# Hausdorff measures of different dimensions are not Borel isomorphic

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**Abstract**

We show that Hausdorff measures of different dimensions are not Borel isomorphic; that is, the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^t)$ are not isomorphic if $s \neq t$, $s, t \in [0, 1]$, where $\mathcal{B}$ is the $\sigma$-algebra of Borel subsets of $\mathbb{R}$ and $\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure. This answers a question of B. Weiss and D. Preiss.

To prove our result, we apply a random construction and show that for every Borel function $f : \mathbb{R} \to \mathbb{R}$ and for every $d \in [0, 1]$ there exists a compact set $C$ of Hausdorff dimension $d$ such that $f(C)$ has Hausdorff dimension $\leq d$.

We also prove this statement in a more general form: If $A \subset \mathbb{R}^n$ is Borel and $f : A \to \mathbb{R}^m$ is Borel measurable, then for every $d \in [0, 1]$ there exists a Borel set $B \subset A$ such that $\dim B = d \cdot \dim A$ and $\dim f(B) \leq d \cdot \dim f(A)$.

## 1 Introduction

The question whether Hausdorff measures of different dimensions are Borel isomorphic or not, has been around for several years. This problem is attributed to B. Weiss and D. Preiss, see also [5]. Let $\mathcal{H}^d$ denote the $d$-dimensional Hausdorff measure and let $\mathcal{B}$ denote the $\sigma$-algebra of Borel subsets of $\mathbb{R}$.

**Theorem 1.1.** *For every $0 \leq d_1 < d_2 \leq 1$ the measure spaces $(\mathbb{R}, \mathcal{B}, \mathcal{H}^{d_1})$ and $(\mathbb{R}, \mathcal{B}, \mathcal{H}^{d_2})$ are not isomorphic. Moreover, there does not exist a Borel bijection $f : \mathbb{R} \to \mathbb{R}$ such that for any Borel set $B \subset \mathbb{R}$*

$$0 < \mathcal{H}^{d_1}(B) < \infty \iff 0 < \mathcal{H}^{d_2}(f(B)) < \infty. \tag{1}$$

The analogous theorem in $\mathbb{R}^n$ holds, too (see Theorem 5.7).

On the other hand, M. Elekes [1] has proved that the continuum hypothesis implies that the measure spaces $(\mathbb{R}, \mathcal{M}_{\mathcal{H}^s}, \mathcal{H}^s)$ and $(\mathbb{R}, \mathcal{M}_{\mathcal{H}^t}, \mathcal{H}^t)$ are isomorphic whenever $s, t \in (0, 1)$, where $\mathcal{M}_{\mathcal{H}^d}$ is the $\sigma$-algebra of measurable sets with respect to $\mathcal{H}^d$.

In the same article, M. Elekes suggests a method to give a partial solution to the Borel isomorphism problem (Theorem 1.1) and asks the following question.

**Question 1.** Fix $0 < \alpha < 1$. Is it true that every Borel function $f : \mathbb{R} \to \mathbb{R}$ is Hlder continuous of exponent $\alpha$ on a set $H_f$ of Hausdorff dimension $1 - \alpha$?

The author of the present article has answered Question 1 in the positive [2].

It is easy to see that the positive answer implies that $t \leq \frac{s}{1-s}$ whenever the $s$-dimensional and the $t$-dimensional Hausdorff measures are Borel isomorphic. Unfortunately this approach does not seem to lead to Theorem 1.1 in its whole generality. Note that $1 - \alpha$ is the best we can have for the dimension of $H_f$, since a typical continuous function is not Hlder continuous of exponent $\alpha$ on any set of dimension larger than $1 - \alpha$, as shown by M. Elekes in [1].

Let $\dim H$ denote the Hausdorff dimension of the set $H$.

**Theorem 1.2.** *Let $f : \mathbb{R} \to \mathbb{R}$ be Borel (or Lebesgue) measurable. For every $0 \leq d \leq 1$ there exists a compact set $C \subset \mathbb{R}$ such that $\dim C = d$ and $\dim f(C) \leq d$.*

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Theorem 1.2 clearly implies Theorem 1.1: Let $f$ be Borel measurable and choose a $d$ for which $d_1 < d < d_2$. By applying Theorem 1.2 we get a compact set $C$ of dimension $d$ with $\dim f(C) \le d$. Since $d_1 < d$, there exists a Borel subset $B$ of $C$ for which $0 < \mathcal{H}^{d_1}(B) < \infty$ (see e.g. [3]). Now $f(B) \subset f(C)$, so it has dimension at most $d$, which implies that $\mathcal{H}^{d_2}(f(B)) = 0$. So $f$ cannot be an isomorphism of the measure spaces ($\mathbb{R}$, $\mathcal{B}$, $\mathcal{H}^{d_1}$) and ($\mathbb{R}$, $\mathcal{B}$, $\mathcal{H}^{d_2}$), and cannot satisfy (1).

To prove Theorem 1.2 it is clearly enough to show the following.

**Theorem 1.3.** *Suppose that $K$ is a compact set of positive Lebesgue measure and $f : K \to \mathbb{R}$ is continuous. For every $0 \le d \le 1$ there exists a compact set $C \subset K$ of Hausdorff dimension $d$ such that $f(C)$ has Hausdorff dimension at most $d$.*

The sketch of the proof is the following. We define a large class of random constructions such that each of them gives a Cantor set $F$ of dimension at most $d$ almost surely (Section 3). Then, for the given $K$ and $f$, we choose a random construction of this class which gives a set $F$ for which $F \subset f(K)$ and $\dim f^{-1}(F) \ge d$ almost surely. This will imply the theorem with a simple additional argument (Section 4).

As it can be expected, Theorem 1.2 has the following generalisation (proved in Section 5).

