M/(s_1, \dots, s_{i-1})M$ for each $i = 1, \dots, n$. Spelling that out to distinguish the initial, the inductive and the final steps: s_1 is regular for M (as above), s_2 is regular for M/s_1M, and ditto all the way to s_n regular for $M/(s_1, \dots, s_{n-1})M$. This includes the condition that $(s_1, \dots, s_n)M \subsetneq M$.

The *I-depth* of M is defined as the maximum length n of a regular sequence $s_1, \dots, s_n$ in I.

If $x \in A$ is a nonzerodivisor of A then the quotient $A/(x)$ comes in a s.e.s. $0 \to A \xrightarrow{x} A \to A/(x) \to 0$ where the first two elements are isomorphic. This corresponds to the idea of cutting an n-dimensional variety V by a hypersurface section. In geometry, this is a really obvious thing to try, but there is a hidden difficulty: the point is to make sure that this is a "clean" cut, meaning that we have the whole ideal of the section (as a geometric locus), and don't need to mop up nilpotents after the cut. The next section discusses how this obvious cutting can fail.

I now give a first introduction to the relation between regular sequences and the Koszul complex, restricted to length 2: if A, I are given and $s_1, s_2 \in I$, their Koszul complex

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \tag{5.10}$$

has $P_0 = A$, $P_1 = 2A$, $P_2 = A$, the first map (s_1, s_2) and second map $\left(\begin{smallmatrix} -s_2 \\ s_1 \end{smallmatrix}\right)$. The complex (5.10) is clearly always defined (the composite is zero).

Proposition 5.5 *(1) Assume (s_1, s_2) is a regular sequence. Then*

$$K(s_1, s_2)_\bullet : \; P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \tag{5.11}$$

is exact at P_1 and P_2. (Regular sequence replaces the assumptions of the introduction that A is a UFD and f, g coprime.)

(2) If s_1 is a regular element then $H_1(K(s_1, s_2)) = 0$ implies that s_2 is regular for A/s_1, so that (s_1, s_2) (in that order) is a regular sequence.

(3) Assume in addition that A, m is local Noetherian, and $s_1, s_2 \in m$. Then $H_1(K(s_1, s_2)) = 0$ also implies s_1 is regular.

(4) The complex $K(s_1, s_2)_\bullet$ is symmetric in s_1, s_2 up to swapping the elements and the signs, so that in the local Noetherian case, (s_1, s_2) a regular sequence implies that (s_2, s_1) is also.

Proof (1) $P_2 \to P_1$ takes $c \in A$ to $(-s_2 c, s_1 c)$, and already the second factor is injective (regardless of s_2).

For exactness at P_1, the homology $H_1(K_\bullet(s_1, s_2))$ computes the module quotient

$$\{(a, b) \mid s_1 a + s_2 b = 0\} \;/\; \{(-s_2 c, s_1 c) \text{ for } c \in A\}. \tag{5.12}$$

Let $(a, b) \in P_1$ with $s_1 a + s_2 b = 0 \in P_0$. The regular sequence assumption is that s_2 is a nonzerodivisor modulo s_1: however $s_1 a + s_2 b = 0 \in P_0$ means that s_2 multiplies the class of b in $A/(s_1)$ to $s_2 b = -s_1 a = 0 \in A/(s_1)$, so b was already in (s_1).

Now set $b = s_1 c$. Then $s_1 a + s_2 b = 0$ gives $s_1(a + s_2 c) = 0$. But s_1 was a nonzerodivisor of A, so in turn $a = -s_2 c$. Thus (a, b) is the image of c under d_2, and the complex is exact at P_1.

(2) Conversely: if $H_1 = 0$, an element b such that $s_1 a + s_2 b = 0$ is $b = s_1 c$, so that if $s_2 b = 0 \in A/(x)$ it follows that b is already a multiple of s_1. Therefore s_2 is a nonzerodivisor for $A/(s_1)$.

(3) Assume $K_\bullet$ is exact at K_1. I claim that an element $a \in A$ with $s_1 a = 0$ is a multiple of s_2. In fact $(a, 0) \in P_1$ is in the kernel of $P_1 \to P_0$, so $H_1 = 0$ gives $(a, 0) = (-s_2 c, s_1 c)$ for some $c \in A$.

$$\ker s_1 = s_2(\ker s_1), \tag{5.13}$$

and in the Noetherian local set-up, Nakayama's lemma implies that $\ker s_1 = 0$ so $a = 0$.

(4) is obvious. $\square$

Example 5.6 Without local, (3) and (4) fail: Take fairly general polynomials $F, G, H \in A = k[x_1, \dots, x_n]$. Set $A = k[x_1, \dots, x_n]/(FG)$ and consider $s_1 = 1 - F$, $s_2 = FH$. Then s_1 is regular provided that $1 - F$ has no common factors with FG.

Next, F is a unit mod s_1 in the quotient $A/(s_1)$, so multiplying by s_2 in $A/(s_1)$ is the same as multiplying by H, and is injective. Thus (s_1, s_2) in that order is a regular sequence. However, s_2 is not regular because $s_2 G = 0$.

If P is a prime ideal containing both of s_1, s_2, then G maps to zero in the local ring A_P. Thus the counterexample goes away in the local setting.

The statement and proof of the proposition applies verbatim with A replaced by an A-module M, and the sequence by

$$M \leftarrow 2M \leftarrow M \leftarrow 0. \tag{5.14}$$

For (3–4) we still require A, m local Noetherian and M finite.

5.5 Examples of depth 0 and depth 1

See Worksheet 3, Part I for Serre's R_1 plus S_2 criterion for normal.

Let A, m be a local ring. Then an A-module M has m-depth zero if and only if every $f \in m$ is a zerodivisor of M. By basic facts on primary decomposition, this happens if and only if m is an associated prime of M, in other words, there exists a nonzero $x \in M$ with $mx = 0$.

1. Embedded point The ideal $I = (xy, y^2) \subset A = k[x, y]$ is a key case of primary decomposition. You can describe I as the functions f that satisfy two conditions

- f vanishes on the x-axis $y = 0$.
- f is singular at $(0,0)$. Equivalently: it has multiplicity ≥ 2. It belongs to m^2 where $m = (x,y)$; it has zero derivatives $\partial f/\partial x = \partial f/\partial y = 0$.

In the quotient A/I, the element y satisfies $y^2 = 0$, so it takes the value zero everywhere. Also $my = 0$, so y is in the ideal away from the origin, but $y \notin I$, so its class is not zero in A/I. It is a little scrap of nilpotent fluff hanging onto the line at 0, but it causes difficulties in different arguments.

The submodule $(y)/I \subset A/I$ is nonzero, but annihilated by m, so is isomorphic to $k = A/m$. Thus $m \in \operatorname{Ass}(A/I)$. Since $my = 0$, every $f \in m$ is a zerodivisor for A/I, so $m\text{-}\operatorname{depth}(A/I) = 0$.

