Lines of polynomials with Galois group $\mathfrak{A}_{2n}$

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Greek **A**lgebra and **N**umber **T**heory Seminar

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I want to report on joint work in progress with Nuno Arala Santos.

In these slides, n denotes a natural number.

This talk presents yet another construction of polynomials with Galois group the alternating group $\mathfrak{A}_n$.

Even more specifically, just the *even* alternating groups $\mathfrak{A}_{2n}$.

(And the odd symmetric groups, by accident.)

It's the journey, not the destination...

Set $K_n = \mathbb{Q}\left(\sqrt{(-1)^{n-1}(2n-1)}\right)$ — an extension of $\mathbb{Q}$ of degree ≤ 2.

Theorem

The Galois group of the polynomial

$$x^{2n} + nx^{2n-1} + t\left(nx + (n-1)^2\right)$$

coincides with $\mathfrak{A}_{2n}$, for most values of $t \in K_n$.

- The polynomials depend linearly on t;
- any number field containing K_n works just as well;
- most is intended in analytic number theory sense;
- for $n \in \{1, 5, 13, 25, \ldots, 2k(k+1)+1, \ldots\}$, we can choose $t \in \mathbb{Q}$;
- the field K_n arises from a simple discriminant computation.

Motivation from enumerative Galois theory

Let $n, H \in \mathbb{N}$ be natural numbers and let $G \subset \mathfrak{S}_n$ be a subgroup.

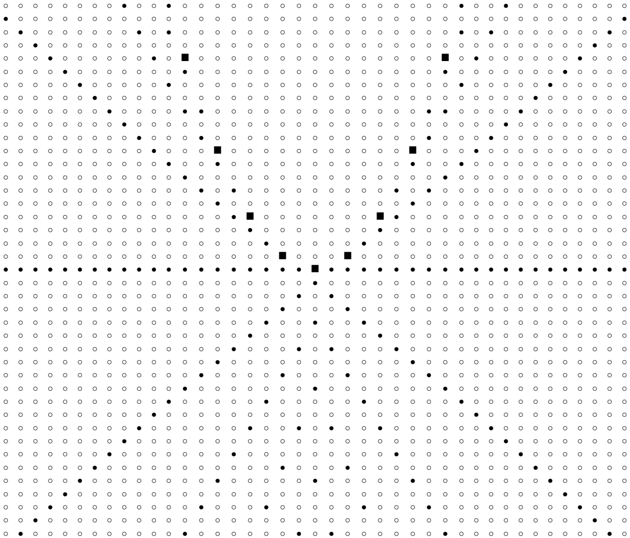
Denote by $F_G(H)$ the number of monic, integer polynomials of degree n with coefficients bounded by H and Galois group conjugate to G:

$$F_G(H) = \# \left\{ f = x^n + a_1 x^{n-1} + \cdots + a_n \;\middle|\; \begin{array}{l} \forall i \in \{1, \ldots, n\},\ |a_i| \leq H \\ \mathrm{Gal}(f) \sim G \end{array} \right\}.$$

The estimate $F_{\mathfrak{S}_n}(H) = H^n + o(H^n)$ is a result of Hilbert.

The error term $o(H^n)$ involves contributions of polynomials whose Galois group is not $\mathfrak{S}_n$.

$n = 2,\ H = 20, \qquad (a, b) \leftrightarrow f(x) = x^2 + ax + b$



$\circ$ Galois group $\mathfrak{S}_2$ contributes to $F_{\mathfrak{S}_2}$

$\bullet$ Galois group $\mathfrak{A}_2$ contributes to $F_{\mathfrak{A}_2}$

$$F_G(H) = \#\left\{ f = x^n + a_1 x^{n-1} + \cdots + a_n \;\middle|\; \begin{array}{l} \forall i \in \{1, \ldots, n\},\ |a_i| \leq H \\ \mathrm{Gal}(f) \sim G \end{array} \right\}$$

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The exponent n comes from a linear family of dimension n.

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Today, I focus on linear families of polynomials with Galois group $\mathfrak{A}_n$.

In fact, on lines of polynomials with Galois group $\mathfrak{A}_n$ (and even n).

Definition

A *line of polynomials* is a family of polynomials of the form $f(x) + tg(x)$, where $f, g \in \mathbb{Q}[x]$ and t is a parameter.

