Higher Algebraic K-Theory

(After Quillen, Thomason and Others)

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Abstract We present an introduction (with a few proofs) to higher algebraic K-theory of schemes based on the work of Quillen, Waldhausen, Thomason and others. Our emphasis is on the application of triangulated category methods in algebraic K-theory.

1 Introduction

These are the expanded notes for a course taught by the author at the Sedano Winter School on K-theory, January 23–26, 2007, in Sedano, Spain. The aim of the lectures was to give an introduction to higher algebraic K-theory of schemes. I decided to give only a quick overview of Quillen's fundamental results [35, 73], and then to focus on the more modern point of view where structure theorems about derived categories of sheaves are used to compute higher algebraic K-groups.

Besides reflecting my own taste, there are at least two other good reasons for this emphasis. First, there is an ever growing number of results in the literature about the structure of triangulated categories. To name only a few of their authors, we refer the reader to the work of Bondal, Kapranov, Orlov, Kuznetsov, Samarkhin, Keller, Thomason, Rouquier, Neeman, Drinfeld, Toen, van den Bergh, Bridgeland, etc. ... The relevance for K-theory is that virtually all results about derived categories translate into results about higher algebraic K-groups. The link is provided by an abstract Localization Theorem due to Thomason and Waldhausen which – omitting hypothesis – says that a "short exact sequence of triangulated categories gives rise to a long exact sequence of algebraic K-groups". The second reason for this emphasis is that an analog of the Thomason–Waldhausen Localization Theorem also holds for many other (co-) homology theories besides K-theory, among which Hochschild homology, (negative, periodic, ordinary) cyclic homology [49], topological Hochschild (and cyclic) homology [2], triangular Witt groups [6] and higher Grothendieck–Witt groups [77]. All K-theory results that are proved using triangulated category methods therefore have analogs in all these other (co-) homology theories.

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Here is an overview of the contents of these notes. Sect. 2 is an introduction to Quillen's fundamental article [73]. Here the algebraic K-theory of exact categories is introduced via Quillen's Q-construction. We state some fundamental theorems, and we state/derive results about the G-theory of noetherian schemes and the K-theory of smooth schemes. The proofs in [73] are all elegant and very well-written, so there is no reason to repeat them here. The only additions I have made are a hands-on proof of the fact that Quillen's Q-construction gives the correct K_0-group, and a description of negative K-groups which is absent in Quillen's work.

Section 3 is an introduction to algebraic K-theory from the point of view of triangulated categories. In Sect. 3.1 we introduce the Grothendieck-group K_0 of a triangulated category, give examples and derive some properties which motivate the introduction of higher algebraic K-groups. In Sect. 3.2 we introduce the K-theory space (and the non-connected $\mathbb{K}$-theory spectrum) of a complicial exact category with weak equivalences via Quillen's Q-construction. This avoids the use of the technically heavier $S_\bullet$-construction of Waldhausen [100]. We state in Theorem 3.2.27 the abstract Localization Theorem mentioned above that makes the link between exact sequences of triangulated categories and long exact sequences of algebraic K-groups. In Sect. 3.3 we show that most of Quillen's results in [73] – with the notable exception of *Dévissage* – can be viewed as statements about derived categories, in view of the Localization Theorem. In Sect. 3.4 we give a proof – based on Neeman's theory of compactly generated triangulated categories – of Thomason's Mayer-Vietoris principle for quasi-compact and separated schemes. In Sect. 3.5 we illustrate the use of triangulated categories in the calculation of the K-theory of projective bundles and of blow-ups of schemes along regularly embedded centers. We also refer the reader to results on derived categories of rings and schemes which yield further calculations in K-theory.

Section 4 is a mere collection of statements of mostly recent results in algebraic K-theory the proofs of which go beyond the methods explained in Sects. 2 and 3.

In Appendix A, Sects. 1 and 2 we assemble results from topology and the theory of triangulated categories that are used throughout the text. In Appendix A, Sect. 3, we explain the constructions and elementary properties of the derived functors we will need. Finally, we give in Appendix A, Sect. 4, a proof of the fact that the derived category of complexes of quasi-coherent sheaves (supported on a closed subset with quasi-compact open complement) on a quasi-compact and separated scheme is compactly generated – a fact used in the proof of Thomason's Mayer-Vietoris principle in Sect. 3.4.

2 The K-Theory of Exact Categories

2.1 The Grothendieck Group of an Exact Category

2.1.1 Exact Categories

An *exact category* [73, Sect. 2] is an additive category $\mathscr{E}$ equipped with a family of sequences of morphisms in $\mathscr{E}$, called *conflations* (or admissible exact sequences),

$$X \xrightarrow{i} Y \xrightarrow{p} Z \qquad (1)$$

satisfying the properties (a)–(f) below. In a conflation (1), the map i is called *inflation* (or admissible monomorphism) and may be depicted as $\rightarrowtail$, and the map p is called *deflation* (or admissible epimorphism) and may be depicted as $\twoheadrightarrow$.

(a) In a conflation (1), the map i is a kernel of p, and p is a cokernel of i.
(b) Conflations are closed under isomorphisms.
(c) Inflations are closed under compositions, and deflations are closed under compositions.
(d) Any diagram $Z \leftarrow X \overset{i}{\rightarrowtail} Y$ with i an inflation can be completed to a cocartesian square

$$\begin{array}{ccc} X & \overset{i}{\rightarrowtail} & Y \\ \downarrow & & \downarrow \\ Z & \overset{j}{\rightarrowtail} & W \end{array}$$

with j an inflation.

(e) Dually, any diagram $X \to Z \overset{p}{\twoheadleftarrow} Y$ with p a deflation can be completed to a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & Z \end{array}$$

with q a deflation.

(f) The following sequence is a conflation

$$X \overset{\binom{1}{0}}{\longrightarrow} X \oplus Y \overset{(0\ 1)}{\longrightarrow} Y.$$

Quillen lists another axiom [73, Sect. 2 Exact categories c)] which, however, follows from the axioms listed above [46, Appendix]. For a detailed account of exact categories including the solutions of some of the exercises below, we refer the reader to [17].

An additive functor between exact categories is called *exact* if it sends conflations to conflations.

Let $\mathscr{A}$, $\mathscr{B}$ be exact categories such that $\mathscr{B} \subset \mathscr{A}$ is a full subcategory. We say that $\mathscr{B}$ is a *fully exact subcategory* of $\mathscr{A}$ if $\mathscr{B}$ is closed under extensions in $\mathscr{A}$ (that is, if in a conflation (1) in $\mathscr{A}$, X and Z are isomorphic to objects in $\mathscr{B}$ then Y is isomorphic to an object in $\mathscr{B}$), and if the inclusion $\mathscr{B} \subset \mathscr{A}$ preserves and detects conflations.

2.1.2 Examples

(a) Abelian categories are exact categories when equipped with the family of conflations (1) where $0 \to X \to Y \to Z \to 0$ is a short exact sequence. Examples of abelian (thus exact) categories are: the category R-Mod of all (left) R-modules where R is a ring; the category R-mod of all finitely generated (left) R-modules where R is a noetherian ring; the category O_X-Mod ($\mathrm{Qcoh}(X)$) of (quasi-coherent) O_X-modules where X is a scheme; the category $\mathrm{Coh}(X)$ of coherent O_X-modules where X is a noetherian scheme.
(b) Let $\mathscr{A}$ be an exact category, and let $\mathscr{B} \subset \mathscr{A}$ be a full additive subcategory closed under extensions in $\mathscr{A}$. Call a sequence (1) in $\mathscr{B}$ a conflation if it is a conflation in $\mathscr{A}$. One checks that $\mathscr{B}$ equipped with this family of conflations is an exact category making $\mathscr{B}$ into a fully exact subcategory of $\mathscr{A}$. In particular, any extension closed subcategory of an abelian category is canonically an exact category.

(c) The category $\mathrm{Proj}(R)$ of finitely generated projective left R-modules is extension closed in the category of all R-modules. Similarly, the category $\mathrm{Vect}(X)$ of vector bundles (that is, locally free sheaves of finite rank) on a scheme X is extension closed in the category of all O_X-modules. In this way, we consider $\mathrm{Proj}(R)$ and $\mathrm{Vect}(X)$ as exact categories where a sequence is a conflation if it is a conflation in its ambient abelian category.
(d) An additive category can be made into an exact category by declaring a sequence (1) to be a conflation if it is isomorphic to a sequence of the form 2.1.1 (f). Such exact categories are referred to as *split exact categories*.
(e) Let $\mathscr{E}$ be an exact category. We let $\mathrm{Ch}\,\mathscr{E}$ be the category of chain complexes in $\mathscr{E}$. Objects are sequences (A,d):

$$\cdots \to A^{i-1} \stackrel{d^{i-1}}{\to} A^i \stackrel{d^i}{\to} A^{i+1} \to \cdots$$

of morphisms in $\mathscr{E}$ such that $d \circ d = 0$. A morphism $f : (A,d_A) \to (B,d_B)$ is a collection of morphisms $f^i : A^i \to B^i$, $i \in \mathbb{Z}$, such that $f \circ d_A = d_B \circ f$. A sequence $(A,d_A) \to (B,d_B) \to (C,d_C)$ of chain complexes is a conflation if $A^i \to B^i \to C^i$ is a conflation in $\mathscr{E}$ for all $i \in \mathbb{Z}$. This makes $\mathrm{Ch}\,\mathscr{E}$ into an exact category.
The full subcategory $\mathrm{Ch}^b\mathscr{E} \subset \mathrm{Ch}\,\mathscr{E}$ of bounded chain complexes is a fully exact subcategory, where a complex (A,d_A) is bounded if $A^i = 0$ for $i >> 0$ and $i << 0$.

It turns out that the examples in Example 2.1.2 (b), (c) are typical as the following lemma shows. The proof can be found in [94, Appendix A] and [46, Appendix A].

2.1.3 Lemma

Every small exact category can be embedded into an abelian category as a fully exact subcategory.

2.1.4 Exercise

Use the axioms Sect. 2.1.1 (a)–(f) of an exact category or Lemma 2.1.3 above to show the following (and their duals).

(a) A cartesian square as in Sect. 2.1.1 (e) with p a deflation is also cocartesian. Moreover, if $X \to Z$ is an inflation, then $W \to Y$ is also an inflation.
(b) If the composition ab of two morphisms in an exact category is an inflation, and if b has a cokernel, then b is also an inflation. This is Quillen's redundant axiom [73, Sect. 2 Exact categories c)].
(c) Given a composition pq of deflations p, q in $\mathscr{E}$, then there is a conflation $\ker(q) \rightarrowtail \ker(pq) \twoheadrightarrow \ker p$ in $\mathscr{E}$.

2.1.5 Definition of K_0

Let $\mathscr{E}$ be a small exact category. The *Grothendieck group* $K_0(\mathscr{E})$ of $\mathscr{E}$ is the abelian group freely generated by symbols $[X]$ for every object X of $\mathscr{E}$ modulo the relation

$$[Y] = [X] + [Z] \text{ for every conflation } X \rightarrowtail Y \twoheadrightarrow Z. \tag{2}$$

An exact functor $F : \mathscr{A} \to \mathscr{B}$ between exact categories induces a homomorphism of abelian groups $F : K_0(\mathscr{A}) \to K_0(\mathscr{B})$ via $[X] \mapsto [FX]$.

2.1.6 Remark

The conflation $0 \rightarrowtail 0 \twoheadrightarrow 0$ implies that $0 = [0]$ in $K_0(\mathscr{E})$. Let $X \xrightarrow{\cong} Y$ be an isomorphism, then we have a conflation $0 \rightarrowtail X \twoheadrightarrow Y$, and thus $[X] = [Y]$ in $K_0(\mathscr{E})$. So $K_0(\mathscr{E})$ is in fact generated by isomorphism classes of objects in $\mathscr{E}$. The split conflation 2.1.1 (f) implies that $[X \oplus Y] = [X] + [Y]$.

2.1.7 Remark (K_0 for Essentially Small Categories)

By Remark 2.1.6, isomorphic objects give rise to the same class in K_0. It follows that we could have defined $K_0(\mathscr{E})$ as the group generated by isomorphism classes of objects in $\mathscr{E}$ modulo the relation 2.1.5 (2). This definition makes sense for any essentially small (= equivalent to a small) exact category. With this in mind, K_0 is also defined for such categories.

2.1.8 Definition

The groups $K_0(R)$, $K_0(X)$, and $G_0(X)$ are the Grothendieck groups of the essentially small exact categories $\mathrm{Proj}(R)$ of finitely generated projective R-modules where R is any ring, of the category $\mathrm{Vect}(X)$ of vector bundles on a scheme X[1], and of the category $\mathrm{Coh}(X)$ of coherent O_X-modules over a noetherian scheme X.

2.1.9 Examples

For commutative noetherian rings, there are isomorphisms

$K_0(\mathbb{Z}) \cong \mathbb{Z}$,	
$K_0(R) \cong \mathbb{Z}$	where R is a local (not necessarily noetherian) ring,
$K_0(F) \cong \mathbb{Z}$	where F is a field,
$K_0(A) \cong \mathbb{Z}^n$	where $\dim A = 0$ and $n = \#\operatorname{Spec} A$,
$K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R)$	where R is connected and $\dim R = 1$,
$K_0(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(X)$	where X is a connected smooth projective curve.

The group $\mathrm{Pic}(R)$ is the Picard group of a commutative ring R, that is, the group of isomorphism classes of rank 1 projective R-modules with tensor product $\otimes_R$ as group law. The isomorphism in the second to last row is induced by the map $K_0(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R)$ sending a projective module P to its rank $\mathrm{rk} P \in \mathbb{Z}$ and its highest non-vanishing exterior power $\Lambda^{\mathrm{rk} P} P \in \mathrm{Pic}(R)$. Similarly for the last isomorphism.

Proof

The first three follow from the fact that any finitely generated projective module over a commutative local ring or principal ideal domain R is free. So, in all these cases $K_0(R) = \mathbb{Z}$.

[1] For this to be the correct K_0-group, one has to make some assumptions about X such as quasi-projective or separated regular noetherian. See Sect. 3.4.

For the fourth isomorphism, in addition we use the decomposition of A into a product $A_1 \times \ldots \times A_n$ of Artinian local rings [3, Theorem 8.7] and the fact $K_0(R \times S) \cong K_0(R) \times K_0(S)$. For the second to last isomorphism, the map $K_0(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R)$ is surjective for any commutative ring R. Injectivity for $\dim R = 1$ follows from Serre's theorem [84, Théorème 1] which implies that a projective module P of rank r over a noetherian ring of Krull dimension d can be written, up to isomorphism, as $Q \oplus R^{r-d}$ for some projective module Q of rank d provided $r \geq d$. For $d = 1$, this means that $P \cong R^{r-1} \oplus \Lambda^r P$.

For the last isomorphism, let $x \in X$ be a closed point with residue field $k(x)$ and $U = X - x$ its open complement. Note that U is affine [41, IV Exercise 1.3]. Anticipating a little, we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(k(x)) & \longrightarrow & K_0(X) & \longrightarrow & K_0(U) & \longrightarrow & 0 \\ & & \big\downarrow \cong & & \big\downarrow & & \big\downarrow \cong & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathrm{Pic}(X) & \longrightarrow & \mathbb{Z} \oplus \mathrm{Pic}(U) & \longrightarrow & 0 \end{array}$$

in which the left and right vertical maps are isomorphisms, by the cases proved above. The top row is a special case of Quillen's localization long exact sequence Theorem 2.3.7 (5) using the fact that $K_0 = G_0$ for smooth varieties (Poincaré Duality Theorem 3.3.5), both of which can be proved directly for G_0 and K_0 without the use of the machinery of higher K-theory. The second row is the sum of the exact sequences $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$ [41, II Proposition 6.5] in view of the isomorphism $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$ for smooth varieties [41, II Corollary 6.16]. □

2.2 Quillen's Q-Construction and Higher K-Theory

In order to define higher K-groups, one constructs a topological space $K(\mathscr{E})$ and defines the K-groups $K_i(\mathscr{E})$ as the homotopy groups $\pi_i K(\mathscr{E})$ of that space. The topological space $K(\mathscr{E})$ is the loop space of the classifying space (Sect. 2.2.2 and Appendix A, Sect. 3) of Quillen's Q-construction. We start with describing the Q-construction.

2.2.1 Quillen's Q-Construction [73, Sect. 2]

Let $\mathscr{E}$ be a small exact category. We define a new category $Q\mathscr{E}$ as follows. The objects of $Q\mathscr{E}$ are the objects of $\mathscr{E}$. A map $X \to Y$ in $Q\mathscr{E}$ is an equivalence class of data $X \overset{p}{\twoheadleftarrow} W \overset{i}{\rightarrowtail} Y$ where p is a deflation and i an inflation. The datum (W, p, i) is equivalent to the datum (W', p', i') if there is an isomorphism $g : W \to W'$ such that $p = p'g$ and $i = i'g$. The composition of $(W, p, i) : X \to Y$ and $(V, q, j) : Y \to Z$ in $Q\mathscr{E}$ is the map $X \to Z$ represented by the datum $(U, p\bar{q}, j\bar{i})$ where U is the pull-back of q along i as in the diagram

$$\begin{array}{ccccccc} X & \overset{p}{\twoheadleftarrow} & W & \overset{\bar{q}}{\twoheadleftarrow} & U & & \\ & & i \big\downarrow & & \big\downarrow \bar{i} & & \\ & & Y & \underset{q}{\twoheadleftarrow} & V & \underset{j}{\rightarrowtail} & Z \end{array}$$

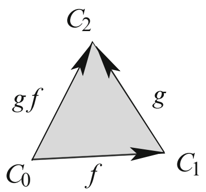
which exists by 2.1.1 (e). The map $\bar{q}$ (and hence $p\bar{q}$ by 2.1.1 (c)) is a deflation by 2.1.1 (e), and the map $\bar{i}$ (and hence $j\bar{i}$) is an inflation by Exercise 2.1.4 (a). The universal property of cartesian squares implies that composition is well-defined and associative (exercise!). The identity map id_X of an object X of $Q\mathscr{E}$ is represented by the datum $(X, 1, 1)$.

2.2.2 The Classifying Space of a Category

To any small category $\mathscr{C}$, one associates a topological space $B\mathscr{C}$ called the *classifying space of* $\mathscr{C}$. This is a CW-complex constructed as follows (for the precise definition, see Appendix A, Sect. 1.3).

- 0-cells are the objects of $\mathscr{C}$.
- 1-cells are the non-identity morphisms attached to their source and target.
- 2-cells are the 2-simplices (see the figure below) corresponding to pairs (f,g) of composable morphisms such that neither f nor g is an identity morphism.

(f,g) :



  The edges f, g and gf which make up the boundary of the 2-simplex (f,g) are attached to the 1-cells corresponding to f, g, and gf. In case $gf = id_{C_0}$, the whole edge gf is identified with the 0-cell corresponding to C_0.
- 3-cells are the 3-simplices corresponding to triples $C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3$ of composable arrows such that none of the maps f_0, f_1, f_2 is an identity morphism. They are attached in a similar way as in the case of 2-cells, etc.

2.2.3 Exercise

Give the CW-structure of the classifying spaces $B\mathscr{C}$ where the category $\mathscr{C}$ is as follows.

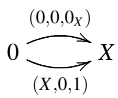
(a) $\mathscr{C}$ is the category with 3 objects A, B, C. The hom sets between two objects of $\mathscr{C}$ contain at most 1 element where the only non-identity maps are $f : A \to B$, $g : B \to C$ and $gf : A \to C$.
(b) $\mathscr{C}$ is the category with 2 objects A, B. The only non-identity maps are $f : A \to B$ and $g : B \to A$. They satisfy $gf = 1_A$ and $fg = 1_B$.

Hint: the category in (a) is the category [2] given in Appendix A, Sect. 1.3. Both categories have contractible classifying space by Lemma A.1.6.

Since we have a category $Q\mathscr{E}$, we have a topological space $BQ\mathscr{E}$. We make the classifying space $BQ\mathscr{E}$ of $Q\mathscr{E}$ into a pointed topological space by choosing a 0-object of $\mathscr{E}$ as base-point. Every object X in $Q\mathscr{E}$ receives an arrow from 0, the map represented by the data $(0,0,0_X)$ where 0_X denotes the zero map $0 \to X$ in $\mathscr{E}$. In particular, the topological space $BQ\mathscr{E}$ is connected, that is, $\pi_0 BQ\mathscr{E} = 0$.

To an object X of $\mathscr{E}$, we associate a loop $l_X = (0,0,0_X)^{-1}(X,0,1)$ based at 0

l_X :



in $BQ\mathscr{E}$ by first "walking" along the arrow $(X,0,1)$ and then back along $(0,0,0_X)$ in the opposite direction of the arrow. This loop thus defines an element $[l_X]$ in $\pi_1 BQ\mathscr{E}$.

2.2.4 Proposition

The assignment which sends an object X to the loop l_X induces a well-defined homomorphism of abelian groups $K_0(\mathscr{E}) \to \pi_1 BQ\mathscr{E}$ which is an isomorphism.

Proof

In order to see that the assignment $[X] \mapsto [l_X]$ yields a well defined group homomorphism $K_0(\mathscr{E}) \to \pi_1 BQ\mathscr{E}$, we observe that we could have defined $K_0(\mathscr{E})$ as the free group generated by symbols $[X]$ for each $X \in \mathscr{E}$, modulo the relation $[Y] = [X][Z]$ for any conflation $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$. The commutativity is forced by axiom Sect. 2.1.1 (f). So, we have to check that the relation $[l_Y] = [l_X][l_Z]$ holds in $\pi_1 BQ\mathscr{E}$. The loops l_X and l_Z are homotopic to the loops

$$0 \underset{(X,0,1)}{\overset{(0,0,0_X)}{\rightrightarrows}} X \xrightarrow{(X,1,i)} Y \quad \text{and} \quad 0 \underset{(Z,0,1)}{\overset{(0,0,0_Z)}{\rightrightarrows}} Z \xrightarrow{(Y,p,1)} Y \quad \text{which are}$$

$$0 \underset{(X,0,i)}{\overset{(0,0,0_Y)}{\rightrightarrows}} Y \quad \text{and} \quad 0 \underset{(Y,0,1)}{\overset{(X,0,i)}{\rightrightarrows}} Y. \quad \text{Therefore,}$$

$[l_X][l_Z] = [(0,0,0_Y)^{-1}(X,0,i)][(X,0,i)^{-1}(Y,0,1)] = [(0,0,0_Y)^{-1}(Y,0,1)] = [l_Y]$, and the map $K_0(\mathscr{E}) \to \pi_1 BQ\mathscr{E}$ is well-defined.

Now, we show that the map $K_0(\mathscr{E}) \to \pi_1 BQ\mathscr{E}$ is surjective. By the Cellular Approximation Theorem [104, II Theorem 4.5], every loop in the CW-complex $BQ\mathscr{E}$ is homotopic to a loop with image in the 1-skeleton of $BQ\mathscr{E}$, that is, it is homotopic to a loop which travels along the arrows of $Q\mathscr{E}$. Therefore, every loop in $BQ\mathscr{E}$ is homotopic to a concatenation $a_n^{\pm 1} a_{n-1}^{\pm 1} \cdots a_2^{\pm 1} a_1^{\pm 1}$ of composable paths $a_i^{\pm 1}$ in $BQ\mathscr{E}$ where the a_i's are maps in $Q\mathscr{E}$, and a (resp. a^{-1}) means "walking in the positive (resp. negative) direction of the arrow a". By inserting trivial loops $g \circ g^{-1}$ with $g = (A,0,1) : 0 \to A$, we see that the loop $a_n^{\pm 1} a_{n-1}^{\pm 1} \cdots a_2^{\pm 1} a_1^{\pm 1}$ represents the element

$$\begin{aligned} &[a_n^{\pm 1} g_{n-1} g_{n-1}^{-1} a_{n-1}^{\pm 1} \cdots g_2 g_2^{-1} a_2^{\pm 1} g_1 g_1^{-1} a_1^{\pm 1}] \\ = &[a_n^{\pm 1} g_{n-1}] \cdot [g_{n-1}^{-1} a_{n-1}^{\pm 1} g_{n-2}] \cdots [g_2^{-1} a_2^{\pm 1} g_1] \cdot [g_1^{-1} a_1^{\pm 1}] \end{aligned}$$

in $\pi_1 BQ\mathscr{E}$ where $g_i = (A_i, 0, 1) : 0 \to A_i$ has target A_i which is the endpoint of the path $a_i^{\pm 1}$ and the starting point of $a_{i+1}^{\pm 1}$. Let $(U_i, 0, j_i) : 0 \to X_i$ be the composition $a_i \circ (Y_i, 0, 1)$ in $Q\mathscr{E}$ where Y_i and X_i are the source and target of a_i, respectively. Then we have $[g_{i+1}^{-1} a_i^{\pm 1} g_i] = [(X_i,0,1)^{-1}(U_i,0,j_i)]^{\pm 1}$. This means that the group $\pi_1 BQ\mathscr{E}$ is generated by loops of the form

$$0 \underset{(U,0,j)}{\overset{(X,0,1)}{\rightrightarrows}} X.$$

In the following sequence of homotopies of loops

$$\begin{aligned} (X,0,1)^{-1}(U,0,j) &\sim (X,0,1)^{-1}(U,1,j) \circ (U,0,1) \\ &\sim (X,0,1)^{-1}(U,1,j) \circ (0,0,0_U) \circ (0,0,0_U)^{-1}(U,0,1) \\ &\sim (X,0,1)^{-1}(0,0,0_X) \circ (0,0,0_U)^{-1}(U,0,1) \\ &= l_X^{-1} \circ l_U, \end{aligned}$$

the first and third homotopies follow from the identities $(U,0,j) = (U,1,j) \circ (U,0,1)$ and $(0,0,0_X) = (U,1,j) \circ (0,0,0_U)$ in $Q\mathscr{E}$. Since $[l_X^{-1} \circ l_U] = [l_X]^{-1}[l_U]$ is in the image of the map $K_0(\mathscr{E}) \to \pi_1 BQ\mathscr{E}$, we obtain surjectivity.

To show injectivity, we construct a map $\pi_1 BQ\mathscr{E} \to K_0(\mathscr{E})$ so that the composition $K_0(\mathscr{E}) \to K_0(\mathscr{E})$ is the identity. To this end, we introduce a little notation. For a group G, we let $\underline{G}$ be the category with one object $*$ and $\mathrm{Hom}(*,*) = G$. Recall from Appendix A, Sect. 1.5 that $\pi_i B\underline{G} = 0$ for $i \neq 1$ and $\pi_1 B\underline{G} = G$ where the isomorphism $G \to \pi_1 B\underline{G}$ sends an element $g \in G$ to the loop l_g represented by the morphism $g : * \to *$. In order to obtain a map $\pi_1 BQ\mathscr{E} \to K_0(\mathscr{E})$, we construct a functor $F : Q\mathscr{E} \to \underline{K_0(\mathscr{E})}$. The functor sends an object X of $Q\mathscr{E}$ to the object $*$ of $\underline{K_0(\mathscr{E})}$. A map $(W,p,i) : X \to Y$ in $Q\mathscr{E}$ is sent to the map represented by the element $[\ker(p)] \in K_0(\mathscr{E})$. Using the notation of 2.2.1, we obtain $F[(V,q,j) \circ (W,p,i)] = F(U,p\bar{q},j\bar{i}) = [\ker(p\bar{q})] = [\ker(\bar{q})] + [\ker(p)] = [\ker(q)] + [\ker(p)] = F(V,q,j) \circ F(W,p,i)$ since, by Exercise 2.1.4 (c), there is a conflation $\ker(\bar{q}) \rightarrowtail \ker(p\bar{q}) \twoheadrightarrow \ker(p)$ and $\ker(\bar{q}) = \ker(q)$ by the universal property of pull-backs. So, F is a functor and it induces a map on fundamental groups of classifying spaces $\pi_1 BQ\mathscr{E} \to \pi_1 \underline{K_0(\mathscr{E})} = K_0(\mathscr{E})$. It is easy to check that the composition $K_0(\mathscr{E}) \to K_0(\mathscr{E})$ is the identity. □

2.2.5 Definition of $K(\mathscr{E})$

Let $\mathscr{E}$ be a small exact category. The *K-theory space of $\mathscr{E}$* is the pointed topological space

$$K(\mathscr{E}) = \Omega BQ\mathscr{E}$$

with base point the constant loop based at $0 \in Q\mathscr{E}$. The *K-groups of $\mathscr{E}$* are the homotopy groups $K_i(\mathscr{E}) = \pi_i K(\mathscr{E}) = \pi_{i+1} BQ\mathscr{E}$ of the K-theory space of $\mathscr{E}$. An exact functor $\mathscr{E} \to \mathscr{E}'$ induces a functor $Q\mathscr{E} \to Q\mathscr{E}'$ on Q-constructions, and thus, continuous maps $BQ\mathscr{E} \to BQ\mathscr{E}'$ and $K(\mathscr{E}) \to K(\mathscr{E}')$ compatible with composition of exact functors. Therefore, the K-theory space and the K-groups are functorial with respect to exact functors between small exact categories. By Proposition 2.2.4, the group $K_0(\mathscr{E})$ defined in this way coincides with the group defined in Sect. 2.1.5.

