Martin Gardner's minimum no-3-in-a-line problem

Alec S. Cooper [*] Oleg Pikhurko [†] John R. Schmitt [‡]
Gregory S. Warrington [§]

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Abstract

In Martin Gardner's October 1976 Mathematical Games column in *Scientific American*, he posed the following problem: "What is the smallest number of [queens] you can put on an [$n \times n$ chessboard] such that no [queen] can be added without creating three in a row, a column, or a diagonal?" We use the Combinatorial Nullstellensatz to prove that this number is at least n. A second, more elementary proof is also offered in the case that n is even.

1 Introduction

In Martin Gardner's October 1976 Mathematical Games column in *Scientific American*, he introduced this combinatorial chessboard problem: What is the minimum number of counters that can be placed on an $n \times n$ chessboard, no three in a line, such that adding one more counter on any vacant square will produce three in a line? He dubbed the problem the *minimum no-3-in-a-line problem.*

Figure 1 shows an 8×8 chessboard with an initial placement of 9 black queens with no three in a line. This placement is maximal, that is, any additional queen will create three in a line. The figure illustrates the corresponding 'three-in-a-line' created when an additional queen, shown in

[*]Department of Mathematics, Middlebury College, Middlebury, VT 05753. Supported by Middlebury College's Undergraduate Collaborative Research Fund.

[†]Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK and Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890. Partially supported by the National Science Foundation, Grant DMS-1100215.

[‡]Department of Mathematics, Middlebury College, Middlebury, VT 05753. Supported in part by National Science Foundation, Grant DMS-0758057 and National Security Agency, Grant H98230-10-1-0173.

[§]Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05401. Supported in part by National Security Agency, Grant H98230-09-1-0023. This work was partially supported by a grant from the Simons Foundation (197419 to GSW).

a distinct shading, is placed in the fourth column and eighth row. This particular placement is also of minimum size (where *size* of a placement is the number of queens in the placement), that is, there is no placement with eight or fewer queens meeting the requirements.

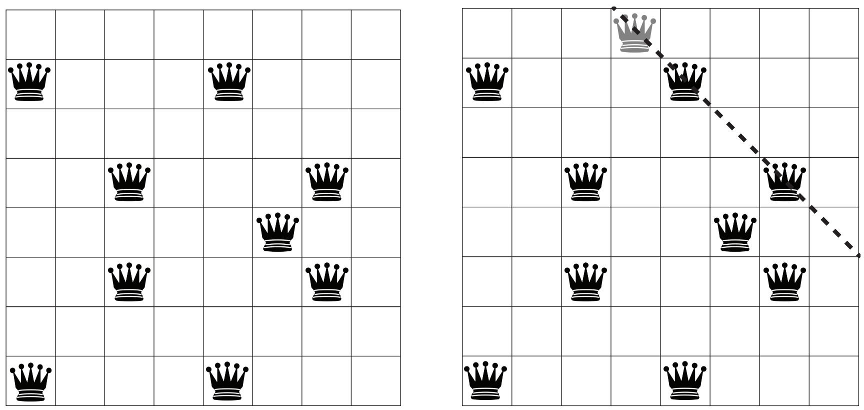

Figure 1: Maximal 8×8 placement

Gardner makes the following observation [8, Chapter 5, pg. 71]:

> If 'line' is taken in the broadest sense — a straight line of any orientation — the problem is difficult. . . The problem is also unsolved if 'line' is restricted to orthogonals and diagonals.

In this paper we provide a lower bound for this latter *queens version* of the problem:

Theorem 1 *For $n \geq 1$, the answer to Gardner's no-3-in-a-line problem is at least n.*

We offer an elementary, ad hoc proof in the case of n even (the approach yields a lower bound of only $n - 1$ when n is odd). This proof, suggested by an argument of John Harris [11], is provided in Section 4.

The proof of Theorem 1 for arbitrary n ultimately relies on the Nullstellensatz. Hilbert's "zero-locus theorem" [12] is a foundational result that connects geometry and algebra. In [2], Alon leverages a special case of Hilbert's theorem to prove a *Combinatorial Nullstellensatz* (reproduced here as Theorem 2) that is ideally suited for obtaining lower bounds on restricted-sum sets and other similar objects (see [14, Chapter 9]).

