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Fano varieties in index one Fano complete intersections

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Abstract Let $X \subset \mathbb{P}^N$ be a smooth complex complete intersection such that $\omega_X \simeq \mathcal{O}_X(-1)$. Let $f : S \to X$ be a generically finite morphism from a smooth projective variety to X. Under some positivity assumption on the anticanonical divisor of S, if $2 \leq \dim S \leq \dim X - 2$ we prove that the deformations of f are contained in a subvariety of codimension at least 2.

1 Introduction

This note is a generalization of the argument given in [1]. All varieties considered are defined over the complex numbers. We give further evidence to the conjecture that general complex hypersurfaces X of degree $n+1$ in $\mathbb{P}^{n+1}$ are not unirational for sufficiently large n. In fact our results apply as soon as $\dim X \geq 4$ and X is a smooth complete intersection of Fano index one. The approach to the question of unirationality is the one suggested in [2]: if X is unirational, then X is covered by rational subvarieties of any dimension smaller than the dimension of X. Our main result is that X cannot be covered by subvarieties with nef anticanonical bundle. More precisely, let $f : S \to X$ be a morphism between smooth complex projective varieties. Let

$$\begin{array}{ccccc} S & \hookrightarrow & \Sigma & \xrightarrow{F} & X \\ \downarrow & & \downarrow{\scriptstyle p} & & \\ \{0\} & \hookrightarrow & Z & & \end{array} \tag{1.1}$$

be a family of deformations of f, i.e. F is a morphism, Z is a (connected) scheme, $\Sigma \to Z$ is a flat proper morphism, and there is a point $0 \in Z$ such that $\Sigma_0 \simeq S$ and $F|_{\Sigma_0} = f$. The following is the main result of this note.

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Theorem 1.1 *Let $X \subset \mathbb{P}^N$ be a smooth complex complete intersection with $\omega_X \simeq \mathcal{O}_X(-1)$. Let $f : S \to X$ be a generically finite morphism, where S is a smooth variety with $\omega_S^\vee$ nef. Assume that* $2 \leq \dim S \leq \dim X - 2$ *and if $\omega_S \simeq \mathcal{O}_S$ also that $f(S) \subset \mathbb{P}^N$ is non-degenerate. Then the image of any F as in* (1.1) *is contained in a subvariety of codimension at least* 2.

A similar result holds if S is a toric variety and f is finite.

The argument is structured as follows. Given any morphism $f : S \to X$ between smooth projective varieties and for all $r \geq 0$, Proposition 2.1 identifies a torsion-free sheaf $\mathcal{F}_r$ on S such that if the deformations of f contain a subvariety of codimension r, then the sheaf $\mathcal{F}_r$ has a non-zero global section. Proposition 2.2 gives some technical assumptions implying that the sheaf $\mathcal{F}_1$ has no global sections. Finally Theorems 2.3 and 2.4 are immediate applications of Proposition 2.2.

2 Proof of the theorem

Given a morphism $f : S \to X$ we let $\mathcal{C}_f := \ker\big(df : \ f^*\Omega_X \to \Omega_S\big)$.

Proposition 2.1 *Let X be a smooth projective variety of dimension n. Let there be given the diagram*

$$\begin{array}{ccccc} S := \Sigma_0 & \hookrightarrow & \Sigma & \xrightarrow{F} & X \\ \downarrow & & \downarrow{\scriptstyle p} & & \\ \{0\} & \hookrightarrow & Z & & \end{array}$$

where $f := F|_S$, $\dim F(S) = h$, the morphism p is smooth, the morphism dF has rank at least $n - r$ at the generic point of S, and $0 \in Z$ is a general point. Then the subsheaf of ${\mathcal{C}_f}^\vee$ generated by global sections has rank at least $n - r - h$. In particular if the sheaf $\big(\bigwedge^{n-r-h} f^*T_X/T_S\big)^{\vee\vee}$ has no global sections, then the deformations of f are contained in a subvariety of codimension at least $r + 1$.