2.2.6 Definition of $K(R)$, $K(X)$, $G(X)$

For a ring R and a scheme X, the K-theory spaces $K(R)$ and $K(X)$ are the K-theory spaces associated with the exact categories $\mathrm{Proj}(R)$ of finitely generated projective R-modules and $\mathrm{Vect}(X)$ of vector bundles on X[2]. For a noetherian scheme X, the G-theory space $G(X)$ is the K-theory space associated with the abelian category $\mathrm{Coh}(X)$ of coherent O_X-modules.

2.2.7 Remark

There is actually a slight issue with the Definition 2.2.6. The categories $\mathrm{Proj}(R)$ and $\mathrm{Vect}(X)$ have too many objects, so many that they do not form a set; the same is true for $\mathrm{Coh}(X)$. But a topological space is a *set* with a topology. Already the 0-skeletons of $BQ\mathrm{Proj}(R)$ and $BQ\mathrm{Vect}(X)$ – which are the collections of objects of $\mathrm{Proj}(R)$ and $\mathrm{Vect}(X)$ – are too large. To get around this problem, one has to choose "small models" of $\mathrm{Proj}(R)$ and $\mathrm{Vect}(X)$ in order to define the K-theory spaces of R and X. More precisely, one has to choose equivalences of

[2] See footnote in Definition 2.1.8.

exact categories $\mathscr{E}_R \simeq \mathrm{Proj}(R)$ and $\mathscr{E}_X \simeq \mathrm{Vect}(X)$ where $\mathscr{E}_R$ and $\mathscr{E}_X$ are small categories, i.e., categories which have a set of objects as opposed to a class of objects. This is possible because $\mathrm{Proj}(R)$ and $\mathrm{Vect}(X)$ only have a *set* of isomorphism classes of objects (Exercise!). Given such a choice of equivalence, one sets $K(R) = K(\mathscr{E}_R)$ and $K(X) = K(\mathscr{E}_X)$. For any other choice of equivalences $\mathscr{E}'_R \simeq \mathrm{Proj}(R)$ and $\mathscr{E}'_X \simeq \mathrm{Vect}(X)$ as above, there are equivalences $\mathscr{E}_R \simeq \mathscr{E}'_R$ and $\mathscr{E}_X \simeq \mathscr{E}'_X$ compatible with the corresponding equivalences with $\mathrm{Proj}(R)$ and $\mathrm{Vect}(X)$ which are unique up to equivalence of functors. It follows from Lemma A.1.6 that these equivalences induce homotopy equivalences of K-theory spaces $K(\mathscr{E}_R) \simeq K(\mathscr{E}'_R)$ and $K(\mathscr{E}_X) \simeq K(\mathscr{E}'_X)$ which are unique up to homotopy. Usually, one avoids these issues by working in a fixed "universe".

In order to reconcile the definition of $K(R)$ given above with the plus-construction given in Cortiñas' lecture [21], we cite the following theorem of Quillen a proof of which can be found in [35].

2.2.8 Theorem ($Q = +$)

There is a natural homotopy equivalence

$$BGL(R)^+ \simeq \Omega_0 BQ\mathrm{Proj}(R),$$

where Ω_0 stands for the connected component of the constant loop in the full loop space. In particular, there are natural isomorphisms for $i \geq 1$

$$\pi_i BGL(R)^+ \cong \pi_{i+1} BQ\mathrm{Proj}(R).$$

Note that the space denoted by $K(R)$ in Cortiñas' lecture [21] is the connected component of 0 of the space we here denote by $K(R)$.

2.2.9 Warning

Some authors define $K(R)$ to be $K_0(R) \times BGL(R)^+$ as functors in R. Strictly speaking, this is wrong: there is no zig–zag of homotopy equivalences between $K_0(R) \times BGL(R)^+$ and $\Omega BQ\mathrm{Proj}(R)$ which is functorial in R.

The problem is not that the usual construction of $BGL(R)^+$ involves choices when attaching 2 and 3-cells. The plus-construction can be made functorial. For instance, Bousfield–Kan's $\mathbb{Z}$-completion $\mathbb{Z}_\infty BGl(R)$ does the job by [10, I 5.5, V 3.3] (compare [10, VII 3.4]). The problem is that one cannot write $K(R)$ functorially as a product of $K_0(R)$ and $BGL(R)^+$. To explain this point, let R be any ring, ΓR be the cone ring of R (see Cortiñas' lecture [21]), and ΣR be the suspension ring of R which is the factor ring $\Gamma R / M_\infty(R)$ of the cone ring by the two-sided ideal $M_\infty R$ of finite matrices. Let $\tilde{R} = \Gamma R \times_{\Sigma R} \Gamma R$. The fibre product square of (unital) rings defining $\tilde{R}$ induces commutative diagrams

$$\begin{array}{ccc} K(\tilde{R}) & \longrightarrow & K(\Gamma R) \\ \downarrow & & \downarrow \\ K(\Gamma R) & \longrightarrow & K(\Sigma R) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_0(\tilde{R}) & \longrightarrow & K_0(\Gamma R) \\ \downarrow & & \downarrow \\ K_0(\Gamma R) & \longrightarrow & K_0(\Sigma R). \end{array}$$

Using Quillen's Q-construction, or other functorial versions of K-theory, one can show that the left square is homotopy cartesian, and that there is a non-unital ring map $R \to \tilde{R}$ which induces isomorphisms in K-theory. If the K-theory space $K(R)$ were the product $K_0(R) \times BGL(R)^+$ in a functorial way, the set $K_0(R)$ considered as a topological space with the discrete topology would be a natural retract of $K(R)$. Since the left-hand square in the diagram above is homotopy cartesian, its retract, the right-hand square, would have to be homotopy cartesian as well. This is absurd since $K_0(\Gamma R) = 0$ for all rings R, and $K_0(\tilde{R}) = K_0(R) \neq 0$ for most rings.

2.3 Quillen's Fundamental Theorems

In what follows we will simply cite several fundamental theorems of Quillen. Their proofs in [73] are very readable and highly recommended.

2.3.1 Serre Subcategories and Exact Sequences of Abelian Categories

Let $\mathscr{A}$ be an abelian category. A *Serre subcategory* of $\mathscr{A}$ is a full subcategory $\mathscr{B} \subset \mathscr{A}$ with the property that for a conflation in $\mathscr{A}$

$$M_0 \rightarrowtail M_1 \twoheadrightarrow M_2, \quad \text{we have} \quad M_1 \in \mathscr{B} \iff M_0 \text{ and } M_2 \in \mathscr{B}.$$

It is easy to see that a Serre subscategory $\mathscr{B}$ is itself an abelian category, and that the inclusion $\mathscr{B} \subset \mathscr{A}$ is fully exact. Given a Serre subcategory $\mathscr{B} \subset \mathscr{A}$, one can (up to set theoretical issues which do not exist when $\mathscr{A}$ is small) construct the quotient abelian category $\mathscr{A}/\mathscr{B}$ which has the universal property of a quotient object in the category of exact categories. The quotient abelian category $\mathscr{A}/\mathscr{B}$ is equivalent to the localization $\mathscr{A}[S^{-1}]$ of $\mathscr{A}$ with respect to the class S of morphisms f in $\mathscr{A}$ for which $\ker(f)$ and $\operatorname{coker}(f)$ are isomorphic to objects in $\mathscr{B}$. The class S satisfies a calculus of fractions (Exercise!) and $\mathscr{A}[S^{-1}]$ has a very explicit description; see Appendix A, Sect. 2.6. We will call $\mathscr{B} \to \mathscr{A} \to \mathscr{A}/\mathscr{B}$ an *exact sequence of abelian categories*. More details can be found in [29, 70].

The following two theorems are proved in [73, Sect. 5 Theorem 5] and [73, Sect. 5 Theorem 4].

2.3.2 Theorem (Quillen's Localization Theorem)

Let $\mathscr{A}$ be a small abelian category, and let $\mathscr{B} \subset \mathscr{A}$ be a Serre subcategory. Then the sequence of topological spaces

$$BQ(\mathscr{B}) \to BQ(\mathscr{A}) \to BQ(\mathscr{A}/\mathscr{B})$$

is a homotopy fibration (see Appendix A, Sect. 1.7 for a definition). In particular, there is a long exact sequence of associated K-groups

$$\cdots \to K_{n+1}(\mathscr{A}) \to K_{n+1}(\mathscr{A}/\mathscr{B}) \to K_n(\mathscr{B}) \to K_n(\mathscr{A}) \to K_n(\mathscr{A}/\mathscr{B}) \to \cdots$$
$$\cdots \to K_0(\mathscr{A}) \to K_0(\mathscr{A}/\mathscr{B}) \to 0.$$

2.3.3 Theorem (*Dévissage*)

Let $\mathscr{A}$ be a small abelian category, and $\mathscr{B} \subset \mathscr{A}$ be a full abelian subcategory such that the inclusion $\mathscr{B} \subset \mathscr{A}$ is exact. Assume that every object A of $\mathscr{A}$ has a finite filtration

$$0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$$

such that the quotients A_i/A_{i-1} are in $\mathscr{B}$. Then the inclusion $\mathscr{B} \subset \mathscr{A}$ induces a homotopy equivalence

$$K(\mathscr{B}) \xrightarrow{\sim} K(\mathscr{A}).$$

In particular, it induces isomorphisms of K-groups $K_i(\mathscr{B}) \cong K_i(\mathscr{A})$.

The following are two applications of Quillen's Localization and *Dévissage* Theorems.

2.3.4 Nilpotent Extensions

Let X be a noetherian scheme and $i : Z \hookrightarrow X$ a closed subscheme corresponding to a nilpotent sheaf of ideals $I \subset O_X$. Assume $I^n = 0$. Then $i_* : \mathrm{Coh}(Z) \to \mathrm{Coh}(X)$ satisfies the hypothesis of the *Dévissage* Theorem because $\mathrm{Coh}(Z)$ can be identified with the subcategory of those coherent sheaves F on X for which $IF = 0$, and every sheaf $F \in \mathrm{Coh}(X)$ has a filtration $0 = I^n F \subset I^{n-1} F \subset \ldots \subset IF \subset F$ with quotients in $\mathrm{Coh}(Z)$. We conclude that i_* induces a homotopy equivalence $G(Z) \simeq G(X)$. In particular:

2.3.5 Theorem

For a noetherian scheme X, the closed immersion $i : X_{red} \hookrightarrow X$ induces a homotopy equivalence of G-theory spaces

$$i_* : G(X_{red}) \xrightarrow{\sim} G(X).$$

2.3.6 *G*-Theory Localization

Let X be a noetherian scheme, and $j : U \subset X$ be an open subscheme with $i : Z \subset X$ being its closed complement $X - U$. Let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the fully exact subcategory of those coherent sheaves F on X which have support in Z, that is, for which $F_{|U} = 0$. Then the sequence

$$\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X) \xrightarrow{j^*} \mathrm{Coh}(U) \qquad (3)$$

is an exact sequence of abelian categories (see Sect. 2.3.8 below). By Theorem 2.3.2, we obtain a homotopy fibration $K\mathrm{Coh}_Z(X) \to K\mathrm{Coh}(X) \to K\mathrm{Coh}(Z)$ of K-theory spaces. For another proof, see Theorem 3.3.2. The inclusion $i_* : \mathrm{Coh}(Z) \subset \mathrm{Coh}_Z(X)$ satisfies *Dévissage* (Exercise!), so we have a homotopy equivalence $K\mathrm{Coh}(Z) \simeq K\mathrm{Coh}_Z(X)$. Put together, we obtain:

2.3.7 Theorem

Let X be a noetherian scheme, and $j : U \subset X$ be an open subscheme with $i : Z \subset X$ being its closed complement $X - U$. Then the following sequence of spaces is a homotopy fibration

$$G(Z) \xrightarrow{i_*} G(X) \xrightarrow{j^*} G(U). \qquad (4)$$

In particular, there is an associated long exact sequence of G-theory groups

$$\cdots G_{i+1}(U) \to G_i(Z) \to G_i(X) \to G_i(U) \to G_{i-1}(Z) \cdots \to G_0(U) \to 0 \qquad (5)$$

2.3.8 *Proof that (3) is an Exact Sequence of Abelian Categories*

As the "kernel" of the exact functor $\mathrm{Coh}(X) \to \mathrm{Coh}(U)$, the category $\mathrm{Coh}_Z(X)$ is automatically a Serre subcategory of $\mathrm{Coh}(X)$. The composition $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X) \to \mathrm{Coh}(U)$ is trivial. Therefore, we obtain an induced functor

$$\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \to \mathrm{Coh}(U) \tag{6}$$

which we have to show is an equivalence.

The functor (6) is essentially surjective on objects. This is because for any $F \in \mathrm{Coh}(U)$, the O_X-module j_*F is quasi-coherent (X is noetherian). Therefore, j_*F is a filtered colimit $\operatorname{colim} G_i$ of its coherent sub-O_X-modules G_i. Every ascending chain of subobjects of a coherent sheaf eventually stops. Therefore, we must have $j^*G_i \cong j^*j_*F = F$ for some i.

The functor (6) is full because for $F, G \in \mathrm{Coh}(X)$, any map $f : j^*F \to j^*G$ in $\mathrm{Coh}(U)$ equals $g_{|U} \circ (t_{|U})^{-1}$ where t and g are maps in a diagram $F \stackrel{t}{\leftarrow} H \stackrel{g}{\to} G$ of coherent O_X-modules. This diagram can be taken to be the pull-back in $\mathrm{Qcoh}(X)$ of the diagram $F \to j_*j^*F \to j_*j^*G \leftarrow G$ with middle map $j^*(f)$ and outer two maps the unit of adjunction maps. The object H is coherent as it is a quasi-coherent subsheaf of the coherent sheaf $F \oplus G$. The unit of adjunction $G \to j_*j^*G$ is an isomorphism when restricted to U. Since j^* is an exact functor of abelian categories, the same is true for its pull back $t : F \to H$.

Finally, the functor (6) is faithful by the following argument. The "kernel category" of this functor is trivial, by construction. This implies that the functor (6) is conservative, i.e., detects isomorphisms. Now, let $f : F \to G$ be a map in $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X)$ such that $j^*(f) = 0$. Then $\ker(f) \to F$ and $G \to \operatorname{coker}(f)$ are isomorphisms when restricted to U. Since the functor (6) is conservative, these two maps are already isomorphisms in $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X)$ which means that $f = 0$.

Since a fully faithful and essentially surjective functor is an equivalence, we are done. □

2.3.9 Theorem (Homotopy Invariance of G-Theory [73, Proposition 4.1])

Let X and P be noetherian schemes and $f : P \to X$ be a flat map whose fibres are affine spaces (for instance, a geometric vector bundle). Then

$$f^* : G(X) \xrightarrow{\sim} G(P)$$

is a homotopy equivalence. In particular, $G_i(X \times \mathbb{A}^1) \cong G_i(X)$.

2.3.10 *K***-Theory of Regular Schemes**

Let X be a regular noetherian and separated scheme. Then the inclusion $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ induces a homotopy equivalence $K\,\mathrm{Vect}(X) \simeq K\,\mathrm{Coh}(X)$, that is,

$$K(X) \xrightarrow{\sim} G(X)$$

(see the Poincaré Duality Theorem 3.3.5; classically it also follows from Quillen's Resolution Theorem 2.3.12 below). Thus, Theorems 2.3.7 and 2.3.9 translate into theorems about $K(X)$ when X is regular, noetherian and separated. For instance, Theorem 2.3.9 together with Poincaré Duality implies that the projection $X \times \mathbb{A}^n \to X$ induces isomorphisms

$$K_i(X) \xrightarrow{\cong} K_i(X \times \mathbb{A}^n)$$

whenever X is regular noetherian separated.

Besides the results mentioned above, Quillen proves two fundamental theorems which are also of interest: the Additivity Theorem [73, Sect. 3 Theorem 2 and Corollary 1] and the Resolution Theorem [73, Sect. 4 Theorem 3]. Both are special cases of the Thomason–Waldhausen Localization Theorem 3.2.23 which is stated below. We simply quote Quillen's theorems here, and in Sect. 3.3 we give a proof of them based on the Localization Theorem. However, we have to mention that the Additivity Theorem is used in the proof of the Localization Theorem.

2.3.11 Theorem (Additivity [73, Sect. 3, Corollary 1])

Let $\mathscr{E}$ and $\mathscr{E}'$ be exact categories, and let

$$0 \to F_{-1} \to F_0 \to F_1 \to 0$$

be a sequence of exact functors $F_i : \mathscr{E} \to \mathscr{E}'$ such that $F_{-1}(A) \rightarrowtail F_0(A) \twoheadrightarrow F_1(A)$ is a conflation for all objects A in $\mathscr{E}$. Then the induced maps on K-groups satisfy

$$F_0 = F_{-1} + F_1 : K_i(\mathscr{E}) \to K_i(\mathscr{E}').$$

2.3.12 Theorem (Resolution [73, Sect. 4])

Let $\mathscr{A} \subset \mathscr{B}$ be a fully exact subcategory of an exact category $\mathscr{B}$. Assume that

(a) if $M_{-1} \rightarrowtail M_0 \twoheadrightarrow M_1$ is a conflation in $\mathscr{B}$ with $M_0, M_1 \in \mathscr{A}$, then $M_{-1} \in \mathscr{A}$, and
(b) for every $B \in \mathscr{B}$ there is an exact sequence

$$0 \to A_n \to A_{n-1} \to \cdots A_0 \to B \to 0$$

with $A_i \in \mathscr{A}$.

Then the inclusion $\mathscr{A} \subset \mathscr{B}$ induces a homotopy equivalence of K-theory spaces

$$K(\mathscr{A}) \xrightarrow{\sim} K(\mathscr{B}).$$

2.4 Negative K-Groups

Besides the positive K-groups, one can also define the negative K-groups K_i with $i < 0$. They extend certain K_0 exact sequences to the right (see Cortiñas' lecture [21]). For rings and additive (or split exact) categories they were introduced by Bass [7] and Karoubi [45]. The treatment for exact categories below follows [81].

2.4.1 Idempotent Completion

Let $\mathscr{A}$ be an additive category, and $\mathscr{B} \subset \mathscr{A}$ be a full additive subcategory. We call the inclusion $\mathscr{B} \subset \mathscr{A}$ *cofinal*, or *equivalence up to factors* if every object of $\mathscr{A}$ is a direct factor of an object of $\mathscr{B}$. If $\mathscr{A}$ and $\mathscr{B}$ are exact categories, we require moreover that the inclusion $\mathscr{B} \subset \mathscr{A}$ is fully exact, that is, the inclusion is extension closed, and it preserves and detects conflations. As an example, the category of (finitely generated) free R-modules is cofinal in the category of (finitely generated) projective R-modules.

Given an additive category $\mathscr{A}$, there is a "largest" category $\tilde{\mathscr{A}}$ of $\mathscr{A}$ such that the inclusion $\mathscr{A} \subset \tilde{\mathscr{A}}$ is cofinal. This is the *idempotent completion* $\tilde{\mathscr{A}}$ of $\mathscr{A}$. An additive category is called *idempotent complete* if for every idempotent map $p = p^2 : A \to A$, there is an isomorphism $A \cong X \oplus Y$ under which the map p corresponds to $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right) : X \oplus Y \to X \oplus Y$. The objects of the idempotent completion $\tilde{\mathscr{A}}$ of $\mathscr{A}$ are pairs (A, p) with A an object of $\mathscr{A}$ and $p = p^2 : A \to A$ an idempotent endomorphism. Maps $(A, p) \to (B, q)$ in $\tilde{\mathscr{A}}$ are maps $f : A \to B$ in $\mathscr{A}$ such that $fp = f = qf$. Composition is composition in $\mathscr{A}$, and $id_{(A,p)} = p$. Every idempotent $q = q^2 : (A, p) \to (A, p)$ corresponds to $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ under the isomorphism $(q, p-q) : (A, q) \oplus (A, p-q) \cong (A, p)$. Therefore, the category $\tilde{\mathscr{A}}$ is indeed idempotent complete. Furthermore, we have a fully faithful embedding $\mathscr{A} \subset \tilde{\mathscr{A}} : A \mapsto (A, 1)$ which is cofinal since the object (A, p) of $\tilde{\mathscr{A}}$ is a direct factor the object $(A, 1)$ of $\mathscr{A}$.

If $\mathscr{E}$ is an exact category, its idempotent completion $\tilde{\mathscr{E}}$ becomes an exact category when we declare a sequence in $\tilde{\mathscr{E}}$ to be a conflation if it is a retract (in the category of conflations) of a conflation of $\mathscr{E}$. For more details, see [94, Appendix A]. Note that the inclusion $\mathscr{E} \subset \tilde{\mathscr{E}}$ is indeed fully exact.

2.4.2 Proposition (Cofinality [36, Theorem 1.1])

Let $\mathscr{A}$ be an exact category and $\mathscr{B} \subset \mathscr{A}$ be a cofinal fully exact subcategory. Then the maps $K_i(\mathscr{B}) \to K_i(\mathscr{A})$ are isomorphisms for $i > 0$ and a monomorphism for $i = 0$. This holds in particular for $K_i(\mathscr{E}) \to K_i(\tilde{\mathscr{E}})$.

2.4.3 Negative K-Theory and the Spectrum $\mathbb{K}(\mathscr{E})$

To any exact category $\mathscr{E}$, one can associate a new exact category $S\mathscr{E}$ (see Sect. 2.4.6), called *the suspension of* $\mathscr{E}$, such that there is a natural homotopy equivalence [81]

$$K(\tilde{\mathscr{E}}) \xrightarrow{\sim} \Omega K(S\mathscr{E}). \tag{7}$$

If $\mathscr{E} = \mathrm{Proj}(R)$ one can take $S\mathscr{E} = \mathrm{Proj}(\Sigma R)$ where ΣR is the suspension ring of R; see Cortiñas' lecture [21]) for the definition of ΣR.

One uses the suspension construction to slightly modify the definition of algebraic K-theory in order to incorporate negative K-groups as follows. One sets $\mathbb{K}_i(\mathscr{E}) = K_i(\mathscr{E})$ for $i \geq 1$, $\mathbb{K}_0(\mathscr{E}) = K_0(\tilde{\mathscr{E}})$ and $\mathbb{K}_i(\mathscr{E}) = K_0(\widetilde{S^{-i}\mathscr{E}})$ for $i < 0$. Since $\mathrm{Vect}(X)$, $\mathrm{Coh}(X)$ and $\mathrm{Proj}(R)$ are all idempotent complete, we have the equalities $\mathbb{K}_0 \mathrm{Vect}(X) = K_0 \mathrm{Vect}(X) = K_0(X)$, $\mathbb{K}_0 \mathrm{Coh}(X) = K_0 \mathrm{Coh}(X) = G_0(X)$ and $\mathbb{K}_0 \mathrm{Proj}(R) = K_0 \mathrm{Proj}(R) = K_0(R)$. In these cases, we have not changed the definition of K-theory; we have merely introduced "negative K-groups" $\mathbb{K}_i$ for $i < 0$. For this reason, we may write $K_i(X)$ and $K_i(R)$ instead of $\mathbb{K}_i(X)$ and $\mathbb{K}_i(R)$ for all $i \in \mathbb{Z}$.

In a fancy language, one constructs a spectrum $\mathbb{K}(\mathscr{E})$ whose homotopy groups are the groups $\mathbb{K}_i(\mathscr{E})$ for $i \in \mathbb{Z}$. The n-th space of this spectrum is $K(S^n\mathscr{E})$, and the structure maps are given by (7). For terminology and basic properties of spectra, we refer the reader to Appendix A, Sect. 1.8.

By Bass' Fundamental Theorem stated in Theorem 3.5.3 below there is a split exact sequence for $i \in \mathbb{Z}$

$$0 \to K_i(R) \to K_i(R[T]) \oplus K_i(R[T^{-1}]) \to K_i(R[T, T^{-1}]) \to K_{i-1}(R) \to 0.$$

One can use this sequence to give a recursive definition of the negative K-groups $K_i(R)$ for $i < 0$, starting with the functor K_0. This was Bass' original definition.

2.4.4 Remark

Although Quillen did not define negative K-groups of exact categories, all K-theory statements in [73] and [35] extend to negative K-theory. The only exceptions are Quillen's Localization Theorem 2.3.2 and the *Dévissage* Theorem 2.3.3. To insure that these two theorems also extend to negative K-theory, we need the abelian categories in question to be noetherian, though it is conjectured that the noetherian hypothesis is unnecessary.

2.4.5 Remark

Not much is known about $\mathbb{K}_i(\mathscr{E})$ when $i < 0$, even though we believe their calculations to be easier than those of $K_i(\mathscr{E})$ when $i \geq 0$. However, we do know the following. We have $K_i(R) = 0$ for $i < 0$ when R is a regular noetherian ring [7]. We have $\mathbb{K}_{-1}(\mathscr{A}) = 0$ for any abelian category $\mathscr{A}$ [82, Theorem 6], and $\mathbb{K}_i(\mathscr{A}) = 0$ for $i < 0$ when $\mathscr{A}$ is a noetherian abelian category [82, Theorem 7]. In particular, $K_{-1}(R) = 0$ for a regular coherent ring R, and $K_i(X) = 0$ for $i < 0$ when X is any regular noetherian and separated scheme. In [19] it is shown that $K_i(X) = 0$ for $i < -d$ when X is a d-dimensional scheme essentially of finite type over a field of characteristic 0, but $K_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$ can be non-zero [75]. For finite-type schemes over fields of positive characteristic, the same is true provided strong resolution of singularities holds over the base field [30]. It is conjectured that $K_i(\mathbb{Z}G) = 0$ for $i < -1$ when G is a finitely presented group [42]. For results in this direction see [56].

2.4.6 Construction of the Suspension $S\mathscr{E}$

Let $\mathscr{E}$ be an exact category. The countable envelope $\mathscr{F}\mathscr{E}$ of $\mathscr{E}$ is an exact category whose objects are sequences

$$A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \dots$$

of inflations in $\mathscr{E}$. The morphism set from a sequence A_* to another sequence B_* is

$$\mathrm{Hom}_{\mathscr{F}\mathscr{E}}(A_*, B_*) = \lim_i \operatorname*{colim}_j \mathrm{Hom}_{\mathscr{E}}(A_i, B_j).$$

A sequence in $\mathscr{F}\mathscr{E}$ is a conflation iff it is isomorphic in $\mathscr{F}\mathscr{E}$ to the sequence of maps of sequences $A_* \to B_* \to C_*$ where $A_i \to B_i \to C_i$ is a conflation in $\mathscr{E}$ for $i \in \mathbb{N}$. Colimits of sequences of inflations exist in $\mathscr{F}\mathscr{E}$, and are exact. In particular, $\mathscr{F}\mathscr{E}$ has exact countable direct sums. There is a fully faithful exact functor $\mathscr{E} \to \mathscr{F}\mathscr{E}$ which sends an object $X \in \mathscr{E}$ to the constant sequence $X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$. For details of the construction see [46, Appendix B] where $\mathscr{F}\mathscr{E}$ was denoted by $\mathscr{E}^{\sim}$.

The suspension $S\mathscr{E}$ of $\mathscr{E}$ is the quotient $\mathscr{F}\mathscr{E}/\mathscr{E}$ of the countable envelope $\mathscr{F}\mathscr{E}$ by the subcategory $\mathscr{E}$. The quotient is taken in the category of small exact categories. The proof of the existence of $\mathscr{F}\mathscr{E}/\mathscr{E}$ and an explicit description is given in [81]. By [81, Theorem 2.1 and Lemma 3.2] the sequence $\mathscr{E} \to \mathscr{F}\mathscr{E} \to S\mathscr{E}$ induces a homotopy fibration $K(\tilde{\mathscr{E}}) \to K(\mathscr{F}\mathscr{E}) \to K(S\mathscr{E})$ of K-theory spaces. Since $\mathscr{F}\mathscr{E}$ has exact countable direct sums, the total space $K(\mathscr{F}\mathscr{E})$ of the fibration is contractible. This yields the homotopy equivalence Sect. 2.4.3 (7).

3 Algebraic K-Theory and Triangulated Categories

3.1 The Grothendieck-Group of a Triangulated Category

Most calculations in the early days of K-theory were based on Quillen's Localization Theorem 2.3.2 for abelian categories together with *Dévissage* Theorem 2.3.3. Unfortunately, not all K-groups are (not even equivalent to) the K-groups of some abelian category, notably $K(X)$ where X is some singular variety. Also, there is no satisfactory generalization of Quillen's Localization Theorem to exact categories which would apply to all situations K-theorists had in mind. This is where triangulated categories come in. They provide a flexible framework that allows us to prove many results which cannot be proved with Quillen's methods alone.

For the rest of this subsection, we will assume that the reader is familiar with Appendix A, Sects. 2.1–2.7.

3.1.1 Definition of $K_0(\mathscr{T})$

Let $\mathscr{T}$ be a small triangulated category. The Grothen-dieck-group $K_0(\mathscr{T})$ of $\mathscr{T}$ is the abelian group freely generated by symbols $[X]$ for every object X of $\mathscr{T}$, modulo the relation $[X]+[Z]=[Y]$ for every distinguished triangle $X \to Y \to Z \to TX$ in $\mathscr{T}$.

3.1.2 Remark

As in Remark 2.1.6, we have $[X]=[Y]$ if there is an isomorphism $f : X \cong Y$ in view of the distinguished triangle $X \xrightarrow{f} Y \to 0 \to TX$. We also have $[X \oplus Y] = [X] + [Y]$ because there is a distinguished triangle $X \to X \oplus Y \to Y \to TX$ which is the direct sum of the distinguished triangles $X \to X \to 0 \to TX$ and $0 \to Y \to Y \to 0$. Moreover, the distinguished triangle $X \to 0 \to TX \to TX$ shows that $[TX] = -[X]$. In particular, every element in $K_0(\mathscr{T})$ can be represented as $[X]$ for some object X in $\mathscr{T}$.