The proof we present in Section 3 using the Combinatorial Nullstellensatz is inspired by a similar proof of Alon and Füredi [3]; their proof gives a result about the number of hyperplanes needed to cover all but one of the vertices of the hypercube (see [2, Theorem 6.3]). We believe that our proof serves as nice illustrative application of the Combinatorial Nullstellensatz.

n	1	2	3	4	5	6	7	8	9
$m_3(n)$	1	4	4	4	6	6	8	**9**	**10**
n	10	11	12	13	14	15	16	17	18
$m_3(n)$	**10**	**12**	12	**[13,14]**	**[14,16]**	**[15,16]**	**[16,18]**	**[17,20]**	**[18,20]**

Table 1: $m_3(n)$, for small values of n. Brackets indicate lower and upper bounds.

We may arrive at lower bounds that are weaker than that promised by Theorem 1 quite quickly. If we make the observation that each of the q queens 'covers' at most $4n-4$ squares and each of the n^2 squares requires either two queens to 'cover' it or one queen to occupy it, a lower bound of $\frac{n}{2}$ follows [1]. (This last observation can be strengthened by noting that only a few queens can cover $4n-4$ squares. However, any queen covers at 'worst' $3n-2$ squares, but still we could not push this line of argument to get us to n.)

Prior to our proof of Theorem 1 we discuss some history of the problem drawn from Gardner's notes and correspondence pertaining to his writing of the *Scientific American* column [7] – this is done in Section 2. Section 3, as mentioned above, presents a proof of Theorem 1 using the Combinatorial Nullstellensatz. We also offer a more elementary proof in Section 4 suggested by an argument of John Harris [11].

2 History

Gardner and some of his readers found good placements via pencil-and-paper; others conducted computer searches. We also conducted computer searches, though with computing power that is better than it was 35 years ago. Collectively, these results are contained in Table 1; $m_3(n)$ denotes the answer to Gardner's no-3-in-a-line problem on an $n \times n$ chessboard. A bold-faced entry in the second row indicates that an improvement was made to previous knowledge.

Theorem 1 is not "tight" for small values of n. The data suggest that for n odd and $n \geq 3$, we should have $m_3(n) \geq n+1$. Our search for good placements was done via brute-force search[1]. As such, and to illustrate the computational challenges involved, our program took around 900 3GHz-CPU hours to confirm that there is no good placement of 11 queens on an 11×11 chessboard; Theorem 1 indicates that this was the smallest size we needed to test. We estimate that the corresponding search for a 13×13 chessboard using our program would require at least 70 thousand 3GHz-CPU hours.

In that October column (and in an addendum to it), Gardner gave a few results on the queens

[1]The C code that performed this search is available in the source package for [5] at `http://arXiv.org`.

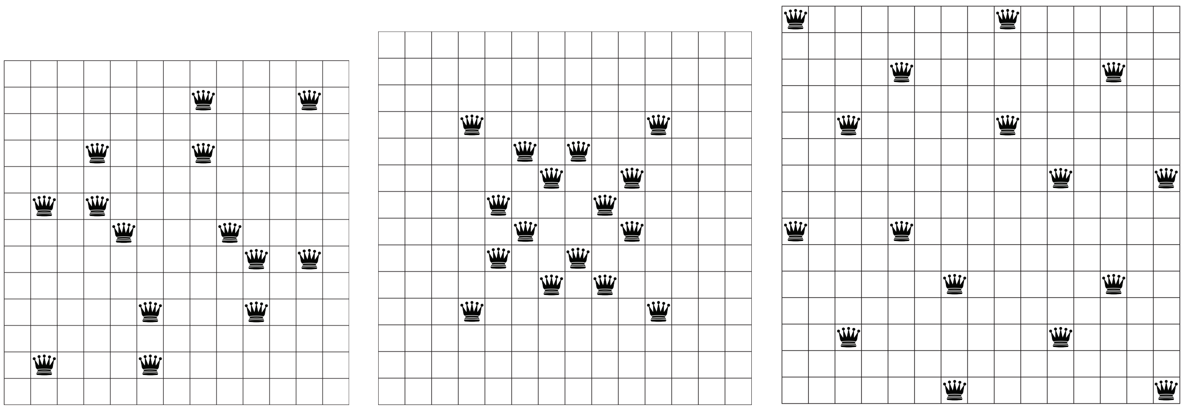

Figure 2: Maximal placements: 14 queens for $n = 13$; 16 queens for $n = 14$ and $n = 15$.

