DELOOPING THE K-THEORY OF EXACT CATEGORIES

MARCO SCHLICHTING

ABSTRACT. We generalize, from additive categories to exact categories, the concept of "Karoubi filtration" and the associated homotopy fibration in algebraic K-theory. As an application, we construct for any idempotent complete exact category $\mathcal{E}$ an exact category $\mathcal{S}\mathcal{E}$ such that $K(\mathcal{E}) \simeq \Omega K(\mathcal{S}\mathcal{E})$.

INTRODUCTION

In [Qui73] Quillen defined higher algebraic K-groups of an exact category $\mathcal{E}$ as homotopy groups of a topological space $K(\mathcal{E}) = \Omega|Q\mathcal{E}|$. It turned out that $K(\mathcal{E})$ is an infinite loop space (see for instance [Seg74], [May72], [Wal85]). Since then K-theory has been regarded as taking values in (-1-connected) spectra rather than spaces.

As far as the K-theory of rings is concerned (for additive categories see [PW89]), Wagoner [Wag72] gave another proof of the infinite loop space structure by showing $\mathrm{BGL}(R)^+ \times K_0(R) \simeq \Omega\mathrm{BGL}(SR)^+$ where SR is the suspension ring of R. The spectrum $\{K(R), K(SR), K(S^2R), ...\}$ obtained in this way is non-connective, in general, and has as negative homotopy groups the negative K-groups introduced by Bass [Bas68] and Karoubi [Kar70].

The purpose of this article is to generalize the above results to exact categories. We construct for any exact category $\mathcal{E}$ an exact category $S\mathcal{E}$, called the suspension of $\mathcal{E}$, such that the K-theory space $K(\tilde{\mathcal{E}})$ of the idempotent completion $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has the same homotopy type as $\Omega K(S\mathcal{E})$. This answers part of problem 5 of the Lake Louise Problem Session [JS89].

We obtain an Ω-spectrum $\mathbb{K}(\mathcal{E})$ by setting $\mathbb{K}(\mathcal{E})_n = K(\widetilde{S^n\mathcal{E}})$. Its positive homotopy groups are the Quillen K-groups of $\mathcal{E}$. Its negative homotopy groups are the negative K-groups of the exact category $\mathcal{E}$ as defined in [Sch]. They coincide with Bass' negative K-groups of a ring R [Bas68] when $\mathcal{E}$ is the category of finitely generated projective R modules; with Karoubi's negative K-groups [Kar70] when $\mathcal{E}$ is a split exact category; with Thomason's negative K-groups [TT90] of a quasi-compact and separated scheme X which admits an ample family of line bundles when $\mathcal{E}$ is the exact category of vector bundles of finite rank over X [Sch, 7.1].

In section 1 we introduce the concept of "left s-filtering" and "right s-filtering" (Definitions 1.3, 1.5). These are conditions on an exact subcategory $\mathcal{A}$ of an exact category $\mathcal{U}$ which enable us to construct a quotient exact category $\mathcal{U}/\mathcal{A}$ (Proposition 1.16). Our concept is a generalization of the concept of "Karoubi filtration" [Kar70], [PW89], [CP97].

In section 2, we show that if $\mathcal{A}$ is an idempotent complete, right (or left) s-filtering subcategory of an exact category $\mathcal{U}$, then there is a homotopy fibration of K-theory spaces (Theorem 2.1)

$$K(\mathcal{A}) \to K(\mathcal{U}) \to K(\mathcal{U}/\mathcal{A}).$$

The homotopy fibration is also induced by an exact sequence of triangulated categories (Proposition 2.6) which implies that the homotopy fibration extends to a homotopy fibration of non-connective $\mathbb{K}$-theory spectra (Theorem 2.10)

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A}),$$

Date: July 1, 2003.
1991 *Mathematics Subject Classification.* 18F25, 55P47.
Key words and phrases. algebraic K-theory, lower algebraic K-theory, infinite loop spaces.

incorporating negative K-groups as defined in [Sch].

In section 3, we construct for any (idempotent complete) exact category $\mathcal{E}$, a left s-filtering embedding $\mathcal{E} \subset \mathcal{F}\mathcal{E}$ of $\mathcal{E}$ into an exact category $\mathcal{F}\mathcal{E}$ such that $K(\mathcal{F}\mathcal{E})$ and $I\!\!K(\mathcal{F}\mathcal{E})$ are contractible (Lemma 3.2). Defining the suspension $\mathcal{S}\mathcal{E}$ to be the quotient exact category $\mathcal{F}\mathcal{E}/\mathcal{E}$ (Definition 3.3), we obtain homotopy equivalences $K(\mathcal{E}) \simeq \Omega K(\mathcal{S}\mathcal{E})$ when $\mathcal{E}$ is idempotent complete, and $I\!\!K(\mathcal{E}) \simeq \Omega I\!\!K(\mathcal{E})$ in general (Theorem 3.4).

1. Filtering subcategories and quotient exact categories

1.1. *Exact categories.* We recall from [Kel96, 4] the definition of an exact category. A sequence $X \to Y \to Z$ in an additive category is called an *exact pair* if $X \to Y$ is a kernel for $Y \to Z$, and $Y \to Z$ is a cokernel for $X \to Y$. In an exact pair, the map $X \to Y$ is called inflation, and the map $Y \to Z$ is called deflation. An *exact category* is an additive category equipped with a class of exact pairs, called conflations, satisfying the axioms Ex0-Ex2$^{\mathrm{op}}$ below.

Ex0 The identity morphism of the zero object is a deflation.
Ex1 A composition of two deflations is a deflation.
Ex1^{op} A composition of two inflations is an inflation.
Ex2 The pull-back of a deflation exists and is a deflation.
Ex2^{op} The push-out of an inflation exists and is an inflation.

These axioms are equivalent to Quillen's original axioms in [Qui73] (cf. [Kel90, appendix]). In [Qui73] inflations were called admissible monomorphisms deflations were called admissible epimorphisms deflations, and conflations were called short exact sequences.

Let $\mathcal{U}$ be an exact category, and let $\mathcal{A} \subset U$ be an extension closed full subcategory. Unless otherwise stated, $\mathcal{A}$ will then be equipped with the induced exact structure. It is the exact category with conflations those sequences in $\mathcal{A}$ which are also conflations in $\mathcal{U}$. One checks that $\mathcal{A}$, equipped with this class of conflations, satisfies the axioms for an exact category.

1.2. *The embedding $\mathcal{E} \subset \mathrm{Lex}\mathcal{E}$.* Let $\mathcal{E}$ be a small exact category and let $\mathrm{Lex}\mathcal{E}$ be the category of left exact additive functors from $\mathcal{E}^{op}$ to $< ab >$, the category of abelian groups. We identify $\mathcal{E}$ with the representable functors via the Yoneda embedding. The category $\mathrm{Lex}\mathcal{E}$ is a Grothendieck abelian category with $\mathrm{Ob}\mathcal{E}$ as generating set of small objects. The Yoneda embedding $\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ is exact, reflects conflations, and is closed under extensions.

If $\mathcal{E}$ is idempotent complete, then the inclusion $\mathcal{E} \subset \mathrm{Lex}\mathcal{E}$ is also closed under kernels of surjections. In particular, given two composable maps f, g in $\mathcal{E}$, if $f \circ g$ is a deflation in $\mathcal{E}$ then so is f.

For details we refer the reader to [Kel90, Appendix A], [TT90, Appendix A].

1.3 Definition. Let $\mathcal{U}$ be an exact category, and let $\mathcal{A} \subset \mathcal{U}$ be an extentsion closed full subcategory. Then the inclusion $\mathcal{A} \subset \mathcal{U}$ is called *right filtering*, and $\mathcal{A}$ is called *right filtering in* $\mathcal{U}$, if

(1) $\mathcal{A}$ is closed under taking admissible subobjects and admissible quotients in $\mathcal{U}$ and if
(2) every map $U \to A$ from an object U of $\mathcal{U}$ to an object A of $\mathcal{A}$ factors through an object B of $\mathcal{A}$ such that the arrow $U \to B$ is a deflation:

$$\begin{array}{ccc} U & \xrightarrow{\ \forall\ } & A \\ & \searrow & \uparrow \\ & & \exists\, B. \end{array}$$

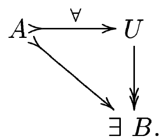
The inclusion $\mathcal{A} \subset \mathcal{U}$ is called *left filtering* if $\mathcal{A}^{op}$ is right filtering in $\mathcal{U}^{op}$.