One would like to relate the Grothendieck-group $K_0(\mathscr{E})$ of an exact category $\mathscr{E}$ to the Grothendieck-group of a triangulated category associated with $\mathscr{E}$. This *rôle* is played by the bounded derived category $D^b(\mathscr{E})$ of $\mathscr{E}$.

3.1.3 The Bounded Derived Category of an Exact Category

Let $\mathrm{Ch}^b \mathscr{E}$ be the exact category of bounded chain complexes in $\mathscr{E}$; see Example 2.1.2 (e). Call a bounded chain complex (A,d) in $\mathscr{E}$ *strictly acyclic* if every differential $d^i : A^i \to A^{i+1}$ can be factored as $A^i \twoheadrightarrow Z^{i+1} \rightarrowtail A^{i+1}$ such that the sequence $Z^i \rightarrowtail A^i \twoheadrightarrow Z^{i+1}$ is a conflation in $\mathscr{E}$ for all $i \in \mathbb{Z}$. A bounded chain complex is called *acyclic* if it is homotopy equivalent to a strictly acyclic chain complex. A map $f : (A,d) \to (B,d)$ is called *quasi-isomorphism* if its cone $C(f)$ (see Appendix A, Sect. 2.5 for a definition) is acyclic.

As a category, the *bounded derived category* $D^b(\mathscr{E})$ is the category

$$D^b(\mathscr{E}) = [\mathrm{quis}^{-1}]\,\mathrm{Ch}^b \mathscr{E}$$

obtained from the category of bounded chain complexes $\mathrm{Ch}^b \mathscr{E}$ by formally inverting the quasi-isomorphisms. A more explicit description of $D^b(\mathscr{E})$ is obtained as follows. Let $\mathscr{K}^b(\mathscr{E})$

be the homotopy category of bounded chain complexes in $\mathscr{E}$. Its objects are bounded chain complexes in $\mathscr{E}$, and maps are chain maps up to chain homotopy. With the same definitions as in Appendix A, Sect. 2.5, the homotopy category $\mathscr{K}^b(\mathscr{E})$ is a triangulated category. Let $\mathscr{K}^b_{ac}(\mathscr{E}) \subset \mathscr{K}^b(\mathscr{E})$ be the full subcategory of acyclic chain complexes. The category $\mathscr{K}^b_{ac}(\mathscr{E})$ is closed under taking cones and shifts T and T^{-1} in $\mathscr{K}^b(\mathscr{E})$. Therefore, it is a full triangulated subcategory of $\mathscr{K}^b(\mathscr{E})$. The bounded derived category of the exact category $\mathscr{E}$ is the Verdier quotient $\mathscr{K}^b(\mathscr{E})/\mathscr{K}^b_{ac}(\mathscr{E})$.

It turns out that distinguished triangles in $D^b(\mathscr{E})$ are precisely those triangles which are isomorphic to the standard triangles constructed as follows. A conflation $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ of chain complexes in $\mathrm{Ch}^b\mathscr{E}$ yields the standard distinguished triangle

$$X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{q\circ s^{-1}} TX$$

in $D^b(\mathscr{E})$ where s is the quasi-isomorphism $C(i) \to C(i)/C(id_X) \cong Z$, and q is the canonical map $C(f) \to TX$ as in Appendix A, Sect. 2.5. For more details, see [48].

3.1.4 Exercise

Let $\mathscr{E}$ be an exact category. Consider the objects of $\mathscr{E}$ as chain complexes concentrated in degree zero. Show that the map $K_0(\mathscr{E}) \to K_0(D^b\mathscr{E})$ given by $[X] \mapsto [X]$ is an isomorphism. *Hint*: The inverse $K_0(D^b\mathscr{E}) \to K_0(\mathscr{E})$ is given by $[A,d] \mapsto \Sigma_i(-1)^i[A^i]$. The point is to show that this map is well-defined.

3.1.5 Definition

A sequence of triangulated categories $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is called *exact* if the composition sends $\mathscr{A}$ to 0, if $\mathscr{A} \to \mathscr{B}$ is fully faithful and coincides (up to equivalence) with the subcategory of those objects in $\mathscr{B}$ which are zero in $\mathscr{C}$, and if the induced functor $\mathscr{B}/\mathscr{A} \to \mathscr{C}$ from the Verdier quotient $\mathscr{B}/\mathscr{A}$ to $\mathscr{C}$ is an equivalence.

3.1.6 Exercise

Let $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ be an exact sequence of triangulated categories. Then the following sequence of abelian groups is exact

$$K_0(\mathscr{A}) \to K_0(\mathscr{B}) \to K_0(\mathscr{C}) \to 0. \tag{8}$$

Hint: Show that the map $K_0(\mathscr{C}) \to \mathrm{coker}(K_0(\mathscr{A}) \to K_0(\mathscr{B}))$ given by $[C] \mapsto [B]$ is well-defined where $B \in \mathscr{B}$ is any object whose image in the category $\mathscr{C}$ is isomorphic to the object C.

How can we decide whether a sequence of exact categories induces an exact sequence of bounded derived categories so that we could apply Exercise 3.1.6 and Theorems 3.2.23 and 3.2.27 below? For this, the following facts are often quite useful.

3.1.7 Some Criteria and Facts

Let $\mathscr{A} \to \mathscr{B}$ be an exact functor between exact categories.

(a) If $\mathscr{B}$ is the localization $\Sigma^{-1}\mathscr{A}$ of $\mathscr{A}$ with respect to a set of maps Σ which satisfies a calculus of left (or right) fractions, then $\mathrm{Ch}^b\mathscr{B}$ is the localization of $\mathrm{Ch}^b\mathscr{A}$ with respect to the set of maps which degree-wise belong to Σ. This set of maps also satisfies a calculus of left (right) fractions, and therefore, $D^b(\mathscr{A}) \to D^b(\mathscr{B})$ is a localization. In particular, $D^b(\mathscr{B})$ is the Verdier quotient of $D^b(\mathscr{A})$ modulo the full triangulated subcategory of objects which are zero in $D^b(\mathscr{B})$.

(b) Suppose that $\mathscr{A}$ is a fully exact subcategory of $\mathscr{B}$. If for any inflation $A \rightarrowtail B$ in $\mathscr{B}$ with $A \in \mathscr{A}$ there is a map $B \to A'$ with $A' \in \mathscr{A}$ such that the composition $A \to A'$ is an inflation in $\mathscr{A}$, then the functor $D^b(\mathscr{A}) \to D^b(\mathscr{B})$ is fully faithful [48, 12.1].

(c) If $\mathscr{A} \to \mathscr{B}$ is a cofinal fully exact inclusion, then $D^b\mathscr{A} \to D^b\mathscr{B}$ is fully faithful and cofinal. If $\mathscr{E}$ is an idempotent complete exact category, then its bounded derived category $D^b\mathscr{E}$ is also idempotent complete [16, Theorem 2.8].

(d) The bounded derived category $D^b\mathscr{E}$ of an exact category $\mathscr{E}$ is generated (as a triangulated category) by the objects of $\mathscr{E}$ (considered as complexes concentrated in degree zero) in the sense that $D^b\mathscr{E}$ is the smallest triangulated subcategory of $D^b\mathscr{E}$ closed under isomorphisms which contains the objects of $\mathscr{E}$ (Exercise!).

We illustrate these facts by giving an example of a sequence of exact categories which induces an exact sequence of bounded derived categories.

3.1.8 Example

Let R be a ring with unit, and let $S \subset R$ be a multiplicative set of central non-zero-divisors in R. Let $\mathscr{H}_S(R) \subset R$-Mod be the full subcategory of those left R-modules which are direct factors of finitely presented S-torsion R-modules of projective dimension at most 1. This category is extension closed in the category of all left R-modules. We consider it as an exact category where a sequence in $\mathscr{H}_S(R)$ is a conflation if it is a conflation of R-modules. Let $\mathscr{P}^1(R) \subset R$-Mod be the full subcategory of those left R-modules M which fit into an exact sequence $0 \to P \to M \to H \to 0$ of R-modules where P is finitely generated projective and $H \in \mathscr{H}_S(R)$. The inclusion $\mathscr{P}^1(R) \subset R$-Mod is closed under extensions and we consider $\mathscr{P}^1(R)$ as a fully exact subcategory of R-Mod. Finally, let $\mathscr{P}'(S^{-1}R) \subset \mathrm{Proj}(S^{-1}R)$ be the full additive subcategory of those finitely generated projective $S^{-1}R$-modules which are localizations of finitely generated projective R-modules.

3.1.9 Lemma

The sequence $\mathscr{H}_S(R) \to \mathscr{P}^1(R) \to \mathscr{P}'(S^{-1}R)$ induces an exact sequence of associated bounded derived categories. Moreover, the inclusion $\mathrm{Proj}(R) \subset \mathscr{P}^1(R)$ induces an equivalence $D^b\mathrm{Proj}(R) \xrightarrow{\simeq} D^b\mathscr{P}^1(R)$ of categories. In particular, there is an exact sequence of triangulated categories

$$D^b\mathscr{H}_S(R) \to D^b\mathrm{Proj}(R) \to D^b\mathscr{P}'(S^{-1}R). \tag{9}$$

For instance, let R be a Dedekind domain and $S \subset R$ be the set of non-zero elements in R. Then $S^{-1}R = K$ is the field of fractions of R, the category $\mathscr{P}'(S^{-1}R)$ is the category of finite dimensional K-vector spaces, and $\mathscr{H}_S(R)$ is the category of finitely generated torsion R-modules.

Proof of Lemma 3.1.9

Let $\mathscr{P}^1_0(R) \subset \mathscr{P}^1(R)$ be the full subcategory of those R-modules M which fit into an exact sequence $0 \to P \to M \to H \to 0$ with P finitely generated projective, and H an S-torsion R-module of projective dimension at most 1. The inclusion $\mathscr{P}^1_0(R) \subset \mathscr{P}^1(R)$ is fully exact and cofinal. By Sect. 3.1.7 (c), the induced triangle functor $D^b\mathscr{P}^1_0(R) \to D^b\mathscr{P}^1(R)$ is fully faithful and cofinal. By the dual of Sect. 3.1.7 (b), the functor $D^b\operatorname{Proj}(R) \to D^b\mathscr{P}^1_0(R)$ is fully faithful. By Sect. 3.1.7 (d), this functor is also essentially surjective (hence an equivalence) since every object of $\mathscr{P}^1_0(R)$ has projective dimension at most 1. The category $\operatorname{Proj}(R)$ is idempotent complete. By Sect. 3.1.7 (c), the same is true for $D^b\operatorname{Proj}(R)$. It follows that the cofinal inclusions $D^b\operatorname{Proj}(R) \subset D^b\mathscr{P}^1_0(R) \subset D^b\mathscr{P}^1(R)$ are all equivalences.

For any finitely generated projective R-modules P and Q, the natural map $S^{-1}\operatorname{Hom}_R(P,Q) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}P, S^{-1}Q)$ is an isomorphism. Therefore, the category $\mathscr{P}'(S^{-1}R)$ is obtained from $\operatorname{Proj}(R)$ by a calculus of right fractions with respect to the multiplication maps $P \to P : x \mapsto sx$ for $s \in S$ and $P \in \operatorname{Proj}(R)$. By Sect. 3.1.7 (a), the functor $D^b\operatorname{Proj}(R) \cong D^b\mathscr{P}^1(R) \to D^b\mathscr{P}'(S^{-1}R)$ is a localization, that is, $D^b\mathscr{P}'(S^{-1}R)$ is the Verdier quotient $D^b\mathscr{P}^1(R)/\mathscr{K}$ where $\mathscr{K} \subset D^b\mathscr{P}^1(R)$ is the full triangulated subcategory of those objects which are zero in $D^b\mathscr{P}'(S^{-1}R)$. The functor $D^b\mathscr{H}_S(R) \to D^b\mathscr{P}^1(R)$ is fully faithful by Sect. 3.1.7 (b), and it factors through $\mathscr{K}$. We have to show that the full inclusion $D^b\mathscr{H}_S(R) \subset \mathscr{K}$ is an equivalence, that is, we have to show that every object E in $\mathscr{K}$ is isomorphic to an object of $D^b\mathscr{H}_S(R)$. Since $D^b\operatorname{Proj}(R) \cong D^b\mathscr{P}^1(R)$, we can assume that E is a complex of projective R-modules. The acyclic complex $S^{-1}E$ is a bounded complex of projective $S^{-1}R$-modules, and thus it is contractible. The degree-wise split inclusion $i : E \rightarrowtail CE$ of E into its cone CE induces a map of contractible complexes $S^{-1}E \rightarrowtail S^{-1}CE$ which a fortiori is degree-wise split injective. A degree-wise split inclusion of contractible complexes always has a retraction; see Example 3.2.6. Applied to the last map we obtain a retraction $r : S^{-1}CE \to S^{-1}E$ in $\operatorname{Ch}^b\mathscr{P}'(S^{-1}R)$. We can write r as a right fraction ps_0^{-1} with $p : CE \to E$ a chain map and $s_0 : CE \to CE : x \mapsto s_0x$ the multiplication by s_0 for some $s_0 \in S$. After localization at S, we have $1 = ri = ps_0^{-1}i = s_0^{-1}pi$, and thus, $s_0 = pi$ since the elements of S are central. By the calculus of fractions, there is an $s_1 \in S$ such that $p \circ i \circ s_1 = s_0 s_1$. Since the set $S \subset R$ consists of non-zero-divisors and since E consists of projective modules in each degree, the morphism $s : E \to E$ given by $x \mapsto sx$ with $s \in S$ is injective. Therefore, we obtain a conflation of chain complexes of R-modules $CE \rightarrowtail E \oplus CE/is_1(E) \twoheadrightarrow E/s_0s_1E$ where the maps $CE \to E$ and $CE/is_1(E) \to E/s_0s_1E$ are induced by p and the other two maps are (up to sign) the natural quotient maps. This shows that in $D^b\mathscr{P}^1(R)$ we have an isomorphism $E \oplus CE/is_1(E) \cong E/s_0s_1E$. In particular, the chain complex E is a direct factor of an object of $D^b\mathscr{H}_S(R)$, namely of E/s_0s_1E. By Sect. 3.1.7 (c), the category $D^b\mathscr{H}_S(R)$ is idempotent complete. Hence, the complex E must be in $D^b\mathscr{H}_S(R)$. □

Lemma 3.1.9 illustrates a slight inconvenience. The definition of the K-theory of $S^{-1}R$ uses all finitely generated projective $S^{-1}R$-modules and not only those lying in $\mathscr{P}'(S^{-1}R)$. Therefore, one would like to replace $D^b\mathscr{P}'(S^{-1}R)$ with $D^b\operatorname{Proj}(S^{-1}R)$ in Lemma 3.1.9, but these two categories are not equivalent, in general. However, the inclusion $D^b\mathscr{P}'(S^{-1}R) \subset D^b\operatorname{Proj}(S^{-1}R)$ is an equivalence up to factors by Sect. 3.1.7 (c). This observation together with Neeman's Theorem 3.4.5 below motivates the following.

3.1.10 Definition

A sequence of triangulated categories $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is *exact up to factors* if the composition is zero, the functor $\mathscr{A} \to \mathscr{B}$ is fully faithful, and the induced functor $\mathscr{B}/\mathscr{A} \to \mathscr{C}$ is an equivalence up to factors. (See Sect. 2.4.1 for the definition of "exact up to factors".)

In this situation, the inclusion $\mathscr{A} \subset \mathscr{A}'$ of $\mathscr{A}$ into the full subcategory $\mathscr{A}'$ of $\mathscr{B}$ whose objects are zero in $\mathscr{C}$ is also an equivalence up to factors [66, Lemma 2.1.33]. Thus, a sequence $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ of triangulated categories is exact up to factors if, up to equivalences up to factors, $\mathscr{A}$ is the kernel category of $\mathscr{B} \to \mathscr{C}$ and $\mathscr{C}$ is the cokernel category of $\mathscr{A} \to \mathscr{B}$.

3.1.11 Example

Keep the hypothesis and notation of Example 3.1.8. The sequence of triangulated categories

$$D^b \mathscr{H}_S(R) \to D^b \operatorname{Proj}(R) \to D^b \operatorname{Proj}(S^{-1}R)$$

is exact up to factors but not exact, in general.

3.1.12 Idempotent Completion of Triangulated Categories

A triangulated category $\mathscr{A}$ is, in particular, an additive category. So we can speak of its idempotent completion $\tilde{\mathscr{A}}$; see Sect. 2.4.1. It turns out that $\tilde{\mathscr{A}}$ can be equipped with the structure of a triangulated category such that the inclusion $\mathscr{A} \subset \tilde{\mathscr{A}}$ is a triangle functor [16]. A sequence in $\tilde{\mathscr{A}}$ is a distinguished triangle if it is a direct factor of a distinguished triangle in $\mathscr{A}$. Note that the triangulated categories $D^b \operatorname{Vect}(X)$, $D^b \operatorname{Coh}(X)$ and $D^b \operatorname{Proj}(R)$ are all idempotent complete by Sect. 3.1.7 (c).

3.1.13 Exercise

(a) Let $\mathscr{A} \subset \mathscr{B}$ be a cofinal inclusion of triangulated categories. Show that the map $K_0(\mathscr{A}) \to K_0(\mathscr{B})$ is injective [91, Corollary 2.3].
(b) Let $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ be a sequence of triangulated categories which is exact up to factors. Then the following sequence of abelian groups is exact

$$K_0(\tilde{\mathscr{A}}) \to K_0(\tilde{\mathscr{B}}) \to K_0(\tilde{\mathscr{C}}). \tag{10}$$

3.1.14 Remark

The statement in Exercise 3.1.13 (a) is part of Thomason's classification of dense subcategories. Call a triangulated subcategory $\mathscr{A} \subset \mathscr{B}$ *dense* if $\mathscr{A}$ is closed under isomorphisms in $\mathscr{B}$ and if the inclusion is cofinal. Thomason's Theorem [91, Theorem 2.1] says that the map which sends a dense subcategory $\mathscr{A} \subset \mathscr{B}$ to the subgroup $K_0(\mathscr{A}) \subset K_0(\mathscr{B})$ is a bijection between the set of dense subcategories of $\mathscr{B}$ and the set of subgroups of $K_0(\mathscr{B})$.

We note that an object of $\mathscr{B}$ of the form $A \oplus A[1]$ is in every dense triangulated subcategory of $\mathscr{B}$ since $[A \oplus A[1]] = [A] - [A] = 0 \in K_0(\mathscr{B})$.

3.2 The Thomason–Waldhausen Localization Theorem

We would like to extend the exact sequence Exercise 3.1.6 (8) to the left, and the exact sequence Exercise 3.1.13 (10) in both directions. However, there is no functor from triangulated categories to spaces (or spectra) which does that and yields Quillen's K-theory of an exact category $\mathscr{E}$ when applied to $D^b\mathscr{E}$ [80, Proposition 2.2]. This is the reason why we need to introduce more structure.

3.2.1 Notation

To abbreviate, we write $\mathrm{Ch}^b(\mathbb{Z})$ for the exact category of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules; see Example 2.1.2 (e). A sequence here is a conflation if it splits in each degree (that is, it is isomorphic in each degree to the sequence Sect. 2.1.1 (f)). There is a symmetric monoidal tensor product

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathrm{Ch}^b(\mathbb{Z}) \to \mathrm{Ch}^b(\mathbb{Z})$$

which extends the usual tensor product of free $\mathbb{Z}$-modules. It is given by the formulas

$$(E \otimes F)^n = \bigoplus_{i+j=n} E^i \otimes F^j, \qquad d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} x \otimes dy. \tag{11}$$

where $|x|$ denotes the degree of x. The unit of the tensor product is the chain complex $\mathbb{1} = \mathbb{Z} \cdot 1_{\mathbb{Z}}$ which is $\mathbb{Z}$ in degree 0 and 0 elsewhere. There are natural isomorphisms $\alpha : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, $\lambda : \mathbb{1} \otimes A \cong A$ and $\rho : A \otimes \mathbb{1} \cong A$ such that certain pentagonal and triangular diagrams commute and such that $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$; see [61, VII.1] and Definition 3.2.2 below. Formally, the sextuplet $(\mathrm{Ch}^b(\mathbb{Z}), \otimes, \mathbb{1}, \alpha, \rho, \lambda)$ is a symmetric monoidal category.

Besides the chain complex $\mathbb{1}$, we have two other distinguished objects in $\mathrm{Ch}^b(\mathbb{Z})$. The complex $C = \mathbb{Z} \cdot 1_C \oplus \mathbb{Z} \cdot \eta$ is concentrated in degrees 0 and -1 where it is the free $\mathbb{Z}$-module of rank 1 generated by 1_C and η, respectively. The only non-trivial differential is $d\eta = 1_C$. In fact, C is a commutative differential graded $\mathbb{Z}$-module with unique multiplication such that 1_C is the unit in C. Furthermore, there is the complex $T = \mathbb{Z} \cdot \eta_T$ which is the free $\mathbb{Z}$-module generated by η_T in degree -1 and it is 0 elsewhere. Note that there is a short exact sequences of chain complexes

$$0 \to \mathbb{1} \rightarrowtail C \twoheadrightarrow T \to 0 : \quad 1_{\mathbb{Z}} \mapsto 1_C, \quad (1_C, \eta) \mapsto (0, \eta_T).$$

3.2.2 Definition

An exact category $\mathscr{E}$ is called *complicial* if it is equipped with a bi-exact tensor product

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathscr{E} \to \mathscr{E} \tag{12}$$

which is associative and unital in the sense that there are natural isomorphisms $\alpha : A \otimes (B \otimes X) \cong (A \otimes B) \otimes X$ and $\lambda : \mathbb{1} \otimes X \cong X$ such that the pentagonal diagrams

$$\begin{array}{ccccc}
A \otimes (B \otimes (C \otimes X)) & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes X) & \xrightarrow{\alpha} & ((A \otimes B) \otimes C) \otimes X \\
\downarrow{\scriptstyle 1 \otimes \alpha} & & & & \downarrow{\scriptstyle \alpha \otimes 1} \\
A \otimes ((B \otimes C) \otimes X) & & \xrightarrow{\qquad\quad \alpha \quad\qquad} & & (A \otimes (B \otimes C)) \otimes X
\end{array}$$

and triangular diagrams

$$\begin{array}{ccc}
A \otimes (\mathbb{1} \otimes X) & \xrightarrow{\alpha} & (A \otimes \mathbb{1}) \otimes X \\
& {\scriptstyle 1 \otimes \lambda} \searrow & \downarrow{\scriptstyle \rho \otimes 1} \\
& & A \otimes X
\end{array}$$

commute for all $A,B,C \in \mathrm{Ch}^b(\mathbb{Z})$ and $X \in \mathscr{E}$. In other words, a complicial exact category is an exact category $\mathscr{E}$ equipped with a bi-exact action of the symmetric monoidal category $\mathrm{Ch}^b(\mathbb{Z})$ on $\mathscr{E}$; see [35, p. 218] for actions of monoidal categories.

For an object X of $\mathscr{E}$, we write CX and TX instead of $C \otimes X$ and $T \otimes X$. Note that there is a functorial conflation $X \rightarrowtail CX \twoheadrightarrow TX$ which is the tensor product of $1\!\!1 \rightarrowtail C \twoheadrightarrow T$ with X. For a morphism $f : X \to Y$ in $\mathscr{E}$, we write $C(f)$ for the push-out of f along the inflation $X \rightarrowtail CX$, and we call it the *cone of* f. As a push-out of an inflation, the morphism $Y \to C(f)$ is also an inflation with the same cokernel TX. This yields the conflation in $\mathscr{E}$

$$Y \rightarrowtail C(f) \twoheadrightarrow TX. \tag{13}$$

3.2.3 Example

Let X be a scheme. The usual tensor product of vector bundles $\otimes_{O_X} : \mathrm{Vect}(X) \times \mathrm{Vect}(X) \to \mathrm{Vect}(X)$ extends to a tensor product

$$\otimes : \mathrm{Ch}^b\,\mathrm{Vect}(X) \times \mathrm{Ch}^b\,\mathrm{Vect}(X) \longrightarrow \mathrm{Ch}^b\,\mathrm{Vect}(X)$$

of bounded chain complexes of vector bundles defined by the same formula as in Sect. 3.2.1 (11). The structure map $p : X \to \mathrm{Spec}\,\mathbb{Z}$ associated with the unique ring map $\mathbb{Z} \to \Gamma(X, O_X)$ induces a symmetric monoidal functor $p^* : \mathrm{Proj}(\mathbb{Z}) \to \mathrm{Vect}(X)$ and thus an action

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathrm{Ch}^b\,\mathrm{Vect}(X) : (M,V) \mapsto p^*M \otimes V$$

which makes $\mathrm{Ch}^b\,\mathrm{Vect}(X)$ into a complicial exact category.

3.2.4 Example

For any exact category $\mathscr{E}$, the category $\mathrm{Ch}^b\,\mathscr{E}$ of bounded chain complexes in $\mathscr{E}$ can be made into a complicial exact category as follows. Write $F(\mathbb{Z})$ for the category of finitely generated free $\mathbb{Z}$-modules where each module is equipped with a choice of a basis. So, we have an equivalence $\mathrm{Ch}^b(\mathbb{Z}) \cong \mathrm{Ch}^b F(\mathbb{Z})$. We define an associative and unital tensor product $F(\mathbb{Z}) \times \mathscr{E} \to \mathscr{E}$ by $\mathbb{Z}^n \otimes X = Xe_1 \oplus \ldots \oplus Xe_n$ where Xe_i stands for a copy of X corresponding the basis element e_i of the based free module $\mathbb{Z}^n = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$. On maps, the tensor product is defined by $(a_{ij}) \otimes f = (a_{ij} f)$. With the usual formulas for the tensor product of chain complexes as in Sect. 3.2.1 (11), this tensor product extends to an associative, unital and bi-exact pairing

$$\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathrm{Ch}^b\,\mathscr{E} \to \mathrm{Ch}^b\,\mathscr{E}$$

making the category of bounded chain complexes $\mathrm{Ch}^b\,\mathscr{E}$ into a complicial exact category.

3.2.5 The Stable Category of a Complicial Exact Category

Let $\mathscr{E}$ be a complicial exact category. Call a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathscr{E}$ a *Frobenius conflation* if for every object $U \in \mathscr{E}$ the following holds: Every map $X \to CU$ extends to a map $Y \to CU$, and every map $CU \to Z$ lifts to a map $CU \to Y$. It is shown in Lemma A.2.16 that $\mathscr{E}$ together with the Frobenius conflations is a Frobenius exact category. That is, it is an exact category which has enough injectives and enough projectives, and where injectives and projectives coincide; see Appendix A, Sect. 2.14. The injective-projective objects are precisely the direct factors of objects of the form CU for $U \in \mathscr{E}$. The *stable category* $\underline{\mathscr{E}}$ of the complicial exact category $\mathscr{E}$ is, by definition, the stable category of the Frobenius exact category $\mathscr{E}$; see Appendix A, Sect. 2.14. It has objects the objects of $\mathscr{E}$, and maps are the homotopy classes of maps in $\mathscr{E}$ where two maps $f,g : X \to Y$ are homotopic if their difference factors through an object of the form CU. As the stable category of a Frobenius exact category, the category $\underline{\mathscr{E}}$ is a triangulated category (Appendix A, Sect. 2.14). Distinguished triangles are those triangles which are isomorphic in $\underline{\mathscr{E}}$ to sequences of the form

$$X \xrightarrow{f} Y \to C(f) \to TX \tag{14}$$

attached to any map $f : X \to Y$ in $\mathscr{E}$ and extended by the sequence (13).

3.2.6 Example

Continuing the Example 3.2.4, the complicial exact category $\mathrm{Ch}^b\mathscr{E}$ of bounded chain complexes of $\mathscr{E}$ has as associated stable category the homotopy category $\mathscr{K}^b\mathscr{E}$ of Sect. 3.1.3. This is because contractible chain complexes are precisely the injective-projective objects for the Frobenius exact structure of $\mathrm{Ch}^b\mathscr{E}$ (Exercise!). See also Lemma A.2.16.

3.2.7 Definition

An *exact category with weak equivalences* is an exact category $\mathscr{E}$ together with a set $w \subset \mathrm{Mor}\,\mathscr{E}$ of morphisms in $\mathscr{E}$. Morphisms in w are called *weak equivalences*. The set of weak equivalences is required to contain all identity morphisms; to be closed under isomorphisms, retracts, push-outs along inflations, pull-backs along deflations, composition; and to satisfy the "two out of three" property for composition: if two of the three maps among a, b, ab are weak equivalences, then so is the third.

3.2.8 Example

Let $\mathscr{E}$ be an exact category. The exact category $\mathrm{Ch}^b\mathscr{E}$ of bounded chain complexes in $\mathscr{E}$ of Example 2.1.2 (e) together with the set quis of quasi-isomorphisms as defined in Sect. 3.1.3 is an exact category with weak equivalences.