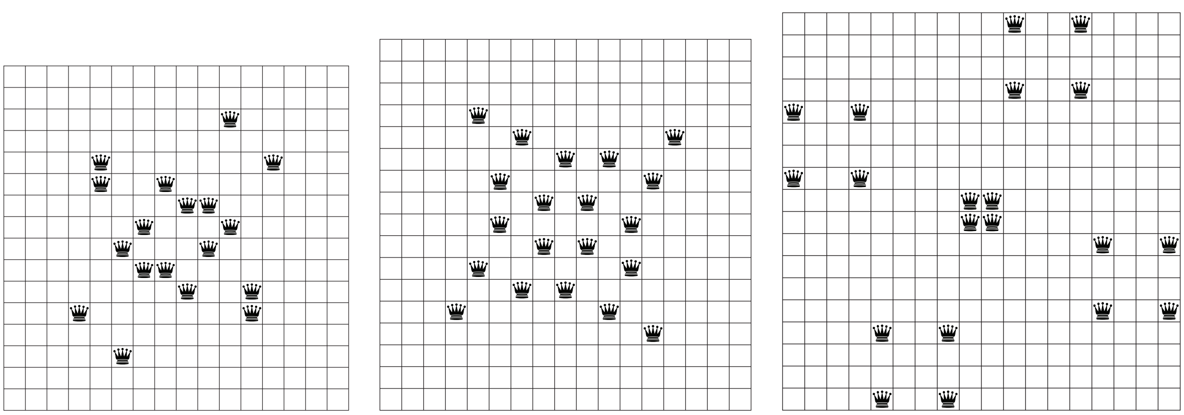

Figure 3: Maximal placements: 18 queens for $n = 16$; 20 queens for $n \in \{17, 18\}$.

version. These included placements of queens on chessboards 3×3 through 12×12, which provided upper bounds on this number. His archives also contain a good placement of 52 queens on a 48×48 chessboard [7]. Gardner also stated that John Harris of Santa Barbara, CA (who, we later learned, was a frequent correspondent of Gardner's) was able to show that the minimum number of queens needed for an $n \times n$ chessboard is at least n, except when n is congruent to 3 modulo 4, in which case it could be one less. Gardner did not supply Harris' argument. These results were the "jumping off" point of our investigations — mostly, we wondered what Harris' argument was.

Subsequent to obtaining our results that confirm and improve upon those of Harris, we were able to obtain copies of Gardner's notes and correspondences concerning this problem [7]. These are a small fraction of the 60 linear feet(!) of notes and correspondences archived at Stanford University that pertain to his writing of the Mathematical Games column. (These materials were a gift to Stanford by Gardner in 2002.) There are numerous carbon-copies of letters that Gardner wrote to other mathematicians, as well as readers, and copies of letters they wrote to him about the problem. Chronologically first is a letter, dated June 2, 1975, that Gardner wrote to the world-renowned John H. Conway. In it, he states that the problem occurred to him while considering a game of the mathematician Stanislaw Ulam (though not the game that commonly goes by the name Ulam's game) — the game had appeared in an earlier column. The game, consists of taking turns "putting a counter on an $n \times n$ [chessboard] until one person wins by getting 3 in line, orthogonally or diagonally." In the weeks that followed were letters to and from Bill Sands (then a Ph.D. student at U. Manitoba, now at U. Calgary), who independently suggested the problem, and John Harris, including one that sketches some ideas for the above-mentioned claim. Subsequent letters from readers (that appeared after the October 1976 column) contained their best solutions to the problem for small chessboards; some of Gardner's notes do the same.

Of course, the reader may be more aware of some related or similarly worded problems. Gardner mentioned one of them in that month's column, the *maximum no-3-in-a-line problem*, that is, what is the maximum number of counters (or queens) one can place on an $n \times n$ chessboard so that there are no three in a line? Here an easy upper bound of $2n$ follows from the pigeonhole principle as each of the n columns may contain at most 2 counters — Guy and Kelly [10] showed that one is 'unlikely' to find any with more than $\sim 1.87n$ queens – this was later corrected to $\sim 1.81n$ queens (see [13, A000769]). Another related problem is the *queens domination problem.* In this problem, one asks for the minimum number of queens needed so that each square of the chessboard is either occupied or attacked. There are two versions of this problem, one where the queens are non-attacking and the other where this restriction is lifted, see [4] and [6] for some results.

3 Proof of Main Theorem via the Combinatorial Nullstellensatz

In this section we prove Theorem 1 using the Combinatorial Nullstellensatz. To begin, we give a brief discussion of the theorem to be applied and its statement.