1.4 *Remark.* Condition (2) already implies that $\mathcal{A} \subset \mathcal{U}$ is closed under taking admissible subobjects.

1.5 Definition. Let $\mathcal{U}$ be an exact category, and let $\mathcal{A} \subset \mathcal{U}$ be an extension closed full subcategory. An inflation $A \to U$ from an object A of $\mathcal{A}$ to an object U of $\mathcal{U}$ is called *special* if there is a deflation $U \to B$ to an object B of $\mathcal{A}$ such that the composition $A \to B$ is an inflation.

The inclusion $\mathcal{A} \subset \mathcal{U}$ is called *right s-filtering* if it is right filtering and if every inflation $A \to U$ from an object A of $\mathcal{A}$ to an object U of $\mathcal{U}$ is special:

(1.6)



The category $\mathcal{A}$ is called *left s-filtering* in $\mathcal{U}$ if $\mathcal{A}^{op}$ is right s-filtering in $\mathcal{U}^{op}$.

1.7 *Example.* Let $\mathcal{A}$ be a Serre subcategory of an abelian category $\mathcal{U}$. Then $\mathcal{A}$ is right and left filtering in $\mathcal{U}$.

1.8 *Example.* Any filtering subcategory in the sense of Karoubi [Kar70] or Pedersen-Weibel [PW89] is left and right s-filtering. This is because split inflations (and split deflations) fit into a diagram 1.6 (dual of 1.6) as we can take $A = B$.

1.9 *Example.* Let R be a noetherian ring. Then the category of left R-modules of finite type is a left s-filtering subcategory of the category of all left R-modules. The filtering property is clear as the inclusion is Serre. For a surjective map $U \to M$ to a finitely generated module M, we lift a finite set of generators for M to U. This generates a finitely generated submodule of U which still surjects onto M.

1.10 *Example.* (compare [Gra76, p. 233], [Car80]). Let R be a ring with unit, and let $S \subset R$ be a multiplicative set of central non zero divisors in R. Let $\mathcal{H}_S(R)$ be the exact category of finitely presented S-torsion left R-modules of projective dimension at most 1. It is an extension closed full subcategory of the category of all left R-modules, and we therefore consider it as an exact category (1.1). Let $\mathcal{P}^1(R)$ be the full subcategory of R-modules whose objects M fit into an exact sequence of R modules $0 \to P \to M \to H \to 0$ with P finitely projective and $H \in \mathcal{H}_S(R)$ One checks that $\mathcal{P}^1(R)$ is closed under extensions in the category of all left R-modules. This gives $\mathcal{P}^1(R)$ the structure of an exact category. Of course, $\mathcal{H}_S(R) \subset \mathcal{P}^1(R)$, and we claim that $\mathcal{H}_S(R)$ is right s-filtering in $\mathcal{P}^1(R)$.

Proof. Let $M \to T$ be a map from an object M of $\mathcal{P}^1(R)$ to an object T of $\mathcal{H}_S(R)$. There is a finitely generated projective module P_0 which contains a finitely generated projective module P_1 with quotient P_0/P_1 isomorphic to T. Furthermore, M has a finitely generated projective submodule P with quotient M/P in $\mathcal{H}_S(R)$. Since P is projective, the induced map $P \to T$ lifts to P_0. Then $X = P \times_{P_0} P_1$ is a subobject of P. Since $S^{-1}R$ is flat over R, the quotient P/X is an S-torsion R-module which, as a quotient of P, is finitely generated. Hence there is an $s \in S$ with $s(P/X) = 0$. This means that $X = P \times_{P_0} P_1$ contains sP. The R-modules sP is isomorphic to P by the injectivity of the multiplication with s and is therefore finitely generated and projective. The quotient M/sP is an object of $\mathcal{H}_S(R)$ since, by the snake lemma, it is an extension of P/sP and M/P. The map $M \to T$ factors through M/sP. This proves the right filtering condition. Given a $\mathcal{P}^1(R)$-inflation $T \to M$, its cokernel $M' = M/T$ lies in $\mathcal{P}^1(R)$ and contains therefore a finitely generated projective R-submodule P such that the quotient M'/P lies in $\mathcal{H}_S(R)$. The inclusion $P \subset M'$ lifts to an inclusion $P \subset M$ whose cokernel M/P is an extension of T and M'/P, and hence lies in $\mathcal{H}_S(R)$. The diagram involving T, M and M/P shows that the inflation $T \to M$ is special. □

The quotient category $\mathcal{P}^1(R)/\mathcal{H}_S(R)$ as defined in 1.14 is equivalent to the category of finitely generated projective $S^{-1}R$-modules which are localizations of finitely generated projective R modules (exercise!). The inclusion $R-\text{proj} \subset \mathcal{P}^1(R)$ satisfies resolution, and thus induces a K-theory equivalence.

Theorem 2.10 yields the well-known long exact sequence for $i \in \mathbb{Z}$ [Gra76], [Car80]

$$K_i(\mathcal{H}_S(R)) \to K_i(R) \to K_i(S^{-1}R) \to K_{i-1}(\mathcal{H}_S(R)) \to K_{i-1}(R).$$

1.11 *Example.* Let $\mathcal{E}$ be an exact category. In section 3 we construct a left s-filtering embedding $\mathcal{E} \subset \mathcal{F}\mathcal{E}$ into an exact category $\mathcal{F}\mathcal{E}$ whose K-theory space is contractible.

1.12 Definition. Let $\mathcal{U}$ be a small exact category and $\mathcal{A} \subset \mathcal{U}$ an extension closed full subcategory. A $\mathcal{U}$-morphism is called a *weak isomorphism* if it is a (finite) composition of inflations with cokernel in $\mathcal{A}$ and deflations with kernel in $\mathcal{A}$. We write $\Sigma_{\mathcal{A}\subset\mathcal{U}}$ for the set of weak isomorphisms.

1.13 Lemma. *If $\mathcal{A}$ is left filtering in $\mathcal{U}$, then the set of weak isomorphisms $\Sigma_{\mathcal{A}\subset\mathcal{U}}$ admits a calculus of left fractions. Dually, if $\mathcal{A}$ is right filtering in $\mathcal{U}$ then the set of weak isomorphisms $\Sigma_{\mathcal{A}\subset\mathcal{U}}$ admits a calculus of right fractions.*

Proof. We will prove the lemma in the left filtering case. We have to verify conditions a)-d) of [GZ67, I.2.2, p.12]. By definition, all identity morphisms are in $\Sigma_{\mathcal{A}\subset\mathcal{U}}$ and the set of weak isomorphisms is closed under composition. Given a diagram $X' \overset{s}{\leftarrow} X \overset{u}{\to} Y$ with $s \in \Sigma_{\mathcal{A}\subset\mathcal{U}}$, we have to show the existence of a diagram $X' \overset{u'}{\to} Y' \overset{t}{\leftarrow} Y$ with $t \in \Sigma_{\mathcal{A}\subset\mathcal{U}}$ such that $u's = tu$. Proceeding by induction, we only have to show the claim for s an inflation with cokernel in $\mathcal{A}$ and for s a deflation with kernel in $\mathcal{A}$. In the first case we take the pushout of s along u. In the second case, the map from the kernel of s to Y factors through an admissible subobject $A \in \mathcal{A}$ of Y (1.3). We set $Y' = Y/A$ and u', t the induced maps. For condition d), we have to show that given a diagram $X' \overset{s}{\to} X \overset{f}{\to} Y$ with $s \in \Sigma_{\mathcal{A}\subset\mathcal{U}}$ and $fs = 0$, there is a $Y \overset{t}{\to} Y' \in \Sigma_{\mathcal{A}\subset\mathcal{U}}$ with $tf = 0$. Again it suffices to proof the claim for s an inflation with cokernel in $\mathcal{A}$ and for s a deflation with kernel in $\mathcal{A}$. In the first case, f factors through $X/X' \in \mathcal{A}$. The map $X/X' \to Y$ factors through an admissible subobject $A \in \mathcal{A}$ of Y (1.3). We set $Y' = Y/A$ and t the induced map $Y \to Y'$. By construction, we have $tf = 0$. In the second case, f is already zero. □

1.14 Definition. Let $\mathcal{A}$ be an extension closed full subcategory of a small exact category $\mathcal{U}$. We write $\mathcal{U}/\mathcal{A}$ for the category $\mathcal{U}[\Sigma^{-1}_{\mathcal{A}\subset\mathcal{U}}]$ which is obtained from $\mathcal{U}$ by formally inverting the weak isomorphisms. By Lemma 1.13, if $\mathcal{A}$ is left filtering (right filtering) in $\mathcal{U}$, then $\mathcal{U}/\mathcal{A}$ is obtained from $\mathcal{U}$ by a calculus of left fractions (right fractions).