3.2.9 Definition

An exact category with weak equivalences $(\mathscr{E}, w)$ is *complicial* if $\mathscr{E}$ is complicial and if the tensor product Definition 3.2.2 (12) preserves weak equivalences in both variables, that is, if f is a homotopy equivalence in $\mathrm{Ch}^b(\mathbb{Z})$ and g is a weak equivalence in $\mathscr{E}$, then $f \otimes g$ is a weak equivalence in $\mathscr{E}$.

3.2.10 Example

The exact category with weak equivalences $(\mathrm{Ch}^b\mathscr{E}, \mathrm{quis})$ of Example 3.2.8 is complicial with action by $\mathrm{Ch}^b(\mathbb{Z})$ as defined in Sect. 3.2.4.

Before we come to the definition of the K-theory space of a complicial exact category, we introduce some notation. Let $\mathbf{E} = (\mathscr{E}, w)$ be a complicial exact category with weak equivalences. We write $\mathscr{E}^w \subset \mathscr{E}$ for the fully exact subcategory of those objects X in $\mathscr{E}$ for which the map $0 \to X$ is a weak equivalence.

3.2.11 Exercise

Show that $\mathscr{E}^w$ is closed under retracts in $\mathscr{E}$. More precisely, let X be an object of $\mathscr{E}$. Show that if there are maps $i: X \to A$ and $p: A \to X$ with $pi = 1_X$ and $A \in \mathscr{E}^w$ then $X \in \mathscr{E}^w$. In particular, objects of $\mathscr{E}$ which are isomorphic to objects of $\mathscr{E}^w$ are in $\mathscr{E}^w$.

3.2.12 Definition of $K(\mathscr{E},w)$, $K(\mathbf{E})$

The *algebraic K-theory space* $K(\mathbf{E}) = K(\mathscr{E},w)$ of a complicial exact category with weak equivalences $\mathbf{E} = (\mathscr{E},w)$ is the homotopy fibre of the map of pointed topological spaces $BQ(\mathscr{E}^w) \to BQ(\mathscr{E})$ induced by the inclusion $\mathscr{E}^w \subset \mathscr{E}$ of exact categories. That is,

$$K(\mathbf{E}) = K(\mathscr{E},w) = F(g) \quad \text{where} \quad g : BQ(\mathscr{E}^w) \to BQ(\mathscr{E})$$

and where $F(g)$ is the homotopy fibre of g as in Appendix A, Sect. 1.7. The *higher algebraic K-groups* $K_i(\mathbf{E})$ of $\mathbf{E}$ are the homotopy groups $\pi_i K(\mathbf{E})$ of the K-theory space of $\mathbf{E}$ for $i \geq 0$.

Exact functors preserving weak equivalences induce maps between algebraic K-theory spaces of complicial exact categories with weak equivalences.

3.2.13 Remark

The K-theory space of an exact category with weak equivalences is usually defined using Waldhausen's $S_\bullet$-construction [100, p. 330, Definition]. A complicial exact category $\mathbf{E} = (\mathscr{E},w)$ has a "cylinder functor" in the sense of [100, Sect. 1.6] obtained as the tensor product with the usual cylinder in $\mathrm{Ch}^b(\mathbb{Z})$ via the action of $\mathrm{Ch}^b(\mathbb{Z})$ on $\mathscr{E}$. Theorem [100, 1.6.4] together with [100, Sect. 1.9] then show that the K-theory space of any complicial exact category with weak equivalences as defined in Definition 3.2.12 is equivalent to the one in [100].

3.2.14 Theorem [94, Theorem 1.11.7]

Let $\mathscr{E}$ be an exact category. The embedding of $\mathscr{E}$ into $\mathrm{Ch}^b\,\mathscr{E}$ as degree-zero complexes induces a homotopy equivalence

$$K(\mathscr{E}) \simeq K(\mathrm{Ch}^b\,\mathscr{E}, \mathrm{quis}).$$

3.2.15 The Triangulated Category $\mathscr{T}(\mathbf{E})$

Let $\mathbf{E} = (\mathscr{E},w)$ be a complicial exact category with weak equivalences. For objects X of $\mathscr{E}^w$ and A of $\mathrm{Ch}^b(\mathbb{Z})$, the object $A \otimes X$ is in $\mathscr{E}^w$ because the map $0 \to A \otimes X$ is a weak equivalence since it is the tensor product $id_A \otimes (0 \to X)$ of two weak equivalences. It follows that we can consider $\mathscr{E}^w$ as a complicial exact category where the action by $\mathrm{Ch}^b(\mathbb{Z})$ is induced from the action on $\mathscr{E}$. For every object $U \in \mathscr{E}$, the object CU is in $\mathscr{E}^w$ because the map $0 \to CU$ is a weak equivalence as it is the a tensor product $(0 \to C) \otimes 1_U$ of two weak equivalences. More generally, every retract of an object of the form CU is in $\mathscr{E}^w$ by Exercise 3.2.11. Therefore, the two Frobenius categories $\mathscr{E}$ and $\mathscr{E}^w$ have the same injective-projective objects. It follows that the inclusion $\mathscr{E}^w \subset \mathscr{E}$ induces a fully faithful triangle functor of associated stable categories $\underline{\mathscr{E}^w} \subset \underline{\mathscr{E}}$.

3.2.16 Exercise

Show that $\underline{\mathscr{E}^w}$ is closed under retracts in $\underline{\mathscr{E}}$. More precisely, let X be an object of $\underline{\mathscr{E}}$. Show that if there are maps $i : X \to A$ and $p : A \to X$ in $\underline{\mathscr{E}}$ with $pi = 1_X$ and $A \in \underline{\mathscr{E}^w}$ then $X \in \underline{\mathscr{E}^w}$. In particular, objects of $\underline{\mathscr{E}}$ which are isomorphic in $\underline{\mathscr{E}}$ to objects of $\underline{\mathscr{E}^w}$ are already in $\underline{\mathscr{E}^w}$.

3.2.17 Definition

The triangulated category $\mathscr{T}(\mathbf{E})$ associated with a complicial exact category with weak equivalences $\mathbf{E} = (\mathscr{E}, w)$ is the Verdier quotient

$$\mathscr{T}(\mathbf{E}) = \underline{\mathscr{E}}/\underline{\mathscr{E}^w}$$

of the inclusion of triangulated stable categories $\underline{\mathscr{E}^w} \subset \underline{\mathscr{E}}$. By construction, distinguished triangles in $\mathscr{T}(\mathbf{E})$ are those triangles which are isomorphic in $\mathscr{T}(\mathbf{E})$ to triangles of the form Sect. 3.2.5 (14).

One easily checks that the canonical functor $\mathscr{E} \to \mathscr{T}(\mathbf{E}) : X \mapsto X$ induces an isomorphism of categories

$$w^{-1}\mathscr{E} \xrightarrow{\cong} \mathscr{T}(\mathbf{E}).$$

Therefore, as a category, the triangulated category $\mathscr{T}(\mathscr{E}, w)$ of $(\mathscr{E}, w)$ is obtained from $\mathscr{E}$ by formally inverting the weak equivalences.

3.2.18 Exercise

Let $(\mathscr{E}, w)$ be a complicial exact category with weak equivalences. Show that a morphism in $\mathscr{E}$ which is an isomorphism in $\mathscr{T}(\mathscr{E}, w)$ is a weak equivalence. *Hint*: Show that (a) if in a conflation $X \rightarrowtail Y \twoheadrightarrow A$ the object A is in $\mathscr{E}^w$ then $X \rightarrowtail Y$ is a weak equivalence, and (b) a map $f : X \to Y$ is an isomorphism in $\mathscr{T}(\mathscr{E}, w)$ iff its cone $C(f)$ is in $\mathscr{E}^w$ by Exercise 3.2.16. Conclude using the conflation $X \rightarrowtail CX \oplus Y \twoheadrightarrow C(f)$.

3.2.19 Remark

A conflation $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in a complicial exact category with weak equivalences (E, w) gives rise to a distinguished triangle

$$X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow TX$$

in $\mathscr{T}(\mathscr{E}, w)$. By definition, this triangle is isomorphic to the standard distinguished triangle Sect. 3.2.5 (14) via the quotient map $C(i) \twoheadrightarrow C(i)/CX \cong Z$ which is weak equivalence in $\mathscr{E}$ and an isomorphism in $\mathscr{T}(\mathscr{E}, w)$.

3.2.20 Example

For an exact category $\mathscr{E}$, the triangulated category $\mathscr{T}(\mathrm{Ch}^b \mathscr{E}, \mathrm{quis})$ associated with the complicial exact category with weak equivalences $(\mathrm{Ch}^b \mathscr{E}, \mathrm{quis})$ of Example 3.2.10 is the usual bounded derived category $D^b\mathscr{E}$ of $\mathscr{E}$ as defined in Sect. 3.1.3.

3.2.21 **Example** (DG Categories)

Let $\mathscr{C}$ be a dg-category ([50], or see Toen's lecture [92]). There is a canonical embedding $\mathscr{C} \subset \mathscr{C}^{pretr}$ of dg-categories of $\mathscr{C}$ into a "pretriangulated dg-category" $\mathscr{C}^{pretr}$ associated with $\mathscr{C}$ [12, Sect. 4.4]. It is obtained from $\mathscr{C}$ by formally adding iterated shifts and cones of objects of $\mathscr{C}$. The homotopy category $\mathrm{Ho}(\mathscr{C}^{pretr})$ of $\mathscr{C}^{pretr}$ is equivalent to the full triangulated subcategory of the derived category $D(\mathscr{C})$ of dg $\mathscr{C}$-modules which is generated by $\mathscr{C}$. The idempotent completion of $\mathrm{Ho}(\mathscr{C}^{pretr})$ is equivalent to the triangulated category of compact objects in $D(\mathscr{C})$ which is sometimes called the *derived category of perfect $\mathscr{C}$-modules*, and it is also equivalent to the homotopy category of the *triangulated hull* of $\mathscr{C}$ mentioned Toen's lecture [92].

Exercise: Show that $\mathscr{C}^{pretr}$ and the triangulated hull of $\mathscr{C}$ can be made into complicial exact categories with weak equivalences such that the associated triangulated categories are the homotopy categories of $\mathscr{C}^{pretr}$ and of the triangulated hull of $\mathscr{C}$.

3.2.22 **Proposition** (Presentation of $K_0(\mathbf{E})$)

Let $\mathbf{E} = (\mathscr{E}, w)$ be a complicial exact category with weak equivalences. Then the map $K_0(\mathbf{E}) \to K_0(\mathscr{T}(\mathbf{E})) : [X] \mapsto [X]$ is well-defined and an isomorphism of abelian groups.

Proof:

By Definition 3.2.12 and Proposition 2.2.4, the group $K_0(\mathbf{E})$ is the cokernel of the map $K_0(\mathscr{E}^w) \to K_0(\mathscr{E})$. By Remark 3.2.19, conflations in $\mathscr{E}$ yield distinguished triangles in $\mathscr{T}(\mathbf{E})$. Therefore, the map $K_0(\mathscr{E}) \to K_0(\mathscr{T}(\mathbf{E}))$ given by $[X] \mapsto [X]$ is well-defined. This map clearly sends $K_0(\mathscr{E}^w)$ to zero. It follows that the map in the Proposition is also well-defined.

Now, we show that the inverse map $K_0(\mathscr{T}(\mathbf{E})) \to K_0(\mathbf{E})$ defined by $[X] \mapsto [X]$ is also well-defined. We first observe that the existence of a weak equivalence $f : X \to Y$ implies that $[X] = [Y]$ in $K_0(\mathbf{E})$. This is because there is a conflation $X \rightarrowtail CX \oplus Y \twoheadrightarrow C(f)$ in $\mathscr{E}$ by the definition of the mapping cone $C(f)$. The objects CX and $C(f)$ are in $\mathscr{E}^w$ which implies $[X] = [Y] \in K_0(\mathbf{E})$. More generally, any two objects which are isomorphic in $\mathscr{T}(\mathbf{E})$ give rise to the same element in $K_0(\mathbf{E})$ because they are linked by a zigzag of weak equivalences by Exercise 3.2.18 and the definition of $\mathscr{T}(\mathbf{E})$. Next, we observe that for every object X of $\mathscr{E}$, the existence of the conflation $X \rightarrowtail CX \twoheadrightarrow TX$ in $\mathscr{E}$ with $CX \in \mathscr{E}^w$ shows that $[X] = -[TX]$ in $K_0(\mathbf{E})$. Finally, every distinguished triangle $A \to B \to C \to TA$ in $\mathscr{T}(\mathbf{E})$ is isomorphic in $\mathscr{T}(\mathbf{E})$ to a triangle of the form Sect. 3.2.5 (14) where $Y \rightarrowtail C(f) \twoheadrightarrow TX$ is a conflation in $\mathscr{E}$. Therefore, we have $[A]-[B]+[C] = [X]-[Y]+[C(f)] = [X]-[Y]+[Y]+[TX] = [X]-[Y]+[Y]-[X] = 0$ in $K_0(\mathbf{E})$, and the inverse map is well-defined. □

Now, we come to the theorem which extends the sequence Exercise 3.1.6 (8) to the left. It is due to Thomason [94, 1.9.8., 1.8.2] based on the work of Waldhausen [100].

3.2.23 **Theorem** (Thomason–Waldhausen Localization, Connective Version)

Given a sequence $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ be of complicial exact categories with weak equivalences. Assume that the associated sequence $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B} \to \mathscr{T}\mathbf{C}$ of triangulated categories is exact. Then the induced sequence of K-theory spaces

$$K(\mathbf{A}) \to K(\mathbf{B}) \to K(\mathbf{C})$$

is a homotopy fibration. In particular, there is a long exact sequence of K-groups

$$\cdots \to K_{i+1}(\mathbf{C}) \to K_i(\mathbf{A}) \to K_i(\mathbf{B}) \to K_i(\mathbf{C}) \to K_{i-1}(\mathbf{A}) \to \cdots$$

ending in $K_0(\mathbf{B}) \to K_0(\mathbf{C}) \to 0$.

The following special case of Theorem 3.2.23 which is important in itself is due to Thomason [94, Theorem 1.9.8].

3.2.24 Theorem (Invariance Under Derived Equivalences)

Let $\mathbf{A} \to \mathbf{B}$ *be a functor of complicial exact categories with weak equivalences. Assume that the associated functor of triangulated categories* $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ *is an equivalence. Then the induced map* $K(\mathbf{A}) \to K(\mathbf{B})$ *of K-theory spaces is a homotopy equivalence. In particular, it induces isomorphisms* $K_i(\mathbf{A}) \cong K_i(\mathbf{B})$ *of K-groups for* $i \geq 0$.

3.2.25 Example

Theorem 3.2.23 applied to Example 3.1.8 yields a homotopy fibration

$$K(\mathscr{H}_S(R)) \to K(R) \to K(\mathscr{P}'(S^{-1}R))$$

of K-theory spaces. As mentioned earlier, one would like to replace $\mathscr{P}'(S^{-1}R)$ with $\mathrm{Proj}(S^{-1}R)$ in the homotopy fibration and its associated long exact sequence of homotopy groups. We can do so by the Cofinality Theorem 2.4.2, and we obtain a long exact sequence of K-groups

$$\cdots \to K_{i+1}(S^{-1}R) \to K_i(\mathscr{H}_S(R)) \to K_i(R) \to K_i(S^{-1}R) \to K_i(\mathscr{H}_S(R)) \to \cdots$$

ending in $\cdots \to K_0(R) \to K_0(S^{-1}R)$. The last map $K_0(R) \to K_0(S^{-1}R)$, however, is not surjective, in general. We have already introduced the negative K-groups of an exact category in Sect. 2.4. They do indeed extend this exact sequence to the right. But this is best understood in the framework of complicial exact categories with weak equivalences.

3.2.26 Negative K-Theory of Complicial Exact Categories

To any complicial exact category with weak equivalences $\mathbf{E}$, one can associate a new complicial exact category with weak equivalences $S\mathbf{E}$, called the *suspension* of $\mathbf{E}$, such that there is a natural map

$$K(\mathbf{E}) \to \Omega K(S\mathbf{E}) \tag{15}$$

which is an isomorphism on π_i for $i \geq 1$ and a monomorphism on π_0 [82]; see the construction in Sect. 3.2.33 below. In fact, $K_1(S\mathbf{E}) = K_0((\mathscr{T}\mathbf{E})^{\sim})$ where $(\mathscr{T}\mathbf{E})^{\sim}$ denotes the idempotent completion of $\mathscr{T}\mathbf{E}$. Moreover, the suspension functor sends sequences of complicial exact categories with weak equivalences whose associated sequence of triangulated categories is exact up to factors to sequences with that same property.

One uses the suspension construction to slightly modify the definition of algebraic K-theory in order to incorporate negative K-groups as follows. One sets $\mathbb{K}_i(\mathbf{E}) = K_i(\mathbf{E})$ for $i \geq 1$, $\mathbb{K}_0(\mathbf{E}) = K_0((\mathscr{T}\mathbf{E})^{\sim})$ and $\mathbb{K}_i(\mathbf{E}) = K_0((\mathscr{T}S^{-i}\mathbf{E})^{\sim})$ for $i < 0$. As in the case of exact categories in Sect. 2.4.3, one constructs a spectrum $\mathbb{K}(\mathbf{E})$ whose homotopy groups are the groups $\mathbb{K}_i(\mathbf{E})$ for $i \in \mathbb{Z}$. The n-th space of this spectrum is $K(S^n\mathbf{E})$, and the structure maps are given by (15).

If the category $\mathscr{T}\mathbf{E}$ is idempotent complete, then we may write $K_i(\mathbf{E})$ instead of $I\!K_i(\mathbf{E})$ for $i \in \mathbb{Z}$. In this case, $I\!K_i(\mathbf{E}) = K_i(\mathbf{E})$ for all $i \geq 0$. Therefore, we have merely introduced negative K-groups without changing the definition of $I\!K_0$.

The following theorem extends the exact sequence Exercise 3.1.13 (10) in both directions. For the definition of a homotopy fibration of spectra, see Appendix A, Sect. 1.8.

3.2.27 Theorem (Thomason–Waldhausen Localization, Non-Connective Version)

Let $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ be a sequence of complicial exact categories with weak equivalences such that the associated sequence of triangulated categories $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B} \to \mathscr{T}\mathbf{C}$ is exact up to factors. Then the sequence of K-theory spectra

$$I\!K(\mathbf{A}) \to I\!K(\mathbf{B}) \to I\!K(\mathbf{C})$$

is a homotopy fibration. In particular, there is a long exact sequence of K-groups for $i \in \mathbb{Z}$

$$\cdots \to I\!K_{i+1}(\mathbf{C}) \to I\!K_i(\mathbf{A}) \to I\!K_i(\mathbf{B}) \to I\!K_i(\mathbf{C}) \to I\!K_{i-1}(\mathbf{A}) \to \cdots$$

3.2.28 Remark

Theorem 3.2.27 is proved in [82, Theorem 9] in view of the fact that for a complicial exact category with weak equivalences $(\mathscr{E}, w)$, the pair $(\mathscr{E}, \mathscr{E}^w)$ is a "Frobenius pair" in the sense of [82, Definition 5] when we equip $\mathscr{E}$ with the Frobenius exact structure.

3.2.29 Theorem (Invariance of $I\!K$-Theory Under Derived Equivalences)

If a functor $\mathbf{A} \to \mathbf{B}$ of complicial exact categories with weak equivalences induces an equivalence up to factors $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ of associated triangulated categories, then it induces a homotopy equivalence of $I\!K$-theory spectra $I\!K(\mathbf{A}) \xrightarrow{\sim} I\!K(\mathbf{B})$ and isomorphisms $I\!K_i(\mathbf{A}) \cong I\!K_i(\mathbf{B})$ of $I\!K$-groups for $i \in \mathbb{Z}$.

3.2.30 Agreement

Let $\mathscr{E}$ be an exact category and $(\mathrm{Ch}^b \mathscr{E}, \mathrm{quis})$ be the associated complicial exact category of bounded chain complexes with quasi-isomorphisms as weak equivalences. There are natural isomorphisms

$$I\!K_i(\mathrm{Ch}^b \mathscr{E}, \mathrm{quis}) \cong I\!K_i(\mathscr{E}) \quad \text{for} \quad i \in \mathbb{Z}$$

between the $I\!K$-groups defined in Sect. 3.2.26 and those defined in Sect. 2.4.3. For $i > 0$, this is Theorem 3.2.14. For $i = 0$, we have $I\!K_0(\mathscr{E}) = K_0(\tilde{\mathscr{E}}) = K_0(D^b(\tilde{\mathscr{E}})) = K_0(D^b(\mathscr{E})^\sim) = I\!K_0(\mathrm{Ch}^b \mathscr{E}, \mathrm{quis})$ by Exercise 3.1.4, Sect. 3.1.7 (c) and Proposition 3.2.22. For $i < 0$, this follows from Theorem 3.2.27, from the case $i = 0$ above, from the fact that the sequence $\mathscr{E} \to \mathscr{F}\mathscr{E} \to S\mathscr{E}$ of Sect. 2.4.6 induces a sequence of associated triangulated categories which is exact up to factors [81, Proposition 2.6, Lemma 3.2] and from the fact that $\mathscr{F}\mathscr{E}$ has exact countable sums which implies $I\!K_i(\mathscr{F}\mathscr{E}) = 0$ for all $i \in \mathbb{Z}$.

3.2.31 Example

From Example 3.1.11, we obtain a homotopy fibration of K-theory spectra

$$\mathbb{K}(\mathscr{H}_S(R)) \to \mathbb{K}(R) \to \mathbb{K}(S^{-1}R)$$

and an associated long exact sequence of K-groups

$$\cdots \to K_{i+1}(S^{-1}R) \to K_i(\mathscr{H}_S(R)) \to K_i(R) \to K_i(S^{-1}R) \to K_i(\mathscr{H}_S(R)) \to \cdots$$

for $i \in \mathbb{Z}$. Here we wrote K_i instead of $\mathbb{K}_i$ since all exact categories in this example as well as their bounded derived categories are idempotent complete.

3.2.32 $\mathbb{K}$-Theory of DG-Categories

The $\mathbb{K}$-theory $\mathbb{K}(\mathscr{C})$ of a dg-category $\mathscr{C}$ is the $\mathbb{K}$-theory associated with the complicial exact category $\mathscr{C}^{pretr}$ with weak equivalences the homotopy equivalences, that is, those maps which are isomorphisms in $D\mathscr{C}$. By construction, $\mathbb{K}_0(\mathscr{C})$ is K_0 of the triangulated category of compact objects in $D\mathscr{C}$. Instead of $\mathscr{C}^{pretr}$, we could have also used the triangulated hull of $\mathscr{C}$ in the definition of $\mathbb{K}(\mathscr{C})$ because both dg-categories are derived equivalent up to factors.

By the Thomason–Waldhausen Localization Theorem 3.2.27, a sequence of dg categories $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ whose sequence $D\mathscr{A} \to D\mathscr{B} \to D\mathscr{C}$ of derived categories of dg-modules is exact induces a homotopy fibration of $\mathbb{K}$-theory spectra

$$\mathbb{K}(\mathscr{A}) \to \mathbb{K}(\mathscr{B}) \to \mathbb{K}(\mathscr{C}).$$

This is because the sequence $\mathscr{T}(\mathscr{A}^{pretr}) \to \mathscr{T}(\mathscr{B}^{pretr}) \to \mathscr{T}(\mathscr{C}^{pretr})$ of triangulated categories whose idempotent completion is the sequence of compact objects associated with $D\mathscr{A} \to D\mathscr{B} \to D\mathscr{C}$ is exact up to factors by Neeman's Theorem 3.4.5 (b) below.

3.2.33 Construction of the Suspension $S\mathbf{E}$

Let $\mathbf{E} = (\mathscr{E}, w)$ be a complicial exact category with weak equivalences. Among others, this means that $\mathscr{E}$ is an exact category, and we can construct its countable envelope $\mathscr{F}\mathscr{E}$ as in Sect. 2.4.6. The complicial structure on $\mathscr{E}$ extends to a complicial structure on $\mathscr{F}\mathscr{E}$ by setting

$$A \otimes (E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots) \;=\; (A \otimes E_0 \hookrightarrow A \otimes E_1 \hookrightarrow A \otimes E_2 \hookrightarrow \cdots)$$

for $A \in \mathrm{Ch}^b(\mathbb{Z})$ and $E_* \in \mathscr{F}\mathscr{E}$. Call a map in $\mathscr{F}\mathscr{E}$ a *weak equivalence* if its cone is a direct factor of an object of $\mathscr{F}(\mathscr{E}^w)$. As usual, we write w for the set of weak equivalences in $\mathscr{F}\mathscr{E}$. The pair $\mathscr{F}\mathbf{E} = (\mathscr{F}\mathscr{E}, w)$ defines a complicial exact category with weak equivalences. The fully exact inclusion $\mathscr{E} \to \mathscr{F}\mathscr{E}$ of exact categories of Sect. 2.4.6 defines a functor $\mathbf{E} \to \mathscr{F}\mathbf{E}$ of complicial exact categories with weak equivalences such that the induced functor $\mathscr{T}(\mathbf{E}) \to \mathscr{T}(\mathscr{F}\mathbf{E})$ of associated triangulated categories is fully faithful.

Now, the *suspension* $S\mathbf{E}$ of $\mathbf{E}$ is the complicial exact category with weak equivalences which has as underlying complicial exact category the countable envelope $\mathscr{F}\mathscr{E}$ and as set of weak equivalences those maps in $\mathscr{F}\mathscr{E}$ which are isomorphisms in the Verdier quotient $\mathscr{T}(\mathscr{F}\mathbf{E})/\mathscr{T}(\mathbf{E})$.

3.2.34 Remark (Suspensions of DG-Categories)

If $\mathscr{C}$ is a dg-category, one can also define its suspension as $\Sigma\mathscr{C} = \Sigma \otimes_{\mathbb{Z}} \mathscr{C}$ where $\Sigma = \Sigma\mathbb{Z}$ is the suspension ring of $\mathbb{Z}$ as in Cortiñas' lecture [21]. The resulting spectrum whose n-th space is $K((\Sigma^n\mathscr{C})^{pretr})$ is equivalent to the spectrum $\mathbb{K}(\mathscr{C})$ as defined in Sect. 3.2.32. The reason is that the sequence $\mathscr{C} \to \Gamma \otimes \mathscr{C} \to \Sigma \otimes \mathscr{C}$ induces a sequence of pretriangulated dg-categories whose associated sequence of homotopy categories is exact up to factors. This follows from [23] in view of the fact that the sequence of flat dg categories $\mathbb{Z}^{pretr} \to \Gamma^{pretr} \to \Sigma^{pretr}$ induces a sequence of homotopy categories which is exact up to factors. Moreover, $\mathbb{K}(\Gamma \otimes \mathscr{C}) \simeq 0$.

Sketch of the proof of Theorem 3.2.27

We first construct the map (15). For that, denote by $\mathscr{E}' \subset \mathscr{F}\mathscr{E}$ the full subcategory of those objects which are zero in $\mathscr{T}S\mathbf{E} = \mathscr{T}(\mathscr{F}\mathbf{E})/\mathscr{T}(\mathbf{E})$. The category $\mathscr{E}'$ inherits the structure of a complicial exact category with weak equivalences from $\mathscr{F}\mathbf{E}$ which we denote by $\mathbf{E}'$. By construction, the sequence $\mathbf{E}' \to \mathscr{F}\mathbf{E} \to S\mathbf{E}$ induces an exact sequence of associated triangulated categories, and hence, a homotopy fibration $K(\mathbf{E}') \to K(\mathscr{F}\mathbf{E}) \to K(S\mathbf{E})$ of K-theory spaces by the Thomason–Waldhausen Localization Theorem 3.2.23. Since $\mathscr{F}\mathscr{E}$ has infinite exact sums which preserve weak equivalences, we have $K(\mathscr{F}\mathbf{E}) \simeq 0$. Therefore, we obtain a homotopy equivalence $K(\mathbf{E}') \xrightarrow{\simeq} \Omega K(S\mathbf{E})$. By construction of $\mathscr{E}'$, the inclusion $\mathscr{E} \subset \mathscr{E}'$ induces a cofinal triangle functor $\mathscr{T}\mathbf{E} \subset \mathscr{T}\mathbf{E}'$; see Appendix A, Sect. 2.7. By Thomason's Cofinality Theorem 3.2.35 below, the map $K(\mathbf{E}) \to K(\mathbf{E}')$ induces isomorphisms on π_i for $i > 0$ and a monomorphism on π_0.

The main step in proving Theorem 3.2.27 consists in showing that for a sequence of complicial exact categories with weak equivalences $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ where $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B} \to \mathscr{T}\mathbf{C}$ is exact up to factors, the suspended sequence $S\mathbf{A} \to S\mathbf{B} \to S\mathbf{C}$ induces a sequence $\mathscr{T}S\mathbf{A} \to \mathscr{T}S\mathbf{B} \to \mathscr{T}S\mathbf{C}$ of associated triangulated categories which is also exact up to factors. This is proved in [82, Theorem 3] for complicial exact categories with weak equivalences whose exact structure is the Frobenius exact structure. The proof for general exact structures is *mutatis mutandis* the same.

3.2.35 Theorem (Cofinality [94, 1.10.1, 1.9.8])

Let $\mathbf{A} \to \mathbf{B}$ *be a functor of complicial exact categories with weak equivalences such that* $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ *is cofinal. Then* $K_i(\mathbf{A}) \to K_i(\mathbf{B})$ *is an isomorphism for* $i \geq 1$ *and a monomorphism for* $i = 0$.