In primary decomposition, we can write

$$I = (y) \cap (x,y)^2, \tag{5.15}$$

but equally well $I = (y) \cap (y^2, x)$ or $(y) \cap (y^2, x - ay)$. (If a curve already contains the x-axis, requiring it to be tangent to any other curve through $(0,0)$ forces it to be singular.)

2. Transverse planes in $\mathbb{A}^4$ Start from two transverse planes

$$X = \mathbb{A}^2_{\langle x,y\rangle} \cup \mathbb{A}^2_{\langle z,t\rangle} \quad \text{with} \quad I_X = (z,t) \cap (x,y) = (xz, xt, yz, yt). \tag{5.16}$$

Set $A = k[X] = k[x,y,z,t]/I_X$ and cut it by a general hyperplane through the origin, say

$$H : (y + t = 0). \tag{5.17}$$

Geometrically, the hyperplane cuts the first $\mathbb{A}^2$ in the line $y = 0$, and the second $\mathbb{A}^2$ in the line $t = 0$. So every point of $H \cap X$ is in $y = t = xz = 0$, which is the line pair $xz = 0$ in $\mathbb{A}^2_{\langle x,z\rangle}$. This is obviously the right answer as far as the set of points is concerned.

However, the ring $\overline{A} = k[X]/(H)$ has an embedded point at the origin $(0,0,0,0)$, or expressed more algebraically, $m = (x,y,z,t) \in \operatorname{Ass}\overline{A}$. Clearly $t = -y$, and one sees that $my = 0$, but $y \neq 0$ in $\overline{A}$, because the ideal I_X does not have any linear entries. So $\overline{A}$ has y as a nilpotent supported at the origin, and $m\text{-}\operatorname{depth} A = 1$.

3. Missing monomial The polynomial ring $k[x,y]$ is the ring of polynomial functions on the plane $\mathbb{A}^2$. The condition $\partial f/\partial x(0,0) = 0$ defines the subring $B \subset k[x,y]$ based by every monomial except x. One sees that it is generated by

$$u = x^2,\ v = x^3,\ w = y,\ z = xy. \tag{5.18}$$

The ideal of relations between u, v, w, z is

$$J = (v^2 - u^3, z^2 - uw^2, uz - vw, vz - u^2w). \tag{5.19}$$

In Magma:

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RR<x,y,u,v,w,z> := PolynomialRing(Rationals(),6);
L := [-u+x^2,-v+x^3,-w+y,-z+xy]; I := Ideal(L); IsPrime(I);
MinimalBasis(EliminationIdeal(I,2));
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Obviously $B \subset k[x, y]$ is an integral domain, so every nonzero element is regular.

The image of $\mathbb{A}^2$ under the polynomial map to $\mathbb{A}^4$ given by $(x, y) \mapsto (u, v, w, z)$ might seem to be a perfectly nice variety $V = V(J) \subset \mathbb{A}^4$ with coordinate ring $B = k[u, v, w, z]/J$, having a little cusp at the origin a bit like the cuspidal cubic we know from primary school. However, the unquiet spirit of the departed monomial x still haunts B and V.

Write $m = (u, v, w, z)$ for the maximal ideal at the origin. Although $x \notin B$, its product with anything in m is in B. Any section of V through the origin is marked by an embedded point, a little nilpotent submodule not accounted for by the restriction of J.

To explain: pass to the quotient ring $B/(f)$ by any nonzero $f \in m$. Consider the missing monomial x. Its product $fx \in A$ by f is in B, but it is *not* a multiple of f in B. Therefore fx maps to a nonzero element $\xi \in B/(f)$. This ξ is annihilated by every element of the maximal ideal $m/(f)$. In fact for $g \in m$, the product $g\xi$ is the class of $gfx = f \cdot gx$ in B.

This means that although B is an integral domain, it only has m-depth 1. The quotient B/f by any $f \in m$ has a nonzero element ξ annihilated by m, so the regular element f does not extend to a regular sequence of length 2 in m.

The maximal ideal $m \subset k[x_1, \dots, x_n]$ viewed as a module over $k[x_1, \dots, x_n]$ also has m-depth $m = 1$ as mentioned at the end of the proof of Theorem 5.1.

4. Macaulay's quartic curve The rational normal curve in $\mathbb{P}^4$ is the image of $\mathbb{P}^1_{\langle u,v \rangle}$ under its 4th Veronese map $(u^4 : u^3v : u^2v^2 : uv^3 : v^4)$. However, omitting the monomial u^2v^2 also embeds $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ by the map $(u^4 : u^3v : uv^3 : v^4)$. The affine cone over this is the subring $B \subset k[u, v]$ generated by the monomials $(x, y, z, w) = (u^4, u^3v, uv^3, v^4)$ related by

$$xw - yz, x^2z - y^3, xz^2 - y^2w, yw^2 - z^3. \tag{5.20}$$

It is interesting to carry out the same arguments as in Example 3 in terms of the missing monomial $u^2v^2 \notin B$ to verify that B also has m-depth 1.

5. Another depth 1 case Consider three 2-planes $P_1, P_2, Q \subset \mathbb{P}^4$ such that P_1, P_2 meet along a line $\mathbb{P}^1$ but Q meets P_1, P_2 transversally at single points. For example, in homogeneous coordinates x, y, z, u, v, take $P_1 = V(x, y)$, $P_2 = V(x, z)$, $Q = V(u, v)$. The ideal of $\Gamma = P_1 \cup P_2 \cup Q$ is

$$(x, y) \cap (x, z) \cap (u, v) = (x, yz) \cap (u, v) = (xu, xv, yzu, yzv).$$

One calculates that there are 4 syzygies between the 4 relations, yoked by a single second syzygy: That is, the ideal has a free resolution

$$P_0 \xleftarrow{M_0} P_1 \xleftarrow{M_1} P_2 \xleftarrow{M_2} P_3 \leftarrow 0$$

where $P_0 = A$, $P_1 = 2A(-2) \oplus 2A(-3)$, $P_2 = A(-3) \oplus 3A(-4)$, $P_3 = A(-5)$ and the matrices are $M_0 = (xu, xv, yzu, yzv)$,

$$M_1 = \begin{pmatrix} v & 0 & yz & 0 \\ -u & 0 & 0 & yz \\ 0 & v & -x & 0 \\ 0 & -u & 0 & -x \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} yz \\ -x \\ -v \\ u \end{pmatrix}.$$

In view of the Auslander–Buchsbaum refinement of Hilbert syzygies, the shape of the free resolution implies that the depth can only be 1. Write $A = k[x, y, z, u, v]/I_\Gamma$ for the homogeneous coordinate ring. Its local ring A_m at the maximal ideal $m = (x, y, z, u, v)$ has m-depth $= 1$. Any linear form not vanishing on a component of Γ is clearly regular. However, as in the above Example 2, after cutting by s_1, the quotient ring has a nilpotent at the origin. In fact, it is the unquiet spirit of the discontinuous function f with $f = 1$ on $P_1 \cup P_2$ and $f = -1$ on Q. This is not in A_m, but its product with any regular element of $m = (x, y, z, u, v)$, because cutting by any hypersurface disjoint from $(0, 0, 0, 1, 0), (0, 0, 0, 0, 1)$ makes $P_1 \cup P_2$ and Q disjoint.