1.15 Definition. Let $\mathcal{A}$ be left or right s-filtering in $\mathcal{U}$. A sequence $X \to Y \to Z$ in $\mathcal{U}/\mathcal{A}$ is called a conflation if it is isomorphic to the image, under the localization functor $\mathcal{U} \to \mathcal{U}/\mathcal{A}$, of a conflation of $\mathcal{U}$.

1.16 Proposition. *Let $\mathcal{A}$ be left or right s-filtering in $\mathcal{U}$. Then the category $\mathcal{U}/\mathcal{A}$, equipped with the set of conflations defined in 1.15, is an exact category. Moreover, exact functors from $\mathcal{U}$ vanishing on $\mathcal{A}$ correspond bijectively to exact functors from $\mathcal{U}/\mathcal{A}$.*

The proof of Proposition 1.16 will occupy the rest of the section. For part 4 of the next lemma we introduce the following notation. Let $\mathcal{A}$ be an exact category. We write $E(\mathcal{A})$ for the exact category of conflations in $\mathcal{A}$. Objects are conflations in $\mathcal{A}$ and maps are commutative diagrams of conflations in $\mathcal{A}$. We have three functors $E(\mathcal{A}) \to \mathcal{A}$ which are evaluation at kernel, evaluation at extension and evaluation at cokernel. A sequence of conflations is a conflation in $E(\mathcal{A})$ if and only if all three evaluations yield conflations in $\mathcal{A}$.

1.17 Lemma. *Let $\mathcal{A}$ be an idempotent complete exact category which is right s-filtering in $\mathcal{U}$.*

(1) *Let $\alpha : A \to B$ be an $\mathcal{A}$-morphism such that there is a $\mathcal{U}$-morphism $\varphi : X \to A$ with $\alpha \circ \varphi$ a deflation. Then α is a deflation.*

(2) *For any diagram $X \overset{u}{\underset{\sim}{\rightarrowtail}} Y \overset{s}{\underset{\sim}{\leftarrowtail}} Z$ of inflations with cokernel in $\mathcal{A}$, there is a diagram of inflations $X \underset{t}{\overset{\sim}{\leftarrowtail}} W \underset{v}{\overset{\sim}{\rightarrowtail}} Z$ with cokernel in $\mathcal{A}$ such that $ut = sv$.*

(3) *For any weak isomorphism s there is an inflation i with cokernel in $\mathcal{A}$ such that $s \circ i$ is also an an inflation with cokernel in $\mathcal{A}$.*
(4) *$E(\mathcal{A})$ is right s-filtering in $E(\mathcal{U})$.*
(5) *For any diagram $X \overset{u}{\rightarrowtail} Y \overset{s}{\underset{\sim}{\leftarrowtail}} Z$ of inflations with* $\operatorname{coker}(s)$ *in $\mathcal{A}$, there is a diagram of inflations $X \underset{t}{\overset{\sim}{\leftarrowtail}} W \underset{v}{\rightarrowtail} Z$ with* $\operatorname{coker}(t)$ *in $\mathcal{A}$ such that $ut = sv$.*
(6) *The set of weak isomorphisms $\Sigma_{\mathcal{A}\subset\mathcal{U}}$ is saturated, i.e., any $\mathcal{U}$-morphism whose image in $\mathcal{U}/\mathcal{A}$ is an isomorphism lies in $\Sigma_{\mathcal{A}\subset\mathcal{U}}$.*
(7) *If in a map $f = (f_0, f_1, f_2)$ of conflations in $\mathcal{U}$, two of the three arrows are weak isomorphisms, then so is the third.*

Proof. For (1), let W be the pull-back of $\alpha \circ \varphi$ along α, and write $r : W \to X$ and $q : W \to A$ for the other two maps in the pull-back diagram. The map α has a kernel in the idempotent completion of $\mathcal{U}$ (1.2) which is also a kernel for r. The map $\varphi : X \to A$ induces a map $s : X \to W$ such that $r \circ s = 1_X$ and $q \circ s = \varphi$. The image of the idempotent $u := 1_W - sr$ of W is a kernel of r. So we have to show that $\operatorname{im}(u)$ is in $\mathcal{A}$. By the right filtering property 1.3, we can write $q \circ u$ as $\gamma \circ p$ with $p : W \to C$ a deflation and C in $\mathcal{A}$. Let $i : \ker(p) \to W$ be a kernel for p. Then $u \circ i = 0$ since W is a pullback and $r \circ u \circ i = r(1_W - sr)i = 0$ and $q \circ u \circ i = \gamma \circ p \circ i = 0$. Therefore, there is a $t : C \to W$ such that $t \circ p = u$. Since $\mathcal{A}$ is idempotent complete, the idempotent $p \circ u \circ t$ of C has image in $\mathcal{A}$. The map $ut : \operatorname{im}(p \circ u \circ t) \to \operatorname{im}(u)$ has inverse pu. Thus $\operatorname{im}(u)$ is in $\mathcal{A}$.

For (2), the cokernels $p : Y \to Y/X$ of u and $r : Y \to Y/Z$ of s have targets Y/X and Y/Z in $\mathcal{A}$. The map $(p, r) : Y \to Y/X \oplus Y/Z$ factors by the right filtering property (1.3) as $(p, r) = (\alpha, \gamma) \circ q$ with $q : Y \to B$ a deflation and B in $\mathcal{A}$. By (1), α and γ are deflations. Let W be a kernel for q. Then the deflations α and γ induce the inflations t and v of the diagram whose cokernels are the kernels of α and γ and lie therefore in $\mathcal{A}$.

For (3), the weak isomorphism s is a composition of inflations with cokernel in $\mathcal{A}$ and deflations with kernel in $\mathcal{A}$. We show the assertion by induction on the number of morphisms of such a decomposition of s. Using (2), we see that it suffices to prove 3 for $s : X \to Y$ a deflation with kernel in $\mathcal{A}$. Let $l : A \to X$ be a kernel of s. Then A is an object of $\mathcal{A}$. Since $\mathcal{A}$ is right s-filtered in $\mathcal{U}$, l is a special inflation. So there are an inflation $j : A \to B$ and a deflation $p : X \to B$ such that $p \circ l = j$. Let $q : B \to C$ denote a cokernel of j, let $r : Y \to C$ denote the map induced by p and let $i : U \to X$ be a kernel for p. Then by the snake lemma r is a deflation and $s \circ i : U \to Y$ is a kernel for r and therefore an inflation with cokernel $C \in \mathcal{A}$.

To prove (4), we first show that any diagram $B_0 \overset{i}{\rightarrowtail} U \overset{\alpha}{\twoheadrightarrow} A_1$ with B_0 and A_1 in $\mathcal{A}$ and $B_0 \to U$ an inflation can be completed into a commutative diagram

(1.18)
$$\begin{array}{ccc} B_0 & \overset{i}{\rightarrowtail} & U \\ \downarrow & \overset{p}{\swarrow} & \downarrow \alpha \\ B_1 & \underset{\beta}{\longrightarrow} & A_1 \end{array}$$

with B_1 in $\mathcal{A}$, $B_0 \to B_1$ an inflation and $U \to B_1$ a deflation. Since i is special, there is a deflation $q : U \to C$ with target in $\mathcal{A}$ such that $q \circ i$ is an inflation. By the filtering property, we can write $(q, \alpha) : U \to C \oplus A_1$ as the composition of a deflation $p : U \to B_1$ and a map $(c, \beta) : B_1 \to C \oplus A_1$. The map $p \circ i$ is an inflation since $c \circ p \circ i = q \circ i$ is and $\mathcal{A}$ is idempotent complete (1.2).

