IDEMPOTENT COMPLETION OF TRIANGULATED CATEGORIES

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Abstract. We show that the idempotent completion of a triangulated category has a natural structure of a triangulated category. The idempotent completion of the bounded derived category of an exact category gives the derived category of the idempotent completion. In particular, the derived category of an idempotent complete exact category is idempotent complete.

Introduction

Our article intends to close a literature gap by providing a proof that the idempotent completion (also called pseudo-abelian hull or karoubianisation) of a triangulated category is naturally a triangulated category (Theorem 1.5).

The question of idempotent completing triangulated categories arises in the construction of the derived category of mixed motives ([4, part I, chapter I definition 2.1.6]). In loc.cit part II chapter II 2.4, Levine proves our theorem for certain derived categories.

The second author's motivation for the article lies in the observation that for a ring R the unbounded derived category $D(R) = D(\mathcal{P}(R))$ in the sense of [5] of the category $\mathcal{P}(R)$ of finitely generated projective R-modules is idempotent complete iff $K_{-1}(R) = 0$. In fact, $K_{-1}(R)$ is the Grothendieck group of the idempotent completion of $D(R)$ as a triangulated category wherefore we need to know that the idempotent completion of a triangulated category has a natural triangulation.

An advantage of idempotent complete triangulated categories over arbitrary ones is that whenever they occur as a full triangulated subcategory of a triangulated category they are épaisse in the latter category (see Rickard's criterion [6, proposition 1.3]).

Many natural triangulated categories are idempotent complete as are for instance the derived categories of perfect complexes over a quasi-separated, quasi-compact scheme (see for example [8]). We add another example by proving in Theorem 2.8 that the bounded derived category of an idempotent complete exact category is idempotent complete. This has been established in the split exact (additive) case in [2]. However, the quotient of an idempotent complete triangulated category by an idempotent complete full (hence épaisse) triangulated subcategory leads in general out of the category of idempotent complete triangulated categories. By theorem 1.5, the latter category can be idempotent completed into a triangulated category.

1. Idempotent completion of arbitrary triangulated categories.

1.1. Definition. An additive category K is said to be *idempotent complete* if any idempotent $e : A \to A$, $e^2 = e$, defines a splitting of A:

$$A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e).$$

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1.2. Definition. Let K be an additive category. The *idempotent completion* of K is the category $\tilde{K}$ defined as follows. Objects of $\tilde{K}$ are pairs (A,e) where A is an object of K and $e : A \to A$ is an idempotent. A morphism in $\tilde{K}$ from (A,e) to (B,f) is a morphism $\alpha : A \to B$ in K such that

$$\alpha\, e = f\, \alpha = \alpha .$$

The assignment $A \mapsto (A,1)$ defines a functor ι from K to $\tilde{K}$. The following result is well-known.

1.3. Proposition. *The category $\tilde{K}$ is additive, the functor $\iota : K \to \tilde{K}$ is additive and $\tilde{K}$ is idempotent complete. Moreover, if L is an additive idempotent complete category and if $F : K \to L$ is an additive functor, then, up to natural equivalence, F factors in a unique way through $\iota : K \to \tilde{K}$.*

1.4. Remark. The functor ι is fully faithful and we shall hereafter think of K as a full subcategory of $\tilde{K}$. We will write "$A \in K$" to mean A is isomorphic to an object of K.

1.5. Theorem. *Let K be a triangulated category. Then $\tilde{K}$ is triangulated in such a way that $\iota : K \to \tilde{K}$ is exact and in such a way that for any exact functor $F : K \to L$, where L is a idempotent complete triangulated category, F factors in a unique way, up to natural equivalence, through $\tilde{K}$, that is: there exists an exact functor $\tilde{F} : \tilde{K} \to L$ such that $F = \tilde{F} \circ \iota$ and any other such factorization of F is naturally equivalent to $\tilde{F}$.*

1.6. Definition. Let K be a triangulated category. Let us denote by $T : K \to K$ its translation functor. Define $T : \tilde{K} \to \tilde{K}$ by $T(A,e) = \big(T(A), T(e)\big)$. Clearly $T \circ \iota = \iota \circ T$.