5.6 Koszul complexes of length 3 and 4

The Koszul complex $K(s_1, s_2, s_3)$ of length 3 is just a bit more involved: it is

$$A \leftarrow 3A \leftarrow 3A \leftarrow A \leftarrow 0 \tag{5.21}$$

with homomorphisms given by the matrices

$$\begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix}, \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}. \tag{5.22}$$

The 3 columns of the first syzygy matrix give the 3 trivial skewsymmetry identities $s_i s_j = s_j s_i$. Moreover, these 3 are linearly dependent in A^3, as expressed by the final 3×1 matrix.

The logic is as in Proposition 5.5: in any case, (5.21) is a complex. If s_1, s_2, s_3 is a regular sequence, it is exact. And the converse under the extra local Noetherian assumptiona. This is treated more formally in Theorem 5.9 below.

As you know, 3 dimensions is special in lots of ways. For example, you were introduced to cross product of 2 vectors in $\mathbb{R}^3$ in applied math. This gives a skew (antisymmetric) bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, which sadly is never mentioned by our algebraists because it is too advanced for 2nd year algebra and just a special case that is too elementary for 4th year courses. In algebra, the right-hand $\mathbb{R}^3$ should really be $\bigwedge^2 \mathbb{R}^3$ (I discuss this formally below). I was interested to read that in particle physics, $\mathbb{R}^3$ has *polar vectors* (e.g. momentum) whereas $\bigwedge^2 \mathbb{R}^3$ has *axial vectors* (e.g. angular momentum).

It is a well-known issue in algebra that there is no good general ordering or signs for the $k \times k$ minors of an $n \times m$ matrix. In (5.22) I ordered the columns vectors of the first syzygy matrix as for cross product of vectors. Dimension 3 is the last time that this rational and elegant choice is available. For $n \geq 4$ this get progressively messier, and we need a better solution.

The Koszul complex for $n = 4$ is

$$0 \leftarrow A \leftarrow 4A \leftarrow 6A \leftarrow 4A \leftarrow A \leftarrow 0 \tag{5.23}$$

with maps $\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \end{pmatrix}$

$$\begin{pmatrix} 0 & s_3 & -s_2 & s_4 & 0 & 0 \\ -s_3 & 0 & s_1 & 0 & s_4 & 0 \\ s_2 & -s_1 & 0 & 0 & 0 & s_4 \\ 0 & 0 & 0 & -s_1 & -s_2 & -s_3 \end{pmatrix}, \begin{pmatrix} s_4 & 0 & 0 & -s_1 \\ 0 & s_4 & 0 & -s_2 \\ 0 & 0 & s_4 & -s_3 \\ 0 & -s_3 & s_2 & 0 \\ s_3 & 0 & -s_1 & 0 \\ -s_2 & s_1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \tag{5.24}$$

Note the block form $[A \mid B]$ and $[{}^tB \backslash\backslash -{}^tA]$.

Similar exercise as to why it is exact.

5.7 Exterior algebra and general Koszul complex

This is taken from Eisenbud [Ei, pp. 427–429]. The exterior algebra provides a neat formal solution to the issue of notation.

As usual A is a ring and M and N are A-modules. I assume that you have the tensor product of modules $M \otimes_A N$ on board.

The *exterior algebra* of N over A is

$$\bigwedge N = \bigoplus_{r \geq 0} \bigwedge^r N \tag{5.25}$$

where the skew (antisymmetric) product $\bigwedge^r N$ is the quotient of the r-fold tensor $N \otimes \cdots \otimes N$ by relations $n \otimes m + m \otimes n = 0$ for all $n, m \in N$. Assume also the relations $n \wedge n = 0$ to dispel any fear of ambiguity. The image of $n \otimes m$ in $\bigwedge^2 N$ in the quotient is written $n \wedge m$. In (5.25), the product of $u \in \bigwedge^a N$ and $v \in \bigwedge^b N$ is $u \wedge v \in W^{a+b} N$, satisfying $v \wedge u = (-1)^{ab} u \wedge v$. In other words, two homogeneous elements of the exterior algebra (5.25) anticommute if a and b are both odd, and commute if either is even.

A popular device with algebraists is to declare that $\bigwedge^2 N$ is the universal A-module having a skew A-bilinear map $N \times N \to \bigwedge^2 N$. As you know, this is the categorical statement that $\bigwedge^2 N$ is the solution to the UMP for skew maps $N \times N$ to an A-module. (Similarly for $\bigwedge^r N$.) Since the algebraic rules (A-bilinear and skew) are laid out in advance, it can be constructed as the A-module of linear combinations $\sum a_{ij} n_i \wedge n_j$ quotiented by those rules only.

This is just a definition; in some cases the "universal" nature of the construction may give undesired consequences – e.g., if N is not a free A-module then $N \otimes N$ or $\bigwedge^2 N$ may have torsion elements that you were not looking for.

For N an A-module and $s \in N$, the Koszul complex $K(s)$ is defined as the graded exterior product $\bigwedge N$ with differential multiplication by s:

$$K(s) : 0 \to A \to N \to \bigwedge^2 N \to \cdots \to \bigwedge^r N \to \cdots \tag{5.26}$$

Each differential $d_r \colon \bigwedge^r \to \bigwedge^{r+1}$ takes $a \mapsto s \wedge a$. The notation is very slick: the composite $d_i \circ d_{i-1}$ of two differentials involves multiplying by $s \wedge s = 0$, so is zero. The construction is coordinate-free, and the definition also highlights the functoriality of the construction.

5.8 Koszul complex $K(s_1, \ldots, s_n, M)$

The only case we use is the free module of rank n

$$N = nA = \bigoplus Ae_i \quad \text{with basis } e_1, \ldots, e_n, \tag{5.27}$$

and $s = \sum s_j e_j$.

Then $\bigwedge N$ is the free module of rank $2^n = \sum_i \binom{n}{i}$: the degree r component $\bigwedge^r N$ is generated by skewnomials

$$\bigwedge^r N = \bigoplus A e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_r} \quad \text{with } 1 \le i_1 < \cdots < i_r \le k. \tag{5.28}$$

The differential $d_s\colon \bigwedge^r N \to \bigwedge^{r+1} N$ is premultiplication by $s = \sum s_j e_j$, that is, $a \mapsto s \wedge a$. Acting on the skewnomial basis it does

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto \sum s_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_r}. \tag{5.29}$$

The formula in (5.29) politely conceals a pile of unsightly notation – this is more-or-less the formula for the $(r+1) \times (r+1)$ minors of a matrix by expanding them along the jth row.