Expectation:
Use special lines of polynomials to prove lower bounds for $F_{\mathfrak{A}_n}(H)$.

Lines of polynomials and Galois groups

Nuno was able to follow this circle of ideas to find lines of polynomials with Galois group $\mathfrak{A}_n$, for odd n.

More generally, this raises the following question.

Let $G \subset \mathfrak{S}_n$ be a subgroup.

Question

Can the generic Galois group of a *line of polynomials* be conjugate to G?

Goal

Find polynomials $f(x), g(x)$ of degree $\leq n$ such that for "most" values $t \in \mathbb{Q}$, the Galois group of the polynomial $f(x) + tg(x)$ is $\mathfrak{A}_n$.

$n = 2$	**Good**	**Bad**	**Good, but misleading?**
f	$x^2 + x$	x^2	x^2
g	$x + 1$	-1	x
$f + tg$	$x^2 + (t+1)x + t$	$x^2 - t^2$	$x^2 + tx$

From lines of polynomials to morphisms $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$

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Step 1. View $f(x), g(x)$ as defining an inclusion

$$\begin{array}{rccc} \iota\colon & \mathbb{Q}(t) & \longrightarrow & \mathbb{Q}(x) \\ & t & \longmapsto & \dfrac{f(x)}{g(x)}. \end{array}$$

Step 2. Ensure that the Galois group of the extension ι is $\mathfrak{A}_n$.

Step 3. Hilbert's Irreducibility Theorem takes care of the "most".

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Algebra	Geometry
an inclusion $$\begin{array}{rccc} \iota : & \mathbb{Q}(t) & \longrightarrow & \mathbb{Q}(x) \\ & t & \longmapsto & \dfrac{f(x)}{g(x)}. \end{array}$$	a morphism $$\begin{array}{rccc} F : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ & [x,1] & \longmapsto & [f(x), g(x)] \end{array}$$ of degree n.

Step 2. Ensure that the Galois group of the extension ι is $\mathfrak{A}_n$.

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Snapshots of polynomials

Aside

How to think of a line of polynomials $f(x) + tg(x)$ very informally.

View $f(x) + tg(x)$ as a *movie*.

Each value of t is a *frame*: a polynomial in x for a fixed value of t.

Each frame, is a *snapshot* of our *characters*: the *roots of the polynomial*.

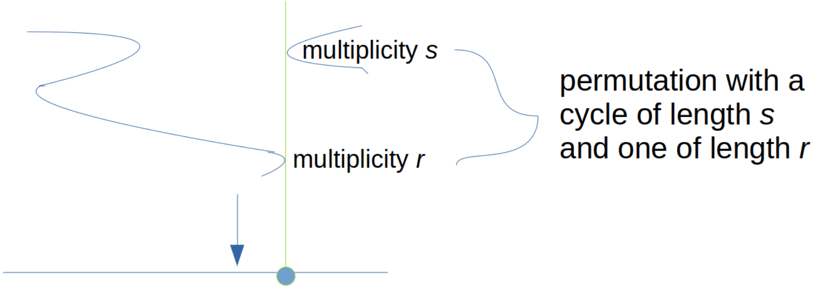
The movie tracks the characters throughout all the snapshots.

Most snapshots are *boring*: these are the values of t for which the polynomial $f(x) + tg(x)$ has distinct roots.

We like snapshots with *interactions*, i.e., where roots come together.

Our aim: reconstruct a whole movie, from a few key scenes.

The Galois group[1] of a morphism $F\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

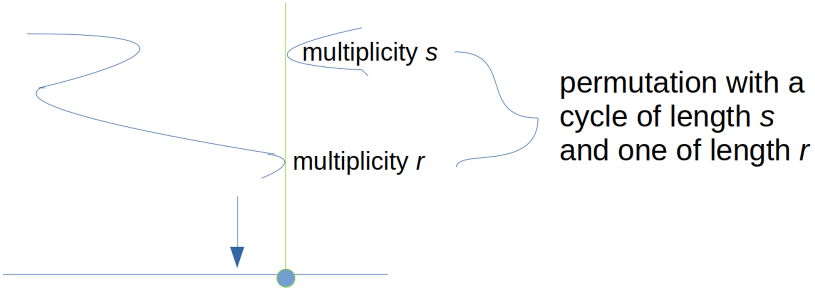


Each fiber consists of n points *counted with multiplicity.*

When $F([x,1]) = [f(x), g(x)]$, the fiber over $[t,1]$ consists of the roots of the equation $f(x) - tg(x) = 0$.