Define a triangle in $\tilde{K}$

$$(\Delta) \qquad A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C \xrightarrow{\ \gamma\ } T(A)$$

to be *exact* when there exist objects A', B', C' of $\tilde{K}$ such that the following triangle is isomorphic in $\tilde{K}$ to the image under ι of an exact triangle of K:

$$A \oplus A' \oplus T^{-1}(C') \xrightarrow{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} B \oplus A' \oplus B' \xrightarrow{\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} C \oplus B' \oplus C' \xrightarrow{\begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(A) \oplus T(A') \oplus C'$$

That is, Δ is exact if it comes from K, up to trivial factors: the above triangle is obtained from Δ by adding the three trivial triangles:

$$A' \xrightarrow{1} A' \longrightarrow 0 \longrightarrow T(A')\ , \qquad 0 \longrightarrow B' \xrightarrow{1} B' \longrightarrow 0 \qquad \text{and} \qquad T^{-1}(C') \longrightarrow 0 \longrightarrow C' \xrightarrow{1} C'.$$

1.7. Theorem. *With the above collection of exact triangles, $\tilde{K}$ is a triangulated category.*

1.8. Proof. We have to check the four axioms of [9, Chap. II, Definition 1.1.1, pp. 93-94].

(TR1). Any triangle isomorphic to an exact triangle is exact directly from the definition. If A is an object of $\tilde{K}$, there exists A' such that $A \oplus A' \in K$ (namely, if $A = (B,e)$ take $A' = (B, 1-e)$ and check that $A \oplus A' = \iota(B)$). Then exactness of $A \oplus A' \xrightarrow{1} A \oplus A' \longrightarrow 0 \longrightarrow T(A) \oplus T(A')$ in K insures exactness of $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow T(A)$ in $\tilde{K}$, by definition. We still have to check that any morphism fits into an exact triangle.

Let $\alpha : A \to B$ be a morphism in $\tilde{K}$. Let A' and B' be such that $A \oplus A' \in K$ and $B \oplus B' \in K$. Let

$$(1) \qquad A \oplus A' \xrightarrow{\ a\ } B \oplus B' \xrightarrow{\ a_1\ } D \xrightarrow{\ a_2\ } T(A \oplus A')$$

be an exact triangle in K, where $a = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$. Now, using (TR3) in K, we complete the following left commutative square into a morphism of exact triangles in K:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{a} & B \oplus B' & \xrightarrow{a_1} & D & \xrightarrow{a_2} & T(A \oplus A') \\
\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow {\scriptstyle p} & & \downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \\
A \oplus A' & \xrightarrow[a]{} & B \oplus B' & \xrightarrow[a_1]{} & D & \xrightarrow[a_2]{} & T(A \oplus A')
\end{array}$$

Of course, p^2 also makes the above diagram commute and so the difference $h := p^2 - p$ has trivial square:

$$h^2 = 0.$$

This is quite classical but let us remind the proof to our reader: from $h\, a_1 = (p^2 - p)\, a_1 = 0$, we can factor h through a_2, *i.e.* there exists $\bar{h} : T(A \oplus A') \to D$ such that $h = \bar{h}\, a_2$ and then $h^2 = \bar{h}\, a_2\, h = 0$ since $a_2\, h = a_2\, (p^2 - p) = 0$.

Applying the trick of lifting idempotents, we set

$$q = p + h - 2\, p\, h.$$

Observe that p and h commute. From $h^2 = 0$, we get $q^2 = p^2 + 2\, p\, h - 4 p^2\, h$ and then replacing p^2 by $p + h$, we have $q^2 = p + h + 2\, p\, h - 4\, p\, h = q$, using again $h^2 = 0$. Clearly, q can replace p in the above diagram, since $h\, a_1 = 0$ and $a_2\, h = 0$. By our computation, q is an idempotent.

Now, let $C = (D, q)$ and $C' = (D, 1 - q)$. Using the isomorphism $D \simeq C \oplus C'$, the above diagram (with q instead of p) becomes:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{a} & B \oplus B' & \xrightarrow{b} & C \oplus C' & \xrightarrow{c} & T(A \oplus A') \\
\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \\
A \oplus A' & \xrightarrow[a]{} & B \oplus B' & \xrightarrow[b]{} & C \oplus C' & \xrightarrow[c]{} & T(A \oplus A')
\end{array}$$

where b and c are necessarily (for the above diagram to commute) of the form:

$$b = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \qquad \text{and} \qquad c = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