In detail, each term $s_j e_j$ of x multiplies the skewnomial. If j equals one of the subscripts i_l, skewsymmetry gives zero. Otherwise, the subscript j is either $< i_1$, or fits between i_l and i_{l+1} for some l, or is $> i_r$, and that term of the skew product is then

$$= (-1)^l s_j e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_j \wedge e_{i_{l+1}} \wedge \cdots \wedge e_{i_r}. \tag{5.30}$$

The ± 1 is the sign of the permutation taking e_j to its rightful place after the first l of the e_i.

I defined the Koszul complex $K(s_1, \ldots, s_n, A)$ for A, but there is also a Koszul complex for an A-module M given by

$$K(s_1, \ldots, s_n, M) = K(s_1, \ldots, s_n, A) \otimes M. \tag{5.31}$$

Since each term of $K(s, A)$ is a direct sum of $\binom{n}{i}$ copies of A, each term of $K(s_1, \ldots, s_n, M)$ is a direct sum of the same number of copies of M.

5.9 The top end of $K(s_1, \ldots, s_n, M)$

The differential of $K(s_1, \ldots, s_n, M)$ is increasing, going from $\bigwedge^r M \to \bigwedge^{r+1} M$. It ends with $\bigwedge^n M \to 0$.

Proposition 5.7 *The cohomology of $K(s_1, \ldots, s_n)$ at the final term equals $A/(s_1, \ldots, s_n)$. In the same way, the top cohomology of $K(s_1, \ldots, s_n, M)$ is $M/(s_1, \ldots, s_n)M$.*

Proof The final term K_n of the complex $K(s_1, \dots, s_n) = \bigwedge^n N$ is the free module of rank 1 Af based by the single skewnomial $f = e_1 \wedge \cdots \wedge e_n$ that involves all the indices $1, \dots, n$. The penultimate term $K_{n-1} = \bigwedge^{n-1} N$ is free of rank n, based by the skewnomials f_i

$$f_i = e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n \quad \text{for } i = 1, \dots, n \tag{5.32}$$

that omit just one index i.

Now the differential d_s applied to f_i gives $(-1)^i s_i f$. This is clear from the above description. Therefore the image of d_s is the submodule of $A = Af$ generated by $(s_1, s_2, \dots, s_n)$. The cohomology $K_n/d_s(K_n)$ is the quotient module $A/(s_1, \dots, s_n)$.

The argument for $K(s_1, \dots, s_n, M)$ is the same: the final term $K(M)_n$ is a single copy $Af \otimes M$ of M; the penultimate term $K(M)_{n-1}$ is the direct sum of n copies of M based by $Af_i \otimes M$, and the differential $d_s\colon K(M)_{n-1} \to K(M)_n$ multiplies the ith summand by s_i, with image $s_i M$. Thus the quotient $K(M)_n/d_s(K(M)_{n-1})$ is as stated. $\square$

[Ma, p. 127] uses a descending notation, where P_k has basis $e_{i_1,\dots,i_k}$ and the differential omits each i one at a time with the appropriate sign change. Relating the two notations is straightforward, and I omit it.

5.10 Tensor product by $K(x)_\bullet$

Let $L_\bullet$ be a complex with differentials $d_L\colon L_i \to L_{i-1}$. For $x \in A$, the basic Koszul complex $K(x)_\bullet$ with entry x is $0 \to A \xrightarrow{x} A \to 0$, with first term A of degree 1 mapping to A of degree 0.

Write $L(x)_\bullet$ for the tensor product $L_\bullet \otimes K(x)_\bullet$ with the 2-term Koszul complex. Since $K(x)$ consists of 2 terms of degree 1 and degree 0, with differential $x\colon A \to A$ decreasing degrees by 1, the tensor product is the following extension of $L_\bullet$ by $L[1]_\bullet$:

$$\begin{array}{lccccccc}
L[1]_\bullet\colon & \cdots \to & L_p & \to & L_{p-1} & \to & L_{p-2} & \to \cdots \\
& & & \searrow & & \searrow & & \\
L_\bullet\colon & \cdots \to & L_{p+1} & \to & L_p & \to & L_{p-1} & \to \cdots
\end{array} \tag{5.33}$$

The top line $L[1]_\bullet$ is the complex obtained by shifting the degree of $L_\bullet$ up by 1: it has L_p in degree $p+1$, that is $L[1]_{p+1} = L_p$, so that the three columns in (5.33) have terms of the same homological degree, respectively $p+1, p, p-1$.

The tensor product $L(x)_\bullet$ is the direct sum of top and bottom rows, with the differential

$$d_\otimes(\xi,\eta) = (d_L\xi + (-1)^p x\eta, d_L\eta) \quad \text{for } \xi \in L_p \text{ and } \eta \in L[x]_p = L_{p-1}. \tag{5.34}$$

Each parallelogram of (5.33) has sloping arrows given by multiplication by x, and the alternating $\pm$ arrange that these anticommute. The condition $d^2 = 0$ for $L(x)_\bullet$ to be a complex follows. (This is the usual argument for tensor product of complexes).

Proposition 5.8 *The tensor product complex $L(x)_\bullet$ fits in a short exact sequence of complexes*

$$0 \to L_\bullet \to L(x)_\bullet \to L[1]_\bullet \to 0. \tag{5.35}$$

The resulting long exact homology sequence does

$$\begin{array}{llll} \cdots \to & H_p(L_\bullet) & \to H_p(L(x)_\bullet) & \to H_p(L[1]_\bullet) \to \\ \xrightarrow{(-1)^{p-1}x} & H_{p-1}(L_\bullet) & \to & \cdots \end{array} \tag{5.36}$$

(The top group on the right $H_p(L[1]_\bullet) = H_{p-1}(L_\bullet)$ and the snake map takes it to itself.)

Moreover, multiplication by x acts by zero on the homology of the tensor product complex. That is, $x \cdot H_p(L(x)) = 0$.

Proof The lower row of (5.33) has no arrows going out of it, so $L_\bullet$ is a subcomplex of $L(x)_\bullet$, with quotient the top row $L[1]_\bullet$. This establishes the s.e.s.

For the boundary map, an element of $H_p(L[1]_\bullet)$ is represented by a cycle $\eta \in L_p$ with $d_L(\eta) = 0$. It is the image of $(0,\eta) \in L(x)_{p+1}$ that has differential $((-1)^{p-1}x\eta, 0)$. This is the assertion of (5.36).