**Theorem 1.4.** *Let $A \subset \mathbb{R}^n$ be a Borel set and $f : A \to \mathbb{R}^m$ Borel measurable. Then for every $0 \le d \le 1$ there exists a Borel set $B \subset A$ such that $\dim B = d \cdot \dim A$ and $\dim f(B) \le d \cdot \dim f(A)$.*

**Notation.** Let $\lambda$ denote the one dimensional Lebesgue measure. For a (Borel) measure $\mu$ let $I_t(\mu)$ denote the $t$-dimensional energy of $\mu$; that is, $I_t(\mu) = \iint |x-y|^{-t}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)$. For Borel measures $\mu_k$ ($k \in \mathbb{N}$) and $\mu$, $\mu_k \to \mu$ denotes that $\mu_k$ weakly converges to $\mu$. Let $\operatorname{supp}\mu$ denote the support of the measure $\mu$.

We denote by $\mathbb{N}$ the set of non-negative integers. We identify each natural number with the set of its predecessors: $n = \{0, 1, \dots, n-1\}$.

By $\operatorname{diam} H$ we mean the diameter of the set $H$. Let $\mathcal{H}^s_\infty$ denote the $s$-dimensional Hausdorff pre-measure; that is, for any $H \subset \mathbb{R}$

$$\mathcal{H}^s_\infty(H) = \inf\Big\{ \sum_{n\in\mathbb{N}} (\operatorname{diam} I_n)^s : \{I_n\}_{n\in\mathbb{N}} \text{ is a sequence of intervals and } H \subset \cup_{n\in\mathbb{N}} I_n \Big\}.$$

# 2 Preliminaries

We start with some (probably well-known) statements which we shall use in the sequel.

**Lemma 2.1.** *Suppose that $\mu$ and $\mu_k$ ($k \in \mathbb{N}$) are probability measures on $\mathbb{R}$ such that $\mu_k \to \mu$. Then $\mu_k \times \mu_k \to \mu \times \mu$.*

*Proof.* We have to show that for every compactly supported continuous function $h : \mathbb{R}^2 \to \mathbb{R}$, $\int_{\mathbb{R}^2} h\, d(\mu_k \times \mu_k) \to \int_{\mathbb{R}^2} h\, d(\mu \times \mu)$. Clearly it is enough to show this for a dense subset of the compactly supported continuous functions. It is well known that functions of the form

$$\sum_{i=1}^n f_i(x) g_i(y) \quad (f, g : \mathbb{R} \to \mathbb{R} \text{ continuous functions with compact support})$$

are dense, so it is enough to check that

$$\int_{\mathbb{R}^2} f(x)g(y)\, d(\mu_k \times \mu_k) \to \int_{\mathbb{R}^2} f(x)g(y) d(\mu \times \mu).$$

By Fubini,

$$\int_{\mathbb{R}^2} f(x)g(y)\, d(\mu_k \times \mu_k) = \int_{\mathbb{R}} f(x)\, d\mu_k(x) \int_{\mathbb{R}} g(y)\, d\mu_k(y)$$

which tends to

$$\int_{\mathbb{R}} f(x)\, d\mu(x) \int_{\mathbb{R}} g(y)\, d\mu(y) = \int_{\mathbb{R}} f(x)g(y)\, d(\mu \times \mu)$$

as $k \to \infty$, using $\mu_k \to \mu$ and Fubini again. □

**Lemma 2.2.** *Suppose that $\mu_k$ ($k \in \mathbb{N}$) are probability measures on $\mathbb{R}$ with support in $[-R, R]$ for some $R > 0$. If $\mu_k \to \mu$ then $I_t(\mu) \le \liminf I_t(\mu_k)$.*

*Proof.* Let $\phi$ be a compactly supported continuous function on the plane which equals 1 on the square $[-R, R]^2$ and for which $0 \le \phi(x, y) \le 1$ everywhere. For each positive integer $i$ define $h_i : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$h_i(x, y) = \phi(x, y) \cdot \min(|x-y|^{-t}, i).$$

Using Lemma 2.1 we have

$$\int h_i(x,y)\,d\mu\,d\mu = \lim_k \int h_i(x,y)\,d\mu_k\,d\mu_k \le \liminf_{k\to\infty} \int |x-y|^{-t}\,d\mu_k\,d\mu_k = \liminf_{k\to\infty} I_t(\mu_k).$$

The support of $\mu\times\mu$ is in $[-R,R]^2$ since the support of $\mu_k$ is in $[-R,R]$ for all $k$, so we have

$$\lim_{i\to\infty} \int h_i(x,y)\,d\mu(x)\,d\mu(y) = \int |x-y|^{-t}\,d\mu(x)\,d\mu(y) = I_t(\mu).$$

Thus $I_t(\mu) \le \liminf_{k\to\infty} I_t(\mu_k)$. $\square$

**Lemma 2.3.** *Let $0<t<1$, $H$ be a compact set in $\mathbb{R}$ and $I=[0,\lambda(H)]$ an interval. Then*

$$\int_H\int_H |x-y|^{-t}\,d\lambda(x)\,d\lambda(y) \le \int_I\int_I |x-y|^{-t}\,d\lambda(x)\,d\lambda(y) = c_t\lambda(H)^{2-t}$$