Let us now compare the two following exact triangles in K:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{a} & B \oplus B' & \xrightarrow{b} & C \oplus C' & \xrightarrow{c} & T(A \oplus A') \\
\downarrow & & \downarrow {\scriptstyle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} & & \exists \downarrow {\scriptstyle \begin{pmatrix} * & * \\ * & \epsilon \end{pmatrix}} & & \downarrow \\
0 & \xrightarrow{} & B \oplus B' & \xrightarrow[1]{} & B \oplus B' & \xrightarrow{} & 0
\end{array}$$

and use (TR3) in K to find the above morphism, and in particular a morphism $\epsilon : C' \to B'$ in $\tilde{K}$ such that $\epsilon\, \beta_2 = \mathrm{Id}_{B'}$. Using the idempotent completeness of $\tilde{K}$, we have

$$C' = B' \oplus E$$

because $B' \simeq \mathrm{Im}(\beta_2\, \epsilon)$ and where $E = \mathrm{Ker}(\beta_2\, \epsilon) = \mathrm{Im}(1 - \beta_2\, \epsilon)$. In this decomposition, the morphism $\gamma_2 : C' = B' \oplus E \to T(A')$ becomes $\gamma_2 = (0\ \delta)$ because $\gamma_2 \beta_2 = 0$ and this for a unique morphism $\delta : E \to T(A')$. Putting all this together, the exact triangle (1) becomes:

$$(2) \qquad A \oplus A' \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} \beta_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} C \oplus B' \oplus E \xrightarrow{\begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix}} T(A) \oplus T(A').$$

Now, similarly, use again (TR3) in K to construct:

$$\begin{array}{ccccccc}
A\oplus A' & \longrightarrow & 0 & \longrightarrow & T(A)\oplus T(A') & \xrightarrow{\;1\;} & T(A)\oplus T(A') \\
{\scriptstyle\begin{pmatrix}0&0\\0&1\end{pmatrix}}\downarrow & & \downarrow & & \exists\,\Big\downarrow{\scriptstyle\begin{pmatrix}*&*\\ *&*\\ *&\varphi\end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix}0&0\\0&1\end{pmatrix}} \\
A\oplus A' & \xrightarrow[\begin{pmatrix}\alpha&0\\0&0\end{pmatrix}]{} & B\oplus B' & \xrightarrow[\begin{pmatrix}\beta_1&0\\0&1\\0&0\end{pmatrix}]{} & C\oplus B'\oplus E & \xrightarrow[\begin{pmatrix}\gamma_1&0&0\\0&0&\delta\end{pmatrix}]{} & T(A)\oplus T(A')
\end{array}$$

and in particular a morphism $\varphi : T(A') \to E$ such that $\delta\,\varphi = \mathrm{Id}_{T(A')}$. As before, we decompose

$$E = T(A') \oplus F$$

where $F = \mathrm{Ker}(\varphi\,\delta) = \mathrm{Im}(1-\varphi\,\delta)$, using the construction of $\tilde{K}$. Our exact triangle in K becomes now:

$$A\oplus A' \xrightarrow{\begin{pmatrix}\alpha&0\\0&0\end{pmatrix}} B\oplus B' \xrightarrow{\begin{pmatrix}\beta_1&0\\0&1\\0&0\\0&0\end{pmatrix}} C\oplus B'\oplus T(A')\oplus F \xrightarrow{\begin{pmatrix}\gamma_1&0&0&0\\0&0&1&0\end{pmatrix}} T(A)\oplus T(A') \tag{3}$$

Now, it is easy to conclude that $F = 0$. For this, consider the endomorphism of the above triangle, given by:

$$\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\right).$$

The reader should check that this is an endomorphism. Now, since the two first one are isomorphisms, so is the third, that is $0 : F \to F$ is an isomorphism.

Triangle (3), with F removed, is a particular case of definition 6. Therefore the following triangle is exact in $\tilde{K}$:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta_1} C \xrightarrow{\gamma_1} T(A).$$

(TR2) is direct from the definition.

(TR3). Consider a commutative (left) square and exact triangles in $\tilde{K}$:

$$\begin{array}{ccccccc}
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & T(A) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle T(\alpha)} \\
X & \xrightarrow[x]{} & Y & \xrightarrow[y]{} & Z & \xrightarrow[z]{} & T(X).
\end{array}$$