If one fiber is defined by $\prod_{i=1}^{s}(x-\alpha_i)^{r_i} = 0$, with distinct $\alpha_1, \ldots, \alpha_s$, then $\operatorname{Gal}(F)$ contains a permutation with cycle structure $(r_1, \ldots, r_s)$.

[1] Properly, they are *monodromy groups*



If one fiber is defined by $\prod_{i=1}^{s}(x-\alpha_i)^{r_i}=0$, with distinct $\alpha_1,\ldots,\alpha_s$, then $\mathrm{Gal}(F)$ contains a permutation with cycle structure $(r_1,\ldots,r_s)$.

The group $\mathrm{Gal}(F)$ is generated by the permutations described above, as you range over all possible fibers.

From a morphism $F\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ to lists of permutations

Except for finitely many points, every multiplicity is equal to 1: sadly, most scenes are boring.

From $F\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ we extract finitely many non-identity permutations $\sigma_1, \ldots, \sigma_h$, such that

- the cycle structure of $\sigma_1, \ldots, \sigma_h$ is determined by the multiplicities;
- $\mathrm{Gal}(F)$ is generated by $\sigma_1, \ldots, \sigma_h$.

The conditions above are *highly* redundant.

For most choices of $\sigma_1, \ldots, \sigma_h \in \mathfrak{S}_n$, there does not exist a morphism $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ with $\sigma_1, \ldots, \sigma_h$ as associated (non-identity) permutations.

An identity in $\mathfrak{S}_n$

Nevertheless, we can realize a very specific choice of permutations.

Two $(n-1)$-cycles whose product is a 3-cycle

Set $\sigma_1 = (1, 2, 3, \ldots, n-2, n-1)$ and $\sigma_2 = (n, n-1, \ldots, 4, 3, 2)$, so that

$$\sigma_1 \circ \sigma_2 = (1, 2, n) = \sigma_3^{-1}$$

We are going to find a line of polynomials with 3 "interesting frames":

- two frames giving the $(n-1)$-cycles σ_1, σ_2 and
- one frame giving the 3-cycle σ_3.

The group generated by $\sigma_1, \sigma_2, \sigma_3$

$$\begin{aligned}
\sigma_1 &= (1, 2, 3, \dots, n-2, n-1) \\
\sigma_2 &= (n, n-1, \dots, 4, 3, 2) \\
\sigma_3 &= (1, n, 2)
\end{aligned}$$

First, we determine the subgroup of $\mathfrak{S}_n$ that σ_1, σ_2 (and σ_3) generate.

Theorem (Jordan)

A transitive subgroup of $\mathfrak{S}_n$ containing an $(n-1)$-cycle and a 3-cycle contains $\mathfrak{A}_n$.

Hence, σ_1 and σ_2 generate $\begin{cases} \mathfrak{S}_n, & \text{if } n \text{ is odd;} \\ \mathfrak{A}_n, & \text{if } n \text{ is even.} \end{cases}$

Theorem (Jordan)

A transitive subgroup of $\mathfrak{S}_n$ containing an $(n-1)$-cycle and a 3-cycle contains $\mathfrak{A}_n$.

Let $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism and $p_1, p_2, p_3 \in \mathbb{P}^1$ be points such that

1. for all $p \in \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$, the fiber $F^{-1}(p)$ has n distinct fibers (the only interesting frames are $p_1, p_2.p_3$);
2. the fibers over the points p_1, p_2 both contribute an $(n-1)$-cycle (each has one point of multiplicity exactly $n-1$);
3. the fiber over the point p_3 contributes a 3-cycle (one point of multiplicity 3, all remaining ones of multiplicity 1).

The group $\mathrm{Gal}(F)$ is generated by two $(n-1)$-cycles and a 3-cycle.

The irreducibility of $\mathbb{P}^1$ implies that $\mathrm{Gal}(F)$ is transitive.