*where $c_t$ is a constant depending only on $t$.*

*Proof.* Let $\varphi : H\to[0,\lambda(H)]$ be the following function:

$$\varphi(h) = \lambda\big((-\infty,h]\cap H\big).$$

Using first the fact that $\varphi$ is a contraction and then that it is a measure preserving transformation between $\lambda|_H$ and $\lambda|_I$, we obtain

$$\begin{aligned}
\int_H\int_H |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y) &\le \int_H\int_H |\varphi(x)-\varphi(y)|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y)\\
&= \int_I\int_I |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y) = \int_{[0,1]}\int_{[0,1]} |\lambda(H)x'-\lambda(H)y'|^{-t}\,\lambda(H)^2\,\mathrm{d}\lambda(x')\,\mathrm{d}\lambda(y')\\
&= \lambda(H)^{2-t}\int_{[0,1]}\int_{[0,1]} |x'-y'|^{-t}\,\mathrm{d}\lambda(x')\,\mathrm{d}\lambda(y') = c_t\lambda(H)^{2-t}
\end{aligned}$$

where $c_t$ is finite if $t<1$. $\square$

# 3 Random construction and upper estimate

Let $M\ge 3$ and $m$ be integers with $2\le m\le M-1$. Let

$$M^{<\omega} = \{(i_0,i_1,\dots,i_{n-1}) : n\in\mathbb{N},\ i_j\in\{0,1,\dots,M-1\}=M\}.$$

We will consider $M^{<\omega}$ as a set of multi-indices and also as the $M$-adic tree with root $\emptyset$, where every node has $M$ children. For an $i\in M^{<\omega}$ let $|i|$ denote the length of the multi-index; that is, the level of the node $i$.

**Definition 1.** A representation of $M^{<\omega}$ is a mapping $\phi$ which maps each node $i$ to a non-trivial compact interval $\phi(i)\subset\mathbb{R}$ such that

- for every node $i$ and its children $ij$ ($j\in M$) we have $\phi(ij)\subset\phi(i)$, and
- for every two distinct $j,j'\in M$, $\phi(ij)$ and $\phi(ij')$ can have at most one point in common.

Now we shall choose a "random $m$-adic subtree" $S$ of $M^{<\omega}$ in the following way. Let $X_i$ ($i\in M^{<\omega}$) be independent random variables with uniform distributions over the set of $m$-element subsets of $M$. That is, for each set $T\subset\{0,1,\dots,M-1\}$ of $m$ elements

$$\mathbb{P}(X_i=T) = \frac{1}{\binom{M}{m}}.$$

Define the random subtree as

$$S = \{(i_0,i_1,\dots,i_{n-1})\in M^{<\omega} : i_j\in X_{(i_0,i_1,\dots,i_{j-1})} \text{ for every } 0\le j\le n-1\}.$$

So $\emptyset\in S$, and for each $i\in S$ exactly $m$ children of $i$ are in $S$. It is easy to see that

$$|\{i\in S : |i|=n\}| = m^n$$

for every $n\in\mathbb{N}$.

Given a representation $\phi$ of $M^{<\omega}$, consider the closed sets $F_i=\phi(i)$ ($i\in M^{<\omega}$) and the random closed sets

$$F^n = \cup\{F_i : i\in S, |i|=n\}\quad (n\in\mathbb{N}).$$

Then $F_\emptyset = F^0\supset F^1\supset F^2\supset\cdots$. Put

$$F = \bigcap_n F^n.$$

We can consider $F$ as the image of the random $m$-adic subtree $S$.

**Proposition 3.1.** *For any representation of $M^{<\omega}$, the random closed set $F$ defined above has Hausdorff dimension at most $\frac{\log m}{\log M}$ almost surely.*

*Proof.* Let $1 > s > \frac{\log m}{\log M}$ be arbitrary and $q = \frac{m}{M^s}$, thus $q < 1$. We cover $F^n$ with those intervals $F_i$, for which $|i| = n$ and $i \in S$. For any $i$ of length $n$ we have

$$\mathbb{P}(i \in S) = \left(\frac{m}{M}\right)^n,$$

hence

$$\mathbb{E}\Big(\sum_{\substack{|i|=n\\ i\in S}} (\operatorname{diam} F_i)^s\Big) = \left(\frac{m}{M}\right)^n \sum_{|i|=n} (\operatorname{diam} F_i)^s.$$

Since the intervals $F_i$ ($|i| = n$) are almost disjoint (two of them can only have one point in common), $\sum_{|i|=n} \operatorname{diam} F_i \leq D \stackrel{\text{def}}{=} \operatorname{diam} F_\emptyset$. Thus, applying Jensen's inequality to the concave function $x \mapsto x^s$, we obtain

$$\left(\frac{m}{M}\right)^n \sum_{|i|=n} (\operatorname{diam} F_i)^s \leq \left(\frac{m}{M}\right)^n M^n \left(\frac{D}{M^n}\right)^s = D^s \left(\frac{m}{M^s}\right)^n = D^s q^n.$$

Therefore

$$\mathbb{E}\big(\mathcal{H}^s_\infty(F)\big) \leq \mathbb{E}\big(\mathcal{H}^s_\infty(F^n)\big) \leq \mathbb{E}\Big(\sum_{\substack{|i|=n\\ i\in S}} (\operatorname{diam} F_i)^s\Big) \leq D^s q^n.$$

Since this is true for every $n$, we get that

$$\mathbb{E}\big(\mathcal{H}^s_\infty(F)\big) = 0,$$

thus $\mathcal{H}^s_\infty(F) = 0$ almost surely, so $\mathcal{H}^s(F) = 0$ almost surely. Because $s > \frac{\log m}{\log M}$ can be chosen arbitrarily, the dimension of $F$ is at most $\frac{\log m}{\log M}$ almost surely. $\square$

# 4 Lower estimate

*Proof of Theorem 1.3.* If there exists an $y \in f(K)$ for which $f^{-1}(y)$ is of positive measure, then we can choose a compact set $C \subset f^{-1}(y)$ of arbitrary Hausdorff dimension $d$ ($0 \leq d \leq 1$), and clearly $f(C) = \{y\}$ has Hausdorff dimension at most $d$. Thus we may assume that for every $y \in f(K)$ the set $f^{-1}(y)$ has Lebesgue measure zero. Without loss of generality we may suppose that $\lambda(K) = 1$.