By definition 1.6, there exists A', B', C', X', Y', Z' in $\tilde{K}$ such that the following triangles are exact in K:

$$\begin{array}{ccccccc}
A\oplus A'\oplus T^{-1}(C') & \xrightarrow{\begin{pmatrix}a&0&0\\0&1&0\\0&0&0\end{pmatrix}} & B\oplus A'\oplus B' & \xrightarrow{\begin{pmatrix}b&0&0\\0&0&1\\0&0&0\end{pmatrix}} & C\oplus B'\oplus C' & \xrightarrow{\begin{pmatrix}c&0&0\\0&0&0\\0&0&1\end{pmatrix}} & T(A)\oplus T(A')\oplus C' \\
\downarrow{\scriptstyle\begin{pmatrix}\alpha&0&0\\0&0&0\\0&0&0\end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix}\beta&0&0\\0&0&0\\0&0&0\end{pmatrix}} & & \exists\,\Big\downarrow{\scriptstyle\begin{pmatrix}\gamma& \\ &*\end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix}T(\alpha)&0&0\\0&1&0\\0&0&0\end{pmatrix}} \\
X\oplus X'\oplus T^{-1}(Z') & \xrightarrow[\begin{pmatrix}x&0&0\\0&1&0\\0&0&0\end{pmatrix}]{} & Y\oplus X'\oplus Y' & \xrightarrow[\begin{pmatrix}y&0&0\\0&0&1\\0&0&0\end{pmatrix}]{} & Z\oplus Y'\oplus Z' & \xrightarrow[\begin{pmatrix}z&0&0\\0&0&0\\0&0&1\end{pmatrix}]{} & T(X)\oplus T(X')\oplus Z'
\end{array}$$

Applying (TR3) for K, we find the above morphism and in particular the morphism $\gamma : C \to Z$ which can be easily verified to fit in a morphism of triangles: (α, β, γ).

So far, we have established that $\tilde{K}$ is a pre-triangulated category, in the sense that it satisfies all the axioms but the octahedron axiom (TR4).

Before establishing the octahedron axiom, we need a couple of lemmas.

1.9. Lemma. *Let L be a pre-triangulated category. Then the direct sum of two exact triangles is exact.*

Confer [9, Corollary 1.2.5] the proof of which does not use the octahedron. □

1.10. Lemma. *Let L be a pre-triangulated category. Suppose that the following triangle is exact:*

$$A \oplus A' \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} T(A) \oplus T(A')$$

then $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ *and* $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} T(A')$ *are exact.*

1.11. Proof of lemma 10. Let us choose exact triangles over u and u':

$$A \xrightarrow{u} B \xrightarrow{x} D \xrightarrow{y} T(A) \qquad \text{and} \qquad A' \xrightarrow{u'} B' \xrightarrow{x'} D' \xrightarrow{y'} T(A')$$

and let us use (TR3) to construct a morphism $f : C \to D$:

$$\begin{array}{ccccccc} A \oplus A' & \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} & B \oplus B' & \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} & C \oplus C' & \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} & T(A) \oplus T(A') \\ \downarrow{\scriptstyle (1 \;\; 0)} & & \downarrow{\scriptstyle (1 \;\; 0)} & & \exists \downarrow{\scriptstyle (f \;\; g)} & & \downarrow{\scriptstyle (1 \;\; 0)} \\ A & \xrightarrow[u]{} & B & \xrightarrow[x]{} & D & \xrightarrow[y]{} & T(A) \end{array}$$

such that $f\,v = x$ and $y\,f = w$ (just ignore g). Similarly, we construct a morphism $f' : C' \to D'$ such that $f'\,v' = x'$ and $y'\,f' = w'$. Now, the direct sum of the two exact triangles over u and u' respectively is exact (Lemma 9). Therefore we have a morphism of exact triangles in L:

$$\begin{array}{ccccccc} A \oplus A' & \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} & B \oplus B' & \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} & C \oplus C' & \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} & T(A) \oplus T(A') \\ \| & & \| & & \downarrow{\scriptstyle \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} & & \| \\ A \oplus A' & \xrightarrow[\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}]{} & B \oplus B' & \xrightarrow[\begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}]{} & D \oplus D' & \xrightarrow[\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}]{} & T(A) \oplus T(A'). \end{array}$$

The "5-lemma" holds as soon as we have a pre-triangulated category and therefore f and f' are isomorphisms. But then, the candidate triangles of the lemma are isomorphic to the ones we chose at the beginning of the proof, respectively with the isomorphisms $(\mathrm{Id}_A, \mathrm{Id}_B, f)$ and $(\mathrm{Id}_{A'}, \mathrm{Id}_{B'}, f')$. □