Next we verify that $E(\mathcal{A})$ is right filtering in $E(\mathcal{U})$. Let $f = (f_0, f_1, f_2)$ be a map from the conflation $(j_X, q_X) = (X_0 \to X_1 \to X_2) \in E(\mathcal{U})$ to the conflation $(j_A, q_A) = (A_0 \to A_1 \to A_2) \in E(\mathcal{A})$. By the right filtering property (1.3) we can write f_0 as $\beta_0 \circ r_0$ such that $r_0 : X_0 \to B_0$ is a deflation with target in $\mathcal{A}$. Let U be a push-out of j_X along r_0 and let $i : B_0 \to U$ and $\bar{r} : X_1 \to U$ be the induced maps. The map i is an inflation and $\bar{r}$ is a deflation. The universal property of push-outs gives a map $\alpha : U \to A_1$ making all possible diagrams commute. By the previous paragraph we find B_1, p and β as in diagram (1.18). The morphism $j_B := p \circ i$ is an inflation. Let $q_B : B_1 \to B_2$ be a cokernel for j_B so that $(j_B, q_B) = (B_0, B_1, B_2)$ is a conflation in $\mathcal{A}$. Then we have induced maps $\beta_2 : B_2 \to A_2$ and $r_2 : X_2 \to B_2$. The map r_2 is a deflation, by the snake lemma applied to the map $(1_{B_0}, p, r_2)$ of conflations. Thus $r = (r_0, p \circ \bar{r}, r_2)$ is a deflation in $E(\mathcal{U})$ and $f = (\beta_0, \beta, \beta_2) \circ r$.

It remains to show that inflations $l = (l_0, l_1, l_2)$ in $E(\mathcal{U})$ from $(j_A, q_A) = (A_0 \to A_1 \to A_2) \in \mathcal{E}(\mathcal{A})$ to $(j_U, q_U) = (U_0 \to U_1 \to U_2) \in \mathcal{E}(\mathcal{U})$ are special. Since l_0 is special in $\mathcal{U}$, there is a deflation $p_0 : U_0 \to B_0$ such that $p_0 \circ l_0$ is an inflation. Let C be the push-out of $p_0 l_0$ along j_A and let V be the push-out of p_0 along j_U. The induced map $C \to V$ is an inflation since it is the push-out of the inflation $U_0 \sqcup_{A_0} A_1 \to U_1$ along $U_0 \sqcup_{A_0} A_1 \to C$. Since $C \to V$ is special in $\mathcal{U}$, we find a deflation $V \to B_1$ such that the composition $C \to B_1$ is an inflation. Write p_1 the composition $U_1 \to B_1$ and $p_2 : U_2 \to B_2 = B_1/B_0$ for the induced map on quotients. Then $p = (p_0, p_1, p_2)$ is a deflation in $E(\mathcal{U})$ such that $p \circ l$ is an inflation in $E(\mathcal{A})$.

We show (5). By (4), the map in $E(\mathcal{U})$ from $(X \to Y \to Y/X) \in E(\mathcal{U})$ to $(1, 0) = (Y/Z \to Y/Z \to 0) \in E(\mathcal{A})$ can be written as the composition of a deflation (p_0, p_1, p_2) with target $A = (A_0 \to A_1 \to A_2)$ in $E(\mathcal{A})$ and a map $A \to (Y/Z \to Y/Z \to 0)$ in $E(\mathcal{A})$. Let $t : W \to X$ be a kernel for p_0, and let $v : W \to Z$ be the induced map on kernels. The map t is an inflation with cokernel A_0 in $\mathcal{A}$. The map v is an inflation as it is the composition of the inflation $W = \ker(p_0) \to \ker(p_1)$ (p is a deflation in $E(U)$) and $\ker(p_1) \to Z$. The last map is an inflation because in the diagram $Y \to A_1 \to Y/Z$, all maps are deflations by construction and (1).

We prove (6). We claim that any idempotent $p : U \to U$ in $\mathcal{U}$ which becomes zero in $\mathcal{U}\ [\Sigma^{-1}_{\mathcal{A}\subset\mathcal{U}}]$ possesses an image lying in $\mathcal{A}$. Since p is zero in $\mathcal{U}\ [\Sigma^{-1}_{\mathcal{A}\subset\mathcal{U}}]$ there is an inflation i with cokernel in $\mathcal{A}$ such that $p \circ i = 0$ by (3) and Lemma 1.13. Let $q : U \to A$ be a cokernel for i. We have $A \in \mathcal{A}$. There is a map $s : A \to U$ such that $p = s \circ q$. The map $q \circ p \circ s$ is an idempotent of A whose image is isomorphic to the image of p. Since $\mathcal{A}$ is idempotent complete, $\mathrm{im}\,(p)$ exists and is in $\mathcal{A}$.

Let $f : X \to Y$ be a $\mathcal{U}$-map which becomes an isomorphism in $\mathcal{U}/\mathcal{A}$, and let gs^{-1} be an inverse to f with $s : Z \to Y$ a weak isomorphism and $g : Z \to X$. We have to show that $f \in \Sigma_{\mathcal{A}\subset\mathcal{U}}$. By the calculus of right fractions and (3) we can assume that s is an inflation with cokernel in $\mathcal{A}$ and $f \circ g = s$ in $\mathcal{U}$. Since s is an inflation, the push-out U of the diagram $X \overset{g}{\leftarrow} Z \overset{s}{\to} Y$ exists in $\mathcal{U}$. Call $t : X \to U$ and $h : Y \to U$ the induced maps. The map f induces a retraction $r : U \to Y$ of h such that $f = r \circ t$. Since h and r are isomorphisms in $\mathcal{U}/\mathcal{A}$, the idempotent $p = 1 - hr$ is zero in $\mathcal{U}/\mathcal{A}$. By the previous paragraph, it has an image in $\mathcal{A}$. It follows that r is isomorphic to the canonical projection $Y \oplus \mathrm{im}\,(p) \to Y$, and hence $r \in \Sigma_{\mathcal{A}\subset\mathcal{U}}$. As $f = r \circ t$ and t is an inflation with cokernel $\mathrm{coker}(s)$ in $\mathcal{A}$ we are done.

It remains to show (7). Let $(f_0, f_1, f_2) : (X_0 \to X_1 \to X_2) \to (Y_0 \to Y_1 \to Y_2)$ be a map of conflations in $\mathcal{U}$. Every map of conflations is a composition of maps of conflations with $f_0 = id$ or $f_2 = id$. By (6) we can thus assume $f_0 = id$ or $f_2 = id$.

Suppose f_0 and f_2 are weak isomorphisms. If $f_0 = id$ then f_1 is a pull-back of f_2, and thus is a weak isomorphism. If $f_2 = id$ then f_1 is a push-out of f_1, and thus is a weak isomorphism.

Suppose f_0 and f_1 are weak isomorphisms. By (3), (2) and (6) we can assume f_0 and f_1 to be inflations with cokernel in $\mathcal{A}$. The map f_2 factors over Y_1/X_0. As a push-out of f_1, $X_2 \to Y_1/X_0$ is an inflation with cokernel in $\mathcal{A}$. The map $Y_1/X_0 \to Y_2$ is a deflation with kernel $\mathrm{coker}(f_0) \in \mathcal{A}$ by the snake lemma applied to the map of conflations from $X_0 \to Y_1 \to Y_1/X_0$ to $Y_0 \to Y_1 \to Y_2$. Thus f_2 is a weak isomorphism.

Finally, suppose f_1 and f_2 are weak isomorphisms. We can assume $f_2 = id$. We claim that if a composition $g \circ h$ of maps in $\mathcal{U}$ is a deflation with h a weak isomorphism, then g is a deflation, and the maps on kernels is also a weak isomorphism. This will finish the proof. It suffices to show the claim in two cases, namely when h is an inflation with cokernel in $\mathcal{A}$ and in the case when h is a deflation with kernel in $\mathcal{A}$. In both case, we only have to show that g is a deflation since the map on kernels will then be a weak isomorphism by the snake lemma. In the first case, the kernel of g in $\mathrm{Lex}\mathcal{U}$ is en extension of $\ker(gh)$ and $\mathrm{coker}(h)$. Since $\mathcal{U} \subset \mathrm{Lex}\mathcal{U}$ is closed under extensions, we have $\ker(g) \in \mathcal{U}$ and g is a deflation in $\mathcal{U}$. In the second case, we have to show that $\ker(h) \to \ker(gh)$ is an inflation in $\mathcal{U}$. More generally, we show that if a composition $a \circ b : A \to X$ of maps in $\mathcal{U}$ with $A \in \mathcal{A}$ is an inflation, then b is also an inflation. Since $A \to X$ is special, we can assume X in $\mathcal{A}$. By the right filtering property and by the assumption of $\mathcal{A}$ being idempotent complete, we can further assume that a is a deflation. But then $\mathrm{coker}(b)$ is en extension of $\ker(a)$ and $\mathrm{coker}(ab)$ in $\mathrm{Lex}\mathcal{U}$. Thus $\mathrm{coker}(b)$ is in $\mathcal{U}$. □

Proof of Proposition 1.16. We will give the proof when $\mathcal{A}$ is idempotent complete. The general case follows from Lemma 1.20 below.