Now we define the particular representation of $M^{<\omega}$ which is adequate for our needs. All the endpoints of the intervals $\phi(i)$ ($i \in M^{<\omega}$) will be contained in $f(K)$. We define $\phi(\emptyset)$ to be the smallest interval which contains $f(K)$. If an interval is already defined, then its $M$ subintervals (its children) are chosen such that their preimages (with respect to $f$) have equal Lebesgue measure: $\frac{1}{M}$ times the Lebesgue measure of the preimage of the interval. Now we give a more precise definition.

Define $\psi : f(K) \to \mathbb{R}$ as

$$\psi(x) = \lambda(\{z \in K : f(z) \leq x\}).$$

Since the inverse image of any point in $f(K)$ has measure zero, this is a continuous increasing function, and its image is the interval $[0, \lambda(K)]$.

For an $i \in M^{<\omega}$ let

$$y_1^i = \max\Big\{y \in f(K) : \psi(y) = \sum_{j=1}^{|i|} \frac{i_{j-1}}{M^j}\Big\},$$

$$y_2^i = \min\Big\{y \in f(K) : \psi(y) = \frac{1}{M^{|i|}} + \sum_{j=1}^{|i|} \frac{i_{j-1}}{M^j}\Big\}.$$

Let $F_i = \phi(i) = [y_1^i, y_2^i]$. It is obvious from the definition that

$$F_{(i_0,\ldots,i_{k-1})} \supset \bigcup_{j=0}^{M-1} F_{(i_0,\ldots,i_{k-1},j)},$$

$$\lambda(f^{-1}(F_i)) = \lambda(\{z \in K : f(z) \in F_i\}) = \frac{1}{M^{|i|}}$$

and that $\phi$ is a representation of $M^{<\omega}$.

Now let $S$ be a random $m$-adic subtree of $M^{<\omega}$, and define the random closed sets

$$F^n = \cup\{F_i : i \in S, |i| = n\} \quad (n \in \mathbb{N}),$$

$$F = \bigcap_{n\in\mathbb{N}} F^n$$

the same way as before. From Proposition 3.1, $F$ has Hausdorff dimension at most $\frac{\log m}{\log M}$ almost surely. Hence $F$ cannot contain an interval, and since all the intervals $\phi(i)$ $(i \in M^{<\omega})$ have their endpoints in $f(K)$, $F \subset f(K)$ almost surely.

Let $G_i = f^{-1}(F_i)$, $G^n = f^{-1}(F^n)$ and $G = f^{-1}(F)$ be (random) compact sets in $K$. Then we also have

$$G^n = \cup\{G_i : i \in S, |i| = n\} \quad (n \in \mathbb{N})$$

and

$$G = \bigcap_{n\in\mathbb{N}} G^n.$$

We claim that $G$ has Hausdorff dimension at least $\frac{\log m}{\log M}$ almost surely. The key point in our construction was that $\lambda(G_i) = \frac{1}{M^{|i|}}$, and we also know that $\lambda(G_i \cap G_{i'}) = 0$ provided that $i \neq i'$ and $|i| = |i'|$. Note that $\lambda(G^k) = \left(\frac{m}{M}\right)^k$.

We define random Borel measures $\mu_k$ on $\mathbb{R}$ by

$$\mu_k(H) = \lambda(H \cap G^k) \cdot \left(\frac{M}{m}\right)^k,$$

or equivalently,

$$\mu_k = \left(\frac{M}{m}\right)^k \cdot \lambda|_{G^k} \quad (k \in \mathbb{N}). \tag{2}$$

Hence $\mu_k$ is a probability measure with support $G^k \subset K$.

Let $0 < t < \frac{\log m}{\log M}$ be fixed. We would like to give an upper bound for the expected value of the $t$-energy of $\mu_k$. To do this at first we need to calculate some basic probability. We know that $\mathbb{P}(i \in S) = \left(\frac{m}{M}\right)^{|i|}$ for every $i \in M^{<\omega}$. How much is $\mathbb{P}(i \in S, i' \in S)$ if $|i| = |i'| = k$? Let $i \wedge i'$ denote the nearest common ancestor of $i$ and $i'$ in the tree $M^{<\omega}$, and let $l = l(i, i') = |i \wedge i|$; that is, $l$ is the largest integer for which $i_0 = i'_0$, $i_1 = i'_1$, $\dots$, $i_{l-1} = i'_{l-1}$ hold $(0 \le l \le k)$.

$$\begin{aligned}\mathbb{P}(i \in S,\, i' \in S) = \mathbb{P}\Big(&(i_j \in X_{(i_0,\dots,i_{j-1})} \text{ for every } 0 \le j \le l-1)\\ &\text{and } (i_l, i'_l \in X_{(i_0,\dots,i_{l-1})})\\ &\text{and } (i_j \in X_{(i_0,\dots,i_{j-1})} \text{ for every } l+1 \le j \le k-1),\\ &\text{and } (i'_j \in X_{(i'_0,\dots,i'_{j-1})} \text{ for every } l+1 \le j \le k-1)\Big).\end{aligned}$$

The random variables $X_i$ are independent, so this probability is

$$= \left(\frac{m}{M}\right)^l \frac{m(m-1)}{M(M-1)} \left(\frac{m}{M}\right)^{k-l-1} \left(\frac{m}{M}\right)^{k-l-1} = \left(\frac{m}{M}\right)^{2k-l-1} \frac{m-1}{M-1} \le \left(\frac{m}{M}\right)^{2k-l} \tag{3}$$

provided that $l < k$, that is, $i \neq i'$, but the upper estimate clearly holds in the case $i = i'$ $(l = k)$ as well.