Jordan's Theorem implies that $\mathrm{Gal}(F)$ contains $\mathfrak{A}_n$.

Want: a morphism $F\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ and 3 points $p_1, p_2, p_3 \in \mathbb{P}^1$ with

1. for all $p \in \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$, the fiber $F^{-1}(p)$ has n distinct fibers;
2. the fibers over the points p_1, p_2 have multiplicities $(n-1, 1)$;
3. the fiber over the point p_3 has multiplicities $(3, 1, 1, \ldots, 1)$.

The conditions above involve points p_1, p_2, p_3 on the target $\mathbb{P}^1$.

Each point p_1, p_2, p_3 in the target $\mathbb{P}^1$ determines a *unique* point q_1, q_2, q_3 in the source $\mathbb{P}^1$:

- q_1 is the point of $\mathbb{P}^1$ of multiplicity $n-1$ in $F^{-1}(p_1)$;
- q_2 is the point of $\mathbb{P}^1$ of multiplicity $n-1$ in $F^{-1}(p_2)$;
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These last three conditions suffice (use the Riemann-Hurwitz formula).

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Summing up

Let $F\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism.

Suppose that q_1, q_2, q_3 are distinct points of $\mathbb{P}^1$ with

1. q_1 is a point of $\mathbb{P}^1$ of multiplicity $n-1$ in its fiber;
2. q_2 is a point of $\mathbb{P}^1$ of multiplicity $n-1$ in its fiber;
3. q_3 is a point of $\mathbb{P}^1$ of multiplicity at least 3 in its fiber.

Then, $\mathrm{Gal}(F)$ contains $\mathfrak{A}_n$.

Moreover, $\mathrm{Gal}(F)$ coincides with $\mathfrak{A}_n$ if and only if n is even.

SMALL PRINT: we must (and will) be careful about the ground field.

We also need to impose linearity.

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Snapshots of a morphism

We look for a morphism with three fibers of the form

- $(x - a_1)^{n-1}(x - b_1)$ — a point of multiplicity $n - 1$ $(a_1 \neq b_1)$;
- $(x - a_2)^{n-1}(x - b_2)$ — another point of multiplicity $n - 1$ $(a_2 \neq b_2)$;
- $(x - a_3)^3 h(x)$ — a point of multiplicity 3 $(\deg h(x) = n - 3)$.

Linearity: the three polynomials above should be linearly dependent.

Further non-degeneracy: we want a_1, a_2, a_3 to be distinct.

We now choose convenient coordinates, via a small geometric detour.

The space of polynomials of degree n

Let k be a field. Let $\mathcal{P}_n$ be the set of polynomials of degree n over k.

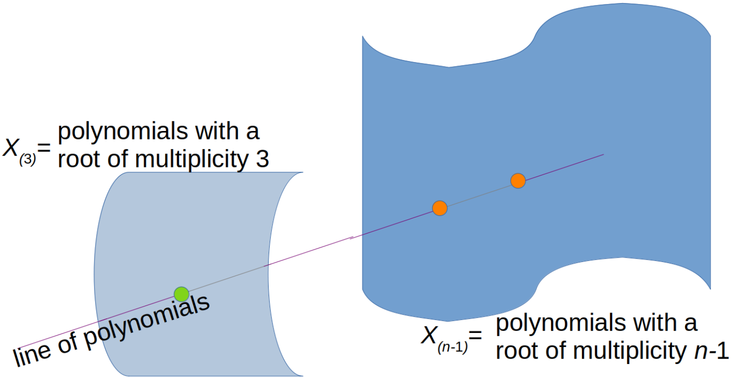
For $i \in \mathbb{N}$, let $X_{(i)} \subset \mathcal{P}_n$ be the set of polynomials with a root of multiplicity at least i:

$$X_{(i)} = \left\{ (x-a)^i h(x) \mid a \in k, \;\; h(x) \in k[x] \right\}.$$

Our players:

$$\begin{aligned} X_{(n-1)} &= \left\{ \text{polynomials with a root of multiplicity } n-1 \right\} \\ X_{(3)} &= \left\{ \text{polynomials with a root of multiplicity } 3 \right\}. \end{aligned}$$