The Fundamental Theorem of Algebra tells us that a degree-t polynomial $f(x)$ contained in a polynomial ring $F[x]$ has at most t zeros. Said another way, for any set S contained in F of cardinality greater than t, there is an element $s \in S$ such that $f(s)$ is nonzero. One may think of this as saying, either a polynomial is zero everywhere or it is zero in very few places. The following theorem, known as the Combinatorial Nullstellensatz, generalizes this fact to polynomials of several variables — it is due to Alon [2, Theorem 1.2]. We may think of it as saying that a multivariable polynomial that isn't zero everywhere has a non-root in a box of large enough volume.

Theorem 2 *[Combinatorial Nullstellensatz, Theorem 1.2 [2]] Let F be an arbitrary field, and let $f = f(x_1, \dots, x_n)$ be a polynomial in $F[x_1, \dots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \dots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, \dots, s_n \in S_n$ so that $f(s_1, \dots, s_n) \neq 0$.*

So that we might precisely state our results, we introduce some definitions and notation. We consider the infinite square $\mathbb{Z}$-lattice a *chessboard* and its vertices as *squares* of the chessboard. A *board* B is a finite subset of the chessboard. Let B_n denote the board $[1, n] \times [1, n]$. As we are interested in the queens version of the problem, the lines that we concern ourselves with have slope $0, +1, -1$, or ∞ and contain vertices of the lattice — so, throughout we use *line* to refer to a line of this type. Any subset S of the infinite square lattice may be considered a *placement of queens*, or *placement* for short, by imagining a queen on each corresponding square of the chessboard. The *size* of a placement S is its cardinality $|S|$. We say that two queens of a placement $\mathcal{Q}$ *define* a line if they lie on the same row, column or diagonal. In such a way, the placement $\mathcal{Q}$ *defines* a set of lines, the set of lines defined by the pairs of $\mathcal{Q}$. Lastly, we call a placement *good* if does not contain 3 queens in a line but loses this property upon the addition of a queen to an unoccupied square.

Let $m_k(n)$ denote the minimum size of a placement on B_n such that there are no k queens in a line but the placement loses this property upon the addition of a queen to an unoccupied square of B_n. As indicated by our title, our focus is on $k = 3$.

We warn the reader that the placements we seek need not have each queen of a placement on a line with another queen. See Figure 4 for an example with $n = 4$.

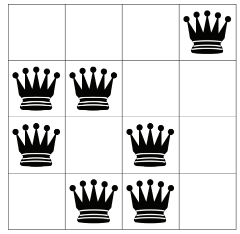

Figure 4: Good placement with one queen not collinear with any other.

We first prove the result for $n = 4k + 1$, where k is a nonnegative integer, as in this case the presentation is cleanest. We next establish the result for $n = 4k$, and we omit the details for the other two cases as these are similar.

Proof of Theorem 1

Let $n = 4k + 1$, where k is a positive integer. (The result is obvious for $k = 0$, i.e. $n = 1$.) Let $\mathcal{Q}$ be a good placement on B_n with size $q = |\mathcal{Q}| \leq 4k$. Our proof will proceed by constructing a polynomial $f(x, y)$ of total degree $8k$ that vanishes on each square $(x, y) \in B_n$. We will then obtain a contradiction through a suitable application of the Combinatorial Nullstellensatz.

We shall construct f as a product of linear factors of three different types. The first type consists of the set of lines defined by $\mathcal{Q}$. Since the placement $\mathcal{Q}$ is good, every unoccupied square of B_n is in the zero locus of at least one line of the first type.

As shown in Figure 4, there may be some queens in $\mathcal{Q}$ not on any defining line. Let $\mathcal{Q}' = \{Q_1, \ldots, Q_{q'}\}$ denote the (possibly empty) subset of queens not collinear with any other queen in $\mathcal{Q}$. For each $Q_i \in \mathcal{Q}'$ we define a new line that passes through the square occupied by Q_i. While we are free to choose any one of the four possible slopes for each line, it will be most convenient to distribute the slopes as evenly as possible. Hence we choose the slope of the i^{th} line to be the j^{th} element of $(0, +1, -1, \infty)$, where $j \equiv i \mod 4$. Every occupied square is in the zero locus of at least one line of either of the first two types.

For each of the four possible slopes there are at most $\left\lfloor \frac{4k-q'}{2} \right\rfloor$ lines of that slope of the first type and at most $\left\lceil \frac{q'}{4} \right\rceil$ lines of that slope of the second type. These quantities sum to at most $2k$. As necessary, define new, distinct lines ("of the third type") in each of the four directions so that there are exactly $2k$ lines of each slope among the three types. (The lines of the third type serve only to facilitate the application of the Combinatorial Nullstellensatz; it is immaterial which squares they vanish on.)