First we show that $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ sends a push-out along an inflation to a cocartesian square. Let

$$Z \xleftarrow{f} X \rightarrowtail^{i} Y \tag{1.19}$$

be a diagram in $\mathcal{U}$ with i an inflation, and let $g : Y \to T$, $h : Z \to T$ be maps in $\mathcal{U}$ such that $gi = hf$ holds in $\mathcal{U}/\mathcal{A}$. Using 1.17 (3) and (5), we see that there is a diagram $Z' \stackrel{f'}{\leftarrow} X' \stackrel{i'}{\to} Y'$ in $\mathcal{U}$ with i' an inflation and a map (z, x, y) from this diagram to the diagram (1.19) above such that $gyi' = hzf'$ and such that x, y and z are inflations with cokernel in $\mathcal{A}$ (we can take $z = id_Z$ and $y = id_Y$). Then the map between the push-outs of the two diagrams is a weak isomorphism by 1.17 (7). Using the calculus of fractions with respect to inflations with cokernel in $\mathcal{A}$, this shows the existence part for the push-out of (1.19) to be cocartesian in $\mathcal{U}/\mathcal{A}$. For the uniqueness part, given a $\mathcal{U}$-map (by the calculus of fractions we can restrict to $\mathcal{U}$-maps) from the $\mathcal{U}$-pushout of (1.19) to some object T which is $\mathcal{U}/\mathcal{A}$-trivial on Y and Z. Using the calculus of fractions with respect to inflations with cokernel in $\mathcal{A}$ and 1.17 (5), we find another push-out diagram along an inflation which maps to the push-out diagram obtained from (1.19) via point-wise weak isomorphisms, such that the new push-out maps $\mathcal{U}$-trivially to T. As the map on push-outs is a weak isomorphism (1.17 (7)), this shows the uniqueness part. Thus $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ sends a push-out along an inflation to a cocartesian square.

Moreover, a calculus of right fractions, and hence $\mathcal{U} \to \mathcal{U}/\mathcal{A}$, preserves all cartesian squares. It follows that conflations in $\mathcal{U}/\mathcal{A}$ are exact pairs.

We check the axioms Ex0-Ex2^{op} of 1.1. Axiom Ex0 is obvious.

For Ex1, we have to show that $\beta \circ ts^{-1} \circ \alpha$ is, up to fractions, a deflation if α and β are deflations in $\mathcal{U}$ and s,t are weak isomorphisms. We can assume s and t to be inflations with cokernel in $\mathcal{A}$ (1.17 (3)). Taking the pullback of α along s, we can assume $s = 1$. Let $t : X \to Y$ and let $i : \ker(\beta) \to Y$ be a kernel for $\beta : Y \to Z$. By Lemma 1.17 (5), there are an inflation $r : U \to \ker(\beta)$ with cokernel in $\mathcal{A}$ and an inflation $j : U \to X$ such that $ir = tj$. Write $p : X \to X/U$ for a cokernel of j and $u : X/U \to Z$ for the map induced by t. The map u is a weak isomorphism (1.17 (7)), and $p \circ \alpha$ is a deflation.

Inflations are closed under composition in $\mathcal{U}/\mathcal{A}$. This follows from the calculus of right fractions w.r.t. inflations with cokernel in $\mathcal{A}$ and 1.17 (5). Thus Ex1^{op} holds.

Any diagram (1.19) in $\mathcal{U}/\mathcal{A}$ with i an inflation in $\mathcal{U}/\mathcal{A}$ is isomorphic to the image of a diagram (1.19) in $\mathcal{U}$ with i an inflation in $\mathcal{U}$. As $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ preserves such push-out diagrams, the push-out of an inflation in $\mathcal{U}/\mathcal{A}$ exists and is an inflation, thus Ex2^{op} holds. The dual argument shows that Ex2 also holds. □

Let $\mathcal{U}$ be an exact category and $\mathcal{A} \subset \mathcal{U}$ a full extension closed subcategory. Then the idempotent completion $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is a full extension closed subcategory of $\tilde{\mathcal{U}}$, and we write $\tilde{\mathcal{U}}^{\mathcal{A}} \subset \tilde{\mathcal{U}}$ for the full subcategory of objects $U \in \tilde{\mathcal{U}}$ for which there is an object $A \in \tilde{\mathcal{A}}$ with $U \oplus A \in \mathcal{U}$. It is immediate to see that $\tilde{\mathcal{U}}^{\mathcal{A}}$ is extension closed in $\tilde{\mathcal{U}}$. This makes $\tilde{\mathcal{U}}^{\mathcal{A}}$ into an exact category.

1.20 Lemma. *If $\mathcal{A} \subset \mathcal{U}$ is right (left) s-filtering, then $\tilde{\mathcal{A}} \subset \tilde{\mathcal{U}}^{\mathcal{A}}$ is also right (left) s-filtering. Moreover, the induced functor $\mathcal{U}/\mathcal{A} \to \tilde{\mathcal{U}}^{\mathcal{A}}/\tilde{\mathcal{A}}$ is an equivalence of categories. Under this equivalence, a sequence in $\mathcal{U}/\mathcal{A}$ is a conflation if and only if it is a conflation in $\tilde{\mathcal{U}}^{\mathcal{A}}/\tilde{\mathcal{A}}$.*

Proof. Left to the reader. □

2. A homotopy fibration

Let $\mathcal{E}$ be an exact category. We denote by $K(\mathcal{E})$ the Quillen K-theory space $\Omega|Q\mathcal{E}|$ of $\mathcal{E}$ [Qui73]. Recall [Wal85, 1.9] that it is homotopy equivalent to Waldhausen's K-theory space $\Omega|iS.\mathcal{E}|$.

2.1 Theorem. *Let $\mathcal{A}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathcal{U}$. Then the sequence of exact categories $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of K-theory spaces*

$$K(\mathcal{A}) \to K(\mathcal{U}) \to K(\mathcal{U}/\mathcal{A}).$$

Proof. For any exact functor $f : \mathcal{A} \to \mathcal{B}$ between categories with cofibrations and weak equivalences, Waldhausen constructs a simplicial category with cofibrations and weak equivalences $S.(f : \mathcal{A} \to \mathcal{B})$ and a homotopy fibration [Wal85, 1.5.5]

$$wS.\mathcal{B} \to wS.S.(f : \mathcal{A} \to \mathcal{B}) \to wS.S.\mathcal{A}. \tag{2.2}$$

In our application f is the inclusion $\mathcal{A} \subset \mathcal{U}$ of an idempotent complete right s-filtering subcategory $\mathcal{A}$ of an exact category $\mathcal{U}$ and w is the set of isomorphisms. We write i instead of w. Then $S.(\mathcal{A} \subset \mathcal{U})$ is a simplicial exact category for which $S_q(\mathcal{A} \subset \mathcal{U})$ is equivalent to the exact category whose objects are sequences $U = (\, U_0 \overset{\sim}{\rightarrowtail} U_1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} U_q \,)$ of inflations with cokernel in $\mathcal{A}$ (labeled as $\overset{\sim}{\rightarrowtail}$). Morphisms are commutative diagrams in $\mathcal{U}$. A sequence $U \to V \to W$ in $S_q(\mathcal{A} \subset \mathcal{U})$ is a conflation if and only if it is pointwise a conflation, *i.e.,* if $U_i \to V_i \to W_i$ is a conflation in $\mathcal{U}$ for $0 \leq i \leq q$.