(TR4) - Octahedron. Let $u : X \to Y$ and $v : Y \to Z$ be two composable morphisms. Let $w = v \circ u$ and chose exact triangles on u, v and w in $\tilde{K}$:

$$(1) \qquad X \xrightarrow{u} Y \xrightarrow{u_1} U \xrightarrow{u_2} T(X)$$

$$(2) \qquad Y \xrightarrow{v} Z \xrightarrow{v_1} V \xrightarrow{v_2} T(Y)$$

$$(3) \qquad X \xrightarrow{w} Z \xrightarrow{w_1} W \xrightarrow{w_2} T(X).$$

Choose A, B and C in $\tilde{K}$ such that $X \oplus A \in K$, $Y \oplus B \in K$ and $Z \oplus C \in K$. Add to (1) the trivial triangles $A \longrightarrow 0 \longrightarrow T(A) \xrightarrow{1} T(A)$ and $0 \longrightarrow B \xrightarrow{1} B \longrightarrow 0$ to obtain the following triangle:

$$X \oplus A \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}} Y \oplus B \xrightarrow{\begin{pmatrix} u_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} U \oplus B \oplus T(A) \xrightarrow{\begin{pmatrix} u_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(X) \oplus T(A). \tag{4}$$

Observe that the first morphism of (4) is in K and therefore fits into an exact triangle of K which is, via ι, an exact triangle of $\tilde{K}$. Those two triangles are isomorphic since $\tilde{K}$ is pre-triangulated. Therefore, (4) is isomorphic to an exact triangle of K.

Similarly, the two following triangles are isomorphic to exact triangles of K:

$$Y \oplus B \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}} Z \oplus C \xrightarrow{\begin{pmatrix} v_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} V \oplus C \oplus T(B) \xrightarrow{\begin{pmatrix} v_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(Y) \oplus T(B) \tag{5}$$

$$X \oplus A \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}} Z \oplus C \xrightarrow{\begin{pmatrix} w_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} W \oplus C \oplus T(A) \xrightarrow{\begin{pmatrix} w_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} T(X) \oplus T(A). \tag{6}$$

Let us put them in an octahedron:

– See (7), Figure –

that can be completed by f and g because it is true in K, which is plain triangulated and then transported by isomorphism to our octahedron. The 0's and 1's appearing in f and g come from the commutativities required by the octahedron axiom. Moreover, we have:

(8) $g_1\, w_1 = v_1$
(9) $w_2\, f_1 = u_2$
(10) $f_1\, u_1 = w_1\, v$
(11) $v_2\, g_1 = T(u)\, w_2$.

From the relation $g\, f = 0$ we obtain:

(12) $g_1\, f_2 + g_2 = 0$
(13) $g_3\, f_1 + f_3 = 0$
(14) $g_3\, f_2 + f_4 + g_4 = 0$.

We shall now use the following endomorphism of $W \oplus C \oplus T(A)$

$$\sigma := \begin{pmatrix} 1 & 0 & -f_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 1 \end{pmatrix}$$

as presented in figure (7), in order to modify our octahedron. Direct computation gives:

$$\begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_3\, f_2 + g_4 + f_4 \\ 0 & 0 & 1 \end{pmatrix} \overset{(14)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly

$$\sigma \cdot \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_3\, f_2 + f_4 + g_4 \\ 0 & 0 & 1 \end{pmatrix} \overset{(14)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which implies that σ is an automorphism with inverse:

$$\sigma^{-1} = \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Let us modify up to isomorphism the foreground triangle of figure (7) by using this automorphism σ to obtain the following candidate triangle:

$$U \oplus B \oplus T(A) \xrightarrow{\sigma f} W \oplus C \oplus T(A) \xrightarrow{g\,\sigma^{-1}} V \oplus C \oplus T(B) \xrightarrow{\begin{pmatrix} T(u_1)\,v_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} T(U) \oplus T(B) \oplus T^2(A) \tag{16}$$

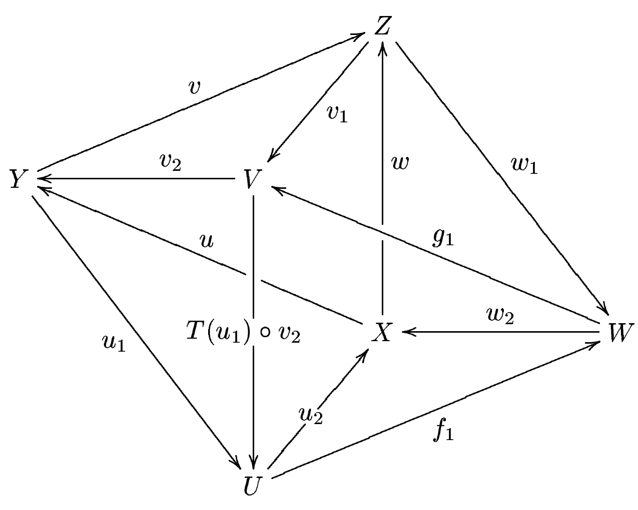
which by its construction is isomorphic to an exact triangle of K. We compute directly