Let $\mathcal{L} = \{L_1, \ldots, L_{8k}\}$ be our set of $8k$ lines and let $l_i = 0$ be the equation in variables x and y

defining L_i. We then define

$$f(x,y) = \prod_{i=1}^{8k} l_i \in \mathbb{R}[x,y]. \tag{1}$$

As desired, the polynomial $f(x,y) = 0$ for every $(x,y) \in B_n$ as every unoccupied square is on a line of the first type and every occupied square is on a line of either the first or second type. By construction, the total degree of f is $8k$. If we group the factors in f according to slope, we see that f can be rewritten as

$$f(x,y) = \prod_{j=1}^{2k} (x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j) \tag{2}$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$. From equation (2) and the binomial theorem we conclude that the coefficient of the top-degree term $x^{4k}y^{4k}$ is $\pm\binom{2k}{k}$, i.e., nonzero.

We now apply Theorem 2 to $f(x,y)$, where $t_1 = t_2 = 4k$ and $S_1 = S_2 = \{1, \dots, 4k+1\}$, to obtain that there are $s_1 \in S_1, s_2 \in S_2$ such that $f(s_1, s_2) \neq 0$. We have reached a contradiction. Therefore, the result holds when n is congruent to 1 modulo 4.

Let us now consider $n = 4k$, where k is a positive integer. We proceed in a similar manner to the above. Again, we consider a good placement on B_n, this time of size $q \leq 4k-1$. Similarly, let $q' = 4r + s$ denote the size of $\mathcal{Q}'$, where r and s are integers with $0 \leq s \leq 3$. As before, we define lines of the first type and the second type, distributing those of the second type as evenly as possible. For each possible slope, the number of lines is at most

$$g(r,s) = \left\lfloor \frac{4k-1-4r-s}{2} \right\rfloor + \left\lceil \frac{4r+s}{4} \right\rceil. \tag{3}$$

For $s \neq 1$, we have $g(r,s) \leq 2k-1$. Likewise, for $s = 1$ and $r > 0$, we have $g(r,s) \leq 2k-1$. For these values, we proceed as before, adding more lines so that there are $2k-1$ of each possible slope. We may now construct a polynomial of degree $8k-4$ and see that the coefficient of the $x^{4k-2}y^{4k-2}$ term is nonzero. As before, applying Theorem 2 we reach a contradiction with $t_1 = t_2 = 4k-2$ and $S_1 = S_2 = \{1, \dots, 4k\}$.

We are left to consider the case where $s = 1$ and $r = 0$, i.e. $q' = 1$. In the process of defining lines of the first type, since $q \leq 4k-1$ and $q' = 1$ we may have $2k-1$ lines of each possible slope. We define one new line of the second type for the single queen in $\mathcal{Q}'$, giving it slope ∞. Finally, we add more lines as necessary so that there are precisely $2k$ with slope ∞ and $2k-1$ for each of the other three slopes. Our polynomial has degree $8k-3$ and we consider the following leading term with nonzero coefficient, $\binom{2k-1}{k}x^{2k}y^{2k-1}(x^2)^{k-1}(-y^2)^k = (-1)^k\binom{2k-1}{k}x^{4k-2}y^{4k-1}$. As before, applying Theorem 2 we reach a contradiction with $t_1 = 4k-2, t_2 = 4k-1$ and $S_1 = S_2 = \{1, \dots, 4k\}$. This completes the proof in this case.

The cases of $n = 4k+2$ and $n = 4k+3$ follow in a similar manner. This completes the proof. ∎

4 A second proof

We now present a second proof to Theorem 1 for the case n is even and obtain a slightly weaker result when n is odd by showing that one needs at least $n-1$ queens; this proof is more elementary than the one given in Section 3. While we arrived at it independently, many of the ideas are to be found in a June 7, 1975 letter of John Harris to Martin Gardner [11]. In the case when $n \equiv 3 \mod 4$, Harris only claimed $n-1$ queens are required. A similar proof for the no-two-in-a-line problem can be found in [15, Chapter 8].

We may refer to a square of B_n by the coordinates (x, y) of its corresponding vertex.