For the final statement, an element of $H_p(L(x))$ is represented by $(\xi,\eta) \in L_p \oplus L_{p-1}$ that is a cycle, so has differential $(d\xi + (-1)^{p-1}x\eta, d\eta) = (0,0)$. That is

$$x\eta = (-1)^p d\xi \quad \text{and} \quad d\eta = 0. \tag{5.37}$$

Its product by x is of course $(x\xi, x\eta)$. Now consider $\pm(0,\xi) \in L_{p+1} \oplus L_p$. Its boundary in $L(x)_\bullet$ consists of the two pieces $x\xi$ and $d\xi = \pm\eta$, so that x times our cycle is a boundary. $\square$

Theorem 5.9 (Ma, Theorem 16.5) *Let A be a Noetherian ring and M a finite A-module.*

(1) Suppose that $s_1, \dots, s_n$ is a regular sequence for M. Then the Koszul complex $K_\bullet(s_1, \dots, s_n, M)$ has $H_0 = M/(s_{1\dots n})$ and $H_p = 0$ for all p with $0 < p \le n$.

(2) If (A, m) is local and $s_1, \dots, s_n \in m$ then a stronger form of the converse holds. Namely, $M \neq 0$ and $H_1(K_\bullet(s_1, \dots, s_n, M)) = 0$ implies that $s_1, \dots, s_n$ is a regular sequence for M.

In the particular case $M = A$, it follows that $K_\bullet(s_1, \dots, s_n, A) \to 0$ is a finite free resolution of the quotient $A/(s_1, \dots, s_n)$, as in the introduction.

Both parts are proved by induction on n, applying Proposition 5.8 with

$$L = K(s_1, \dots, s_{n-1}, M) \quad \text{and} \quad L(s_n) = K(s_1, \dots, s_n, M). \tag{5.38}$$

Proof of (1) We assume $s_1, \dots, s_n$ is a regular sequence, so by induction we can assume that $K(s_1, \dots, s_{n-1}, M)$ is exact except at H_0, where $H_0(K(s_{1,\dots,n-1}, M)) = M/((s_1, \dots, s_{n-1})M)$. Everything we need now comes from Proposition 5.8.

For $p \ge 2$, the homology H_p of the extended complex $L(s_n)$ is sandwiched between two groups that are zero by induction. For $p = 1$ the end of the long exact sequence (5.36) includes

$$\begin{aligned} 0 = H_1(L) \to H_1(K(s_1, \dots, s_n, M)) \to H_0(L) \\ \xrightarrow{\pm s_n} H_0(L) \to H_0(K(s_1, \dots, s_n)) \to 0. \end{aligned} \tag{5.39}$$

Since s_n is regular for $M/((s_1, \dots, s_{n-1})M)$, this implies $K(s_1, \dots, s_n, M)$ is exact at H_1 and has $H_0 = M/((s_1, \dots, s_n)M)$, which proves (1).

Proof of (2) Since the $s_i \in m$ and $M \neq 0$, Nakayama's lemma gives $M/((s_1, \dots, s_n)M) \neq 0$ and also $M/((s_1, \dots, s_i)M) \neq 0$ for $i < n$).

The first point is to derive $H_1(K(s_1, \dots, s_{n-1}, M)) = 0$ from the assumption $H_1(K(s_1, \dots, s_n, M)) = 0$. This allow us to assume by induction that $s_1, \dots, s_{n-1}$ is a regular sequence. In fact, the long exact sequence (5.36), has $H_1(L) \xrightarrow{\pm s_n} H_1(L) \to H_1(K(s_1, \dots, s_n, M)) = 0$ (the snake from $H_2(L[1]_\bullet)$ to exactly the same group $H_1(L_\bullet)$, then the assumption $H_1 = 0$. Thus the snake map is surjective, so $H_1(L) = s_n H_1(L)$, and Nakayama's lemma implies that $H_1(L) = 0$.

Now we know that $s_1, \dots, s_{n-1}$ is a regular sequence, and the same long exact sequence continues with

$$0 \to H_0(L) \xrightarrow{\pm s_n} H_0(L) \to H_0(K(s_1, \dots, s_n, M)) \to 0. \tag{5.40}$$

Therefore s_n is a nonzerodivisor for $H_0(L) = H_0(K(s_1, \dots, s_{n-1}), M) = M/((s_1, \dots, s_{n-1})M)$. $\square$

Appendix: Tensor product of complexes $(L_\bullet, d_L) \otimes (M_\bullet, d_M)$

I assume that we have on board the tensor product of modules: for a ring A and modules M, N there is a tensor product A-module $M \otimes_A N$ that has the UMP for A-bilinear maps $M \times N \to$ an A-module U. In applications, a main case is when M, N are free with bases e_i, f_j, in which case $M \otimes_A N$ is free with basis $e_i \otimes f_j$.

Given two complexes $(L_\bullet, d_L)$ and $(M_\bullet, d_M$ with (differentials decreasing degrees), the double complex is the array $\{L_i \otimes_A M_j\}$ with the two differentials $d_L \otimes 1_M$ decreasing i and $1_L \otimes d_M$ decreasing j.

The tensor product complex $(L \otimes M)_\bullet$ is the single complex obtained from the double complex by summing the modules over i, j with $i + j = k$, that is,

$$(L \otimes M)_k = \bigoplus_{i+j=k} (L_i \otimes M_j). \tag{5.41}$$

The differential d_k has alternating ± 1

$$d_k = \sum_{i+j=k} d_{L_i} + (-1)^j d_{M_j}. \tag{5.42}$$

Here the $(-1)^j$ introduces one minus sign in each square, so that instead of commuting, the arrows now anticommute, which together with the complex conditions $(d_L)^2 = (d_M)^2 = 0$ makes $d_k \circ d_{k-1} = 0$, so that the sum is a complex to make the sum a complex.

5.11 Regular local ring

Theorem 5.10 *Let A, m, k be a Noetherian local ring, and $n = \dim A$. Then A is* regular *if*

(i) The associated graded ring $\operatorname{Gr} A = \bigoplus_k m^{k-1}/m^k$ *is isomorphic to the polynomial ring $k[t_1, \dots, t_n]$.*

(ii) m/m^2 has dimension n as a k-vector space.

(iii) The maximal ideal m is generated by n elements.

(i–iii) also imply:
The maximal ideal m is generated by a regular sequence.

This is easy: (i) implies (ii) is obvious. (ii) implies (iii) follows as usual from Nakayama's lemma: if $x_1, \dots, x_n \in m$ generate m/m^2 then they also generate m. For (iii), if $x_1, \dots, x_n$ generate m then polynomials of degree d base m^d/m^{d+1}. A linear dependence between them would imply that $\dim A < n$ (by the Hilbert series characterisation of dimension), so that $\operatorname{Gr} A$ is the symmetric k-algebra on $x_1, \dots, x_n$.