By (2), for any $i$ of length $k$ we have

$$\mu_k|_{G_i} = \begin{cases} \left(\frac{M}{m}\right)^k \cdot \lambda|_{G_i} & \text{if } \ i \in S\\ 0 & \text{if } \ i \notin S.\end{cases} \tag{4}$$

Applying first that $\operatorname{supp}\mu_k = G^k$ is contained in $\cup_{|i|=k} G_i$, and then (4) and (3),

$$\begin{aligned}\mathbb{E}\, I_t(\mu_k) = \mathbb{E}\left(\iint |x-y|^{-t}\, \mathrm{d}\mu_k(x)\, \mathrm{d}\mu_k(y)\right) = \mathbb{E}\left(\sum_{|i|=|i'|=k} \int_{G_i}\int_{G_{i'}} |x-y|^{-t}\, \mathrm{d}\mu_k(x)\, \mathrm{d}\mu_k(y)\right) =\\ = \sum_{|i|=|i'|=k} \mathbb{E}\left(\int_{G_i}\int_{G_{i'}} |x-y|^{-t}\, \mathrm{d}\mu_k(x)\, \mathrm{d}\mu_k(y)\right) =\\ = \sum_{|i|=|i'|=k} \mathbb{P}(i \in S,\, i' \in S) \int_{G_i}\int_{G_{i'}} |x-y|^{-t} \left(\frac{M}{m}\right)^k \left(\frac{M}{m}\right)^k \mathrm{d}\lambda(x)\, \mathrm{d}\lambda(y) =\\ \le \sum_{|i|=|i'|=k} \left(\frac{m}{M}\right)^{2k-l(i,i')} \left(\frac{M}{m}\right)^{2k} \int_{G_i}\int_{G_{i'}} |x-y|^{-t}\, \mathrm{d}\lambda(x)\, \mathrm{d}\lambda(y) =\end{aligned}$$

$$= \sum_{|i|=|i'|=k} \left(\frac{M}{m}\right)^{l(i,i')} \int_{G_i}\int_{G_{i'}} |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y). \quad (5)$$

We denoted the nearest common ancestor of $i$ and $i'$ by $i \wedge i'$, let us also use the brief notation $h \le i \wedge i'$ if $h$ is a common ancestor of $i$ and $i'$. Starting with (5) and then applying Lemma 2.3,

$$\begin{aligned}
\mathbb{E}\, I_t(\mu_k) &\le \sum_{|i|=|i'|=k} \left(\frac{M}{m}\right)^{l(i,i')} \int_{G_i}\int_{G_{i'}} |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y) \\
&= \sum_{l=0}^{k} \left(\frac{M}{m}\right)^{l} \sum_{\substack{h\\|h|=l}} \sum_{\substack{i,i'\\ h=i\wedge i'\\ |i|=|i'|=k}} \int_{G_i}\int_{G_{i'}} |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y) \\
&\le \sum_{l=0}^{k} \left(\frac{M}{m}\right)^{l} \sum_{\substack{h\\|h|=l}} \sum_{\substack{i,i'\\ h\le i\wedge i'\\ |i|=|i'|=k}} \int_{G_i}\int_{G_{i'}} |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y) \\
&= \sum_{l=0}^{k} \left(\frac{M}{m}\right)^{l} \sum_{\substack{h\\|h|=l}} \int_{G_h}\int_{G_h} |x-y|^{-t}\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y)
\end{aligned}$$

$$\le \sum_{l=0}^{k} \left(\frac{M}{m}\right)^{l} \sum_{\substack{h\\|h|=l}} c_t \lambda(G_h)^{2-t} = \sum_{l=0}^{k} \left(\frac{M}{m}\right)^{l} \sum_{\substack{h\\|h|=l}} c_t \left(\frac{1}{M^l}\right)^{2-t} = \sum_{l=0}^{k} \left(\frac{M}{m}\right)^{l} M^l c_t \left(\frac{1}{M^l}\right)^{2-t}$$

$$= \sum_{l=0}^{k} c_t \left(\frac{M^t}{m}\right)^{l} \le \sum_{l=0}^{\infty} c_t \left(\frac{M^t}{m}\right)^{l} \stackrel{\text{def}}{=} c(t,M,m),$$

where $c(t,M,m)$ is finite whenever $\frac{M^t}{m} < 1$, that is, $t < \frac{\log m}{\log M}$.

By Fatou's lemma,

$$\mathbb{E} \liminf_{k\to\infty} I_t(\mu_k) \le \liminf_{k\to\infty} \mathbb{E}\, I_t(\mu_k) \le c(t,M,m),$$

thus $\liminf_{k\to\infty} I_t(\mu_k)$ is almost surely finite.

Since the probability measures $\mu_k$ are supported on the same compact set $K$, every sequence of them has a weakly convergent subsequence. So we can choose a sequence of integers $k_j$ such that

$$\lim_{j\to\infty} I_t(\mu_{k_j}) = \liminf_{k\to\infty} I_t(\mu_k)$$

and that $\mu_{k_j}$ is weakly convergent. Let $\mu = \lim_{j\to\infty} \mu_{k_j}$.

Since $\operatorname{supp} \mu_{k_j} = G^{k_j}$ and $G^0 \supset G^1 \supset G^2 \supset \cdots$, the weak limit $\mu$ is supported on $\bigcap_j G^{k_j} = G$. Applying Lemma 2.2,

$$I_t(\mu) \le \liminf_{j\to\infty} I_t(\mu_{k_j}) = \liminf_{k\to\infty} I_t(\mu_k),$$

which is almost surely finite. Therefore the compact set $G$ almost surely carries a measure $\mu$ with finite $t$-energy, for any $t < \frac{\log m}{\log M}$. Thus the Hausdorff dimension of the set $G$ is at least $\frac{\log m}{\log M}$ almost surely.