3.3 Quillen's Fundamental Theorems Revisited

The results of this subsection are due to Quillen [73]. However, we give proofs based on the Thomason–Waldhausen Localization Theorem. This has the advantage that the same results hold for other cohomology theories such as Hochschild homology, (negative, periodic, ordinary) cyclic homology, triangular Witt-groups, hermitian K-theory *etc.* where the analog of the Thomason–Waldhausen Localization Theorem also holds.

3.3.1 *G*-Theory Localization (Revisited)

It is not true that every exact sequence of abelian categories induces an exact sequence of associated bounded derived categories. For a counter example, see [49, 1.15 Example (c)] where the abelian categories are even noetherian and artinian. However, the exact sequence of abelian categories Sect. 2.3.6 (3) which (together with *Dévissage*) gives rise to the *G*-theory fibration Theorem 2.3.7 (4) does induce an exact sequence of triangulated categories. So, at least in this case, we can apply the Thomason–Waldhausen Localization Theorem.

3.3.2 Theorem

Let X be a noetherian scheme, $j : U \subset X$ be an open subscheme and $Z = X - U$ be the closed complement. Then the exact sequence of abelian categories Sect. 2.3.6 (3) induces an exact sequence of triangulated categories

$$D^b\,\mathrm{Coh}_Z(X) \to D^b\,\mathrm{Coh}(X) \xrightarrow{j^*} D^b\,\mathrm{Coh}(U).$$

In particular, it induces a homotopy fibration in K-theory

$$K\,\mathrm{Coh}_Z(X) \to K\,\mathrm{Coh}(X) \xrightarrow{j^*} K\,\mathrm{Coh}(U).$$

Proof [49, 1.15 Lemma and Example b)]:

Since the sequence Sect. 2.3.6 (3) is an exact sequence of abelian categories, the functor $\mathrm{Coh}(X) \to \mathrm{Coh}(U)$ is a localization by a calculus of fractions. By Sect. 3.1.7 (a), the functor $D^b\,\mathrm{Coh}(X) \to D^b\,\mathrm{Coh}(U)$ is a localization of triangulated categories. Therefore, j^* induces an equivalence $D^b\,\mathrm{Coh}(X)/D^b_Z\,\mathrm{Coh}(X) \cong D^b\,\mathrm{Coh}(U)$ where $D^b_Z\,\mathrm{Coh}(X) \subset D^b\,\mathrm{Coh}(X)$ denotes the full triangulated subcategory of those complexes whose cohomology is supported in Z, or equivalently, which are acyclic over U.

The functor $D^b\,\mathrm{Coh}_Z(X) \to D^b\,\mathrm{Coh}(X)$ is fully faithful by an application of Sect. 3.1.7 (b). To check the hypothesis of Sect. 3.1.7 (b), let $N \rightarrowtail M$ be an inclusion of coherent O_X-modules with $N \in \mathrm{Coh}_Z(X)$. If $I \subset O_X$ denotes the ideal sheaf of the reduced subscheme $Z_{red} \subset X$ associated with Z, then a coherent O_X-module E has support in Z iff $I^n E = 0$ for some $n \in \mathbb{N}$. By the Artin–Rees Lemma [3, Corollary 10.10] which also works for noetherian schemes with the same proof, there is an integer $c > 0$ such that $N \cap I^n M = I^{n-c}(N \cap I^c M)$ for $n \geq c$. Since N has support in Z, the same is true for $N \cap I^c M$, and we find $N \cap I^n M = I^{n-c}(N \cap I^c M) = 0$ for n large enough. For such an n, the composition $N \subset M \to M/I^n M$ is injective, and we have $M/I^n M \in \mathrm{Coh}_Z(X)$. Hence, the functor $D^b\,\mathrm{Coh}_Z(X) \to D^b_Z\,\mathrm{Coh}(X)$ is fully faithful. It is essentially surjective – hence an equivalence – since both categories are generated as triangulated categories by $\mathrm{Coh}_Z(X)$ considered as complexes concentrated in degree zero.

The homotopy fibration of K-theory spaces follows from the Thomason–Wald–hausen Localization Theorem 3.2.23. □

3.3.3 Remark

The exact sequence of triangulated categories in Theorem 3.3.2 also induces a homotopy fibration of non-connective K-theory spectra $\mathbb{K}\,\mathrm{Coh}_Z(X) \to \mathbb{K}\,\mathrm{Coh}(X) \to \mathbb{K}\,\mathrm{Coh}(U)$ by Theorem 3.2.27. But this does not give us more information since the negative K-groups of noetherian abelian categories such as $\mathrm{Coh}_Z(X)$, $\mathrm{Coh}(X)$ and $\mathrm{Coh}(U)$ are all trivial; see Remark 2.4.5.

3.3.4 Dévissage

The *Dévissage* Theorem 2.3.3 does not follow from the Thomason–Waldhausen Localization Theorem 3.2.27 since *Dévissage* does not hold for Hochschild homology [49, 1.11]. Yet Theorem 3.2.27 holds when K-theory is replaced with Hochschild homology.

Recall that a noetherian scheme X is called *regular* if all its local rings $O_{X,x}$ are regular local rings for $x \in X$. For the definition and basic properties of regular local rings, see [3, 57, 102].

3.3.5 Theorem (Poincaré Duality, [73, Sect. 7.1])

Let X be a regular noetherian separated scheme. Then the fully exact inclusion $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ *of vector bundles into coherent O_X-modules induces an equivalence of triangulated categories*

$$D^b\,\mathrm{Vect}(X) \cong D^b\,\mathrm{Coh}(X).$$

In particular, it induces a homotopy equivalence $K(X) \xrightarrow{\sim} G(X)$.

Proof:

We show below that every coherent sheaf F on X admits a surjective map $V \twoheadrightarrow F$ of O_X-modules where V is a vector bundle. This implies that the dual of criterion Sect. 3.1.7 (b) is satisfied, and we see that $D^b\,\mathrm{Vect}(X) \to D^b\,\mathrm{Coh}(X)$ is fully faithful. The existence of the surjection also implies that every coherent sheaf F admits a resolution

$$\cdots \to V_i \to V_{i-1} \to \cdots \to V_0 \to F \to 0$$

by vector bundles V_i. By Serre's Theorem [102, Theorem 4.4.16], [57, Theorem 19.2], for every point $x \in X$, the stalk at x of the image E_i of the map $V_i \to V_{i-1}$ is a free $O_{X,x}$-module when $i = \dim O_{X,x}$. Since E_i is coherent, there is an open neighborhood U_x of x over which the sheaf E_i is free and $i = \dim O_{X,x}$. Then E_i is locally free on U_x for all $i \geq \dim O_{X,x}$. Since X is quasi-compact, finitely many of the U_x's suffice to cover X, and we see that E_i is locally free on X for $i \gg 0$. The argument shows that we can truncate the resolution of F at some degree $i \gg 0$, and we obtain a finite resolution of F by vector bundles. Since $D^b\,\mathrm{Coh}(X)$ is generated by complexes concentrated in degree 0, the last statement implies that $D^b\,\mathrm{Vect}(X) \to D^b\,\mathrm{Coh}(X)$ is also essentially surjective, hence an equivalence. By Agreement and Invariance under derived equivalences (Theorems 3.2.24 and 3.2.14), we have $K(X) \simeq G(X)$.

To see the existence of a surjection $V \twoheadrightarrow F$, we can assume that X is connected, hence integral. The local rings $O_{X,x}$ are regular noetherian, hence UFD's. This implies that for any closed $Z \subset X$ of pure codimension 1, there is a line bundle $\mathscr{L}$ and a section $s : O_X \to \mathscr{L}$ such that $Z = X - X_s$ where X_s is the non-vanishing locus $\{x \in X |\ s_x : O_{X,x} \cong \mathscr{L}_x\} \subset X$ of s; see [41, Propositions II 6.11, 6.13]. Since any proper closed subset of X is in such a Z, the open subsets X_s indexed by pairs $(\mathscr{L}, s)$ form a basis for the topology of X where $\mathscr{L}$ runs through the line bundles of X and $s \in \Gamma(X, \mathscr{L})$.

For $a \in F(X_s)$, there is an integer $n \geq 0$ such that $a \otimes s^n \in \Gamma(X_s, F \otimes \mathscr{L}^n)$ extends to a global section of $F \otimes \mathscr{L}^n$; see [41, Lemma 5.14]. This global section defines a map $\mathscr{L}^{-n} \to F$ of O_X-modules such that $a \in F(X_s)$ is in the image of $\mathscr{L}^{-n}(X_s) \to F(X_s)$. It follows that there is a surjection $\bigoplus \mathscr{L}_i \twoheadrightarrow F$ from a sum of line bundles $\mathscr{L}_i$ to F. Since F is coherent and X is quasi-compact, finitely many of the $\mathscr{L}_i$'s are sufficient to yield a surjection. □

3.3.6 **Additivity** (Revisited)

Let $\mathscr{E}$ be an exact category, and let $E(\mathscr{E})$ denote the exact category of conflations in $\mathscr{E}$. Objects are conflations $A \rightarrowtail B \twoheadrightarrow C$ in $\mathscr{E}$, and maps are commutative diagrams of conflations. A sequence of conflations is called exact if it is exact at the A, B and C-spots. We define exact functors

$$\begin{aligned}
\lambda :\ & \mathscr{E} \to E(\mathscr{E}) : A \mapsto (A \overset{1}{\rightarrowtail} A \twoheadrightarrow 0)\\
\rho :\ & E(\mathscr{E}) \to \mathscr{E} : (A \rightarrowtail B \twoheadrightarrow C) \mapsto A\\
L :\ & E(\mathscr{E}) \to \mathscr{E} : (A \rightarrowtail B \twoheadrightarrow C) \mapsto C\\
R :\ & \mathscr{E} \to E(\mathscr{E}) : C \mapsto (0 \rightarrowtail C \overset{1}{\twoheadrightarrow} C).
\end{aligned}$$

Note that λ and L are left adjoint to ρ and R. The unit and counit of adjunctions induce natural isomorphisms $id \xrightarrow{\cong} \rho\lambda$ and $LR \xrightarrow{\cong} id$ and a functorial conflation $\lambda\rho \rightarrowtail id \twoheadrightarrow RL$.

3.3.7 **Theorem** (Additivity)

The sequence (λ, L) of exact functors induces an exact sequence of triangulated categories

$$D^b\mathscr{E} \to D^bE(\mathscr{E}) \to D^b\mathscr{E}.$$

The associated homotopy fibrations of K-theory spaces and spectra split via (ρ, R), and we obtain homotopy equivalences

$$(\rho, L) : K(E(\mathscr{E})) \xrightarrow{\sim} K(\mathscr{E}) \times K(\mathscr{E}) \quad \textit{and} \quad (\rho, L) : \mathbb{K}(E(\mathscr{E})) \xrightarrow{\sim} \mathbb{K}(\mathscr{E}) \times \mathbb{K}(\mathscr{E})$$

with inverses $\lambda \oplus R$.

Proof:

The functors ρ, λ, L and R are exact. Therefore, they induce triangle functors $D^b\rho$, $D^b\lambda$, D^bL and D^bR on bounded derived categories. Moreover, $D^b\lambda$ and D^bL are left adjoint to $D^b\rho$ and D^bR. The unit and counit of adjunctions $id \xrightarrow{\cong} D^b\rho \circ D^b\lambda$ and $D^bL \circ D^bR \xrightarrow{\cong} id$ are isomorphisms. The natural conflation $\lambda\rho \rightarrowtail id \twoheadrightarrow RL$ induces a functorial distinguished triangle $D^b\lambda \circ D^b\rho \to id \to D^bR \circ D^bL \to D^b\lambda \circ D^b\rho[1]$. By Appendix A, Exercise 2.8, this implies that the sequence of triangulated categories in the theorem is exact. The statements about K-theory follow from the Thomason–Waldhausen Localization Theorems 3.2.23 and 3.2.27. □

Proof of Additivity 2.3.11

The exact sequence of functors $\mathscr{E} \to \mathscr{E}'$ in Theorem 2.3.11 induces an exact functor $F_\bullet : \mathscr{E} \to E(\mathscr{E}')$. Let $M : E(\mathscr{E}') \to \mathscr{E}'$ denote the functor sending a conflation $(A \rightarrowtail B \twoheadrightarrow C)$ to B. By the Additivity Theorem 3.3.7, the composition of the functors

$$E(\mathscr{E}') \xrightarrow{(\rho,L)} \mathscr{E}' \times \mathscr{E}' \xrightarrow{\lambda\oplus R} E(\mathscr{E}')$$

induces a map on K-theory spaces and spectra which is homotopic to the identity functor. Therefore, the two functors

$$\mathscr{A} \xrightarrow{F_\bullet} E(\mathscr{E}') \xrightarrow{M} \mathscr{E}' \quad \text{and} \quad \mathscr{A} \xrightarrow{F_\bullet} E(\mathscr{E}') \xrightarrow{(\rho,L)} \mathscr{E}' \times \mathscr{E}' \xrightarrow{\lambda\oplus R} E(\mathscr{E}') \xrightarrow{M} \mathscr{E}'$$

induce homotopic maps on K-theory spaces and spectra. But these two functors are F_0 and $F_{-1} \oplus F_1$. □

3.3.8 Proposition (Resolution Revisited)

Under the hypothesis of the Resolution Theorem 2.3.12, the inclusion $\mathscr{A} \subset \mathscr{B}$ of exact categories induces an equivalence of triangulated categories

$$D^b(\mathscr{A}) \xrightarrow{\simeq} D^b(\mathscr{B}).$$

In particular, it induces homotopy equivalences of K-theory spaces and spectra

$$K(\mathscr{A}) \xrightarrow{\sim} K(\mathscr{B}) \quad \textit{and} \quad \mathbb{K}(\mathscr{A}) \xrightarrow{\sim} \mathbb{K}(\mathscr{B}).$$

Proof:

The hypothesis Theorem 2.3.12 (a) and (b) imply that the dual of criterion Sect. 3.1.7 (b) is satisfied, and the functor $D^b(\mathscr{A}) \to D^b(\mathscr{B})$ is fully faithful. Finally, the hypothesis Theorem 2.3.12 (b) implies that the triangle functor is also essentially surjective; see Sect. 3.1.7 (d). □

3.4 Thomason's Mayer-Vietoris Principle

Any reasonable cohomology theory for schemes should come with a Mayer–Vietoris long exact sequence for open covers. For K-theory this means that for a scheme $X = U \cup V$ covered by two open subschemes U and V, we should have a long exact sequence of K-groups

$$\cdots \to K_{i+1}(U \cap V) \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to K_{i-1}(X) \to \cdots$$

for $i \in \mathbb{Z}$. Surprisingly, the existence of such an exact sequence was only proved by Thomason [94] about 20 years after the introduction of higher algebraic K-theory by Quillen. Here, the use of derived categories is essential. For a regular noetherian separated scheme X, the exact sequence also follows from Quillen's Localization Theorem 2.3.7 together with Poincaré Duality Theorem 3.3.5. The purpose of this subsection is to explain the ideas that go into proving Thomason's Mayer-Vietoris exact sequence. Details of some proofs are given in Appendix A, Sects. 3 and 4.

If we defined $\mathbb{K}(X)$ naively as the K-theory $\mathbb{K}\,\mathrm{Vect}(X)$ of vector bundles, we would not have such a long exact sequence, in general. For that reason, one has to use perfect complexes instead of vector bundles in the definition of K-theory. For a quasi-projective scheme or a regular noetherian separated scheme, this is the same as vector bundle K-theory; see Proposition 3.4.8. Thomason proves the Mayer–Vietoris exact sequence for general quasi-compact and quasi-separated schemes; see Remark 3.4.13. Here, we will only give definitions and proofs for quasi-compact and separated schemes. This allows us to work with complexes of quasi-coherent sheaves as opposed to complexes of O_X-modules which have quasi-coherent cohomology. This is easier and it is sufficient for most applications.

For the following, the reader is advised to be acquainted with the definitions and statements in Appendix A, Sect. 3. For a quasi-compact and separated scheme X, we denote by $\mathrm{Qcoh}(X)$ the category of quasi-coherent sheaves on X, by $D\,\mathrm{Qcoh}(X)$ its unbounded derived category (see Appendix A, Sect. 3.1), and for a closed subset $Z \subset X$ we denote by $D_Z\,\mathrm{Qcoh}(X)$ the full subcategory of $D\,\mathrm{Qcoh}(X)$ of those complexes which are acyclic when restricted to $X - Z$.

3.4.1 Perfect Complexes

Let X be a quasi-compact and separated scheme. A complex (A,d) of quasi-coherent O_X-modules is called *perfect* if there is a covering $X = \bigcup_{i\in I} U_i$ of X by affine open subschemes $U_i \subset X$ such that the restriction of the complex $(A,d)_{|U_i}$ to U_i is quasi-isomorphic to a bounded complex of vector bundles for $i \in I$. The fact that this is independent of the chosen affine cover follows from Appendix A, Sect. 4. Let $Z \subset X$ be a closed subset of X with quasi-compact open complement $X - Z$. We write $\mathrm{Perf}_Z(X) \subset \mathrm{Ch}\,\mathrm{Qcoh}(X)$ for the full subcategory of perfect complexes on X which are acyclic over $X - Z$. The inclusion of categories of complexes is extension closed, and we can consider $\mathrm{Perf}_Z X$ as a fully exact subcategory of the abelian category $\mathrm{Ch}\,\mathrm{Qcoh}(X)$. As in Sect. 3.2.3, the ordinary tensor product of chain complexes makes $(\mathrm{Perf}_Z(X), \mathrm{quis})$ into a complicial exact category with weak equivalences. It is customary to write $D\,\mathrm{Perf}_Z(X)$ for $\mathscr{T}(\mathrm{Perf}_Z(X), \mathrm{quis})$ and $\mathrm{Perf}(X)$ for $\mathrm{Perf}_X(X)$.

3.4.2 Definition

Let X be a quasi-compact and separated scheme, and let $Z \subset X$ be a closed subset of X with quasi-compact open complement $X - Z$. The *K-theory spectrum of X with support in Z* is the $\mathbb{K}$-theory spectrum

$$\mathbb{K}(X \text{ on } Z) = \mathbb{K}(\mathrm{Perf}_Z(X), \mathrm{quis})$$

of the complicial exact category $(\mathrm{Perf}_Z(X), \mathrm{quis})$ as defined in Sect. 3.4.1. In case $Z = X$, we simply write $\mathbb{K}(X)$ instead of $\mathbb{K}(X \text{ on } Z)$. It follows from Proposition 3.4.6 below that the triangulated categories $D\,\mathrm{Perf}_Z(X)$ are idempotent complete. Therefore, we may write $K_i(X \text{ on } Z)$ instead of $\mathbb{K}_i(X \text{ on } Z)$ for the K-groups of X with support in Z and $i \in \mathbb{Z}$.

In order to be able to say anything about the K-theory of perfect complexes, we need to understand, to a certain extend, the structure of the triangulated categories $D\,\mathrm{Perf}_Z(X)$. Lemma 3.4.3 and Proposition 3.4.6 summarize what we will need to know.

3.4.3 Lemma

Let X be a quasi-compact and separated scheme and $Z \subset X$ be a closed subset with quasi-compact open complement $j : U = X - Z \subset X$.

(a) *The following sequence of triangulated categories is exact*

$$D_Z\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(U).$$

(b) *Let $g : V \subset X$ be a quasi-compact open subscheme such that $Z \subset V$. Then the restriction functor is an equivalence of triangulated categories*

$$g^* : D_Z\,\mathrm{Qcoh}(X) \xrightarrow{\simeq} D_Z\,\mathrm{Qcoh}(V).$$

Proof:

For (a), the restriction $j^* : D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(U)$ has a right adjoint $Rj_* : D\,\mathrm{Qcoh}(U) \to D\,\mathrm{Qcoh}(X)$ which for $E \in D\,\mathrm{Qcoh}(U)$ is $Rj_*E = j_*I$ where $E \xrightarrow{\sim} I$ is a $\mathscr{K}$-injective resolution of E. The counit of adjunction $j^*Rj_* \to 1$ is an isomorphism in $D\,\mathrm{Qcoh}(U)$ since for

$E \in D\,\mathrm{Qcoh}(U)$ it is $j^*Rj_*E = j^*j_*I \xrightarrow{\cong} I \xleftarrow{\sim} E$. The claim now follows from general facts about triangulated categories; see Exercise 2.8(a) in Appendix A.

For (b), note that the functor $Rg_* : D\,\mathrm{Qcoh}(V) \to D\,\mathrm{Qcoh}(X)$ sends $D_Z\mathrm{Qcoh}(V)$ into $D_Z\mathrm{Qcoh}(X)$. This follows from the Base-Change Lemma A.3.7 since for a complex $E \in D_Z\mathrm{Qcoh}(V)$, the lemma says $(Rg_*E)_{|X-Z} = R\bar{g}_*(E_{|V-Z}) = 0$ where $\bar{g} : V - Z \subset X - Z$. The unit and counit of adjunction $1 \to Rg_* \circ g^*$ and $g^*Rg_* \to 1$ are isomorphisms in the triangulated category of complexes supported in Z because for such complexes this statement only needs to check when restricted to V where it trivially holds since $Z \subset V$. □

The reason why Lemma 3.4.3 is so useful lies in the theory of compactly generated triangulated categories and the fact that the categories $D_Z\mathrm{Qcoh}(X)$ are indeed compactly generated when X and $X - Z$ are quasi-compact and separated. See 3.4.6 below.

3.4.4 Compactly Generated Triangulated Categories

References are [65] and [64]. Let $\mathscr{A}$ be a triangulated category in which all set indexed direct sums exist. An object A of $\mathscr{A}$ is called *compact* if the canonical map

$$\bigoplus_{i\in I}\mathrm{Hom}(A,E_i) \to \mathrm{Hom}(A,\bigoplus_{i\in I}E_i)$$

is an isomorphism for any set of objects E_i in $\mathscr{A}$ and $i \in I$. Let $\mathscr{A}^c \subset \mathscr{A}$ be the full subcategory of compact objects. It is easy to see that $\mathscr{A}^c$ is an idempotent complete triangulated subcategory of $\mathscr{A}$.

A set S of compact objects is said to *generate* $\mathscr{A}$, or $\mathscr{A}$ is *compactly generated* (by S), if for every object $E \in \mathscr{A}$ we have

$$\mathrm{Hom}(A,E) = 0 \ \ \forall A \in S \quad \Longrightarrow \quad E = 0.$$

3.4.5 Theorem (Neeman [64])

(a) Let $\mathscr{A}$ be a compactly generated triangulated category with generating set S of compact objects. Then $\mathscr{A}^c$ is the smallest idempotent complete triangulated subcategory of $\mathscr{A}$ containing S.

(b) Let $\mathscr{R}$ be a compactly generated triangulated category, $S_0 \subset \mathscr{R}^c$ be a set of compact objects closed under taking shifts. Let $\mathscr{S} \subset \mathscr{R}$ be the smallest full triangulated subcategory closed under formation of coproducts in $\mathscr{R}$ which contains the set S_0. Then $\mathscr{S}$ and $\mathscr{R}/\mathscr{S}$ are compactly generated triangulated categories with generating sets S_0 and the image of $\mathscr{R}^c$ in $\mathscr{R}/\mathscr{S}$. Moreover, the functor $\mathscr{R}^c/\mathscr{S}^c \to \mathscr{R}/\mathscr{S}$ induces an equivalence between the idempotent completion of $\mathscr{R}^c/\mathscr{S}^c$ and the category of compact objects in $\mathscr{R}/\mathscr{S}$.

The following proposition will be proved in Appendix A, Sect. 4.

3.4.6 Proposition

Let X be a quasi-compact and separated scheme, and let $Z \subset X$ be a closed subset with quasi-compact open complement $U = X - Z$. Then the triangulated category $D_Z\mathrm{Qcoh}(X)$ is compactly generated with category of compact objects the derived category of perfect complexes $D\,\mathrm{Perf}_Z(X)$.

In many interesting cases, the K-theory of perfect complexes is equivalent to the K-theory of vector bundles. This is the case for quasi-projective schemes and for regular noetherian separated schemes both of which are examples of schemes with an ample family of line bundles.

3.4.7 Schemes with an Ample Family of Line Bundles

A quasi-compact scheme X has an *ample family of line bundles* if there is a finite set $L_1,\dots,L_n$ of line bundles with global sections $s_i \in \Gamma(X,L_i)$ such that the non-vanishing loci $X_{s_i} = \{x \in X | s_i(x) \neq 0\}$ form an open affine cover of X. See [94, Definition 2.1], [85, II 2.2.4].

Any quasi-compact open (or closed) subscheme of a scheme with an ample family of line bundles has itself an ample family of line bundles, namely the restriction of the ample family to the open (or closed) subscheme. Any scheme which is quasi-projective over an affine scheme has an ample line-bundle. A fortiori it has an ample family of line-bundles. Every separated regular noetherian scheme has an ample family of line bundles. This was shown in the proof of Poincaré Duality Theorem 3.3.5. For more on schemes with an ample family of line-bundles, see [15, 85], [94, 2.1.2] and Appendix A, Sect. 4.2.

3.4.8 Proposition [94, Corollary 3.9]

Let X be a quasi-compact and separated scheme which has an ample family of line bundles. Then the inclusion of bounded complexes of vector bundles into perfect complexes $\mathrm{Ch}^b\,\mathrm{Vect}(X) \subset \mathrm{Perf}(X)$ *induces an equivalence of triangulated categories* $D^b\,\mathrm{Vect}(X) \cong D\,\mathrm{Perf}(X)$. *In particular,*

$$\mathbb{K}\,\mathrm{Vect}(X) \simeq \mathbb{K}(X).$$

Proof (see also Appendix A, Proposition 4.7 (a))

Since X has an ample family of line bundles, every quasi-coherent sheaf F on X admits a surjective map $\bigoplus \mathscr{L}_i \to F$ from a direct sum of line bundles to F. The argument is the same as in the last paragraph in the proof of Theorem 3.3.5. This implies that the dual of criterion Sect. 3.1.7 (b) is satisfied, and we have fully faithful functors $D^b\,\mathrm{Vect}(X) \subset D^b\,\mathrm{Qcoh}(X) \subset D\,\mathrm{Qcoh}(X)$. This also implies that the compact objects $\mathrm{Vect}(X)$ generate $D\,\mathrm{Qcoh}(X)$ as a triangulated category with infinite sums. Since $D^b\,\mathrm{Vect}(X)$ is idempotent complete [16], the functor $D^b\,\mathrm{Vect}(X) \to D\,\mathrm{Perf}(X)$ is an equivalence by Theorem 3.4.5 (a) and Proposition 3.4.6. □

3.4.9 Theorem (Localization)

Let X be a quasi-compact and separated scheme. Let $U \subset X$ be a quasi-compact open subscheme with closed complement $Z = X - U$. Then there is a homotopy fibration of $\mathbb{K}$-theory spectra

$$\mathbb{K}(X \text{ on } Z) \longrightarrow \mathbb{K}(X) \longrightarrow \mathbb{K}(U).$$

In particular, there is a long exact sequence of K-groups for $i \in \mathbb{Z}$

$$\cdots \to K_{i+1}(U) \to K_i(X \text{ on } Z) \to K_i(X) \to K_i(U) \to K_{i-1}(X \text{ on } Z) \to \cdots$$

Proof:

In view of the Thomason-Waldhausen Localization Theorem 3.2.27, we have to show that the sequence of complicial exact categories with weak equivalences

$$(\mathrm{Perf}_Z(X), \mathrm{quis}) \to (\mathrm{Perf}(X), \mathrm{quis}) \to (\mathrm{Perf}(U), \mathrm{quis})$$

induces a sequence of associated triangulated categories

$$D\,\mathrm{Perf}_Z(X) \to D\,\mathrm{Perf}(X) \to D\,\mathrm{Perf}(U) \tag{16}$$

which is exact up to factors. By Proposition 3.4.6, the sequence (16) is the sequence of categories of compact objects associated with the exact sequence of triangulated categories in Lemma 3.4.3 (a). The claim now follows from Neeman's Theorem 3.4.5 (b). □

3.4.10 Theorem (Zariski Excision)

Let $j : V \subset X$ be a quasi-compact open subscheme of a quasi-compact and separated scheme X. Let $Z \subset X$ be a closed subset with quasi-compact open complement such that $Z \subset V$. Then restriction of quasi-coherent sheaves induces a homotopy equivalence of $\mathbb{K}$-theory spectra

$$\mathbb{K}(X \text{ on } Z) \xrightarrow{\sim} \mathbb{K}(V \text{ on } Z).$$

In particular, there are isomorphisms of K-groups for all $i \in \mathbb{Z}$

$$K_i(X \text{ on } Z) \xrightarrow{\cong} K_i(V \text{ on } Z).$$

Proof:

By the Invariance Of K-theory Under Derived Equivalences Theorem 3.2.29, it suffices to show that the functor of complicial exact categories with weak equivalences

$$(\mathrm{Perf}_Z(X), \mathrm{quis}) \to (\mathrm{Perf}_Z(V), \mathrm{quis})$$

induces an equivalence of associated triangulated categories. This follows from Lemma 3.4.3 (b) in view of Proposition 3.4.6. □

3.4.11 Remark

There is a more general excision result where open immersions are replaced with flat maps [94, Theorem 7.1]. It is also a consequence of an equivalence of triangulated categories.