Proof

The claim is easily checked for $n = 1$, so we assume $n \geq 2$ for the proof. Let $\mathcal{Q}$ be a good placement of size q on B_n. For this proof we will distinguish between the lines of slope 0 or ∞ defined by $\mathcal{Q}$ and those of slope ± 1. To this end, set $U \subseteq B_n$ to be the set of squares left uncovered by a line of slope 0 or ∞ and set $\mathcal{Q}'' \subseteq \mathcal{Q}$ to be those queens not involved in defining a line of slope 0 or ∞. (Note that squares in U may still be occupied by a queen in $\mathcal{Q}''$.) Write $q'' = |\mathcal{Q}''|$. For any index $i \in \{1, \ldots, n\}$ (respectively $j \in \{1, \ldots, n\}$) let $C_i = \{(i, k) \in U : 1 \leq k \leq n\}$ (respectively $R_j = \{(k, j) \in U : 1 \leq k \leq n\}$).

The sets C_i and R_j keep track of the squares in U for each column and row. Let $a < b$ be the minimum and maximum indices, respectively, for which $C_i \neq \emptyset$. Set c to be the number of the C_i that are nonempty. Define $a' < b'$ and r analogously for the sets R_j. Note that $c, r \geq n - \frac{q-q''}{2}$. In particular, $c \leq 1$ or $r \leq 1$ requires $q \geq 2(n-1)$. We therefore assume for the rest of the proof that $r, c \geq 2$. Without loss of generality, we may assume $b - a \geq b' - a'$ as otherwise we may rotate the placement by 90°. Figure 5 illustrates the various definitions of a 13-queen good placement on a 10×10 chessboard.

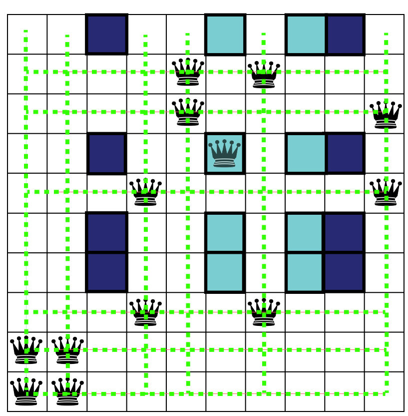

Figure 5: $q = 13$, $q'' = 1$, $c = r = 4$. Squares of U are shaded. Dark shading indicates those that are also in $C_3 \cup C_9$. The pale-shaded queen indicates the single queen in $\mathcal{Q}''$.

As $\mathcal{Q}$ is good, the squares of $C_a \cup C_b$ are either occupied or 'attacked' via a pair of queens that would define a line of slope ± 1. By definition, $|\mathcal{Q} \cap C_a| \leq 1$, $|\mathcal{Q} \cap C_b| \leq 1$; and so $|\mathcal{Q} \cap (C_a \cup C_b)| \leq \min\{q'', 2\}$. There is at most one line of slope $+1$ that attacks two squares of $C_a \cup C_b$ (the line would be a diagonal of the 'rectangle' formed by $C_a \cup C_b \cup R_{a'} \cup R_{b'}$). Likewise, there is at most one line of slope -1 that attacks two squares of $C_a \cup C_b$. Each of the other lines of slope ± 1 defined by $\mathcal{Q}$ attack at most one square of $C_a \cup C_b$. The placement $\mathcal{Q}$ must therefore define at least $2r - 2 - \min\{q'', 2\}$ lines of slope ± 1. Furthermore,

$$2r - 2 - \min\{q'', 2\} \geq 2\left(n - \frac{q - q''}{2}\right) - 2 - q'' = 2n - q - 2. \tag{4}$$

Note that the q queens of $\mathcal{Q}$ can define at most q lines of slope ± 1. Thus, $q \geq 2n - q - 2$, and so $q \geq n - 1$.

We now restrict n to be even and we will reach a contradiction by assuming that $q \leq n - 1$. As $n - 1$ is odd, there are at most $\frac{n-2}{2}$ lines of each possible slope defined by the placement $\mathcal{Q}$. In particular, there are a total of at most $n - 2$ lines of slopes ± 1.

If $q'' = 0$, then $r \geq n - \frac{n-2}{2} = \frac{n}{2} + 1$, and so $2r - 2 \geq n$. So, we need at least n lines of slope ± 1 — a contradiction.

If $q'' > 0$, then $r \geq n - \frac{q-q''}{2} \geq n - \frac{(n-1)-q''}{2} = \frac{n}{2} + \frac{q''+1}{2}$. We have $2r - 2 - \min\{q'', 2\} \geq 2(\frac{n}{2} + \frac{q''+1}{2}) - 2 - q'' = n - 1$, and so we need at least $n - 1$ lines of slopes ± 1 — again, a contradiction. ∎

5 Acknowledgments

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