By Proposition 3.1, almost surely both of the inequalities $\dim F \le \frac{\log m}{\log M}$ and $\dim G \ge \frac{\log m}{\log M}$ hold. Hence there exists a compact set $G \subset K$ such that $\dim G \ge \frac{\log m}{\log M}$ and $\dim f(G) \le \frac{\log m}{\log M}$.

For $d = 0$ or $d = 1$ the statement of the theorem is trivial, so let $0 < d < 1$ be arbitrary. Let

$$E = \left\{ \frac{\log m}{\log M} : M \ge 3,\ 2 \le m \le M-1 \right\},$$

this is a countable dense set in $(0,1)$. We constructed compact sets $G_e$ for every $e \in E$ such that $G_e$ is of dimension at least $e$ and $f(G_e)$ is of dimension at most $e$. Let $G = \cup_{e<d} G_e$. Clearly $G$ is a Borel set of dimension at least $d$, and $f(G) = \cup_{e<d} f(G_e)$ is of dimension at most $d$. It is well known that $G$ contains compact subsets $C_n$ of dimension at least $d - \frac{1}{n}$, and clearly we can require that $C_n$ have diameter at most $\frac{1}{n}$. Let $C$ be the closure of $\cup_n C_n$, then $C \setminus \cup_n C_n$ is at most one point. Thus $C \subset K$, $\dim C = d$, and clearly $\dim f(C) \le d$ for the compact set $C$, which proves the theorem. ∎

# 5 Generalisation of Theorem 1.2

In this section we shall prove Theorem 1.4. Our first step is to extend Theorem 1.2 in the following way:

**Claim 5.1.** *Let $f : [0,1] \to \mathbb{R}$ be a Borel function. For every $0 \le d \le 1$ there exists a compact set $C \subset \mathbb{R}$ such that $\dim C = d$ and $\dim f(C) \le d \cdot \dim f([0,1])$.*

This is a strengthening of Theorem 1.2 if the image set of $f$ has dimension smaller than 1. To prove Claim 5.1, it is clearly enough to show the following:

**Claim 5.2.** *Suppose that $K \subset \mathbb{R}$ is a compact set of positive Lebesgue measure and $f : K \to \mathbb{R}$ is continuous. For every $0 \le d \le 1$ there exists a compact set $C \subset K$ such that $\dim C = d$ and $\dim f(C) \le d \cdot \dim f(K)$.*

To prove this claim we modify the upper estimate we presented in Section 3.

**Definition 2.** Let $\phi$ be a representation of $M^{<\omega}$. The support of this representation is the set

$$K_\phi = \bigcap_{k=0}^{\infty} \bigcup \{\phi(i) : i \in M^{<\omega}, |i| = k\},$$

and the dimension of the representation is the dimension of $K_\phi$.

Recall that if $S$ is a random $m$-adic subtree of $M^{<\omega}$, then we define the random set

$$F = \bigcap_{k=0}^{\infty} \bigcup \{\phi(i) : i \in S, |i| = k\}.$$

**Proposition 5.3.** *For any representation of $M^{<\omega}$ of dimension $\beta$, the random closed set $F$ defined above has Hausdorff dimension at most $\frac{\log m}{\log M} \cdot \beta$ almost surely.*

*Proof.* The case $\beta = 1$ is already proved in Proposition 3.1, so we may assume that $\beta < 1$ and thus $K_\phi$ (the support of the representation) is a nowhere dense compact and perfect set. Hence considering any infinite branch in $M^{<\omega}$, the diameter of the corresponding intervals tends to zero.

It is easy to see that for each $i \in M^{<\omega}$,

$$\mathbb{P}(i \notin S \text{ and } \phi(i) \cap F \neq \emptyset) = 0,$$

thus

$$\mathbb{P}(\phi(i) \cap F \neq \emptyset) = \mathbb{P}(i \in S) = \left(\frac{m}{M}\right)^{|i|}. \tag{6}$$

Fix any $\beta < t < 1$. Since $\mathcal{H}^t(K_\phi) = 0$, for any $\varepsilon > 0$ we can choose a finite collection of disjoint open intervals $\mathcal{I}$ covering $K_\phi$ such that $\sum_{I \in \mathcal{I}} (\operatorname{diam} I)^t < \varepsilon$, and that each interval $I \in \mathcal{I}$ intersects $K_\phi$.

Fix an $I \in \mathcal{I}$ temporarily. Consider the longest multi-index $i \in M^{<\omega}$ for which $\phi(i) \supset I \cap K_\phi$.

At first let us suppose that $i$ has a child $i_I$ for which $\phi(i_I) \subset I$. Set $l_I = |i|$, thus $|i_I| = l_I + 1$. From (6) we obtain

$$\mathbb{P}(I \cap F \neq \emptyset) \le \mathbb{P}(\phi(i) \cap F \neq \emptyset) = \left(\frac{m}{M}\right)^{l_I}.$$

Now suppose that $i$ has no child $i_I$ for which $\phi(i_I) \subset I$. Then it is easy to check that $i$ has two children $i_1$ and $i_2$ such that

$$I \cap K_\phi \subset \phi(i_1) \cup \phi(i_2) \quad \text{and} \quad \phi(i_j) \cap I \cap K_\phi \neq \emptyset \quad (j = 1, 2). \tag{7}$$