$$\sigma f = \begin{pmatrix} 1 & 0 & -f_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 & f_2 \\ f_3 & 0 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ g_3\,f_1 + f_3 & 0 & g_3\,f_2 + f_4 + g_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by (13) and (14). Similarly, we have:

$$g\,\sigma^{-1} \overset{(15)}{=} \begin{pmatrix} g_1 & 0 & g_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_1 & 0 & g_1\,f_2 + g_2 \\ 0 & 1 & g_3\,f_2 + f_4 + g_4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by (12) and (14). Putting all this together, we obtain the following picture in $\tilde{K}$:



in which all commutativities to be an octahedron are satisfied (use relations 8,9,10,11). The only point is to check that the triangle

$$U \xrightarrow{f_1} W \xrightarrow{g_1} V \xrightarrow{T(u_1)\,v_2} T(U)$$

is exact. But this is immediate from the exact triangle (16), the explicit computations of σf and $g\,\sigma^{-1}$ and from definition 1.6. □

1.12. Remark. It is easy to check that if K was satisfying the enriched version of the octahedron axiom described in [1, Remark 1.1.13, p. 25-26], then so does $\tilde{K}$. This is left to the reader.

1.13. Proof of theorem 1.5. This is now an easy consequence. Clearly, by construction of the triangulation on $\tilde{K}$, the functor $\iota : K \to \tilde{K}$ is exact. Existence and uniqueness of the factorization comes from the additive case and we are left with proving that $\tilde{F}$ is exact.

Let Δ be an exact triangle in $\tilde{K}$. By definition 1.6, there exist (trivial) exact triangles Δ_1, Δ_2, Δ_3, (which are necessarily mapped by $\tilde{F}$ to trivial exact triangles of L), and an exact triangle Δ' in K such that

$$\iota(\Delta') \simeq \Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_3$$

in $\tilde{K}$. Applying $\tilde{F}$, which is additive, to this isomorphism we obtain an isomorphism in L and the left hand side is $F(\Delta')$ which is exact by hypothesis. It suffices to apply lemma 1.10 to the right hand side to obtain that $\tilde{F}(\Delta)$ is exact which is the claim. □

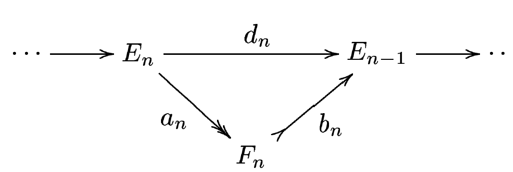
2. Derived categories of idempotent complete exact categories.

The fact that the localization of certain triangulated categories (e.g. perfect complexes) are not idempotent complete forced Thomason in [8] to introduce negative K-theory of schemes. It is therefore an important problem to decide whether a given triangulated category is idempotent complete or not (see also our introduction). The content of this section is to prove that the bounded derived category of an idempotent complete exact category is idempotent complete. For instance, the bounded derived category of an abelian category is idempotent complete (See Corollary 2.10).

2.1. Background. For the basic notion of exact categories and their derived categories, the reader is referred to Keller [3] or to Neeman [5]. Let us recall shortly what we will need hereafter.

(1) An *exact category* is an additive category $\mathcal{E}$ with a collection of *exact sequences* $\{E \rightarrowtail F \twoheadrightarrow G\}$ where the first morphisms $E \rightarrowtail F$ appearing in those exact sequences are called *admissible monomorphisms* and the second ones *admissible epimorphisms.* They have to satisfy a couple of very natural axioms:
   (1) If $E \rightarrowtail F \twoheadrightarrow G$ is an exact sequence of $\mathcal{E}$, then $E \rightarrowtail F$ is a kernel of $F \twoheadrightarrow G$ and $F \twoheadrightarrow G$ is a cokernel of $E \rightarrowtail F$.
   (2) Any split sequence $E \to E \oplus F \to F$ (with usual maps) is exact. Any sequence isomorphic to an exact sequence is exact.
   (3) Admissible monomorphisms are closed under composition and push-out along any morphism. Admissible epimorphisms are closed under composition and pullback along any morphism.