If x_1 maps to t_1 is (i), then x_1 is a nonzerodivisor of A. Applying this to $A/(x_1)$ and using induction gives that $x_1, \dots, x_n$ is a regular sequence. □

Remark 5.11 The simple-minded statement and proof I gave of Theorem 5.1 extends readily to the case of A a regular local ring of dimension n. As in the above proof, we can always pass to $K_0 = \ker\{P_0 \to M\}$ that is torsion-free (because it is a submodule of a free module). Then any element $x \in m \setminus m^2$ can be used in place of x_n in the argument of Lemma 5.3 decreasing the dimension by 1.

However, this is not quite enough to prove the Auslander–Buchsbaum form of the theorem in general in the case when the residue field $k = A/m$ is finite: it might happen that $m\text{-depth}\, M > 0$ but every linear form in $x_1, \dots, x_n$ is a zerodivisor of M. This needs some characterisations of depth in terms of homological algebra and some more work. See [Ma] and [Ei].

Addenda

5.12 Projective modules

I mostly use finite free modules $F = \bigoplus Ae_i$ (I also write nA as above). Projective is a mild generalisation of free, and projective modules appear everywhere in the literature. The main case of interest is finite modules over local rings (or graded rings), when projective is equivalent to free.

Definition 5.12 An A-module P is *projective* if every homomorphism from P to a quotient module M/L lifts to M. To spell that out, if $p\colon M \twoheadrightarrow N$ is a surjective homomorphism and $g\colon P \to N$ a homomorphism, there exists $h\colon P \to M$ such that g is the composite $g = ph$. The picture:

$$\begin{tikzcd} M \arrow[r, "p"] & N \arrow[r] & 0 \\ & P \arrow[ul, "h"] \arrow[u, "g"'] & \end{tikzcd}$$

given p and g, there exists h that makes the triangle commute.

The condition that P is projective is equivalent to $\mathrm{Hom}(P,-)$ an exact functor: if

$$0 \to L \to M \to N \to 0 \text{is a s.e.s.} \Rightarrow \mathrm{Hom}(P,M) \twoheadrightarrow \mathrm{Hom}(P,N).$$

This says that a homorphism to N can be lifted via M, which is the projective assumption. Please think about it as an easy exercise.

Proposition 5.13 (1) *A free module P is projective.*

(2) *An A-module P is projective if and only if P is a direct summand of a free module.*

(3) *A finite projective module P over a local ring (A,m) is free. Therefore, a finite projective module is locally free: its localisation P_p at each $p \in \operatorname{Spec} A$ is free.*

(4) *Suppose that A is graded in positive degrees. Then a finite graded A-module P that is projective is also free.*

(5) *The converse of (3). Let M be a finite module over a Noetherian ring whose localisation M_m is a free A_m-module at every maximal ideal m of A. Then M is projective.*

(1) A free module F is projective: take a basis $F = \bigoplus Ae_i$. Then $M \to N$ is surjective, so $f(e_i)$ is the image of some $v_i \in M$ and the map $P \to M$ taking $e_i \mapsto v_i$ is defined and does everything required. This works because there are no A-linear relations between the e_i, so we can map then to any elements of M we choose. The argument is exactly the same as for vector spaces.

(2) In fact let $u_i \in P$ be a generating set; set $F = \bigoplus Ae_i$ for the free module with basis e_i enumerated by the same set as u_i, and consider the short exact sequence

$$0 \to K \to F \to P \to 0, \tag{5.43}$$

where $F \to P$ takes $e_i \mapsto u_i$ and K is the kernel. If P is projective and $i\colon P \to P$ the identity map, then a lift $g\colon P \to F$ splits the s.e.s., so that $F = K \oplus P$. Now $g \circ f\colon P \to P$ is the identity, whereas $f \circ g\colon M \to M$ is idempotent and projects F to the kernel K. And conversely. Please think about this if you haven't met it before.

(3) If A, m, k is a local ring, a finite projective module P is free by an obvious application of Nakayama's lemma: in fact $V = P/mP$ is a finite dimensional k-vector space. Choose $e_i \in P$ that map to a basis of V. Nakayama's lemma implies that the e_i generate P, that is, $\bigoplus Ae_i \to P$ is surjective. Then P is a direct summand of the free module $F = \bigoplus Ae_i$. Moreover, the complementary summand is zero because the number of e_i equals the dimension of V.

Another proof of (3): A minimal (finite) set of generators of P gives a surjective homomorphism $f\colon F = nA \twoheadrightarrow P$. The projective assumption gives a lift $g\colon P \to F$ of f, so that $F = g(P) \oplus K$, with $K = \ker f$. However, a relation between the generators only has coefficients in m: an invertible coefficient would mean the generators are not minimal. Then $K \subset m \cdot (nA)$, so $K \subset mK$. Then $K = 0$ by Nakayama's lemma.

Matsumura [Ma, p. 10–11] proves the same assertion not assuming P finite by a transfinite induction (due to Kaplansky).

(4) This is a minor variation on the same proof, using induction on the degree in place of Nakayama's lemma.

(5) [Ei] and [Ma] both run the same proof, based on compatibility between localisation and Hom.[2]

Proposition 5.14 (localisation and Hom) see [Ei], Prop 2.10 and [Ma], Th 7.1. *Let A be a ring and B an A-algebra.* $\mathrm{Hom}_A(M, N)$ *is an A-module, so it makes sense to write* $B \otimes_A \mathrm{Hom}_A(M, N)$.

Now there is a B-module homomorphism

$$B \otimes_A \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N).$$

Moreover, if B is flat over A (e.g. $B = S^{-1}A$ a localisation), and M is finitely presented, it is an isomorphism.

Finitely presented means M has a presentation $M \leftarrow b_0 A \leftarrow b_1 A$ with b_0 generators and b_1 relations holding between them. This is automatic if A is Noetherian and M is finite.

The localisation at m at a maximal ideal m of A is S^{-1} where $S = A \setminus m$. If $N_1 \to N_2$ is surjective, write $\mathrm{Hom}_A(P, N_1) \to \mathrm{Hom}_A(P, N_2) \to K \to 0$ for its cokernel. I claim $K = 0$.

[2] To do: I would like to see a complete proof from first principles.

Localisation S^{-1} is an exact functor. so that

$$S^{-1}\operatorname{Hom}_{S^{-1}A}(S^{-1}P, S^{-1}N_1) \to S^{-1}\operatorname{Hom}_{S^{-1}A}(S^{-1}P, S^{-1}N_2) \to S^{-1}K \to 0$$

is still exact. However, in (5) we assume that $S^{-1}P = P_m$ is a free A_m module, so that the localisation $S^{-1}K = K_m$ of K is zero at every m. This implies that $K = 0$ (because if $K \neq 0$ is has an associated prime P, so $x \in K$ with $\operatorname{Ann} x$ contained in a maximal ideal). $\square$

Example 5.15 (Counterexample: projective but not free) Let $\mathcal{O}_K$ is the ring of integers of a number field $K/\mathbb{Q}$. Every localisation of $\mathcal{O}_K$ is a DVR, so that an ideal I is always locally free. An ideal $I \subset \mathcal{O}_K$ is a free $\mathcal{O}_K$-module if and only if it is principal, and if the class group of K is nonzero, there are ideals I that are not principal.