Let $i''_j$ be one of the nearest descendants of $i_j$ for which $\phi(i''_j) \subset I$ holds $(j = 1, 2)$. Let $i'_j$ be the parent of $i''_j$ $(j = 1, 2)$. It is easy to see that

$$\phi(i_j) \cap I \cap K_\phi = \phi(i'_j) \cap I \cap K_\phi \quad (j = 1, 2), \tag{8}$$

since otherwise we have $\phi(i'_j) \subset I$ or $\phi(h) \subset I$ for a sibling $h$ of $i'_j$, contradicting the choice of $i''_j$. By (7) and (8) we obtain

$$I \cap K_\phi \subset \phi(i'_1) \cup \phi(i'_2). \tag{9}$$

Set $l_I = \min(|i'_1|, |i'_2|)$, and let $i_I$ be $i''_j$ $(j = 1$ or $2)$ such that $|i_I| = l_I + 1$. From (9) and (6) we obtain that

$$\mathbb{P}\big(I \cap F \neq \emptyset\big) \le \mathbb{P}\big(\phi(i'_1) \cap F \neq \emptyset \text{ or } \phi(i'_2) \cap F \neq \emptyset\big) \le \left(\frac{m}{M}\right)^{|i'_1|} + \left(\frac{m}{M}\right)^{|i'_2|} \le 2\left(\frac{m}{M}\right)^{l_I}.$$

Thus, for all $I \in \mathcal{I}$, we defined $l_I \in \mathbb{N}$ and $i_I$ of length $l_I + 1$ such that $\emptyset \neq \phi(i_I) \subset I$ and

$$\mathbb{P}(I \cap F \neq \emptyset) \leq 2\left(\frac{m}{M}\right)^{l_I}. \tag{10}$$

Since the intervals $I \in \mathcal{I}$ are disjoint, the nodes $i_I$ form an anti-chain in $M^{<\omega}$; that is, none of them is an ancestor of any other. Thus

$$\sum_{I\in\mathcal{I}} \frac{1}{M^{|i_I|}} \leq 1,$$

hence

$$\sum_{I\in\mathcal{I}} \frac{1}{M^{l_I}} \leq M. \tag{11}$$

Let $s = \frac{\log m}{\log M} \cdot t$, hence $s < t$ and $m^{t/s} = M$. Now cover the random set $F \subset K_\phi$ with those intervals $I \in \mathcal{I}$ which intersect $F$. By (10),

$$\begin{aligned}\mathbb{E}(\mathcal{H}^s_\infty(F)) \leq \sum_{I\in\mathcal{I}} \mathbb{P}(I \cap F \neq \emptyset) \cdot (\operatorname{diam} I)^s \leq \sum_{I\in\mathcal{I}} 2\left(\frac{m}{M}\right)^{l_I} \cdot (\operatorname{diam} I)^s \\ = 2c \sum_{I\in\mathcal{I}} \frac{\left(m^{l_I \cdot t/s} \cdot (\operatorname{diam} I)^t\right)^{s/t}}{cM^{l_I}},\end{aligned} \tag{12}$$

where we choose $c$ so that $\sum_{I\in\mathcal{I}} \frac{1}{cM^{l_I}} = 1$ holds, hence $c \leq M$ by (11). Applying Jensen's inequality to the concave function $x \mapsto x^{s/t}$ and using $m^{t/s} = M$ we have

$$\begin{aligned}2c\sum_{I\in\mathcal{I}} \frac{\left(m^{l_I \cdot t/s} \cdot (\operatorname{diam} I)^t\right)^{s/t}}{cM^{l_I}} \leq 2c\left(\sum_{I\in\mathcal{I}} \frac{m^{l_I\cdot t/s} \cdot (\operatorname{diam} I)^t}{c \cdot M^{l_I}}\right)^{s/t} = 2c\left(\sum_{I\in\mathcal{I}} \frac{1}{c} \cdot (\operatorname{diam} I)^t\right)^{s/t} \\ = 2c^{1-s/t}\left(\sum_{I\in\mathcal{I}} (\operatorname{diam} I)^t\right)^{s/t} \leq 2M\left(\sum_{I\in\mathcal{I}} (\operatorname{diam} I)^t\right)^{s/t} \leq 2M\varepsilon^{s/t} \leq 2M\varepsilon.\end{aligned} \tag{13}$$

Because $\varepsilon$ was arbitrarily small, by (12) and (13) we obtain that $\mathbb{E}(\mathcal{H}^s_\infty(F)) = 0$ for every $s > \beta \cdot \frac{\log m}{\log M}$, since $\beta < t < 1$ was arbitrary. This implies that the dimension of $F$ is at most $\beta \cdot \frac{\log m}{\log M}$ almost surely. □

*Proof of Claim 5.2.* In the proof of Theorem 1.3 in Section 4 we used a representation $\phi$ which had its support in $f(K)$. So that proof with Proposition 5.3 (instead of Proposition 3.1) instantly gives a compact set $C \subset K$ of Hausdorff dimension $d$ such that $f(C)$ has Hausdorff dimension at most $d \cdot \dim f(K)$ (instead of $d$). □

**Claim 5.4.** *Let $A \subset \mathbb{R}$ be compact, $f : A \to \mathbb{R}$ Borel, $\dim A > 0$, and $0 < s < \dim A$. For every $0 \leq d \leq 1$ there exists a Borel set $B \subset A$ such that*

$$\dim B \geq d \cdot s \quad \textit{and} \quad \dim f(B) \leq d \cdot \dim f(A).$$