3.4.12 Theorem (Mayer–Vietoris for Open Covers)

Let $X = U \cup V$ be a quasi-compact and separated scheme which is covered by two open quasi-compact subschemes U and V. Then restriction of quasi-coherent sheaves induces a homotopy cartesian square of $\mathbb{K}$-theory spectra

$$\begin{array}{ccc} \mathbb{K}(X) & \longrightarrow & \mathbb{K}(U) \\ \downarrow & & \downarrow \\ \mathbb{K}(V) & \longrightarrow & \mathbb{K}(U\cap V). \end{array}$$

In particular, we obtain a long exact sequence of K-groups for $i \in \mathbb{Z}$

$$\cdots \to K_{i+1}(U\cap V) \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U\cap V) \to K_{i-1}(X) \to \cdots$$

Proof:

By the Localization Theorem 3.4.9, the horizontal homotopy fibres of the square are $\mathbb{K}(X\ on\ Z)$ and $\mathbb{K}(V\ on\ Z)$ with $Z = X - U = U - U\cap V \subset V$. The claim follows from Zariski-excision 3.4.10.

□

3.4.13 Remark (Separated Versus Quasi-Separated)

Thomason proves Theorems 3.4.9, 3.4.10 and 3.4.12 for quasi-compact and quasi-separated schemes. A scheme X is quasi-separated if the intersection of any two quasi-compact open subsets of X is quasi-compact. For instance, any scheme whose underlying topological space is noetherian is quasi-separated. Of course, every separated scheme is quasi-separated.

In the generality of quasi-compact and quasi-separated schemes X one has to work with perfect complexes of *O_X-modules* rather than with perfect complexes of *quasi-coherent sheaves*. The reason is that the Base-Change Lemma A.3.7 – which is used at several places in the proofs of Lemma 3.4.3 (b) and Proposition 3.4.6 – does not hold for $D\,\mathrm{Qcoh}(X)$ when X is quasi-compact and quasi-separated, in general.

Verdier gives a counter example in [85, II Appendice I]. He constructs a quasi-compact and quasi-separated scheme Z (whose underlying topological space is even noetherian), a covering $Z = U \cup V$ of Z by open affine subschemes $j : U = \mathrm{Spec} A \hookrightarrow Z$ and V, and an injective A-module I such that for its associated sheaf $\tilde{I}$ on U, the natural map

$$j_*\tilde{I} = Rj_{\mathrm{Qcoh}_*}\tilde{I} \xrightarrow{\not\cong} Rj_{Mod*}\tilde{I} \tag{17}$$

is not a quasi- isomorphism where j_{Qcoh_*} and j_{Mod*} are j_* on the category of quasi-coherent modules and O_Z-modules, respectively. If Rj_{Qcoh_*} did satisfy the Base-Change Lemma, then the map (17) would be a quasi- isomorphism on U and V hence a quasi- isomorphism on Z, contradicting (17). Verdier also shows that for this scheme, the forgetful functor

$$D\,\mathrm{Qcoh}(Z) \to D_{qc}(O_Z\text{-Mod})$$

from the derived category of quasi-coherent modules to the derived category of complexes of O_Z-modules with quasi-coherent cohomology is not fully faithful. In particular, it is not an equivalence contrary to the situation when Z is quasi-compact and separated [14, Corollary 5.5], [4, Proposition 1.3].

3.5 Projective Bundle Theorem and Regular Blow-Ups

We will illustrate the use of triangulated categories in the calculation of higher algebraic K-groups with two more examples: in the proof of the Projective Bundle Formula 3.5.1 and in the (sketch of the) proof of the Blow-up Formula 3.5.4. There is, of course, much more to say about triangulated categories in K-theory. For instance, Example 3.1.11 has been generalized in [67] to stably flat non-commutative Cohn localizations $R \to S^{-1}R$ from which one can derive Waldhausen's calculations in [99]; see [74]. As another example, Swan's calculation of the K-theory of a smooth quadric hypersurface $Q \subset \mathbb{P}^n_k$ [89] can be derived from Kapranov's description of $D\,\mathrm{Perf}(Q)$ given in [44]. For certain homogeneous spaces, see [44, 52, 76]. After all, any statement about the structure of triangulated categories translates into a statement about higher algebraic K-groups via the Thomason–Waldhausen Localization Theorem 3.2.27.

3.5.1 Theorem (Projective Bundle Theorem [73, Sect. 8 Theorem 2.1])

Let X be a quasi-compact and separated scheme, and let $\mathscr{E} \to X$ be a geometric vector bundle over X of rank $n+1$. Let $p : \mathbb{P}\mathscr{E} \to X$ be the associated projective bundle with twisting sheaf $O_{\mathscr{E}}(1)$. Then we have an equivalence

$$\prod_{l=0}^{n} O_{\mathscr{E}}(-l) \otimes Lp^* : \prod_{l=0}^{n} \mathbb{K}(X) \xrightarrow{\sim} \mathbb{K}(\mathbb{P}\mathscr{E}).$$

For the proof we will need the following useful lemma which is a special case of Proposition A.4.7 (a).

3.5.2 Lemma

Let X be a scheme with an ample line-bundle L. Then the category $D^b\,\mathrm{Vect}(X)$ is generated – as an idempotent complete triangulated category – by the set $L^{\otimes k}$ of line-bundles for $k < 0$.

Proof of the Projective Bundle Theorem 3.5.1

By the Mayer–Vietoris Theorem 3.4.12, the question is local in X. Therefore, we may assume that $X = \mathrm{Spec}\,A$ is affine and that $p : \mathbb{P}\mathscr{E} \to X$ is the canonical projection $\mathrm{Proj}(A[T_0, \dots, T_n]) = \mathbb{P}^n_A \xrightarrow{p} \mathrm{Spec}\,A$. In this case, X and $\mathbb{P}\mathscr{E}$ have an ample line-bundle A and $O(1)$, and their derived categories of perfect complexes agree with the bounded derived categories of vector bundles by Proposition 3.4.8. Since the twisting sheaf $O(1)$ is ample, we can apply Lemma 3.5.2, and we see that the triangulated category $D\,\mathrm{Vect}(\mathbb{P}^n)$ is generated – as an idempotent complete triangulated category – by the family of line bundles $\{O_{\mathbb{P}^n}(-l) |\, l \geq 0\}$. Consider the polynomial ring $S = A[T_0, \dots, T_n]$ as a graded ring with $\deg T_i = 1$. The sequence $T_0, \dots, T_n$ is a regular sequence in S. Therefore, the Koszul complex $\bigotimes_{i=0}^{n}(\, S(-1) \xrightarrow{T_i} S\,)$ induces an exact sequence of graded S-modules

$$0 \to S(-n-1) \to \bigoplus_{1}^{n+1} S(-n) \to \bigoplus_{1}^{\binom{n+1}{2}} S(-n+1) \to \cdots \to \bigoplus_{1}^{n+1} S(-1) \to S \to A \to 0.$$

Taking associated sheaves, we obtain an exact sequence of vector bundles on $\mathbb{P}^n_A$

$$0 \to O(-n-1) \to \bigoplus_1^{n+1} O(-n) \to \bigoplus_1^{\binom{n+1}{2}} O(-n+1) \to \cdots \to \bigoplus_1^{n+1} O(-1) \to O_{\mathbb{P}^n} \to 0.$$

This shows that $D^b\,\mathrm{Vect}(\mathbb{P}^n_A)$ is generated as an idempotent complete triangulated category by $O(-n),...,O(-1),O_{\mathbb{P}^n}$. For $i \leq j$, let $D^b_{[i,j]} \subset D^b\,\mathrm{Vect}(\mathbb{P}^n)$ be the full idempotent complete triangulated subcategory generated by $O(l)$ where $i \leq l \leq j$. We have a filtration

$$0 \subset D^b_{[0,0]} \subset D^b_{[-1,0]} \subset ... \subset D^b_{[-n,0]} = D^b\,\mathrm{Vect}(\mathbb{P}^n).$$

The unit of adjunction $F \to Rp_*Lp^*F$ is a quasi-isomorphism for $F = A$ because $A \to H^0(Rp_*Lp^*A) = H^0(Rp_*O_{\mathbb{P}^n}) = H^0(\mathbb{P}^n, O_{\mathbb{P}^n})$ is an isomorphism and $H^i(\mathbb{P}^n, O_{\mathbb{P}^n}) = 0$ for $i \neq 0$ [38, Proposition III 2.1.12]. Since $D^b\,\mathrm{Proj}(A)$ is generated as an idempotent complete triangulated category by A, we see that the unit of adjunction $F \to Rp_*Lp^*F$ is a quasi-isomorphism for all $F \in D^b\,\mathrm{Proj}(A)$. This implies that $Lp^* = p^* : D^b\,\mathrm{Proj}(A) \to D^b\,\mathrm{Vect}(\mathbb{P}^n)$ is fully faithful and, hence, an equivalence onto its image $D^b_{[0,0]}$. Since $O(l)$ is an invertible sheaf, we obtain equivalences $O(-l) \otimes Lp^* : D^b\,\mathrm{Proj}(A) \to D^b_{[-l,-l]}$.

By the calculation of the cohomology of the projective space $\mathbb{P}^n_A$ (*loc.cit.*), we have $H^*(\mathbb{P}^n_A, O(-k)) = 0$ for $k = 1,...,n$. Therefore, the homomorphism sets in $D^b\,\mathrm{Vect}(\mathbb{P}^n_A)$ satisfy

$$\mathrm{Hom}(O(-j)[r], O(-l)[s]) = H^{s-r}(\mathbb{P}^n_A, O(-l+j)) = 0$$

for $0 \leq j < l \leq n$. This implies that the composition

$$D^b_{[-l,-l]} \subset D^b_{[-l,0]} \to D^b_{[-l,0]}/D^b_{[-l+1,0]}$$

is an equivalence; see Exercise A.2.8 (b).

To finish the proof, we simply translate the statements about triangulated categories above into statements about K-theory. For $i \leq j$, let $\mathrm{Ch}^b_{[i,j]} \subset \mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^n)$ be the full subcategory of those chain complexes which lie in $D^b_{[i,j]}$. Write w for the set of maps in $\mathrm{Ch}^b_{[-l,0]}$ which are isomorphisms in the quotient triangulated category $D^b_{[-l,0]}/D^b_{[-l+1,0]}$. By construction, the sequence

$$(\mathrm{Ch}^b_{[-l+1,0]}, \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, w) \tag{18}$$

induces an exact sequence of associated triangulated categories, and by Theorem 3.2.27, it induces a homotopy fibration in $I\!\!K$-theory for $l = 1,...,n$. We have seen that the composition

$$O(-l) \otimes p^* : (\mathrm{Ch}^b\,\mathrm{Proj}(A), \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, \mathrm{quis}) \to (\mathrm{Ch}^b_{[-l,0]}, w)$$

induces an equivalence of associated triangulated categories. By Theorem 3.2.29, the composition induces an equivalence in $I\!\!K$-theory. It follows that the $I\!\!K$-theory fibration associated with (18) splits, and we obtain a homotopy equivalence

$$(O(-l) \otimes p^*, 1) : I\!\!K(A) \times I\!\!K(\mathrm{Ch}^b_{[-l+1,0]}, \mathrm{quis}) \xrightarrow{\sim} I\!\!K(\mathrm{Ch}^b_{[-l,0]}, \mathrm{quis})$$

for $l = 1,...,n$. Since $\mathrm{Ch}^b_{[-n,0]} = \mathrm{Ch}^b\,\mathrm{Vect}(\mathbb{P}^n_A)$, this implies the theorem. □

3.5.3 Theorem (Bass' Fundamental Theorem)

Let X be a quasi-compact and separated scheme. Then there is a split exact sequence for all $n \in \mathbb{Z}$

$$0 \to \mathbb{K}_n(X) \to \mathbb{K}_n(X[T]) \oplus \mathbb{K}_n(X[T^{-1}]) \to \mathbb{K}_n(X[T,T^{-1}]) \to \mathbb{K}_{n-1}(X) \to 0.$$

Proof

The projective line $\mathbb{P}^1_X$ over X has a standard open covering given by $X[T]$ and $X[T^{-1}]$ with intersection $X[T,T^{-1}]$. Thomason's Mayer-Vietoris Theorem 3.4.12 applied to this covering yields a long exact sequence

$$\to \mathbb{K}_n(\mathbb{P}^1_X) \xrightarrow{\beta} \mathbb{K}_n(X[T]) \oplus \mathbb{K}_n(X[T^{-1}]) \to \mathbb{K}_n(X[T,T^{-1}]) \to \mathbb{K}_{n-1}(\mathbb{P}^1_X) \to$$

By the Projective Bundle Theorem 3.5.1, the group $\mathbb{K}_n(\mathbb{P}^1_X)$ is $\mathbb{K}_n(X) \oplus \mathbb{K}_n(X)$ with basis $[O_{\mathbb{P}^1}]$ and $[O_{\mathbb{P}^1}(-1)]$. Making a base-change, we can write $\mathbb{K}_n(\mathbb{P}^1_X)$ as $\mathbb{K}_n(X) \oplus \mathbb{K}_n(X)$ with basis $[O_{\mathbb{P}^1}]$ and $[O_{\mathbb{P}^1}] - [O_{\mathbb{P}^1}(-1)]$. Since on $X[T]$ and on $X[T^{-1}]$ the two line-bundles $O_{\mathbb{P}^1}$ and $O_{\mathbb{P}^1}(-1)$ are isomorphic, the map β in the long Mayer–Vietoris exact sequence above is trivial on the direct summand $K(X)$ corresponding to the base element $[O_{\mathbb{P}^1}] - [O_{\mathbb{P}^1}(-1)]$. The map β is split injective on the other summand $K(X)$ corresponding to the base element $[O_{\mathbb{P}^1}]$. Therefore, the long Mayer–Vietoris exact sequence breaks up into shorter exact sequences. These are the exact sequences in the theorem. The splitting of the map $\mathbb{K}_n(X[T,T^{-1}]) \to \mathbb{K}_{n-1}(X)$ is given by the cup product with the element $[T] \in K_1(\mathbb{Z}[T,T^{-1}])$. □

The following theorem is due to Thomason [90]. For the (sketch of the) proof given below, we follow [19, Sect. 1].

3.5.4 Theorem (Blow-Up Formula)

Let $i : Y \subset X$ be a regular embedding of pure codimension d with X quasi-compact and separated. Let $p : X' \to X$ be the blow-up of X along Y and $j : Y' \subset X'$ the exceptional divisor. Write $q : Y' \to Y$ for the induced map. Then the square of $\mathbb{K}$-theory spectra

$$\begin{array}{ccc} \mathbb{K}(X) & \xrightarrow{Li^*} & \mathbb{K}(Y) \\ {\scriptstyle Lp^*}\downarrow & & \downarrow{\scriptstyle Lq^*} \\ \mathbb{K}(X') & \xrightarrow[Lj^*]{} & \mathbb{K}(Y') \end{array}$$

is homotopy cartesian. Moreover, there is a homotopy equivalence

$$\mathbb{K}(X') \simeq \mathbb{K}(X) \times \prod_{1}^{d-1} \mathbb{K}(Y).$$

Proof (sketch)

To simplify the argument, we note that the question as to whether the square of $\mathbb{K}$-theory spectra is homotopy cartesian is local in X by Thomason's Mayer-Vietoris Theorem 3.4.12. Therefore, we can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/I$ are affine and that $I \subset A$ is an ideal generated by a regular sequence $x_1,...,x_d$ of length d. In this case, all schemes X, X', Y, Y' have an ample line-bundle. By Proposition 3.4.8, the K-theory of perfect complexes on X, X', Y, Y' agrees with the vector bundle K-theory on those schemes. Furthermore, the map $Y' \to Y$ is the canonical projection $\mathbb{P}^{d-1}_Y \to Y$.

Let $S = \bigoplus_{i\geq 0} I^n$. Then we have $X' = \operatorname{Proj} S$ and $Y' = \operatorname{Proj} S/IS$. The exact sequence $0 \to IS = S(1) \to S \to S/IS \to 0$ of graded S-modules induces an exact sequence of sheaves $0 \to O_{X'}(1) \to O_{X'} \to j_*O_{Y'} \to 0$ on X' and an associated distinguished triangle $O_{X'}(1) \to O_{X'} \to Rj_*O_{Y'} \to O_{X'}(1)[1]$ in $D^b\operatorname{Vect}(X')$. Restricted to Y', this triangle becomes the following distinguished triangle: $O_{Y'}(1) \to O_{Y'} \to j^*Rj_*O_{Y'} \to O_{Y'}(1)[1]$. Since $O_{Y'}(1) \to O_{Y'}$ is the zero map (as $Y' = \mathbb{P}^{d-1}_Y$), we have an isomorphism $j^*Rj_*O_{Y'} \cong O_{Y'} \oplus O_{Y'}(1)[1]$ in $D^b\operatorname{Vect}(Y')$. This shows that Lj^* respects the filtration of triangulated subcategories

$$\begin{array}{ccccccccc} D^0_{X'} & \subset & D^1_{X'} & \subset & \cdots & \subset & D^{d-1}_{X'} & = & D^b\operatorname{Vect}(X') \\ \downarrow{\scriptstyle Lj^*} & & \downarrow{\scriptstyle Lj^*} & & & & \downarrow{\scriptstyle Lj^*} & & \\ D^0_{Y'} & \subset & D^1_{Y'} & \subset & \cdots & \subset & D^{d-1}_{Y'} & = & D^b\operatorname{Vect}(Y') \end{array}$$

defined by setting $D^l_{X'}$ and $D^l_{Y'}$ to be the full idempotent complete triangulated subcategories of $D^b\operatorname{Vect}(X')$ and $D^b\operatorname{Vect}(Y')$ generated by $O_{X'}$ and the complexes $O_{X'}(-k)\otimes Rj_*O_{Y'}$ for $k = 1,...,l$ in the first case, and by $O_{Y'}(-k)$ for $k = 0,...,l$ in the second case. The fact that $D^{d-1}_{Y'} = D^b\operatorname{Vect}(Y')$ was shown in the proof of Theorem 3.5.1. A similar argument – using the ampleness of $O_{X'}(1)$ and the Koszul complex associated with the regular sequence $x_1,...,x_d \in S$ – shows that we have $D^{d-1}_{X'} = D^b\operatorname{Vect}(X')$. One checks that, on associated graded pieces, Lj^* induces equivalences of triangulated categories for $l = 1,...,d-1$

$$Lj^* : D^l_{X'}/D^{l-1}_{X'} \xrightarrow{\simeq} D^l_{Y'}/D^{l-1}_{Y'}. \tag{19}$$

A cohomology calculation shows that the units of adjunction $E \to Rp_*Lp^*E$ and $F \to Rq_*Lq^*F$ are isomorphisms for $E = O_X$ and $F = O_Y$. Since X and Y are affine, the triangulated categories $D^b\operatorname{Vect}(X)$ and $D^b\operatorname{Vect}(Y)$ are generated (up to idempotent completion) by O_X and O_Y. Therefore, the units of adjunction $E \to Rp_*Lp^*E$ and $F \to Rq_*Lq^*F$ are isomorphisms for all $E \in D^b\operatorname{Vect}(X)$ and $F \in D^b\operatorname{Vect}(Y)$. It follows that Lp^* and Lq^* are fully faithful and induce equivalences onto their images

$$Lp^* : D^b\operatorname{Vect}(X) \xrightarrow{\simeq} D^0_{X'} \quad \text{and} \quad Lq^* : D^b\operatorname{Vect}(Y) \xrightarrow{\simeq} D^0_{Y'}. \tag{20}$$

This finishes the triangulated category background.

In order to prove the $\mathbb{K}$-theory statement, define categories $\operatorname{Ch}^l_{X'} \subset \operatorname{Ch}^b\operatorname{Vect}(X')$ and $\operatorname{Ch}^l_{Y'} \subset \operatorname{Ch}^b\operatorname{Vect}(Y')$ as the fully exact complicial subcategories of those complexes which lie in $D^l_{X'}$ and $D^l_{Y'}$, respectively. Then $\mathscr{T}(\operatorname{Ch}^l_{X'},\mathrm{quis}) = D^l_{X'}$ and $\mathscr{T}(\operatorname{Ch}^l_{Y'},\mathrm{quis}) = D^l_{Y'}$. The functor j^* respects the filtration of exact categories with weak equivalences

$$\begin{array}{ccccccc}
(\mathrm{Ch}^0_{X'},\mathrm{quis}) & \subset & (\mathrm{Ch}^1_{X'},\mathrm{quis}) & \subset & \cdots & \subset & (\mathrm{Ch}^{d-1}_{X'},\mathrm{quis}) \\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & & & \downarrow{\scriptstyle j^*} \\
(\mathrm{Ch}^0_{Y'},\mathrm{quis}) & \subset & (\mathrm{Ch}^1_{Y'},\mathrm{quis}) & \subset & \cdots & \subset & (\mathrm{Ch}^{d-1}_{Y'},\mathrm{quis}).
\end{array} \tag{21}$$

If we denote by quis^l the set of maps in Ch^l which are isomorphisms in D^l/D^{l-1}, then $\mathscr{T}(\mathrm{Ch}^l,\mathrm{quis}^l) = D^l/D^{l-1}$. By the Theorem on Invariance Of $\mathbb{K}$-theory Under Derived Equivalences 3.2.29, the equivalence (19) yields an equivalence of $\mathbb{K}$-theory spectra $j^* : \mathbb{K}(\mathrm{Ch}^l_{X'},\mathrm{quis}^l) \xrightarrow{\simeq} \mathbb{K}(\mathrm{Ch}^l_{Y'},\mathrm{quis}^l)$ for $l = 1,\ldots,d-1$. The sequence $(\mathrm{Ch}^{l-1},\mathrm{quis}) \to (\mathrm{Ch}^l,\mathrm{quis}) \to (\mathrm{Ch}^l,\mathrm{quis}^l)$ induces a homotopy fibration of $\mathbb{K}$-theory spectra by the Thomason–Waldhausen Localization Theorem 3.2.27. Therefore, all individual squares in (21) induce homotopy cartesian squares of $\mathbb{K}$-theory spectra. As a composition of homotopy cartesian squares, the outer square also induces a homotopy cartesian squares of $\mathbb{K}$-theory spectra. By (20), the outer square of (21) yields the $\mathbb{K}$-theory square in the theorem.

The formula for $\mathbb{K}(X')$ in terms of $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ follows from the fact that $Lp^* : \mathbb{K}(X) \to \mathbb{K}(X')$ is split injective with retraction given by Rp_* and the fact that the cofibre of Lp^* is the cofibre of $Lq^* : \mathbb{K}(Y) \to \mathbb{K}(Y')$ which is given by the Projective Bundle Theorem 3.5.1. □

4 Beyond Triangulated Categories

4.1 Statement of Results

Of course, not all results in algebraic K-theory can be obtained using triangulated category methods. In this subsection we simply state some of these results. For more overviews on a variety of topics in K-theory, we refer the reader to the K-theory handbook [24].

4.1.1 Brown–Gersten–Quillen Spectral Sequence [73]

Let X be a noetherian scheme, and write $X^p \subset X$ for the set of points of codimension p in X. There is a filtration $0 \subset \ldots \subset \mathrm{Coh}^2(X) \subset \mathrm{Coh}^1(X) \subset \mathrm{Coh}^0(X) = \mathrm{Coh}(X)$ of $\mathrm{Coh}(X)$ by the Serre abelian subcategories $\mathrm{Coh}^i(X) \subset \mathrm{Coh}(X)$ of those coherent sheaves whose support has codimension $\geq i$. This filtration together with Quillen's Localization and *Dévissage* Theorems leads to the Brown–Gersten–Quillen (BGQ) spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow G_{-p-q}(X).$$

If X is regular and of finite type over a field, inspection of the differential d_1 yields an isomorphism

$$E_2^{p,-p} \cong \mathrm{CH}^p(X)$$

where $\mathrm{CH}^p(X)$ is the Chow-group of codimension p cycles modulo rational equivalence as defined in [28].

4.1.2 Gersten's Conjecture and Bloch's Formula

The Brown–Gersten–Quillen spectral sequence yields a complex

$$0 \to G_n(X) \to \bigoplus_{x \in X^0} K_n(k(x)) \xrightarrow{d_1} \bigoplus_{x \in X^1} K_{n-1}(k(x)) \xrightarrow{d_1} \cdots$$

The Gersten conjecture says that this complex is exact for $X = \operatorname{Spec} R$ where R is a regular local noetherian ring. The conjecture is proved in case R (is regular local noetherian and) contains a field [69] building on the geometric case proved in [73]. For other examples of rings satisfying the Gersten conjecture, see [86]. For K-theory with finite coefficients, Gersten's conjecture holds for the local rings of a smooth variety over a discrete valuation ring [32].

As a corollary, Quillen [73] obtains for a regular scheme X of finite type over a field a calculation of the E_2-term of the BGQ-spectral sequence as $E_2^{p,q} \cong H^p_{Zar}(X, \mathscr{K}_{-q,X})$, and he obtains Bloch's formula

$$\mathrm{CH}^p(X) \cong H^p_{Zar}(X, \mathscr{K}_{p,X})$$

where $\mathscr{K}_{p,X}$ denotes the Zariski sheaf associated with the presheaf $U \mapsto K_p(U)$.

4.1.3 Computation of $K(\mathbb{F}_q)$

Quillen computed the K-groups of finite fields in [72]. They are given by the formulas $K_0(\mathbb{F}_q) \cong \mathbb{Z}$, $K_{2n}(\mathbb{F}_q) = 0$ for $n > 0$ and $K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$ for $n > 0$.

4.1.4 The Motivic Spectral Sequence

Let X be a smooth scheme over a perfect field. Then there is a spectral sequence [27], [55]

$$E_2^{p,q} = H^{p-q}_{mot}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

where $H^p_{mot}(X, \mathbb{Z}(q))$ denotes the motivic cohomology of X as defined in [62,98]. It is proved in *loc.cit.* that this group is isomorphic to Bloch's higher Chow group $\mathrm{CH}^q(X, 2q-p)$ as defined in [13]. Rationally, the spectral sequence collapses and yields an isomorphism [13,53]

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_i \mathrm{CH}^i(X, n)_{\mathbb{Q}}.$$

4.1.5 Milnor K-Theory and the Bloch–Kato Conjecture

Let F be a commutative field. The Milnor K-theory $K^M_*(F)$ of F is the graded ring generated in degree 1 by symbols $\{a\}$ for $a \in F^\times$ a unit in F, modulo the relations $\{ab\} = \{a\} + \{b\}$ and $\{c\} \cdot \{1-c\} = 0$ for $c \neq 1$. One easily computes $K^M_0(F) = \mathbb{Z}$ and $K^M_1(F) = F^\times$. Since $K_1(F) = F^\times$, since Quillen's K-groups define a graded ring $K_*(F)$ which is commutative in the graded sense, and since the Steinberg relation $\{c\} \cdot \{1-c\} = 0$ holds in $K_2(F)$, we obtain a morphism $K^M_*(F) \to K_*(F)$ of graded rings extending the isomorphisms in degrees 0 and 1 above. Matsumoto's Theorem says that this map is also an isomorphism in degree 2, that is, the map $K^M_2(F) \to K_2(F)$ is an isomorphism; see [59].

In a similarly way, the ring structure on motivic cohomology yields a map

$$K^M_n(F) \to H^n_{mot}(F, \mathbb{Z}(n))$$

from Milnor K-theory to motivic cohomology. This map is an isomorphism for all $n \in \mathbb{N}$ and any field F by results of Nesterenko–Suslin [68] and Totaro [93].

Let $m = p^\nu$ be a prime power with p different from the characteristic of F, and let F_s be a separable closure of F. Then we have an exact sequence of Galois modules $1 \to \mu_m \to F_s^\times \xrightarrow{m} F_s^\times \to 1$ where μ_m denotes the group of m-th roots of unity. The first boundary map in the associated long exact sequence of étale cohomology groups induces a map $F^\times \to H^1_{et}(F, \mu_m)$. Using the multiplicative structure of étale cohomology, this map extends to a map of graded rings $K^M_*(F) \to H^*_{et}(F, \mu_m^{\otimes *})$ which induces the "norm residue homomorphism"

$$K^M_n(F)/m \to H^n_{et}(F, \mu_m^{\otimes n}).$$

The Bloch–Kato conjecture [11] for the prime p says that this map is an isomorphism for all n. The conjecture for $m = 2^\nu$ was proved by Voevodsky [96], and proofs for $m = p^\nu$ odd have been announced by Rost and Voevodsky.

As a consequence of the Bloch–Kato conjecture, Suslin and Voevodsky show in [88] (see also [33]) that the natural map from motivic cohomology with finite coefficients to étale cohomology is an isomorphism in a certain range:

$$H^i_{mot}(X, \mathbb{Z}/m(j)) \xrightarrow{\cong} H^i_{et}(X, \mu_m^{\otimes j}) \quad \text{for} \quad i \le j \ \text{and} \ m = p^\nu \tag{22}$$

where X is a smooth scheme over a field F of characteristic $\neq p$. For $i = j+1$, this map is still injective. If $\operatorname{char} F = p$, Geisser and Levine show in [32] that

$$H^i_{mot}(F, \mathbb{Z}/p^\nu(j)) = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad K^M_n(F)/p^\nu \cong K_n(F, \mathbb{Z}/p^\nu).$$

4.1.6 Quillen–Lichtenbaum

The Bloch–Kato conjecture implies the Quillen–Lich–tenbaum conjecture. Let X be a smooth quasi-projective scheme over the complex numbers $\mathbb{C}$. A comparison of the motivic spectral sequence 4.1.4 with the Atiyah–Hirzebruch spectral sequence converging to complex topological K-theory using the isomorphisms (22) and Grothendieck's isomorphism between étale cohomology with finite coefficients and singular cohomology implies an isomorphism

$$K^{alg}_n(X, \mathbb{Z}/m) \xrightarrow{\cong} K^{top}_n(X_{\mathbb{C}}, \mathbb{Z}/m) \quad \text{for} \quad n \ge \dim X - 1$$

between the algebraic K-theory with finite coefficients of X and the topological complex K-theory of the analytic topological space $X_{\mathbb{C}}$ of complex points associated with X; see for instance [71, Theorem 4.1]. For schemes over fields F other than the complex numbers, there is an analogous isomorphism where topological K-theory is replaced with étale K-theory and $\dim X$ with $cd_m X$ provided $\operatorname{char} F \nmid m$; see for instance [54, Corollary 13.3].