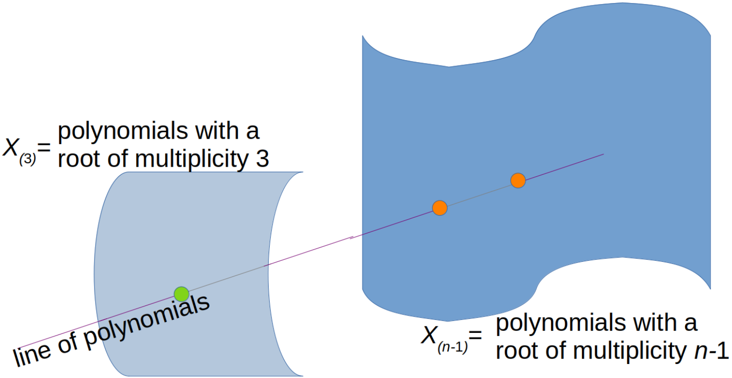
Geometric picture



Besides the repeated roots loci $X_{(n-1)}$ and $X_{(3)}$, we want a line meeting $X_{(n-1)}$ twice and $X_{(3)}$ once.

We are after a secant line to $X_{(n-1)}$ that also meets $X_{(3)}$.

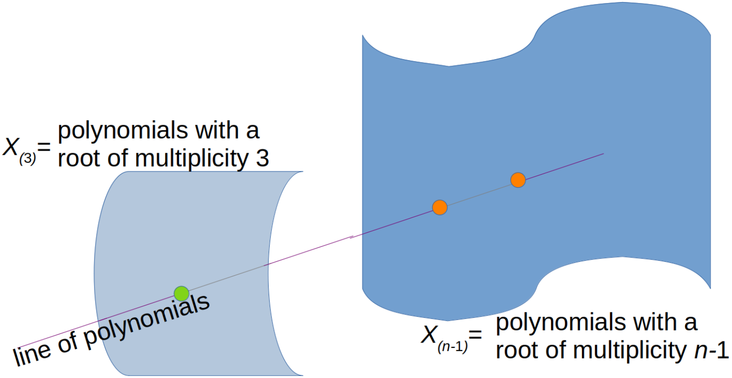
Parametrizing the line of polynomials



We can choose two coordinates:

- the coordinate x of the polynomial;
- the coordinate t along the line of polynomials.

Each coordinate has 3 parameters, since it is a coordinate on $\mathbb{P}^1$.
For a little flexibility, we only fix 5 parameters.



We parametrize the line of polynomials with parameter t so that

- the two intersection points with $X_{(n-1)}$ are $t = 0$ and $t = \infty$;
- the intersection point with $X_{(3)}$ is a flexible $t = t_0$.

We parametrize the polynomials with the variable x so that

- the polynomial for $t = 0$ has $x = 0$ as multiple root;
- the polynomial for $t = \infty$ has $x = \infty$ as multiple root;
- the polynomial for $t = t_0$ has $x = 1$ as multiple root.

With these choices, the candidate polynomial is

$$f_t(x) = x^{n-1}(x - b_1) - t(x - b_2).$$

- For $t = 0$, the polynomial $x^{n-1}(x - b_1)$ has an $(n-1)$st root at 0;
- For $t = \infty$, the polynomial $x - b_2$ has an $(n-1)$st root at ∞.

For $t = t_0$, we obtain the polynomial

$$f_{t_0}(x) = x^{n-1}(x - b_1) - t_0(x - b_2)$$

and we impose a triple root at $x = 1$.

Fixing $x^{n-1}(x-b_1)-t(x-b_2)$

For $t = t_0$, we obtain the polynomial

$$x^{n-1}(x-b_1)-t_0(x-b_2)$$

and we impose a triple root at $x = 1$.

$$\begin{aligned} f_{t_0}(x) &= x^{n-1}(x-b_1)-t_0(x-b_2) \\ f'_{t_0}(x) &= (n-1)x^{n-2}(x-b_1)+x^{n-1}-t_0 \\ f''_{t_0}(x) &= (n-1)\left((n-2)x^{n-3}(x-b_1)+x^{n-2}\right) \end{aligned}$$

$$\left.\begin{aligned} 0 = f_{t_0}(1) &= (1-b_1)-t_0(1-b_2) \\ 0 = f'_{t_0}(1) &= (n-1)(1-b_1)+1-t_0 \\ 0 = f''_{t_0}(1) &= (n-1)\left((n-2)(1-b_1)+1\right) \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} b_1 &= \tfrac{n}{n-2} \\ t_0 &= -\tfrac{n}{n-2} \\ b_2 &= \tfrac{n-2}{n} \end{aligned}\right.$$

Set $r_n = \dfrac{n}{n-2}$.