All of this applies equally well to the divisorial sheaf $\mathcal{O}_C(D)$ of a divisor on a nonsingular affine algebraic curve C.

5.13 Injective modules

The definition and the existence of injective modules are needed for constructions in homological algebra, but for most purposes, we don't pretend to know what they are – in any case, they are hard to work with as explicit constructions. When we use injective modules in constructions, they often do not seem to contain any tangible information, and do not relate closely to the modules they resolve. I outline here some details on their existence and properties, mostly taken from Charles A. Weibel, An introduction to homological algebra, CUP 1994.

Definition 5.16 An A-module I is *injective* if $\operatorname{Hom}_A(-, I)$ is an exact functor. This means that given an inclusion $M_1 \subset M_2$ between A-modules and a homomorphism $e\colon M_1 \to I$, there exists an extension of e to a homomorphism $f\colon M_2 \to I$, with f restricted to M_1 equal to e.

In other words, for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in the category of A-modules, the sequence $0 \to \operatorname{Hom}(M_3, I) \to \operatorname{Hom}(M_2, I) \to \operatorname{Hom}(M_1, I) \to 0$ is split.

Example 5.17 A k-vector space V is an injective k-module: if $U \subset W$ are vector spaces, a k-linear map $U \to V$ extends to $W \to V$ (by choosing a complementary basis). This is basic linear algebra, together with Zorn's lemma if U, W are infinite dimensional.

Example 5.18 The *inverse p-torsion module* $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. It is the Abelian group generated by $\frac{1}{p^n}$ for all $n \geq 0$, and related by $p \cdot \frac{1}{p^n} = \frac{1}{p^{n-1}}$ and $p^n = 0$ for $n \geq 0$. Notice that it is p-divisible. Check that it is also n-divisible for n coprime to p. We took notice of $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ earlier in the course as an Artinian module that is not Noetherian.

There is an inverse π-torsion module $A[\frac{1}{\pi}]/A$ for a DVR with parameter π constructed in the same way, that is also injective. For example, A could be the localisation $k[x]_0$ or completion $k[\![x]\!]$ with $\pi = x$.

Summary: Every $\mathbb{Z}$-module M embeds into a product of the inverse p-torsion modules $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ (usually an infinite product). We can view any ring A as a $\mathbb{Z}$-algebra; then for an injective $\mathbb{Z}$-module I, the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(A, I)$ becomes an A-module *under premultiplication*, and is an injective A-module. We can view an A-module M as a $\mathbb{Z}$-module and embed it into an injective $\mathbb{Z}$-module I. An inclusion of M into an injective A-module is then provided by the tautological identity

$$\mathrm{Hom}_{\mathbb{Z}}(M, I) = \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, I)).$$

Proposition 5.19 (Baer's criterion) *Let I be an A-module. Assume that for every ideal $\mathfrak{a}$ of A, every homomorphism $e\colon \mathfrak{a} \to I$ extends to a homomorphism $f\colon A \to I$ from the whole of A. Then I is an injective A-module.*

For example, if $A = \mathbb{Z}$, the assumption says that every element $i \in I$ is divisible by every nonzero $n \in \mathbb{Z}$.

Proof Suppose $M \subsetneq N$ is an inclusion of A-modules and $\varphi\colon M \to I$ is given. To extend φ to a new element $b \in N \setminus M$, define the ideal $\mathfrak{a} \subset A$ and the homomorphism $e\colon \mathfrak{a} \to I$ by setting

$$\mathfrak{a} = \{x \in A \mid xb \in M\} \subset A \quad \text{and} \quad e(x) = \varphi(xb) \quad \text{for } x \in \mathfrak{a}.$$

By our assumption, this e extends from $\mathfrak{a}$ to the whole of A as a homomorphism $f\colon A \to I$, with $f(x) = \varphi(xb)$ for $x \in \mathfrak{a}$. Now extend φ to the bigger submodule $M + Ab \subset N$ taking b to $\varphi'(b) = f(1)$. That is, set

$$\varphi'\colon M + Ab \to I \quad \text{by} \quad \varphi'(m + yb) = \varphi(m) + yf(1)$$

This is well defined: a different choice $m + yb = m' + y'b$ would give

$$(y - y')b = m' - m \in M,$$

so $y - y' \in \mathfrak{a}$ and $(y - y')f(1) = e((y - y')b) = \varphi(m' - m)$.

Now a standard application of Zorn's lemma: the set $\Sigma = \{M, \varphi\}$ of submodules $M \subset N$ and homomorphisms $\varphi\colon M \to I$ is inductive, so has a maximal element that can only be an extension of φ to the whole of N. $\square$

Corollary 5.20 *An Abelian group (a $\mathbb{Z}$-module) is injective if and only if it is divisible. Hence $\mathbb{Q}$ and $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ are injective $\mathbb{Z}$-module.*

I call the Z-modules $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ the inverse p-torsion module, by analogy with Macaulay's inverse monomials treated earlier in the course. Any injective is a direct product of these.

Exercise 5.21 Prove that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct product of one copy each of $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ taken over all the integer primes p.

Show that if A is a DVR with local parameter z the module $(A[1/z])/A$ is an injective A-module.

The proof that injective embedding of modules always exist start with the following result for $\mathbb{Z}$-modules

Lemma 5.22 *For a $\mathbb{Z}$-module M and nonzero $m \in M$, there exists a prime p and a homomorphism*

$$f\colon M \to (\mathbb{Z}[\tfrac{1}{p}])/\mathbb{Z} \quad \textit{with } f(m) \neq 0.$$

Proof The annihilator of a nonzero element $m \in M$ is an ideal of $\mathbb{Z}$ that must be $(n) \subset \mathbb{Z}$ for $n = 0$ or $n \geq 2$, so that the subgroup $\mathbb{Z} \cdot m \subset M$ generated by m is isomorphic to $\mathbb{Z}/(n)$. Choose a prime $p \mid n$ and a surjective map $\mathbb{Z}/n \twoheadrightarrow \mathbb{Z}/p$. If $n = 0$ then any prime p works. Compose with an embedding $\mathbb{Z}/p\mathbb{Z} \hookrightarrow (\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ and extend from $\mathbb{Z} \cdot m$ to the whole of M using the fact that $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$ is injective. $\square$

Corollary 5.23 *Consider the set of all homomorphisms from M to the injective modules $(\mathbb{Z}[\frac{1}{p}])/\mathbb{Z}$. The direct product of all these homomorphisms is an embedding of M into an injective $\mathbb{Z}$-module.*