2.3 Lemma. *The quotient map $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy equivalence*

$$iS.S.(\mathcal{A} \subset \mathcal{U}) \to iS.S.(0 \subset \mathcal{U}/A).$$

Assuming the lemma for a moment, we finish the proof of Theorem 2.1 as follows. We have a commutative diagram

$$\begin{array}{ccccc} iS.\mathcal{A} & \longrightarrow & iS.\mathcal{U} & \longrightarrow & iS.S.(\mathcal{A} \subset \mathcal{U}) \\ & & \downarrow & & \downarrow \wr \\ & & iS.\mathcal{U}/A & \overset{\sim}{\longrightarrow} & iS.S.(0 \subset \mathcal{U}/A) \end{array}$$

in which the first horizontal line is the homotopy fibration (2.2). The map $iS.\mathcal{U}/A \to iS.S.(0 \subset \mathcal{U}/A)$ is a homotopy equivalence (for instance by appealing again to 2.2), and $iS.S.(\mathcal{A} \subset \mathcal{U}) \to iS.S.(0 \subset \mathcal{U}/A)$ is a homotopy equivalence by Lemma 2.3.

Proof of Lemma 2.3. We will use that the order doesn't matter when realizing multi-simplicial sets and that a map of simplicial categories is a homotopy equivalence if it is degree-wise a homotopy equivalence. Let $\mathcal{C}$ be a small category, and let w be a set of morphisms in $\mathcal{C}$ closed under composition and containing all isomorphisms. We write $w\mathcal{C}$ for the category which has the same objects as $\mathcal{C}$ and where a morphism is a $\mathcal{C}$-morphism lying in w. Moreover, we write $\mathcal{N}_p^w\mathcal{C}$ for the category whose objects are sequences $C_0 \to C_1 \to \cdots \to C_p$ of maps lying in w and whose morphisms are commutative diagrams in $\mathcal{C}$. The usual face and degeneracy maps for nerves give $p \mapsto \mathcal{N}_p^w\mathcal{C}$ the structure of a simplicial category. If we consider $w\mathcal{C}$ as a constant simplicial category, then the canonical map of simplicial categories $w\mathcal{C} \to w\mathcal{N}_*^{iso}\mathcal{C}$, given by $C \mapsto (C \overset{id}{\to} C \overset{id}{\to} \ldots \overset{id}{\to} C)$ on objects, is a homotopy equivalence, since it is degree-wise an equivalence of categories.

Let $\mathcal{A}$ be an idempotent complete right s-filtering exact subcategory of $\mathcal{U}$. We write w_q for the set of inflations in $S_q\mathcal{U}$ with cokernel in $S_q\mathcal{A}$. The category $iS_pS_q(\mathcal{A} \subset \mathcal{U})$ is equivalent to $i\mathcal{N}_q^{w_p}S_p\mathcal{U}$. By realizing in different orders, we see that for fixed p, the simplicial categories $i\mathcal{N}_*^{w_p}S_p\mathcal{U}$ and $w_p\mathcal{N}_*^{iso}S_p\mathcal{U}$ have isomorphic realizations. As pointed out before, $w_pS_p\mathcal{U} \to w_p\mathcal{N}_*^{iso}S_p\mathcal{U}$ is a homotopy equivalence of simplicial categories. The same applies to the trivial (right s-filtering) inclusion $0 \subset \mathcal{U}/A$. So the lemma is proven once we see that for every p the functor $f_p : w_pS_p\mathcal{U} \to iS_p(\mathcal{U}/A)$ is a homotopy equivalence. If $\mathcal{A}$ is idempotent complete, then $S_n\mathcal{A}$ is also idempotent complete. Moreover, if $\mathcal{A} \subset \mathcal{U}$ is right s-filtering, so are $S_n\mathcal{A} \to S_n\mathcal{U}$ for all $n \in \mathbb{N}$. For $n = 0$ both categories are trivial, for $n = 1$ this is the hypothesis, for $n=2$ this is 1.17 (4), for $n \geq 3$ the argument is similar to $n=2$, and we omit the details. Thus it suffices to prove that $f = f_1 : w_1\mathcal{U} \to i(\mathcal{U}/\mathcal{A})$ is a homotopy equivalence. For every object X of $i(\mathcal{U}/\mathcal{A})$, the category $(f \downarrow X)$ is non-empty since $\mathcal{U}$ and $\mathcal{U}/\mathcal{A}$ have the same objects. The category $(f \downarrow X)$ is cofiltered by 1.17 (3) and the calculus of fractions. Thus $(f \downarrow X)$ is contractible. So Quillen's Theorem A [Qui73] applies, and f_1 is a homotopy equivalence. □

We will give a different proof of Theorem 2.1 which also includes negative K-groups.

Let $\mathcal{E}$ be an exact category. The category $Ch^b(\mathcal{E})$ of bounded chain complexes (X^*, d_X^*) in $\mathcal{E}$ is an exact category in which a sequence of chain complexes $X^* \to Y^* \to Z^*$ is a conflation if it is a conflation in each degree, *i.e.,* if $X^n \to Y^n \to Z^n$ is a conflation in $\mathcal{E}$ for all $n \in \mathbb{Z}$.

2.4 Lemma. *Let $\mathcal{A}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathcal{U}$. Then $Ch^b(\mathcal{A})$ is right s-filtering in $Ch^b(\mathcal{U})$ and the induced exact functor $Ch^b(\mathcal{U})/Ch^b(\mathcal{A}) \to Ch^b(\mathcal{U}/\mathcal{A})$ is an equivalence of exact categories.*

Proof. The proof follows from the calculus of fractions and Lemma 1.17. We omit the details. □

2.5. Let $\mathcal{E}$ be an exact category, and let $D^b(\mathcal{E})$ be its bounded derived category. It is the category of bounded chain complexes in $\mathcal{E}$ localized with respect to chain maps whose cones are homotopy equivalent to an acyclic complex [Kel96]. Recall that a complex (E^*, d^*) is acyclic if the differentials d^i admit factorizations $E^i \to Z^{i+1} \to E^{i+1}$ such that $Z^i \to E^i \to Z^{i+1}$ is a conflation in $\mathcal{E}$ for all $i \in \mathbb{Z}$. The category $D^b(\mathcal{E})$ is a triangulated category. A chain complex is zero in $D^b(\mathcal{E})$ if and only if it is a direct factor in $Ch^b(\mathcal{E})$ of an acyclic chain complex (exercise!). For more details we refer the reader to [Kel96].

In the special case of an additive category $\mathcal{U}$ and a right and left s-filtering subcategory $\mathcal{A}$ the following Proposition is implicit in [CP97].

2.6 Proposition. *Let $\mathcal{A}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathcal{U}$. Then in the sequence of bounded derived categories*

$$D^b(\mathcal{A}) \to D^b(\mathcal{U}) \to D^b(\mathcal{U}/\mathcal{A}), \tag{2.7}$$

the first functor is fully faithful, $D^b(\mathcal{A})$ is épaisse in $D^b(\mathcal{U})$, the composition of the two functors is trivial, and the induced functor from the Verdier quotient $D^b(\mathcal{U})/D^b(\mathcal{A})$ to $D^b(\mathcal{U}/\mathcal{A})$ is an equivalence of categories.

Proof. Since every inflation from an object of $\mathcal{A}$ to an object of $\mathcal{U}$ is special, condition C2 of [Kel96, 12.1] is fulfilled, and thus the functor $D^b(\mathcal{A}) \to D^b(\mathcal{U})$ is fully faithful. The category $D^b(\mathcal{A})$ is idempotent complete because $\mathcal{A}$ is idempotent complete [BS01]. In particular, $D^b(\mathcal{A})$ is closed under taking direct factors in $D^b(\mathcal{U})$, hence $D^b(\mathcal{A})$ is épaisse in $D^b(\mathcal{U})$ [Ric89, 1.3]. By Lemma 2.4 and definition 1.14, $Ch^b(\mathcal{U}/\mathcal{A})$ is a localization of $Ch^b(\mathcal{U})$. It follows that $D^b(\mathcal{U}/\mathcal{A})$ is a localization of $D^b(\mathcal{U})$. Therefore, it remains to show that every a chain complex

$$U^*: \qquad 0 \longrightarrow U^0 \xrightarrow{d^0} U^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} U^n \longrightarrow 0$$

of $Ch^{\mathrm{b}}(\mathcal{U})$, which is zero in $D^b(\mathcal{U}/\mathcal{A})$, is isomorphic in $D^b(\mathcal{U})$ to a chain complex of $Ch^{\mathrm{b}}(\mathcal{A})$.