4.1.7 Computation of $K(\mathbb{Z})$

Modulo the Bloch–Kato conjecture for odd primes (which is announced as proven by Rost and Voevodsky) and the Vandiver conjecture, the K-groups of $\mathbb{Z}$ for $n \ge 2$ are given as follows [51, 60, 103]

n mod 8	1	2	3	4	5	6	7	0
$K_n(\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_k$	$\mathbb{Z}/2w_{2k}$	0	$\mathbb{Z}$	$\mathbb{Z}/c_k$	$\mathbb{Z}/w_{2k}$	0

where k is the integer part of $1+\frac{n}{4}$, and the numbers c_k and w_{2k} are the numerator and denominator of $\frac{B_k}{4k}$ with B_k the k-th Bernoulli number. The B_k's are the coefficients of the power series

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{t^{2k}}{(2k)!}$$

The Vandiver is still wide open, though it seems to be hard to come by a counter example; see [101, Remark on p. 159] for a discussion of the probability for finding such a counter example. The Vandiver conjecture is only used in the calculation of $K_{2m}(\mathbb{Z})$. It is in fact equivalent to $K_{4m}(\mathbb{Z}) = 0$ for all $m > 0$. In contrast, the calculation of $K_{2m+1}(\mathbb{Z})$ is independent of the Vandiver conjecture but it does use the Bloch–Kato conjecture.

4.1.8 Cdh Descent [19]

The following is due to Häsemeyer [39]. Let k be a field of characteristic 0, and write Sch_k for the category of separated schemes of finite type over k. Let F be a contravariant functor from Sch_k to the category of spectra (or chain complexes of abelian groups). Let $Y \to X \leftarrow X'$ be maps of schemes in Sch_k and $Y' = Y \times_X X'$ be the fibre product. Consider the following square of spectra (or chain complexes)

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(X') & \longrightarrow & F(Y') \end{array} \qquad (23)$$

obtained by functoriality of F. Suppose that F satisfies the following.

(a) *Nisnevich Descent*. Let $f: X' \to X$ be an étale map and $Y \to X$ be an open immersion. Assume that f induces an isomorphism $f: (X'-Y')_{red} \cong (X-Y)_{red}$. Then the square (23) is homotopy cartesian.
(b) *Invariance under nilpotent extensions*. The map $X_{red} \to X$ induces an equivalence $F(X) \simeq F(X_{red})$.
(c) *Excision for ideals*. Let $f: R \to S$ be a map of commutative rings, $I \subset R$ be an ideal such that $f: I \to f(I)$ is an isomorphism and $f(I)$ is an ideal in S. Consider $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R/I$, $X' = \operatorname{Spec} S$, $Y' = \operatorname{Spec} S/f(I)$ and the induced maps between them. Then (23) is homotopy cartesian.
(d) *Excision for blow-ups along regularly embedded centers*. Let $Y \subset X$ be a regular embedding of pure codimension. A closed immersion is regular of pure codimension d if, locally, its ideal sheaf is generated by a regular sequence of length d. Let X' be the blow-up of X along Y and $Y' \subset X'$ be the exceptional divisor. Then (23) is homotopy cartesian.

If a functor F satisfies (a)–(d), then the square (23) is homotopy cartesian for any *abstract blow-up square* in Sch_k. A fibre square of schemes as above is called abstract blow-up if $Y \subset X$ is a closed immersion, $X' \to X$ is proper and $X' - Y' \to X - Y$ is an isomorphism.

A functor F is said to satisfy *cdh-descent* if it satisfies Nisnevich descent (see (a) above) and if it sends abstract blow-up squares to homotopy cartesian squares. Thus, a functor for which (a)–(d) hold satisfies cdh-descent for separated schemes of finite type over a field of characteristic 0.

Example (Infinitesimal K-theory [19])

By Remark 3.4.11, Theorems 3.2.27 and 3.5.4, $\mathbb{K}$-theory satisfies (a) and (d). But neither (b) nor (c) hold for $\mathbb{K}$-theory. The same is true for cyclic homology and its variants since (a) and (d) are formal consequences of the Localization Theorem 3.2.27. Therefore, the homotopy fibre K^{inf} of the Chern character $\mathbb{K} \to HN$ from $\mathbb{K}$-theory to negative cyclic homology satisfies (a) and (d). By a theorem of Goodwillie [34], K^{inf} satisfies (b), and by a theorem of Cortiñas [22], K^{inf} satisfies (c). Therefore, infinitesimal K-theory K^{inf} satisfies cdh-descent in characteristic 0.

This was used in [19] to prove that $K_i(X) = 0$ for $i < -d$ when X is a d-dimensional scheme essentially of finite type over a field of characteristic 0. Moreover, we have $K_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$.

Examples

Cdh-descent in characteristic 0 also holds for homotopy K-theory KH [39], periodic cyclic homology HP [19] and stabilized Witt groups [78].

4.1.9 Homotopy Invariance and Vorst's Conjecture

Recall from Sect. 2.3.10 that algebraic K-theory is homotopy invariant for regular rings. More precisely, if R is a commutative regular noetherian ring, then the inclusion of constant polynomials $R \to R[T_1,\dots,T_n]$ induces for all $i \in \mathbb{Z}$ an isomorphism on K-groups

$$K_i(R) \xrightarrow{\cong} K_i(R[T_1,\dots,T_n]). \qquad (24)$$

In fact, the converse – a (special case of a) conjecture of Vorst [97]– is true in the following sense [20]. Let R be (a localization of) a ring of finite type over a field of characteristic zero. If the map (24) is an isomorphism for all $n \in \mathbb{N}$ and all $i \in \mathbb{Z}$ (in fact $i = 1 + \dim R$ suffices), then R is a regular ring.

A Appendix

A.1 Background from Topology

In this appendix we recall the definition of a simplicial set and of a classifying space of a category. Details can be found for instance in [25, 31, 58, 102]. We also recall in Sect. A.1.7, the definition of a homotopy fibration and in Sect. A.1.8, the definition of a spectrum.

A.1.1 Simplicial Sets

Let Δ be the category whose objects are the ordered sets $[n] = \{0,1,2,\dots,n\}$ for $n \geq 0$. A morphism in this category is an order preserving map of sets. Composition in Δ is composition of maps of sets. For $i = 0,\dots,n$ the unique order preserving injective maps $d_i : [n-1] \to [n]$ which leave out i are called *face maps*. For $j = 0,\dots,n-1$ the unique order preserving surjective maps $s_j : [n] \to [n-1]$ for which the pre-image of $j \in [n-1]$ contains two elements are called *degeneracy maps*. Every map in Δ is a composition of face and degeneracy maps. Thus, Δ is generated by face and degeneracy maps modulo some relations which the reader can find in the references cited above.

A *simplicial set* is a functor $X : \Delta^{op} \to$ Sets where Sets stands for the category of sets. Thus, for every integer $n \geq 0$ we are given a set X_n, and for every order preserving map $\theta : [n] \to [m]$ we are given a map of sets $\theta^* : X_m \to X_n$ such that $(\theta \circ \sigma)^* = (\sigma)^* \circ (\theta)^*$. Since Δ is generated by face and degeneracy maps, it suffices to specify θ^* for face and degeneracy maps and to check the relations alluded to above. A map of simplicial sets $X \to Y$ is a natural transformation of functors.

A *cosimplicial space* is a functor $\Delta \to$ Top where Top stands for the category of compactly generated Hausdorff topological spaces. A Hausdorff topological space is compactly generated if a subset is closed iff its intersection with every compact subset is closed in that compact subset. Every compact Hausdorff space and every CW-complex is compactly generated. For details, see [61, VIII.8], [104, I.4]. The standard cosimplicial space is the functor $\Delta_* : \Delta \to$ Top where

$$\Delta_n = \{(t_0, ..., t_n) \in \mathbb{R}^n \mid t_i \geq 0,\ \ t_0 + \cdots t_n = 1\} \subset \mathbb{R}^n$$

is equipped with the subspace topology coming from $\mathbb{R}^n$.

An order preserving map $\theta : [n] \to [m]$ defines a continuous map

$$\theta_* : \Delta_n \to \Delta_m : (s_0, ..., s_n) \mapsto (t_0, ..., t_m) \ \text{ with } \ t_i = \sum_{\theta(j)=i} s_j$$

such that $(\theta \circ \sigma)_* = \theta_* \circ \sigma_*$. The space Δ_n is homeomorphic to the usual n-dimensional ball with boundary $\partial\Delta_n = \bigcup_{0 \leq i \leq n} (d_i)_* \Delta_{n-1} \subset \Delta_n$ homeomorphic to the $n-1$-dimensional sphere.

The *topological realization* of a simplicial set X is the quotient topological space

$$|X| = \bigsqcup_{j \geq 0} X_j \times \Delta_j / \sim$$

where the equivalence relation $\sim$ is generated by $(\theta^* x, t) = (x, \theta_* t)$ for $x \in X_j$, $t \in \Delta_i$ and $\theta : [i] \to [j]$. A simplex $x \in X_n$ is called *non-degenerate* if $x \notin s_j^* X_{n-1}$ for all $j = 0, ...n-1$. Write $X_n^{nd} \subset X_n$ for the set of non-degenerate n-simplices. Let $|X|_n \subset |X|$ be the image of $\bigsqcup_{n \geq j \geq 0} X_j \times \Delta_j$ in $|X|$. Note that $|X|_0 = X_0$. One checks that the square

$$\begin{array}{ccc}
X_n^{nd} \times \partial\Delta_n & \hookrightarrow & X_n^{nd} \times \Delta_n \\
\downarrow & & \downarrow \\
|X|_{n-1} & \hookrightarrow & |X|_n
\end{array}$$

is cocartesian. Therefore, the space $|X|_n$ is obtained from $|X|_{n-1}$ by attaching exactly one n-cell Δ_n along its boundary $\partial\Delta_n$ for each non-degenerate n-simplex in X. In particular, $|X| = \bigcup_{n \geq 0} |X|_n$ has the structure of a CW-complex.

If X and Y are simplicial sets, the product simplicial set $X \times Y$ has n-simplices $X_n \times Y_n$ with structure maps given by $\theta^*(x, y) = (\theta^* x, \theta^* y)$. A proof of the following proposition can be found in [25, Proposition 4.3.15].

A.1.2 Proposition

For simplicial sets X and Y the projection maps $X \times Y \to X$ and $X \times Y \to Y$ induce a map of topological spaces $|X \times Y| \to |X| \times |Y|$ which is a homeomorphism provided the cartesian product $|X| \times |Y|$ is taken in the category of compactly generated Hausdorff topological spaces.

A.1.3 The Classifying Space of a Category

Consider the ordered set $[n]$ as a category whose objects are the integers 0, 1, ..., n. There is a unique map $i \to j$ if $i \le j$. Then a functor $[n] \to [m]$ is nothing else than an order preserving map. Thus, we can consider Δ as the category whose objects are the categories $[n]$ for $n \ge 0$, and where the morphisms in Δ are the functors $[n] \to [m]$.

Let $\mathscr{C}$ be a small category. Its nerve is the simplicial set $N_*\mathscr{C}$ whose n-simplices $N_n\mathscr{C}$ are the functors $[n] \to \mathscr{C}$. A functor $\theta : [n] \to [m]$ defines a map $N_m\mathscr{C} \to N_n\mathscr{C}$ given by $F \mapsto F \circ \theta$. We have $(\theta \circ \sigma)^* = (\sigma)^* \circ (\theta)^*$ and $N_*\mathscr{C}$ is indeed a simplicial set. An n-simplex in $N_*\mathscr{C}$, that is, a functor $[n] \to \mathscr{C}$, is nothing else than a string of composable arrows

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} C_n \tag{25}$$

in $\mathscr{C}$. The face map d_i^* deletes the object C_i and, if $i \neq 0, n$, it composes the maps f_{i-1} and f_i. The degeneracy map s_i doubles C_i and it inserts the identity map 1_{C_i}. In particular, the n-simplex (25) is non-degenerate iff none of the maps f_i is the identity map for $i = 0, \ldots, n-1$.

The *classifying space $B\mathscr{C}$* of a small category $\mathscr{C}$ is the topological realization

$$B\mathscr{C} = |N_*\mathscr{C}|$$

of the nerve simplicial set $N_*\mathscr{C}$ of $\mathscr{C}$. Any functor $\mathscr{C} \to \mathscr{C}'$ induces maps $N_*\mathscr{C} \to N_*\mathscr{C}'$ and $B\mathscr{C} \to B\mathscr{C}'$ on associated nerves and classifying spaces.

The classifying space construction commutes with products. This is because a functor $[n] \to \mathscr{C} \times \mathscr{C}'$ is the same as a pair of functors $[n] \to \mathscr{C}$, $[n] \to \mathscr{C}'$. Therefore, we have $N_*(\mathscr{C} \times \mathscr{C}') = N_*\mathscr{C} \times N_*\mathscr{C}'$ and $B(\mathscr{C} \times \mathscr{C}') = B\mathscr{C} \times B\mathscr{C}'$ by Proposition A.1.2.

A.1.4 Example $B[1]$

The nerve of the category $[1]$ has two non-degenerate 0-simplices, namely the objects 0 and 1. It has exactly one non-degenerate 1-simplex, namely the map $0 \to 1$. All other simplices are degenerate. Thus, the classifying space $B[1]$ of $[1]$ is obtained from the two point set $\{0,1\}$ by attaching a 1-cell Δ_1 along its boundary $\partial\Delta_1$. The attachment is such that the two points of $\partial\Delta_1$ are identified with the two points $\{0,1\}$. We see that $B[1]$ is homeomorphic to the usual interval $\Delta_1 \cong [0,1]$.

A.1.5 Example $B\underline{G}$

For a group G, we let $\underline{G}$ be the category with one object $*$ and where $\mathrm{Hom}(*,*) = G$. Then $\pi_i B\underline{G} = 0$ for $i \neq 1$ and $\pi_1 B\underline{G} = G$ where the isomorphism $G \to \pi_1 B\underline{G}$ sends an element $g \in G$ to the loop l_g represented by the morphism $g : * \to *$. For details, see for instance [102, Exercise 8.2.4, Example 8.3.3].

A.1.6 Lemma

A natural transformation $\eta : F_0 \to F_1$ between functors $F_0, F_1 : \mathscr{C} \to \mathscr{C}'$ induces a homotopy $BF_0 \simeq BF_1$ between the associated maps on classifying spaces $BF_0, BF_1 : B\mathscr{C} \to B\mathscr{C}'$. In particular, an equivalence of categories $\mathscr{C} \to \mathscr{C}'$ induces a homotopy equivalence $\mathscr{B}\mathscr{C} \to \mathscr{B}\mathscr{C}'$.

Proof:

A natural transformation $\eta : F_0 \to F_1$ defines a functor $H : [1] \times \mathscr{C} \to \mathscr{C}'$ which sends the object (i,X) to $F_i(X)$ where $i = 0,1$ and $X \in \mathscr{C}$. There are two types of morphisms in $[1] \times \mathscr{C}$, namely (id_i, f) and $(0 \to 1, f)$ where $i = 0,1$ and $f : X \to Y$ is a map in $\mathscr{C}$. They are sent to $F_i(f)$ for $i = 0,1$ and to $\eta_Y F_0(f) = F_1(f)\eta_X$, respectively. It is easy to check that H is indeed a functor. Now, H induces a map $[0,1] \times B\mathscr{C} = B[1] \times B\mathscr{C} = B([1] \times \mathscr{C}) \to B\mathscr{C}'$ on classifying spaces whose restrictions to $\{0\} \times B\mathscr{C}$ and $\{1\} \times B\mathscr{C}$ are BF_0 and BF_1. Thus, BF_0 and BF_1 are homotopic maps.

If $F : \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories, then there are a functor $G : \mathscr{C}' \to \mathscr{C}$ and natural isomorphisms $FG \cong 1$ and $1 \cong GF$. Thus, the map $BG : B\mathscr{C}' \to B\mathscr{C}$ is a homotopy inverse of BF. □

A.1.7 Homotopy Fibres and Homotopy Fibrations

Let $g : Y \to Z$ be a map of pointed topological spaces. The homotopy fibre $F(g)$ of g is the pointed topological space

$$F(g) = \{(\gamma, y) |\ \gamma : [0,1] \to Z \ \text{ s.t. }\ \gamma(0) = *, \gamma(1) = g(y)\} \subset Z^{[0,1]} \times Y$$

with base-point the pair $(*,*)$ where the first $*$ is the constant path $t \mapsto *$ for $t \in [0,1]$. There is a continuous map of pointed spaces $F(g) \to Y$ given by $(\gamma, y) \mapsto y$ which fits into a natural long exact sequence of homotopy groups [104, Corollary IV.8.9]

$$\cdots \to \pi_{i+1} Z \to \pi_i F(g) \to \pi_i Y \to \pi_i Z \to \pi_{i-1} F(g) \to \cdots \tag{26}$$

ending in $\pi_0 Y \to \pi_0 Z$. For more details, see [104, Chap. I.7].

A sequence of pointed spaces $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that the composition is the constant map to the base-point of Z is called *homotopy fibration* if the natural map $X \to F(g)$ given by $x \mapsto (*, f(x))$ is a homotopy equivalence. In this case, there is a long exact sequence of homotopy groups as in (26) with X in place of $F(g)$.

A.1.8 Spectra and Homotopy Cartesian Squares of Spectra

A *spectrum* is a sequence $E_0, E_1, E_2, \ldots$ of pointed topological spaces together with pointed maps $\sigma_i : E_i \to \Omega E_{i+1}$ called *bonding maps* or *structure maps*. The spectrum (E, σ) is called Ω*-spectrum* if the bonding maps σ_i are homotopy equivalences for all $i \in \mathbb{N}$. For $i \in \mathbb{Z}$, the homotopy group $\pi_i E$ of the spectrum (E, σ) is the colimit

$$\pi_i E = \operatorname{colim}(\pi_{i+l} \Omega^{k-l} E_k \xrightarrow{\sigma} \pi_{i+l} \Omega^{k-l+1} E_{k+1} \xrightarrow{\sigma} \pi_{i+l} \Omega^{k-l+2} E_{k+2} \to \cdots).$$

This colimit is independent of k and l as long as $i + l \geq 0$ and $k \geq l$. Thus, it also makes sense for $i < 0$. If $(\mathscr{E}, \sigma)$ is an Ω-spectrum, then $\pi_i E = \pi_i E_0$ for $i \geq 0$ and $\pi_i E = \pi_0 E_{-i}$ for $i < 0$.

A map of spectra $f : (E, \sigma) \to (E', \sigma')$ is a sequence of pointed maps $f_i : E_i \to E'_i$ such that $\sigma'_i f_i = (\Omega f_{i+1})\sigma_i$. The map of spectra is called *equivalence of spectra* if it induces an isomorphism on all homotopy groups π_i for $i \in \mathbb{Z}$. The homotopy fibre $F(f)$ of a map of spectra $f : (E, \sigma) \to (E', \sigma')$ is the sequence of pointed topological spaces $F(f_0), F(f_1), F(f_2), \ldots$ together with bonding maps $F(f_i) \to \Omega F(f_{i+1}) = F(\Omega f_{i+1})$ between the homotopy fibres of f_i and Ωf_{i+1} given by the maps σ_i and σ'_i. Taking a colimit over the exact sequences (26) yields the exact sequence of abelian groups for $i \in \mathbb{Z}$

$$\cdots \to \pi_{i+1} E' \to \pi_i F(f) \to \pi_i E \to \pi_i E' \to \pi_{i-1} F(f) \to \cdots \tag{27}$$

A sequence of spectra $E'' \to E \xrightarrow{f} E'$ is a *homotopy fibration* if the composition $E'' \to E'$ is (homotopic to) the zero spectrum (the spectrum with all spaces a point), and the induced map $E'' \to F(f)$ is an equivalence of spectra. In this case, we can replace $F(f)$ by E'' in the long exact sequence (27). A commutative square of spectra

$$\begin{array}{ccc} E_{00} & \xrightarrow{f_0} & E_{01} \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} \\ E_{10} & \xrightarrow{f_1} & E_{11} \end{array}$$

is called *homotopy cartesian* if the induced map $F(f_0) \to F(f_1)$ on horizontal homotopy fibres (or equivalently, the map $F(g_0) \to F(g_1)$ on vertical homotopy fibres) is an equivalence of spectra. From the exact sequence (27) and the equivalence $F(f_0) \xrightarrow{\sim} F(f_1)$, we obtain a long exact sequence of homotopy groups of spectra for $i \in \mathbb{Z}$

$$\cdots \to \pi_{i+1}(E_{11}) \to \pi_i(E_{00}) \to \pi_i(E_{01}) \oplus \pi_i(E_{10}) \to \pi_i(E_{11}) \to \pi_{i-1}(E_{00}) \to \cdots$$

For more on spectra, see [1, III], [9, 43, 79].

A.2 Background on Triangulated Categories

Our main references here are [48, 66, 95].

A.2.1 Definition

A *triangulated category* is an additive category $\mathscr{A}$ together with an auto-equivalence[3] $T : \mathscr{A} \to \mathscr{A}$ and a class of sequences

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \tag{28}$$

of maps in $\mathscr{A}$ called *distinguished triangles*. They are to satisfy the axioms TR1 – TR4 below.

TR1. Every sequence of the form (28) which is isomorphic to a distinguished triangle is a distinguished triangle. For every object A of $\mathscr{A}$, the sequence $A \xrightarrow{1} A \to 0 \to TA$ is a distinguished triangle. Every map $u : X \to Y$ in $\mathscr{A}$ is part of a distinguished triangle (28).

TR2. A sequence (28) is distinguished if and only if $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is a distinguished triangle.

TR3. For any two distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$ and for any pair of maps $f : X \to X'$ and $g : Y \to Y'$ such that $gu = u'f$ there is a map $h : Z \to Z'$ such that $hv = v'g$ and $(Tf)w = w'h$.

TR4. Octahedron axiom, see [48, 95] and in Sect. A.2.2 below.

In a distinguished triangle (28) the object Z is determined by the map u up to (non-canonical) isomorphism. We call Z "the" *cone* of u.

---

[3] We may sometimes write $A[1]$ instead of TA especially when A is a complex.

A.2.2 Good Maps of Triangles and the Octahedron Axiom

A useful reformulation of the octahedron axiom TR4 (which we haven't stated...) is as follows [66, Definition 1.3.13 and Remark 1.4.7]. Call a map of distinguished triangles

$$\begin{array}{ccccccc} A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & TA_0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle Ta_0} \\ B_0 & \xrightarrow{b_0} & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & TB_0 \end{array} \tag{29}$$

good if the mapping cone (in the sense of complexes)

$$B_0 \oplus A_1 \xrightarrow{\begin{pmatrix} b_0 & f_1 \\ 0 & -a_1 \end{pmatrix}} B_1 \oplus A_2 \xrightarrow{\begin{pmatrix} b_1 & f_2 \\ 0 & -a_2 \end{pmatrix}} B_2 \oplus TA_0 \xrightarrow{\begin{pmatrix} b_2 & Tf_0 \\ 0 & -Ta_0 \end{pmatrix}} TB_0 \oplus TA_1 \tag{30}$$

is a distinguished triangle. The reformulation of the octahedron axiom [63, Theorem 1.8] says that in a triangulated category every commutative diagram

$$\begin{array}{ccccccc} A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & TA_0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle Ta_0} \\ B_0 & \xrightarrow{b_0} & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & TB_0 \end{array}$$

in which the rows are distinguished triangles can be completed into a good morphism of distinguished triangles.

We will need the following special case below. If in a good map of distinguished triangles as in (29) the map f_2 is an isomorphism then the triangle

$$A_0 \xrightarrow{\begin{pmatrix} -f_0 \\ a_0 \end{pmatrix}} B_0 \oplus A_1 \xrightarrow{(b_0\ f_1)} B_1 \xrightarrow{a_2 f_2^{-1} b_1} TA_0$$

is distinguished. This is because, in case f_2 is an isomorphism, this triangle is a direct factor of the distinguished triangle obtained by rotating via TR2 the distinguished triangle (30). Therefore, it is a distinguished triangle itself [16, Lemma 1.6].

A.2.3 Definition

Let $\mathscr{R}$ and $\mathscr{S}$ be triangulated categories. A *triangle functor* [48, Sect. 8] from $\mathscr{R}$ to $\mathscr{S}$ is a pair (F,φ) where $F : \mathscr{R} \to \mathscr{S}$ is an additive functor and $\varphi : FT \xrightarrow{\cong} TF$ is a natural isomorphism such that for any distinguished triangle (28) in $\mathscr{R}$, the triangle $FX \to FY \to FZ \to TFX$ given by the maps $(Fu, Fv, \varphi_X Fw)$ is distinguished in $\mathscr{S}$. Triangle functors can be composed in the obvious way.

If a triangle functor has an adjoint, then the adjoint can be made into a triangle functor in a canonical way [66, Lemma 5.3.6], [47, 6.7]. In particular, if a triangulated category has infinite sums, then an arbitrary direct sum of distinguished triangles is a distinguished triangle.

A.2.4 Exercise

Let $F : \mathscr{S} \to \mathscr{T}$ be a triangle functor. If the functor is conservative (that is, a map f in $\mathscr{S}$ is an isomorphism iff $F(f)$ is) and full, then F is fully faithful.

A.2.5 Example (The Homotopy Category of an Additive Category)

Let $\mathscr{A}$ be an additive category. We denote by $\mathscr{K}(\mathscr{A})$ the homotopy category of chain complexes in $\mathscr{A}$. Its objects are the chain complexes in $\mathscr{A}$. Maps in $\mathscr{K}(\mathscr{A})$ are chain maps up to chain homotopy. The category $\mathscr{K}(\mathscr{A})$ is a triangulated category where a sequence is a distinguished triangle if it is isomorphic in $\mathscr{K}(\mathscr{A})$ to a cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{j} C(f) \xrightarrow{q} TX.$$

Here, $C(f)$ is the *mapping cone* of the chain map $f : X \to Y$ which is $C(f)^i = Y^i \oplus X^{i+1}$ in degree i and has differential $d^i = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}$. The object TX is the *shift* of X which is $(TX)^i = X^{i+1}$ in degree i and has differential $d^i = -d_X^{i+1}$. The maps $j : Y \to C(f)$ and $q : C(f) \to TX$ are the canonical inclusions and projections in each degree.

A.2.6 Calculus of Fractions

Let $\mathscr{C}$ be a category and $w \subset \operatorname{Mor}\mathscr{C}$ be a class of morphisms in $\mathscr{C}$. The *localization of $\mathscr{C}$ with respect to w* is the category obtained from $\mathscr{C}$ by formally inverting the morphisms in w. This is a category $\mathscr{C}[w^{-1}]$ together with a functor $\mathscr{C} \to \mathscr{C}[w^{-1}]$ which satisfies the following universal property. For any functor $\mathscr{C} \to \mathscr{D}$ which sends maps in w to isomorphisms, there is a unique functor $\mathscr{C}[w^{-1}] \to \mathscr{D}$ such that the composition $\mathscr{C} \to \mathscr{C}[w^{-1}] \to \mathscr{D}$ is the given functor $\mathscr{C} \to \mathscr{D}$. In general, the category $\mathscr{C}[w^{-1}]$ may or may not exist. It always exists if $\mathscr{C}$ is a small category.

If the class w satisfies a "calculus of right (or left) fractions", there is an explicit description of $\mathscr{C}[w^{-1}]$ as we shall explain now. A class w of morphisms in a category $\mathscr{C}$ is said to satisfy a *calculus of right fractions* if (a) – (c) below hold.

(a) The class w is closed under composition. The identity morphism 1_X is in w for every object X of $\mathscr{C}$.
(b) For all pairs of maps $u : X \to Y$ and $s : Z \to Y$ such that $s \in w$, there are maps $v : W \to Z$ and $t : W \to X$ such that $t \in w$ and $sv = ut$.
(c) For any three maps $f, g : X \to Y$ and $s : Y \to Z$ such that $s \in w$ and $sf = sg$, there is a map $t : W \to X$ such that $t \in w$ and $ft = gt$.

If the class w satisfies the dual of (a) – (c) then it is said to satisfy a *calculus of left fractions*. If w satisfies both, a calculus of left and right fractions, then w is said to satisfy a *calculus of fractions*.