Now for modules over a general ring A. I will spare you the worries about left and right modules when A is noncommutative – they can be overcome if you care. A ring A is a $\mathbb{Z}$-algebra. If M is a $\mathbb{Z}$-module, the $\mathbb{Z}$-module $\operatorname{Hom}_{\mathbb{Z}}(A, M)$ acquires the *premultiplication* A-module structure. This is a key issue at many points in duality theory. Namely, $x \in A$ acts on $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$ by $f \mapsto f \circ x$, where $f \circ x$ first does multiplication by x while we are still in A, followed by the map f. That is, it is the map $a \mapsto f(xa)$ for $x \in R$. One checks the following points:

(1) For an A-module M and a $\mathbb{Z}$-module N the following identity holds:

$$\operatorname{Hom}_{\mathbb{Z}}(M, N) = \operatorname{Hom}_A(M, \operatorname{Hom}_{\mathbb{Z}}(A, N)).$$

The l.-h.s. is just maps of Abelian groups. On the right, we map to the bigger module $\operatorname{Hom}_{\mathbb{Z}}(A, N)$, but we have to obey all the A-linearity conditions.

(2) If I is an injective $\mathbb{Z}$-module then $\operatorname{Hom}_{\mathbb{Z}}(A, I)$ is an injective A-module.

(3) It follows that for any A-module M, the product of all homomorphisms $M \to \operatorname{Hom}_{\mathbb{Z}}(A, (\mathbb{Z}[1/p])/\mathbb{Z})$ embeds M by an A-homomorphism into an injective A-module.

There is a final point for algebraic geometers: let F be a sheaf of $\mathcal{O}_X$-modules over a ringed space X. For $P \in X$, the stalk F_P is an $\mathcal{O}_{X,P}$-module. Embed each stalk F_P into an injective $\mathcal{O}_{X_P}$-module I_P (a separate choice for each P, not requiring any continuity or compatibility at different P). This defines an $\mathcal{O}_X$-homomorphism of F into the sheaf of discontinuous sections of the injective sheaf $\bigsqcup I_P$.

5.14 Projective dimension and injective dimension

Proposition 5.24 *1. Let M be an A-module. Equivalent conditions:*

(a) $\mathrm{Ext}^{d+1}(M,N)=0$ *for every A-module N.*

(b) M has a projective resolution of length $\le d$.

2. Let N be an A-module. Equivalent conditions:

(a) $\mathrm{Ext}^{d+1}(M,N)=0$ *for every A-module M.*

(b) N has an injective resolution of length $\le d$.

(c) $\mathrm{Ext}^{d+1}(A/P,N)=0$ *for every $P\in\operatorname{Spec}A$.*

The two parts are categorically dual. In both (a) $\Rightarrow$ (b) is trivial, and (b) $\Rightarrow$ (a) very simple.

If $P_0\leftarrow P_1\leftarrow\cdots\leftarrow P_d\leftarrow P_{d+1}\leftarrow\cdots$ is a projective resolution of M, set $Q_d=\operatorname{im}P_{d+1}$ and tag the tail of $P_\bullet$ onto Q_d to give a projective resolution $Q_d\leftarrow P_{d+1}\leftarrow\cdots$. Then the homology $H_{d+1}(\mathrm{Hom}(P_\bullet),M)$ computes both $\mathrm{Ext}^{d+1}(M,N)=0$ and $\mathrm{Ext}^1(Q_d,N)=0$.

Therefore $\mathrm{Ext}^1(Q_d,N)=0$ for all n, so that Q_d is projective, and $P_0\leftarrow P_1\leftarrow\cdots\leftarrow Q_d\leftarrow 0$ is a projective resolution of M of length d, which proves (a).

6 To do list

No human endeavour is ever complete, and that applies especially to lecture notes. Here are some thing I would like to patch up in future drafts.

Chapter 5 treats Regular sequences and the Koszul complex and Chapter 6 Homological algebra methods, Cohen–Macaulay and Gorenstein. [3]

Aim and summary of Chapter 6 A Noetherian local ring A,m,k is Cohen–Macaulay if there exists a regular sequence $s_1,\dots,s_n$ in m of length $n=\dim A$. This means that we can cut down A to an Artinian (0-dimensional) local ring, where each step is $A\to\overline{A}=A/(s)$ with $s\in m$ a nonzerodivisor.

In additions to direct arguments, Chapter 6 treats this notion via the Ext groups, viewed both

[3] This material needs tidying up, but it is mostly there in some form.

(I) as covariant functors, meaning $\operatorname{Ext}^i_A(M,-)$ with M fixed (typically, $M = A/m = k$), usually calculated in terms of a projective or free resolution of M.

(II) as contravariant functors $\operatorname{Ext}^i_A(-,N)$ with fixed N, calculated in terms of an injective resolution of N.

The following sample argument illustrates the relevance of the Ext functors: suppose $A = k[x,y]$ and $M = A/(x,y)$. Then M only contains torsion elements (annihilated by x and y), whereas $N_0 = A$ is torsion-free. It follows that $\operatorname{Hom}_A(M, N_0) = 0$. If I replace the second argument by $N_1 = A/xA$, then N_1 has $\operatorname{Ass} N_1 = (x)$. No element of N_1 is annihilated by y, and again $\operatorname{Hom}(M, N_1) = 0$. However, if I replace the second argument by $N_2 = A/(xA + yA)$, I finally get a module with $\operatorname{Hom}(M, N_2) \neq 0$. We can chase this map back through standard exact sequence of Exts to $\operatorname{Ext}^1(M, N_1)$ and then on to $\operatorname{Ext}^2(M, N_0)$.

To do: The discussion of depth 0 and 1 relates to Serre's criterion: normal iff R_1 plus S_2. This is currently worked out in 2024 Worksheet 3, Part I.

Replace graded polynomial ring by regular local ring. The full form of the Hilbert Syzygies theorem and its Auslander–Buchsbaum refinement still works, but requires more involved arguments.

Serre's characterisation: a local Noetherian ring is regular if and only if is has finite projective dimension.

The calculations involved in the structure sheaf $\mathcal{O}_X$ of the affine scheme $\operatorname{Spec} A = X, \mathcal{O}_X$, together with the associated quasicoherent sheaf $\mathcal{F}$ of an A-module M, the equivalence of categories between A-modules and quasicoherent sheaves on X. Serre's theorem $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

More details on projective A-modules, and the proof that locally free implies projective. The treatment of Eisenbud (Ex 4.11(b) p.137, depending on Prop. 2.10, p.68) and Matsumura [Th 7.12, p.52] both work by saying that $\operatorname{Hom}_A(M,N)$ is compatible with localisation $A \to S^{-1}A$, but I want to do it in the spirit of affine schemes and homs between quasicoherent sheaves.