We first treat the case when $U^* \in Ch^b(\mathcal{U})$ is acyclic in $Ch^b(\mathcal{U}/\mathcal{A})$. We will construct a sequence of chain complexes and chain maps

$$U^* = U_0^* \overset{\sim}{\twoheadleftarrow\!\!\!<} V_0^* \xrightarrow{\simeq} U_1^* \overset{\sim}{\twoheadleftarrow\!\!\!<} V_1^* \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} U_n^* \overset{\sim}{\twoheadleftarrow\!\!\!<} V_n^* \xrightarrow{\simeq} 0 \tag{2.8}$$

where $\overset{\sim}{\rightarrowtail}$ denotes an inflation of chain complexes with cokernel in $Ch^b(\mathcal{A})$ and $\xrightarrow{\simeq}$ a quasi-isomorphism in $Ch^b(\mathcal{U})$. If we have constructed (2.8), then U^* is isomorphic in $D^b(\mathcal{U})$ to an object of $Ch^b(\mathcal{A})$.

Let $f : X \to Y$ be a map in $\mathcal{U}$ which becomes an inflation in $\mathcal{U}/\mathcal{A}$. Then there are $\mathcal{U}$-inflations $X' \to X$ and $Y \to Y'$ with cokernel in $\mathcal{A}$ such that the composition $X' \to Y'$ is an inflation in $\mathcal{U}$. This follows from 1.17 (3). By the right filtering property and 1.17 (1), the map $X' \to Y$ is an inflation in $\mathcal{U}$. Thus for any $\mathcal{U}$-map $f : X \to Y$ which is an inflation in $\mathcal{U}/\mathcal{A}$, there is a $\mathcal{U}$-inflation $X' \to X$ with cokernel in $\mathcal{A}$ such that $X' \to Y$ is an inflation in $\mathcal{U}$.

We apply this argument to $d^0 : U^0 \to U^1$ which is a $\mathcal{U}/\mathcal{A}$-inflation because U^* is $\mathcal{U}/\mathcal{A}$-acyclic. We obtain a $\mathcal{U}$-inflation $\alpha : V^0 \overset{\sim}{\rightarrowtail} U^0$ such that $d^0 \circ \alpha$ is an inflation in $\mathcal{U}$. Let V_0^* be the chain complex

$$V_0^* : \qquad 0 \longrightarrow V^0 \overset{d^0\circ\alpha}{\rightarrowtail} U^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} U^n \longrightarrow 0.$$

The map

$$\begin{array}{cccccccccccc}
U^* : & & 0 & \longrightarrow & U^0 & \xrightarrow{d^0} & U^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & U^n & \longrightarrow 0 \\
\wr\uparrow & & & & \wr\uparrow\alpha & & \| & & & & \| & \\
V_0^* : & & 0 & \longrightarrow & V^0 & \overset{d^0\circ\alpha}{\rightarrowtail} & U^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & U^n & \longrightarrow 0
\end{array}$$

is an inflation of chain complexes with cokernel in $Ch^b(\mathcal{A})$. Let U_1^* be the chain complex

$$0 \longrightarrow U^1/V^0 \xrightarrow{\bar{d^1}} U^2 \xrightarrow{d^2} U^3 \xrightarrow{d^3} \cdots \xrightarrow{d^{n-1}} U^n \longrightarrow 0$$

where $\bar{d^1}$ is the map which is induced by d^1 on the quotient U^1/V^0. Then the map of chain complexes

$$\begin{array}{cccccccccccccc}
V_0^* : & 0 & \longrightarrow & V^0 & \overset{d^0\circ\alpha}{\rightarrowtail} & U^1 & \xrightarrow{d^1} & U^2 & \xrightarrow{d^2} & U^3 & \xrightarrow{d^3} \cdots & \xrightarrow{d^{n-1}} & U^n & \longrightarrow 0 \\
\downarrow & & & \downarrow & & \downarrow & & \| & & \| & & & \| & \\
U_1^* : & 0 & \longrightarrow & 0 & \longrightarrow & U^1/V^0 & \xrightarrow{\bar{d^1}} & U^2 & \xrightarrow{d^2} & U^3 & \xrightarrow{d^3} \cdots & \xrightarrow{d^{n-1}} & U^n & \longrightarrow 0
\end{array}$$

is a quasi-isomorphism since it is a deflation with contractible kernel. Now U_1^* is a shorter chain complex than U^* which is also acyclic in $Ch^b(\mathcal{U}/\mathcal{A})$. We repeat the construction to obtain (2.8).

We treat the general case. Let U be a chain complex in $\mathcal{U}$ which is zero in $D^b(\mathcal{U}/\mathcal{A})$. Then there is an acyclic chain complex X in $\mathcal{U}/\mathcal{A}$ which contains U as a direct factor in $Ch^b(\mathcal{U}/\mathcal{A})$. It follows from Lemma 2.4, that $Ch^b(\mathcal{U}) \to Ch^b(\mathcal{U}/\mathcal{A})$ is essentially surjective on objects. Thus we can assume X to be a chain complex in $\mathcal{U}$ which is acyclic in $\mathcal{U}/\mathcal{A}$. By the previous paragraph, X is isomorphic in $D^b(\mathcal{U})$ to an object of $D^b(\mathcal{A})$. Using Lemma 2.4, the calculus of fractions and Lemma 1.17 (3) applied to $Ch^b(\mathcal{A}) \subset Ch^b(\mathcal{U})$, we can express the fact that U is a direct factor of X in $Ch^b(\mathcal{U}/\mathcal{A})$ by the existence of the following commutative diagram in $Ch^b(\mathcal{U})$

$$\begin{array}{ccc}
V & \longrightarrow & X \\
\wr\downarrow & & \downarrow \\
U & \overset{\sim}{\rightarrowtail} & W.
\end{array}$$

Since X is zero in the Verdier quotient $D^b(\mathcal{U})/D^b(\mathcal{A})$, and since inflations of chain complexes in $\mathcal{U}$ with cokernel in $Ch^b(\mathcal{A})$ are isomorphisms in the Verdier quotient, it follows that U, V, W are zero in $D^b(\mathcal{U})/D^b(\mathcal{A})$. As $D^b(\mathcal{A})$ is épaisse, this means that U is isomorphic in $D^b(\mathcal{U})$ to an object of $D^b(\mathcal{A})$. □

2.9. For $\mathcal{E}$ an exact category, we have constructed in [Sch] a spectrum $\mathbb{K}(\mathcal{E})$ such that $\pi_i\mathbb{K}(\mathcal{E})$ are Quillen's K-groups $K_i(\mathcal{E})$ of $\mathcal{E}$ for $i > 0$, $\pi_0\mathbb{K}(\mathcal{E})$ is K_0 of the idempotent completion of $\mathcal{E}$, and $\pi_i\mathbb{K}(\mathcal{E})$ are the negative K-groups of $\mathcal{E}$ for $i < 0$. We write $\mathbb{K}_i(\mathcal{E})$ for $\pi_i\mathbb{K}(\mathcal{E})$.

2.10 Theorem. *Let $\mathcal{A}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathcal{U}$. Then the sequence of exact categories $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of spectra*

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A}).$$

In particular, there is a long exact sequence of abelian groups for $i \in \mathbb{Z}$

$$\cdots \to \mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{U}) \to \mathbb{K}_i(\mathcal{U}/\mathcal{A}) \to \mathbb{K}_{i-1}(\mathcal{A}) \to \mathbb{K}_{i-1}(\mathcal{U}) \to \cdots$$

Proof. It follows from Proposition 2.6 that (2.7) is exact in the sense of [Sch, 1.1]. The Theorem is a consequence of [Sch, 11.10, 11.13]. For $i \geq 0$ the long exact sequence also follows from Theorem 2.1 or from Proposition 2.6 and [TT90]. □

3. The suspension of an exact category

3.1. *Countable envelopes.* Let $\mathcal{E}$ be an idempotent complete exact category. We will construct a left s-filtering embedding $\mathcal{E} \subset \mathcal{F}\mathcal{E}$ with $K(\mathcal{F}\mathcal{E})$ and $I\!K(\mathcal{F}\mathcal{E})$ contractible (3.2).