- We found a morphism $F_n \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ given by

$$\begin{array}{rccc} F_n \colon & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ & [x,y] & \longmapsto & \left[x^{n-1}(x - r_n y),\ (x - r_n^{-1} y) y^{n-1}\right]. \end{array}$$

- The rational function associated to the morphism F_n is

$$x^{n-1} \frac{x - r_n}{x - r_n^{-1}}.$$

- The line of polynomials associated to the morphism F_n is

$$x^n - r_n x^{n-1} - tx + tr_n^{-1}.$$

Theorem

The Galois group associated to the line of polynomials

$$x^n - r_n x^{n-1} - tx + tr_n^{-1}$$

contains $\mathfrak{A}_n$. It coincides with $\mathfrak{A}_n$ if n is even.

Caring for the ground field

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The Galois group associated to the line of polynomials

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So far, we have disregarded issues about the ground field.

The result above computes the Galois group of the field extension

$$\begin{array}{ccc} k(t) & \subset & k(x) \\ t & \mapsto & x^{n-1}\dfrac{x - r_n}{x - r_n^{-1}} \end{array}$$

when k is algebraically closed and $\operatorname{char} k \nmid n(n-1)(n-2)$.

Discriminants

Recall that the Galois group of a square-free polynomial of degree n is contained in $\mathfrak{A}_n$ if and only if its discriminant is a square.

Theorem

The Galois group associated to the line of polynomials

$$a_{n,t}(x) = x^{2n} + nx^{2n-1} + t\left(nx + (n-1)^2\right)$$

coincides with $\mathfrak{A}_{2n}$.

The discriminant $\Delta_n(t) = \operatorname{disc} a_{n,t}(x)$
of $a_{n,t}(x)$ is a polynomial in t and is a square in $\overline{\mathbb{Q}}(t)$.

$\Delta_n(t)$ divided by its leading coefficient, is a square in $\mathbb{Q}[t]$.

Hence, there is no need to go all the way to $\overline{\mathbb{Q}}$: adding a square root of the leading coefficient of $\Delta_n(t)$ is enough.

Set $K_n = \mathbb{Q}\left(\sqrt{(-1)^{n-1}(2n-1)}\right)$ — an extension of $\mathbb{Q}$ of degree ≤ 2.

Theorem

The Galois group of the polynomial

$$x^{2n} + nx^{2n-1} + t\left(nx + (n-1)^2\right)$$

coincides with $\mathfrak{A}_{2n}$*, for most values of* $t \in K_n$*.*

The discriminant of the polynomial in the theorem is

$$(-1)^{n-1} n^{2n} (2n-1)^{2n-1} t^{2n-2} \left(t + (n-1)^{2n-2}\right)^2 =$$

$$(-1)^{n-1}(2n-1)\left(n^n (2n-1)^{n-1} t^{n-1} \left(t + (n-1)^{2n-2}\right)\right)^2$$

Summary

We searched for families of polynomials with generic Galois group $\mathfrak{A}_n$.

We found the linear families

$$a_{n,t}(x) = x^{2n} + nx^{2n-1} + t\left(nx + (n-1)^2\right)$$

over an at most quadratic extension of $\mathbb{Q}$.

However, the methods used are fairly flexible.

Future directions

Do lines of $\mathfrak{A}_{2n}$ exist over $\mathbb{Q}$, not over a quadratic extension?
For what subgroups of $\mathfrak{S}_n$ do lines of polynomials exist?

We would like to quantify rates of growth of numbers of polynomials with given Galois groups and coefficients of bounded size.

High dimensional linear families of polynomials with a given generic Galois group provide potentially interesting lower bounds.

This feeds into *enumerative Galois theory*.

The methods that we use go via classical projective geometry of coincident root loci.

Employ more tools from algebraic geometry to address questions motivated by analytic number theory.

Thank you!!

Questions?