*Proof.* It is well-known (see e.g. [3]) that for every $s < \dim A$ there exist a probability measure $\nu$ with $\operatorname{supp} \nu \subset A$ and a positive constant $c$ such that for every $x, y \in A$ we have

$$\nu([x, y]) \leq c\,|x - y|^s. \tag{14}$$

Let us define the continuous function $\psi : A \to [0, 1]$ and the Borel function $\chi : [0, 1] \to A$ by setting

$$\psi(x) = \nu\big((-\infty, x]\big),$$
$$\chi(y) = \min\{x : \psi(x) = y\}.$$

Thus $\psi \circ \chi$ is the identity of $[0, 1]$. It is easy to check that (14) implies that for every set $H \subset [0, 1]$,

$$\dim \chi(H) \geq s \cdot \dim H. \tag{15}$$

Apply Claim 5.1 to the Borel function $f \circ \chi : [0, 1] \to \mathbb{R}$. We get that for every $0 \leq d \leq 1$ there exists a compact set $C \subset [0, 1]$ such that

$$\dim C = d \quad \text{and} \quad \dim f(\chi(C)) \leq d \cdot \dim f(A).$$

Put $B = \chi(C)$. (This is a Borel set, since $B = \psi^{-1}(C) \cap \{x \in A : \chi(\psi(x)) = x\}$.) Applying (15),

$$\dim B \geq d \cdot s \quad \text{and} \quad \dim f(B) \leq d \cdot \dim f(A),$$

which proves the claim. □

**Claim 5.5.** *Let $A \subset \mathbb{R}$ be a Borel set and let $f : A \to \mathbb{R}$ be Borel. For every $0 \le d \le 1$ there exists a Borel set $B \subset A$ such that $\dim B = d \cdot \dim A$ and $\dim f(B) \le d \cdot \dim f(A)$.*

*Proof.* We may suppose that $\dim A > 0$. For every sufficiently large positive integer $n$ choose a compact set $A_n \subset A$ of dimension $\ge \dim A - \frac{1}{n}$, and apply Claim 5.4 to $A_n$ and $s = \dim A - \frac{2}{n} > 0$. We obtain a Borel set $B_n \subset A_n$ for which

$$\dim B_n \ge d \cdot (\dim A - \tfrac{2}{n}) \quad \text{and} \quad \dim f(B_n) \le d \cdot \dim f(A_n) \le d \cdot \dim f(A).$$

Now any Borel subset of $\cup_n B_n$ of dimension $d \cdot \dim A$ is an appropriate choice for $B$. $\square$

**Lemma 5.6.** *For each positive integer $n$ there exists a Borel set $B_n \subset \mathbb{R}$ and a Borel bijection $p_n : B_n \to \mathbb{R}^n$ such that for every set $H \subset B_n$ we have*

$$\dim p_n(H) = n \cdot \dim H,$$

*moreover, for every $0 \le d \le 1$ and $H \subset B_n$,*

$$0 < \mathcal{H}^d(H) < \infty \iff 0 < \mathcal{H}^{d \cdot n}(p_n(H)) < \infty.$$

*Proof.* For $x \in \mathbb{R}$ let $d_k(x) \in \{0, 1, \dots, 9\}$ $(k \in \mathbb{Z})$ denote the digits of $x$ in the decimal number system; that is,

$$x = \sum_{k \in \mathbb{Z}} d_k(x) \cdot 10^k,$$

where $d_k(x) = 0$ if $k \ge k_0$ for some $k_0$, and $\liminf_{k \to \infty} d_{-k}(x) \ne 9$. Let

$$B_n = \{x \in \mathbb{R} : \forall j \in \{0, 1, \dots, n-1\} \ \liminf_{i \to \infty} d_{j-ni}(x) \ne 9\},$$

$$p_n^j(x) = \sum_{i \in \mathbb{Z}} d_{j+ni}(x) \cdot 10^i \quad (j \in \{0, 1, \dots, n-1\})$$

and

$$p_n(x) = (p_n^0(x), p_n^1(x), \dots, p_n^{n-1}(x)).$$

Hence $p_n$ is a Borel bijection between $B_n$ and $\mathbb{R}^n$. It is easy to check that $p_n$ satisfies all the requirements, see [4, Theorem 49] and its proof for a hint. $\square$

*Proof of Theorem 1.4.* Suppose that $A \subset \mathbb{R}^n$ is a Borel set and $f : A \to \mathbb{R}^m$ is Borel measurable. Let $d \in [0, 1]$ be arbitrary. Let $p_n$ and $p_m$ be as in Lemma 5.6. Applying Claim 5.5 to the Borel set $p_n^{-1}(A) \subset \mathbb{R}$ and Borel mapping

$$p_m^{-1} \circ f \circ p_n|_{p_n^{-1}(A)} : p_n^{-1}(A) \to p_m^{-1}(f(A))$$

we obtain a Borel set $B \subset p_n^{-1}(A)$ such that

$$\dim B = d \cdot \dim p_n^{-1}(A) \quad \text{and} \quad \dim p_m^{-1} \circ f \circ p_n(B) \le d \cdot \dim p_m^{-1}(f(A)).$$

Using Lemma 5.6 four times we get that

$$\dim p_n(B) = d \cdot \dim(A) \quad \text{and} \quad \dim f(p_n(B)) \le d \cdot \dim f(A)$$

hold for the Borel set $p_n(B) \subset A$. $\square$

Let $\mathcal{B}_n$ denote the $\sigma$-algebra of Borel subsets of $\mathbb{R}^n$. Lemma 5.6 implies that the generalisation of Theorem 1.1 in $\mathbb{R}^n$ holds.

**Theorem 5.7.** *For every $0 \le d_1 < d_2 \le n$ the measure spaces $(\mathbb{R}^n, \mathcal{B}_n, \mathcal{H}^{d_1})$ and $(\mathbb{R}^n, \mathcal{B}_n, \mathcal{H}^{d_2})$ are not isomorphic. Moreover, there does not exist a Borel bijection $f : \mathbb{R}^n \to \mathbb{R}^n$ such that for any Borel set $B \subset \mathbb{R}^n$*

$$0 < \mathcal{H}^{d_1}(B) < \infty \iff 0 < \mathcal{H}^{d_2}(f(B)) < \infty.$$

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