Proof Let $\pi : \Omega_\Sigma \to \Omega_{\Sigma|Z}$ be the cokernel of $dp : p^*\Omega_Z \to \Omega_\Sigma$ and let $\mathcal{K}$ be the kernel of the morphism $\pi \circ dF : F^*\Omega_X \to \Omega_{\Sigma|Z}$. There is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & F^*\Omega_X & \longrightarrow & \Omega_{\Sigma|Z} & & \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle dF} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & p^*\Omega_Z & \xrightarrow{dp} & \Omega_\Sigma & \xrightarrow{\pi} & \Omega_{\Sigma|Z} & \longrightarrow & 0 \end{array}$$

where the morphism u exists by the universal property of the kernel. The rank of the sheaf $\mathcal{K}$ is at least $n - h$ and since the kernel of dF has rank at most r at the generic point of S, it follows that the image of u has rank at least $d := n - r - h$.

The image of $\pi \circ dF$ is torsion-free since it is a subsheaf of the locally free sheaf $\Omega_{\Sigma|Z}$. By generic freeness and the fact that $0 \in Z$ is general it follows that $\mathcal{K}|_S \simeq \mathcal{C}_f$. Restricting the morphism u to S, we obtain a morphism $\mathcal{C}_f \to \Omega_{Z,0} \otimes \mathcal{O}_S \simeq \mathcal{O}_S^{\dim Z}$ whose image has rank at least d. Since non-zero morphisms of sheaves $\mathcal{C}_f \to \mathcal{O}_S$ correspond to non-zero global sections of ${\mathcal{C}_f}^\vee$ up to scaling, we deduce that $\mathrm{H}^0(S, {\mathcal{C}_f}^\vee)$ spans a subsheaf of ${\mathcal{C}_f}^\vee$ of rank at least d.

For the last statement let $\mathcal{N} = f^*T_X/T_S$. Dualize the sequence $T_S \to f^*T_X \to \mathcal{N} \to 0$ to deduce that $\mathcal{C}_f \simeq \mathcal{N}^\vee$; thus ${\mathcal{C}_f}^\vee \simeq (\mathcal{N}^\vee)^\vee$ and we conclude. □

Proposition 2.2 *Let $X \subset \mathbb{P}^N$ be a smooth complex projective variety of dimension n with $\omega_X \simeq \mathcal{O}_X(-1)$. Let $f : S \to X$ be a generically finite morphism, where S is a smooth projective variety of dimension k, with $2 \le k \le n-2$. Assume either that the sheaf ω_S has no non-zero global sections or that $\omega_S \simeq \mathcal{O}_S$ and $f(S) \subset \mathbb{P}^N$ is non-degenerate. If $h^1\big(S, f^*\mathcal{I}_X \otimes \omega_S(1)\big) = 0$, then the image of any F as in (1.1) is contained in a subvariety of codimension at least 2.*

Proof Using Proposition 2.1, it is enough to show that $\big(\bigwedge^{n-1-k} f^*T_X/T_S\big)^{\vee\vee}$ has no non-zero global sections. Consider first the double dual of the morphism

$$\bigwedge^k T_S \otimes \bigwedge^{n-1-k} f^*T_X/T_S \longrightarrow \bigwedge^{n-1} f^*T_X.$$

On the open dense set where f is smooth to its image, this morphism is clearly injective; moreover, the target sheaf is locally free and hence (tensoring with ω_S) we obtain an injective morphism of sheaves

$$\left(\bigwedge^{n-1-k} f^*T_X/T_S\right)^{\vee\vee} \longrightarrow \omega_S \otimes \bigwedge^{n-1} f^*T_X.$$