Let $\mathcal{F}\mathcal{E}$ be the countable envelope of $\mathcal{E}$ [Kel90, Appendix B] (denoted by $\mathcal{E}^{\sim}$ in *loc.cit*). We review definitions and basic properties from *loc.cit.* It is an exact category whose objects are sequences $A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \ldots$ of inflations in $\mathcal{E}$. The morphism set from a sequence A_* to B_* is $\lim_i \operatorname{colim}_j \hom_{\mathcal{E}}(A_i, B_j)$. The functor colim : $\mathcal{F}\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ which sends a sequence A_* to $\operatorname{colim}_i A_i$ is fully faithful and extension closed and thus induces an exact structure on $\mathcal{F}\mathcal{E}$ by declaring a sequence in $\mathcal{F}\mathcal{E}$ to be a conflation if it is a conflation in $\mathrm{Lex}\mathcal{E}$. It turns out that a sequence in $\mathcal{F}\mathcal{E}$ is a conflation iff it is isomorphic to the maps of sequences $A_* \to B_* \to C_*$ with $A_i \to B_i \to C_i$ a conflation in $\mathcal{E}$. Therefore, the exact structure does not depend on the embedding $\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ and $\mathcal{F}$ defines a functor from exact categories to exact categories.

Colimits of sequences of inflations in $\mathcal{F}\mathcal{E}$ exist in $\mathcal{F}\mathcal{E}$ and are exact. In particular, $\mathcal{F}\mathcal{E}$ has exact, countable direct sums.

There is a fully faithful exact functor $\mathcal{E} \to \mathcal{F}\mathcal{E}$ which sends an object X of $\mathcal{E}$ to the constant sequence $X \stackrel{id}{\to} X \stackrel{id}{\to} X \stackrel{id}{\to} \cdots$. The functor is fully faithful, exact and reflects exactness because $\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ and colim : $\mathcal{F}\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ are fully faithful, exact and reflect exactness.

3.2 Lemma. *Let $\mathcal{E}$ be an idempotent complete exact category. The fully faithful exact functor $\mathcal{E} \to \mathcal{F}\mathcal{E}$ makes $\mathcal{E}$ into a left s-filtering subcategory of $\mathcal{F}\mathcal{E}$. Moreover, $K(\mathcal{F}\mathcal{E})$ and $I\!K(\mathcal{F}\mathcal{E})$ are contractible.*

Proof. Given objects X of $\mathcal{E}$ and $Y = (Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots)$ of $\mathcal{F}\mathcal{E}$, by the definition of maps in $\mathcal{F}\mathcal{E}$, any morphism from X to Y factors over some Y_i. The sequence $Y_i \to Y \to (Y_{i+1}/Y_i \hookrightarrow Y_{i+2}/Y_i \hookrightarrow Y_{i+3}/Y_i \hookrightarrow \cdots)$ is exact in $\mathrm{Lex}\mathcal{E}$ and thus is a conflation in $\mathcal{F}\mathcal{E}$. It follows that $\mathcal{E} \subset \mathcal{F}\mathcal{E}$ is left filtering.

Given objects X of $\mathcal{E}$ and Y of $\mathcal{F}\mathcal{E}$ as above, and given a surjection $Y = \operatorname{colim} Y_i \to X$ in $\mathrm{Lex}\mathcal{E}$. The object X is small in $\mathrm{Lex}\mathcal{E}$, that is $\hom(X, \operatorname{colim} U_i) = \operatorname{colim} \hom(X, U_i)$ for filtered colimits. This implies that there is a Y_j such that the composition $Y_j \to Y = \operatorname{colim} Y_i \to X$ is still surjective. As $\mathcal{E}$ is idempotent complete, the embedding $\mathcal{E} \subset \mathrm{Lex}\mathcal{E}$ is closed under kernels of surjections. Thus the surjection $Y_j \to X$ is a deflation in $\mathcal{E}$. As pointed out before, the map $Y_j \to Y$ is an inflation in $\mathcal{F}\mathcal{E}$. Therefore, any deflation $Y \to X$ from an object of $\mathcal{F}\mathcal{E}$ to an object of $\mathcal{E}$ is special.

The inclusion $\mathcal{E} \to \mathcal{F}\mathcal{E}$ is closed under taking admissible quotient objects because of the left filtering property. The inclusion is also closed under taking admissible subobjects because it is closed under taking admissible quotient objects and because the inclusion $\mathcal{E} \subset \mathrm{Lex}\mathcal{E}$ is closed under taking kernels of surjections.

The space $K(\mathcal{F}\mathcal{E})$ and the spectrum $I\!K(\mathcal{F}\mathcal{E})$ are contractible because $\mathcal{F}\mathcal{E}$ has countable exact coproducts.

□

3.3 Definition. Let $\mathcal{E}$ be an exact category, and let $\widetilde{\mathcal{E}}$ be its idempotent completion. We define the *suspension* $\mathcal{S}\mathcal{E}$ of $\mathcal{E}$ to be the quotient exact category $(\mathcal{F}\widetilde{\mathcal{E}})/\widetilde{\mathcal{E}}$.

3.4 Theorem. *Let $\mathcal{E}$ be an exact category. Then there is a homotopy equivalence of spectra*

$$I\!K(\mathcal{E}) \simeq \Omega I\!K(\mathcal{S}\mathcal{E}).$$

In particular, the negative K-groups of $\mathcal{E}$ satisfy $I\!K_{-n}(\mathcal{E}) = K_0(\widetilde{\mathcal{S}^n\mathcal{E}})$ for $n \geq 0$.

If $\mathcal{E}$ is an idempotent complete exact category, then there is a homotopy equivalence of K-theory spaces

$$K(\mathcal{E}) \simeq \Omega K(\mathcal{S}\mathcal{E}).$$

Proof. We have $\mathbb{K}(\mathcal{E}) \simeq \mathbb{K}(\widetilde{\mathcal{E}})$ [Sch]. So we can assume in both cases that $\mathcal{E}$ is idempotent complete. Since $K(\mathcal{F}\mathcal{E})$ and $\mathbb{K}(\mathcal{F}\mathcal{E})$ are contractible, the Theorem follows from Lemma 3.2 and Theorem 2.10 (or Theorem 2.1 for the second part). The "in particular" follows from the fact that $\mathbb{K}_0(\mathcal{E}) = K_0(\widetilde{\mathcal{E}})$ for any exact category $\mathcal{E}$. □

References

[Bas68] Hyman Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[BS01] Paul Balmer and Marco Schlichting. Idempotent completion of triangulated categories. *Journal of Algebra*, 236:819–834, 2001.

[Car80] David W. Carter. Localization in lower algebraic K-theory. *Comm. Algebra*, 8(7):603–622, 1980.

[CP97] M. Cárdenas and E. K. Pedersen. On the Karoubi filtration of a category. *K-Theory*, 12(2):165–191, 1997.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). pages 217–240. Lecture Notes in Math., Vol. 551, 1976.

[GZ67] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Springer-Verlag New York, Inc., New York, 1967. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35.

[JS89] J. F. Jardine and V. P. Snaith, editors. *Algebraic K-theory: connections with geometry and topology*, volume 279 of *NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences*, Dordrecht, 1989. Kluwer Academic Publishers Group.

[Kar70] Max Karoubi. Foncteurs dérivés et K-théorie. In *Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie (1967/68)*, volume 136 of *Lecture Notes in Mathematics*, pages 107–186, Berlin, 1970. Springer.

[Kel90] Bernhard Keller. Chain complexes and stable categories. *Manuscripta Math.*, 67(4):379–417, 1990.

[Kel96] Bernhard Keller. Derived categories and their uses. In *Handbook of algebra, Vol. 1*, pages 671–701. North-Holland, Amsterdam, 1996.

[May72] J. P. May. *The geometry of iterated loop spaces.* Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[PW89] Erik K. Pedersen and Charles A. Weibel. K-theory homology of spaces. In *Algebraic topology (Arcata, CA, 1986)*, volume 1370 of *Lecture Notes in Math.*, pages 346–361. Springer, Berlin, 1989.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Ric89] Jeremy Rickard. Derived categories and stable equivalence. *J. Pure Appl. Algebra*, 61(3):303–317, 1989.

[Sch] Marco Schlichting. Negative K-theory of derived categories. Preprint available at http://www.math.uiuc.edu/K-theory/.

[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Wag72] J. B. Wagoner. Delooping classifying spaces in algebraic K-theory. *Topology*, 11:349–370, 1972.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, pages 318–419. Springer, Berlin, 1985.

Marco Schlichting, Universität GHS Essen, Fb 6 Mathematik, 45117 Essen, Germany
E-mail address: marco.schlichting@uni-essen.de