   The additional "obscure" axiom invoked by Quillen in the original version is known to be superfluous. In order to avoid set theoretical difficulties we suppose our exact categories to be small.
(2) Any exact category $\mathcal{E}$ can be embedded as a full subcategory in an abelian category $\mathcal{A}$ in such a way that a sequence in $\mathcal{E}$ is exact iff it is exact in $\mathcal{A}$. If $\mathcal{E}$ is idempotent complete (or even less), this embedding can be chosen in a way that any map in $\mathcal{E}$ which becomes an epimorphism in $\mathcal{A}$ was already an admissible epimorphism in $\mathcal{E}$. For further details we refer the reader to appendix A of [8].
(3) For an exact category $\mathcal{E}$, we let $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ be the category whose objects are bounded resp. bounded below complexes with homological indexing and whose morphisms are chain maps up to chain homotopy. These are triangulated categories. An *acyclic complex* is a complex E_* whose differentials decompose as



   in such a way that $F_n \overset{b_n}{\rightarrowtail} E_{n-1} \overset{a_{n-1}}{\twoheadrightarrow} F_{n-1}$ is an exact sequence of $\mathcal{E}$ for all $n \in \mathbb{Z}$. A map of chain complexes is a *quasi-isomorphism* if its cone is homotopy equivalent to an acyclic complex.

The full subcategory of $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ consisting of acyclic complexes is a full triangulated subcategory of $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ (see [5, lemma 1.1]). The *bounded derived category* $\mathrm{D_b}(\mathcal{E})$ resp. *bounded below derived category* $\mathrm{D}_+(\mathcal{E})$ of $\mathcal{E}$ is the quotient of the triangulated category $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ by the full triangulated subcategory of acyclic complexes, i.e. it is the localization of $\mathcal{K}_b(\mathcal{E})$ resp. of $\mathcal{K}_+(\mathcal{E})$ with respect to quasi-isomorphisms. The respective localizations are obtained by a calculus of fractions. Furthermore, $\mathrm{D_b}(\mathcal{E})$ is a full triangulated subcategory of $\mathrm{D}_+(\mathcal{E})$.

If the exact category $\mathcal{E}$ is idempotent complete then a complex E_* is acyclic iff it has trivial homology computed in the ambient abelian category of 2.1 (2), a quasi-isomorphism is a chain map whose cone is acyclic, the triangulated subcategories of acyclic complexes are isomorphism closed and épaisses (see [5, lemma 1.2]), and the set of quasi-isomorphisms is saturated.

2.2. Lemma. *Let K be a small triangulated category. If $K_0(\tilde{K}) = 0$ then K is idempotent complete.*

2.3. Proof. This follows easily from Landsburg's criterion ([7, theorem 2.4]) identifying the objects of a triangulated category which give the same class in K_0, see also [7, theorem 2.1]. □

2.4. Lemma. *Let $\mathcal{E}$ be an exact category. The derived category of bounded below complexes $\mathrm{D}_+(\mathcal{E})$ is idempotent complete.*

2.5. Proof. By the previous lemma, we only have to check that $K_0(\widetilde{\mathrm{D}_+(\mathcal{E})}) = 0$. The functor

$$S = \bigoplus_{k\geq 0} T^{2k} : \mathrm{D}_+(\mathcal{E}) \to \mathrm{D}_+(\mathcal{E})$$

is well defined on bounded below complexes and chain maps. It passes to the derived category and prolongs to a functor $S : \widetilde{\mathrm{D}_+(\mathcal{E})} \to \widetilde{\mathrm{D}_+(\mathcal{E})}$. There is a natural equivalence $T^2 \circ S \oplus id \simeq S$. Since the functor T^2 induces the identity on K_0, it follows that $K_0(S) + K_0(id) = K_0(S)$, so $K_0(id) = 0$, hence $K_0(\widetilde{\mathrm{D}_+(\mathcal{E})}) = 0$. This is a variant of the usual "Eilenberg swindle". □

2.6. Lemma. *Let $\mathcal{E}$ be an idempotent complete exact category.*

(1) *Let $M = \cdots \to E_n \xrightarrow{d_n} E_{n-1} \longrightarrow \cdots$ be a (not necessarily bounded) acyclic complex. Then d_n has a kernel in $\mathcal{E}$ for all $n \in \mathbb{Z}$ and the truncation*

$$\sigma_{\leq n}(M) = \cdots \longrightarrow 0 \longrightarrow \ker(d_n) \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots$$

is a complex in $\mathcal{E}$ which is quasi-isomorphic to M by the natural map $M \to \sigma_{\leq n}(M)$.