Thus it is enough to show that the last sheaf has no global sections. We have $\bigwedge^{n-1} T_X \simeq \omega_X^\vee \otimes \Omega_X^1 \simeq \Omega_X^1(1)$ and the normal sequence of X in $\mathbb{P}^N$

$$0 \longrightarrow \mathcal{I}_X|_X \longrightarrow \Omega^1_{\mathbb{P}^N}|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Restrict to S and tensor with $\omega_S(1)$ to obtain

$$0 \longrightarrow \mathcal{I}_X|_X \otimes \omega_S(1) \longrightarrow \Omega^1_{\mathbb{P}^N} \otimes \omega_S(1) \longrightarrow \Omega^1_X \otimes \omega_S(1) \longrightarrow 0.$$

Thus, thanks to our assumptions, it is enough to show that $h^0\big(S, \omega_S \otimes \Omega^1_{\mathbb{P}^N}(1)\big) = 0$. Consider the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^N} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1)^{N+1} \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0$$

and tensor with $\omega_S \otimes \omega_X^\vee \simeq \omega_S(1)$ to deduce the short exact sequence

$$0 \longrightarrow \omega_S(1) \otimes \Omega^1_{\mathbb{P}^N} \longrightarrow \omega_S^{N+1} \longrightarrow \omega_S(1) \longrightarrow 0.$$

If ω_S has no non-zero global section, the result follows. If ω_S has a non-zero global section, it follows that $\omega_S \simeq \mathcal{O}_S$, since $\omega_S^\vee$ is nef. In this case, the morphism $\mathrm{H}^0\big(S, \omega_S^{N+1}\big) \to \mathrm{H}^0\big(S, \omega_S(1)\big)$ is injective since the image of S in $\mathbb{P}^N$ in non-degenerate, and hence again $\mathrm{H}^0\big(S, \omega_S(1) \otimes \Omega^1_{\mathbb{P}^N}\big) = 0$, and we conclude. □

Theorem 2.3 *Let $X \subset \mathbb{P}^N$ be a smooth complex complete intersection with $\omega_X \simeq \mathcal{O}_X(-1)$. Let $f : S \to X$ be a generically finite morphism from a smooth variety with $\omega_S^\vee$ nef to X. Assume that $2 \le \dim S \le \dim X - 2$ and if $\omega_S \simeq \mathcal{O}_S$ also that $f(S)$ is non-degenerate. Then the image of any F as in (1.1) is contained in a subvariety of codimension at least 2.*

Proof Thanks to Proposition 2.2 and Serre duality the result follows if we show that $h^{\dim S-1}\big(S, f^*\mathcal{N}_X(-1)\big) = 0$. Replacing $\mathbb{P}^N$ by the linear subspace spanned by X we may assume that $X \subset \mathbb{P}^N$ is non-degenerate. We have $\mathcal{N}_{X|\mathbb{P}^N}(-1) \simeq \mathcal{O}_X(d_1-1) \oplus \cdots \oplus \mathcal{O}_X(d_s-1)$, with $d_i \ge 2$. Thus the sheaf $f^*\mathcal{N}_{X|\mathbb{P}^N}(-1) \otimes \omega_S^\vee$ is a direct sum of big and nef line bundles since $\omega_S^\vee$ is nef. The required vanishing follows from the Kawamata–Viehweg Vanishing Theorem (cf. [3], Theorem 9.1.18). □

Theorem 2.4 *Let $X \subset \mathbb{P}^N$ be a smooth complex complete intersection with $\omega_X \simeq \mathcal{O}_X(-1)$. Let $f : S \to X$ be a finite morphism from a smooth projective toric variety to X. Assume that* $2 \leq \dim S \leq \dim X - 2$. *Then the image of any F as in* (1.1) *is contained in a subvariety of codimension at least* 2.

Proof Proceed as before: the sheaf $f^*\mathcal{N}_{X|\mathbb{P}^N}(-1)$ is a direct sum of ample line bundles and the required vanishing follows from [4]. □

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