(2) *Let $s : L \to N$ be a morphism of complexes of $\mathcal{E}$. Suppose that s is a quasi-isomorphism. Suppose further that one of L or N is bounded above. Call the other one X, then X has the following property: for some $n_0 \in \mathbb{Z}$ all its boundary maps d_n have a kernel in $\mathcal{E}$ for $n \geq n_0$. In particular, $\sigma_{\leq n_0}X$ is a complex in $\mathcal{E}$ and the natural chain map $X \to \sigma_{\leq n_0}X$ is a quasi-isomorphism.*

2.7. Proof. (1). This follows from the definition of M acyclic, see 2.1. (3).

(2). Apply (1) to the mapping cone of s which is acyclic by idempotent completeness of $\mathcal{E}$ (see 2.1. (3)) and observe that for n big enough the boundary of this mapping cone is the boundary of the one of the two complexes which is not supposed bounded above. For the last statement use the description of quasi-isomorphisms for idempotent complete exact categories given in 2.1. (3). □

2.8. Theorem. *Let $\mathcal{E}$ be an idempotent complete exact category. Then $\mathrm{D_b}(\mathcal{E})$ is idempotent complete.*

2.9. Proof. Since $\mathcal{E}$ is idempotent complete, there exists a embedding $\mathcal{E} \hookrightarrow \mathcal{A}$ as described in point 2.1, part (2).

Let (M, p) be a bounded complex equipped with an idempotent $p = p^2 : M \to M$ in $\mathrm{D_b}(\mathcal{E})$. By lemma 2.2, there exist two bounded below complexes $P, Q \in \mathrm{D}_+(\mathcal{E})$ and an isomorphism in $\mathrm{D}_+(\mathcal{E})$

$$f : M \xrightarrow{\sim} P \oplus Q$$

such that p becomes $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $P \oplus Q$. The isomorphism f can be represented by a fraction of quasi-isomorphism:

$$M \xrightarrow{s} R \xleftarrow{t} P \oplus Q$$

for some bounded below complex $R = \cdots \longrightarrow R_n \xrightarrow{d_n} R_{n-1} \longrightarrow \cdots$. Here we use that the set of quasi-isomorphisms is saturated (see 2.1. (3)). Applying lemma 2.6 (2) to M, R and s we have that $\ker d_n \in \mathcal{E}$ for n greater or equal than some $n_0 \in \mathbb{Z}$. Now apply the same lemma to $P \oplus Q$, $\sigma_{\leq n_0}(R)$ and the quasi-isomorphism $P \oplus Q \xrightarrow{\sim} R \xrightarrow{\sim} \sigma_{\leq n_0}(R)$. This proves that for n big enough, the kernel of the boundary map of $P \oplus Q$ is in $\mathcal{E}$. But in the abelian category $\mathcal{A}$, this kernel is the sum of the corresponding kernels in P and Q. Since $\mathcal{E}$ is idempotent complete, it forces both of them to be in $\mathcal{E}$.

Put $\tilde{P}$ and $\tilde{Q}$ to be the corresponding truncations. Then we have an isomorphism

$$M \simeq \tilde{P} \oplus \tilde{Q}$$

in $\mathrm{D_b}(\mathcal{E})$ which carries the idempotent p to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $\tilde{P} \oplus \tilde{Q}$, due to the functoriality of the truncations described above. □

2.10. Corollary. *Let $\mathcal{A}$ be an abelian category. Then $\mathrm{D_b}(\mathcal{A})$ is idempotent complete.*

2.11. Proof. An abelian category is idempotent complete. □

2.12. Corollary. *Let $\mathcal{E}$ be an exact category. Then* $\widetilde{\mathrm{D_b}(\mathcal{E})} \cong \mathrm{D_b}(\widetilde{\mathcal{E}})$.

2.13. Proof. The inclusion of exact categories $\mathcal{E} \subset \widetilde{\mathcal{E}}$ induces a full inclusion of triangulated categories $\mathrm{D_b}(\mathcal{E}) \subset \mathrm{D_b}(\widetilde{\mathcal{E}})$. It is easily seen that every object of $\mathrm{D_b}(\widetilde{\mathcal{E}})$ is a direct summand of an object of $\mathrm{D_b}(\mathcal{E})$. The corollary then follows from theorem 2.8. □

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