If a class w of maps in a category $\mathscr{C}$ satisfies a calculus of right fractions, then the localized category $\mathscr{C}[w^{-1}]$ has the following description. Objects are the same as in $\mathscr{C}$. A map $X \to Y$ in $\mathscr{C}[w^{-1}]$ is an equivalence class of data $X \xleftarrow{s} M \xrightarrow{f} Y$ written as a *right fraction* fs^{-1} where f and s are maps in $\mathscr{C}$ such that $s \in w$. The datum fs^{-1} is equivalent to the datum $X \xleftarrow{t} N \xrightarrow{g} Y$ iff there are map $\bar{s} : P \to N$ and $\bar{t} : P \to M$ such that $\bar{s}$ (or $\bar{t}$) is in w and such that $s\bar{t} = \bar{s}t$ and $f\bar{t} = g\bar{s}$. The composition $(fs^{-1})(gt^{-1})$ is defined as follows. By (b) above, there are maps h and r in $\mathscr{C}$ such that $r \in w$ and $sh = gr$. Then $(fs^{-1})(gt^{-1}) = (fh)(tr)^{-1}$. In this description it is not clear whether $\operatorname{Hom}_{\mathscr{C}[w^{-1}]}(X,Y)$ is actually a set. However, it is a set if $\mathscr{C}$ is a small category. But in general, this issue has to be dealt with separately.

A.2.7 Verdier Quotient

Let $\mathscr{A}$ be a triangulated category and $\mathscr{B} \subset \mathscr{A}$ be a full triangulated subcategory. The class w of maps whose cones are isomorphic to objects in $\mathscr{B}$ satisfies a calculus of fractions. The *Verdier quotient* $\mathscr{A}/\mathscr{B}$ is, by definition, the localized category $\mathscr{A}[w^{-1}]$. It is a triangulated category where a sequence is a distinguished triangle if it is isomorphic to the image of a distinguished triangle of $\mathscr{A}$ under the localization functor $\mathscr{A} \to \mathscr{A}[w^{-1}]$; see [95], [66, Sect. 2]. If $\mathscr{B}' \subset \mathscr{A}$ denotes the full subcategory of those objects which are zero in the Verdier quotient $\mathscr{A}/\mathscr{B}$, then we have $\mathscr{B} \subset \mathscr{B}'$, the category $\mathscr{B}'$ is a triangulated category and every object of $\mathscr{B}'$ is a direct factor of an object of $\mathscr{B}$ [66, 2.1.33].

A.2.8 Exercise

The following exercises are variations on a theme called "Bousfield localization"; see [66, Sect. 9].

(a) Let $L : \mathscr{S} \to \mathscr{T}$ be a triangle functor which has a right adjoint R such that the counit of adjunction $LR \to id$ is an isomorphism. Let $\lambda : \mathscr{R} \subset \mathscr{S}$ be the full subcategory of $\mathscr{S}$ of those objects which are zero in $\mathscr{T}$. Then the sequence $\mathscr{R} \to \mathscr{S} \to \mathscr{T}$ is an exact sequence of triangulated categories (in the sense of Definition 3.1.5). Furthermore, the inclusion $\lambda : \mathscr{R} \subset \mathscr{S}$ has a right adjoint $\rho : \mathscr{S} \to \mathscr{R}$, and the counit and unit of adjunction fit into a functorial distinguished triangle in $\mathscr{S}$

$$\lambda\rho \to 1 \to RL \to \lambda\rho[1].$$

(b) Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{T}_0, \mathscr{T}_1 \subset \mathscr{T}$ be full triangulated subcategories. Assume that $\mathrm{Hom}(A_0, A_1) = 0$ for all objects $A_0 \in \mathscr{T}_0$ and $A_1 \in \mathscr{T}_1$. If $\mathscr{T}$ is generated as a triangulated category by the union of $\mathscr{T}_0$ and $\mathscr{T}_1$, then the composition $\mathscr{T}_1 \subset \mathscr{T} \to \mathscr{T}/\mathscr{T}_0$ is an equivalence. Moreover, an inverse induces a left adjoint $\mathscr{T} \to \mathscr{T}/T_0 \cong \mathscr{T}_1$ to the inclusion $\mathscr{T}_1 \subset \mathscr{T}$.

(c) Let $\mathscr{A} \xrightarrow{\lambda} \mathscr{B} \xrightarrow{L} \mathscr{C}$ be a sequence of triangle functors. Assume that λ and L have right adjoints ρ and R such that the unit $1 \to \rho\lambda$ and counit $LR \to 1$ are isomorphisms. Assume furthermore that for every object B of $\mathscr{B}$ the unit and counit of adjunction extend to a distinguished triangle in $\mathscr{B}$

$$\lambda\rho(B) \to B \to RL(B) \to \lambda\rho(B)[1].$$

Then the sequence of triangulated categories (λ, L) is exact.

A.2.9 Example (The Derived Category of an Abelian Category)

Let $\mathscr{A}$ be an abelian category. Its *unbounded derived category* $D(\mathscr{A})$ is obtained from the category $\mathrm{Ch}\mathscr{A}$ of chain complexes in $\mathscr{A}$ by formally inverting the quasi-isomorphisms. Recall that a chain map $f : A \to B$ is a quasi-isomorphism if it induces isomorphisms $H^i(f) : H^iA \to H^iB$ in cohomology for all $i \in \mathbb{Z}$ where for a chain complex (C, d) we have $H^iC = \ker d^i / \mathrm{im}\, d^{i-1}$. Since homotopy equivalences are quasi-isomorphisms, the category $D(\mathscr{A})$ is also obtained from the homotopy category $\mathscr{K}(\mathscr{A})$ by formally inverting the quasi-isomorphisms. Let $\mathscr{K}_{ac}(\mathscr{A}) \subset \mathscr{K}(\mathscr{A})$ be the full subcategory of acyclic chain complexes. This is the category of those chain complexes C for which $H^iC = 0$ for all $i \in \mathbb{Z}$. The inclusion

$\mathcal{K}_{ac}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$ is closed under taking cones. Furthermore, a chain complex A is acyclic iff TA is. Therefore, $\mathcal{K}_{ac}(\mathcal{A})$ is a full triangulated subcategory of $\mathcal{K}(\mathcal{A})$. Since a map is a quasi-isomorphism iff its cone is acyclic, we see that the category $D(\mathcal{A})$ is the Verdier quotient $\mathcal{K}(\mathcal{A})/\mathcal{K}_{ac}(\mathcal{A})$. In particular, the category $D(\mathcal{A})$ is a triangulated category (provided it exists, that is, provided it has small homomorphism sets).

There are versions $D^b\mathcal{A}$, $D^+\mathcal{A}$, $D^-\mathcal{A}$ of $D\mathcal{A}$ which are obtained from the category of bounded, bounded below, bounded above chain complexes in $\mathcal{A}$ by formally inverting the quasi-isomorphisms. Again, they are the Verdier quotients $\mathcal{K}^{b+-}(\mathcal{A})/\mathcal{K}^{b+-}_{ac}(\mathcal{A})$ of the corresponding homotopy categories by the homotopy category of acyclic chain complexes.

A.2.10 Exercise

Let $\mathcal{A}$ be an abelian category. Show that the obvious triangle functors $D^b\mathcal{A}, D^+\mathcal{A}, D^-\mathcal{A} \to D\mathcal{A}$ are fully faithful. Hint: Use the existence of the truncation functors $\tau^{\geq n} : D\mathcal{A} \to D^+\mathcal{A}$ and $\tau^{\leq n} : D\mathcal{A} \to D^-\mathcal{A}$ which for a complex E are the quotient complex $\tau^{\geq n}E = \cdots 0 \to \operatorname{coker} d^{n-1} \to E^{n+1} \to \cdots$ and the subcomplex $\tau^{\leq n}E = \cdots \to E^{n-1} \to \ker(d^n) \to 0 \to \cdots$ of E; see [8, Exemple 1.3.2].

A.2.11 The Derived Category of a Grothendieck Abelian Category

Recall that a *Grothendieck abelian category* is an abelian category $\mathcal{A}$ in which all set-indexed colimits exist, where filtered colimits are exact and which has a generator. An object U is a generator of $\mathcal{A}$ if for every object X of $\mathcal{A}$ there is a surjection $\bigoplus_I U \to X$ with I some index set. A set of objects is called *set of generators* if their direct sum is a generator. The unbounded derived category $D\mathcal{A}$ of a Grothendieck abelian category has small hom sets [102, Remark 10.4.5], [5, 26].

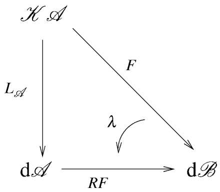
For a Grothendieck abelian category $\mathcal{A}$, the derived category $D\mathcal{A}$ has the following explicit description. Following [87], a complex $I \in \mathrm{Ch}\mathcal{A}$ is called *$\mathcal{K}$-injective* if for every map $f : X \to I$ and every quasi-isomorphism $s : X \to Y$ there is a unique map (up to homotopy) $g : Y \to I$ such that $gs = f$ in $\mathcal{K}(\mathcal{A})$. This is equivalent to the requirement that $\mathrm{Hom}_{\mathcal{K}\mathcal{A}}(A, I) = 0$ for all acyclic chain complexes A. For instance, a bounded below chain complex of injective objects in $\mathcal{A}$ is $\mathcal{K}$-injective. But $\mathcal{K}$-injective chain complexes do not need to consist of injective objects (for instance, every contractible chain complex is $\mathcal{K}$-injective), nor does an unbounded chain complex of injective objects need to be $\mathcal{K}$-injective.

In a Grothendieck abelian category, every chain complex has a $\mathcal{K}$-injective resolution [5, 26]. This means that for every chain complex X in $\mathcal{A}$ there is a quasi-isomorphism $X \to I$ where I is a $\mathcal{K}$-injective complex. Let $\mathcal{K}_{inj}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$ be the full subcategory of all $\mathcal{K}$-injective chain complexes. This is a triangulated subcategory. By definition, a quasi-isomorphism $I \xrightarrow{\sim} X$ from a $\mathcal{K}$-injective complex I to an arbitrary complex X always has a retraction up to homotopy. Therefore, the composition of triangle functors $\mathcal{K}_{inj}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful. This composition is also essentially surjective because every chain complex in $\mathcal{A}$ has a $\mathcal{K}$-injective resolution. Therefore, the triangle functor $\mathcal{K}_{inj}(\mathcal{A}) \to D(\mathcal{A})$ is an equivalence.

A.2.12 Right Derived Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. The functor induces a triangle functor $\mathcal{K}\mathcal{A} \to \mathcal{K}\mathcal{B}$ between the homotopy categories of unbounded chain complexes in $\mathcal{A}$ and $\mathcal{B}$. Denote by $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$ the localization triangle functors $\mathcal{K}\mathcal{A} \to D\mathcal{A}$

and $\mathscr{K}\mathscr{B} \to D\mathscr{B}$. Furthermore, denote by $F : \mathscr{K}\mathscr{A} \to D\mathscr{B}$ the composition of $\mathscr{K}\mathscr{A} \to \mathscr{K}\mathscr{B}$ with the localization functor $L_{\mathscr{B}}$. The *right-derived functor of* F is a pair (RF, λ) as in the diagram



where $RF : D\mathscr{A} \to D\mathscr{B}$ is a triangle functor and $\lambda : F \to RF \circ L_{\mathscr{A}}$ is a natural transformation of triangle functors which has the following universal property. For any pair (G, γ) where $G : D\mathscr{A} \to D\mathscr{B}$ is a triangle functor and $\gamma : F \to G \circ L_{\mathscr{A}}$ is a natural transformation of triangle functors, there is a unique natural transformation of triangle functors $\eta : RF \to G$ such that $\gamma = \eta \circ \lambda$. Of course, the pair (RF, λ) is uniquely determined by the universal property up to isomorphisms of natural transformations of triangle functors.

If $F : \mathscr{A} \to \mathscr{B}$ is any additive functor between Grothendieck abelian categories, then the right derived functor (RF, λ) of F always exists. For $E \in D\mathscr{A}$, it is given by $RF(E) = F(I)$ where $E \to I$ is a $\mathscr{K}$-injective resolution of E. The natural transformation λ at E is the image $FE \to FI$ under F of the resolution map $E \to I$. More generally, one has the following.

A.2.13 Exercise

Let $F : \mathscr{K}\mathscr{A} \to D\mathscr{B}$ be a triangle functor. Assume that there is a triangle endofunctor $G : \mathscr{K}\mathscr{A} \to \mathscr{K}\mathscr{A}$ such that $FG : \mathscr{K}\mathscr{A} \to D\mathscr{B}$ sends quasi-isomorphisms to isomorphisms. Assume furthermore that there is a natural quasi-isomorphism $\lambda : id \xrightarrow{\sim} G$ such that the two natural transformations $G\lambda$ and λ_G of functors $G \to GG$ satisfy $FG\lambda = F\lambda_G$. Then the pair $(FG, F\lambda)$ represents the right derived functor of F.

In the remainder of the subsection, we collect some basic facts about Frobenius exact categories and their triangulated stable categories. They constitute the framework for the complicial exact categories considered in the text.

A.2.14 Frobenius Exact Categories

An object P in an exact category $\mathscr{E}$ is called *projective* if for every deflation $q : Y \twoheadrightarrow Z$ and every map $f : P \to Z$ there is a map $g : P \to Y$ such that $f = qg$. An exact category $\mathscr{E}$ has *enough projectives* if for every object E of $\mathscr{E}$ there is a deflation $P \twoheadrightarrow E$ with P projective. Dually, an object I in $\mathscr{E}$ is called *injective* if for every inflation $j : X \rightarrowtail Y$ and every map $f : X \to I$ there is a map $g : Y \to I$ such that $f = gj$. An exact category $\mathscr{E}$ has *enough injectives* if for every object E of $\mathscr{E}$ there is an inflation $E \rightarrowtail I$ with I injective.

An exact category $\mathscr{E}$ is called *Frobenius exact category* if it has enough injectives and enough projectives, and an object is injective iff it is projective. Call two maps $f, g : X \to Y$ in a Frobenius exact category $\mathscr{E}$ *homotopic* if their difference factors through a projective-injective object. Homotopy is an equivalence relation. The *stable category* $\underline{\mathscr{E}}$ of a Frobenius exact category $\mathscr{E}$ is the category whose objects are the objects of $\mathscr{E}$ and whose maps are the homotopy classes of maps in $\mathscr{E}$. The stable category of a Frobenius exact category is a triangulated category as follows. To define the shift $T : \underline{\mathscr{E}} \to \underline{\mathscr{E}}$, we choose for every object X of $\mathscr{E}$ an inflation $X \rightarrowtail I(X)$ into an injective object, and we set $TX = I(X)/X$. Distinguished triangles in the stable category $\underline{\mathscr{E}}$ are those triangles which are isomorphic in $\underline{\mathscr{E}}$ to sequences of the form

$$X \xrightarrow{f} Y \to I(X) \sqcup_X Y \to I(X)/X$$

where $f : X \to Y$ is any map in $\mathscr{E}$. For more details, we refer the reader to [48] and [40, Sect. 9].

A.2.15 Complicial Exact Categories as Frobenius Categories

Recall from Sect. 3.2.1 the bounded complex of free $\mathbb{Z}$-modules $C = \mathbb{Z} \cdot 1_C \oplus \mathbb{Z} \cdot \eta$ where 1_C and η have degrees 0 and -1, respectively. There is a degree-wise split inclusion of chain complexes $i : \mathbb{1} = \mathbb{Z} \rightarrowtail C$ defined by $1 \mapsto 1_C$. Similarly, denote by $P \in \mathrm{Ch}^b(\mathbb{Z})$ the complex $P = Hom(C, \mathbb{1})$ which is concentrated in degrees 0 and 1 where it is a free $\mathbb{Z}$-module of rank 1. There is a degree-wise split surjection $p : P \to \mathbb{1} = \mathbb{Z}$ defined by $f \mapsto f(1_C)$.

Let $\mathscr{E}$ be a complicial exact category. This means that $\mathscr{E}$ comes equipped with an action by the category $\mathrm{Ch}^b(\mathbb{Z})$ of bounded complexes of free $\mathbb{Z}$-modules of finite rank; see Definition 3.2.2. We have natural inflations $i_E = i \otimes 1_E : E \rightarrowtail CE$ and natural deflations $p_E = p \otimes 1_E : PE \twoheadrightarrow E$ for every object E of $\mathscr{E}$. Call an inflation $j : X \rightarrowtail Y$ in $\mathscr{E}$ *Frobenius inflation* if for every object $U \in \mathscr{E}$ and every map $f : X \to CU$ there is a map $g : Y \to CU$ such that $f = gj$. Similarly, call a deflation $q : Y \twoheadrightarrow Z$ in $\mathscr{E}$ *Frobenius deflation* if for every object $U \in \mathscr{E}$ and every map $f : CU \to Z$ there is a map $g : CU \to Y$ such that $f = qg$.

A.2.16 Lemma

Let $\mathscr{E}$ be a complicial exact category.

(a) For every object E of $\mathscr{E}$, the natural inflation $i_X : X \rightarrowtail CX$ is a Frobenius inflation, and the natural deflation $p_X : PX \twoheadrightarrow X$ is a Frobenius deflation.
(b) Frobenius inflations (deflations) are closed under composition.
(c) Frobenius inflations (deflations) are preserved under push-outs (pull-backs)
(d) Split injections (surjections) are Frobenius inflations (deflations).
(e) For a conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathscr{E}$, the map $X \rightarrowtail Y$ is a Frobenius inflation iff the map $Y \twoheadrightarrow Z$ is a Frobenius deflation.
(f) The category $\mathscr{E}$ equipped with the Frobenius conflations as defined in Sect. 3.2.5 is a Frobenius exact category. In this exact structure, an object is injective (projective) iff it is a direct factor of an object of the form CU with $U \in \mathscr{E}$.

Proof:

For (a), we note that C is a commutative dg $\mathbb{Z}$-algebra with unique multiplication $\mu : C \otimes C \to C$ and unit map $i : \mathbb{1} \to C : 1 \mapsto 1_C$. Let $f : X \to CU$ be a map in $\mathscr{E}$. We define the map $f' : CX \to CU$ as the composition $CX \stackrel{1 \otimes f}{\to} CCU \stackrel{\mu \otimes 1}{\to} CU$. Then we have $f' i_X = (\mu \otimes 1_U) \circ (1_C \otimes f) \circ (i \otimes 1_X) = (\mu \otimes 1_U) \circ (i \otimes 1_C \otimes 1_U) \circ f = f$ since the composition $C \stackrel{i \otimes 1}{\to} C \otimes C \stackrel{\mu}{\to} C$ is the identity. This shows that i_X is a Frobenius inflation. The proof that $PX \to X$ is a Frobenius deflation is similar using the fact that $P = Hom(C, \mathbb{1})$ is a co-algebra. Sections (b), (c) and (d) are clear.

For (e), we note that the map $CX \to TX$ is a Frobenius deflation. This is because this map is isomorphic to $PTX \to TX$ via the isomorphism $C \to Hom(C,T) = PT$ which is adjoint to $C \otimes C \xrightarrow{\mu} C \to T$. Let $X \rightarrowtail Y$ be a Frobenius inflation. By definition, there is a map $Y \to CX$ such that the composition $X \to Y \to CX$ is the canonical Frobenius inflation $i_X : X \rightarrowtail CX$. Passing to quotients, we see that $Y \twoheadrightarrow Z$ is a pull-back of $CX \twoheadrightarrow TX$. Since the latter is a Frobenius deflation, we can apply (c), and we see that $Y \twoheadrightarrow Z$ is a Frobenius deflation as well. The other implication in (e) is dual. For (f), we note that (a) – (e) imply that $\mathscr{E}$ together with the Frobenius conflations is an exact category. By definition, objects of the form CU are injective and projective for the Frobenius exact structure, hence any of its direct factors is injective and projective. For an object I of $\mathscr{E}$ which is injective in the Frobenius exact structure, the Frobenius inflation $I \rightarrowtail CI$ has a retraction since, by the definition of injective objects, the map $1 : I \to I$ to the injective I extends to CI. Therefore, the injective object I is a direct factor of CI. Similarly for projective objects. □

A.3 The Derived Category of Quasi-Coherent Sheaves

A.3.1 Separated Schemes and Their Quasi-Coherent Sheaves

Let X be a quasi-compact and separated scheme. In the category $\mathrm{Qcoh}(X)$ of quasi-coherent O_X-modules, all small colimits exist and filtered colimits are exact (as they can be calculated locally on quasi-compact open subsets). Every quasi-coherent O_X-module is a filtered colimit of its quasi-coherent submodules of finite type [37, 9.4.9]. Therefore, the set of quasi-coherent O_X-modules of finite type forms a set of generators for $\mathrm{Qcoh}(X)$. Hence, the category $\mathrm{Qcoh}(X)$ is a Grothendieck abelian category. In particular, its derived category $D\,\mathrm{Qcoh}(X)$ exists, and it has an explicit description as in Example A.2.9.

A.3.2 Examples of Hom-Sets in $D\,\mathrm{Qcoh}(X)$

For a complex E of quasi-coherent O_X-modules, the set of homomorphisms $\mathrm{Hom}(O_X, E)$ in the triangulated category $D\,\mathrm{Qcoh}(X)$ is given by the formula

$$\mathrm{Hom}(O_X, E) = H^0(Rg_* E)$$

where $g : X \to \mathrm{Spec}\,\mathbb{Z}$ is the structure map of X. We can see this by replacing E with a $\mathscr{K}$-injective resolution $E \xrightarrow{\sim} I$. Then both sides are $H^0(I(X))$.

More generally, for a vector bundle A on X, the homomorphism set $\mathrm{Hom}(A, E)$ in $D\,\mathrm{Qcoh}(X)$ can be calculated as above using the equality

$$\mathrm{Hom}(A, E) = \mathrm{Hom}(O_X, E \otimes A^\vee)$$

where $A^\vee$ is the dual sheaf $Hom(A, O_X)$ of A. Again, we can see this by choosing a K-injective resolution $E \xrightarrow{\sim} I$ of E and noting that $E \otimes A^\vee \xrightarrow{\sim} I \otimes A^\vee$ is a $\mathscr{K}$-injective resolution of $E \otimes A^\vee$ when A (and thus $A^\vee$) is a vector bundle.

A.3.3 The Čech Resolution

Let X be a quasi-compact scheme, and let $\mathscr{U} = \{U_0,...,U_n\}$ be a finite cover of X by quasi-compact open subsets $U_i \subset X$. For a $k+1$ tuple $\underline{i} = (i_0,...,i_k)$ such that $0 \le i_0,...,i_k \le n$, write $j_{\underline{i}} : U_{\underline{i}} = U_{i_0} \cap ... \cap U_{i_k} \subset X$ for the open immersion of the intersection of the corresponding U_i's. Let F be a quasi-coherent O_X module. We consider the sheafified Čech complex $\check{C}(\mathscr{U},F)$ associated with the cover $\mathscr{U}$ of X. In degree k it is the quasi-coherent O_X-module

$$\check{C}(\mathscr{U},F)_k = \bigoplus_{\underline{i}} j_{\underline{i}*} j_{\underline{i}}^* F$$

where the indexing set is taken over all $k+1$-tuples $\underline{i} = (i_0,...,i_k)$ such that $0 \le i_0 < \cdots < i_k \le n$. The differential $d_k : \check{C}(\mathscr{U},F)_k \to \check{C}(\mathscr{U},F)_{k+1}$ for the component $\underline{i} = (i_0,...,i_{k+1})$ is given by the formula

$$(d_k(x))_{\underline{i}} = \sum_{l=0}^{k+1} (-1)^l \, j_{\underline{i}*} j_{\underline{i}}^* \, x_{(i_0,...,\hat{i}_l,...,i_{k+1})}.$$

Note that the complex $\check{C}(\mathscr{U},F)$ is concentrated in degrees $0,...,n$.

The units of adjunction $F \to j_{i*} j_i^* F$ define a map $F \to \check{C}(\mathscr{U},F)_0 = \bigoplus_{i=0}^n j_{i*} j_i^* F$ into the degree zero part of the Čech complex such that $d_0(F) = 0$. Therefore, we obtain a map of complexes of quasi-coherent O_X-modules $F \to \check{C}(\mathscr{U},F)$. This map is a quasi-isomorphism for any quasi-coherent O_X-module F as can be checked by restricting the map to the open subsets U_i of the cover $\mathscr{U}$ of X.

More generally, if F is a complex, then $\check{C}(\mathscr{U},F)$ is a bicomplex, and we can consider its total complex $\operatorname{Tot}\check{C}(\mathscr{U},F)$ which, by a slight abuse of notation, we will still denote by $\check{C}(\mathscr{U},F)$. The map $F \to \check{C}(\mathscr{U},F)$ is a map of bicomplexes. Taking total complexes, we obtain a natural quasi-isomorphism of complexes of quasi-coherent O_X-modules

$$\lambda_F : F \xrightarrow{\sim} \check{C}(\mathscr{U},F). \tag{31}$$

which is called the *Čech resolution of* F associated with the open cover $\mathscr{U}$.

A.3.4 Exercise

Let X be a scheme and $\mathscr{U}$ be a finite open cover of X. Write $\check{C}$ for the functor $F \mapsto \check{C}(\mathscr{U},F)$ from complexes of quasi-coherent sheaves to itself defined in Sect. A.3.3 above. Show that for any complex F of quasi-coherent sheaves on X, the following two maps $\check{C}(F) \to \check{C}(\check{C}(F))$ are chain homotopic:

$$\check{C}(\lambda_F) \sim \lambda_{\check{C}(F)}.$$

A.3.5 Explicit Description of Rg_*

Let $g : X \to Y$ be a map of quasi-compact schemes such that there is a finite cover $\mathscr{U} = \{U_0,...,U_n\}$ of X with the property that the restrictions $g_{\underline{i}} : U_{\underline{i}} \to Y$ of g to all finite intersections $U_{\underline{i}} = U_{i_0} \cap ... \cap U_{i_k}$ of the U_i's are affine maps where $\underline{i} = (i_0,...,i_k)$ and $i_0,...,i_k \in \{0,...,n\}$. If X and Y are quasi-compact and separated such a cover always exists. In this case, any cover $\mathscr{U} = \{U_0,...,U_n\}$ of X by affine open subschemes $U_i \subset X$ such that each U_i maps into an open affine subscheme of Y has this property.

Using the cover $\mathscr{U}$ instead of $\mathscr{K}$-injective resolutions, one can construct the right-derived functor $Rg_* : D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(Y)$ of $g_* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ as follows. By assumption, for every $k+1$-tuple $\underline{i} = (i_0, \ldots, i_k)$, the restriction $g_{\underline{i}} = g \circ j_{\underline{i}} : U_{\underline{i}} \to Y$ of g to $U_{\underline{i}}$ is an affine map. Therefore, the functor

$$g_*\check{C}(\mathscr{U}) : \mathrm{Qcoh}(X) \to \mathrm{Ch}\,\mathrm{Qcoh}(Y) : F \mapsto g_*\check{C}(\mathscr{U}, F)$$

is exact. Taking total complexes, this functor extends to a functor on all complexes

$$g_*\check{C}(\mathscr{U}) : \mathrm{Ch}\,\mathrm{Qcoh}(X) \to \mathrm{Ch}\,\mathrm{Qcoh}(Y) : F \mapsto g_*\check{C}(\mathscr{U}, F)$$

which preserves quasi-isomorphisms as it is exact and sends acyclics to acyclics. This functor is equipped with a natural quasi-isomorphism given by the Čech resolution

$$\lambda_F : F \xrightarrow{\sim} \check{C}(\mathscr{U}, F).$$

By Exercises A.2.13 and A.3.4, the right derived functor Rg_* of g_* is represented by the pair $(g_*\check{C}(\mathscr{U}), g_*\lambda)$.

A.3.6 Lemma

Let $g : X \to Y$ be a map of quasi-compact and separated schemes. Then for any set E_i, $i \in I$, of complexes of quasi-coherent O_X-modules, the following natural map of complexes of O_Y-modules is a quasi-isomorphism

$$\bigoplus_I Rg_*(E_i) \xrightarrow{\sim} Rg_*\left(\bigoplus_I E_i\right).$$

Proof:

This follows from the explicit construction of Rg_* given in Sect. A.3.5, for which the map in the lemma is already an isomorphism in $\mathrm{Ch}\,\mathrm{Qcoh}(Y)$. □

A.3.7 Lemma (Base-Change for Open Immersions)

Let $X = U \cup V$ be a quasi-compact separated scheme which is covered by two quasi-compact open subschemes U and V. Denote by $j : U \hookrightarrow X$, $j_V : U \cap V \hookrightarrow V$, $i : V \hookrightarrow X$, $i_U : U \cap V \hookrightarrow U$ the corresponding open immersions. Then for every complex E of quasi-coherent O_U-modules, the natural map

$$i^* \circ Rj_*E \xrightarrow{\cong} Rj_{V*} \circ i_U^* E$$

of complexes of quasi-coherent O_V-modules is an isomorphism in $D\,\mathrm{Qcoh}(V)$.

Proof

We first make the following remark. For a quasi-coherent O_U-module M, we have the canonical map $i^* j_* M \to j_{V*} i_U^* M$ which is adjoint to the map $j_V^* i^* j_* M = i_U^* j^* j_* M \to i_U^* M$ obtained by applying i_U^* to the counit map $j^* j_* M \to M$. Calculating sections over open subsets, we see that the map $i^* j_* M \to j_{V*} i_U^* M$ is an isomorphism for every quasi-coherent O_U-module M.

For the proof of the lemma, recall that U is quasi-compact. Therefore we can find a finite open affine cover $\mathscr{U} = \{U_0, \ldots, U_n\}$ of U. For this cover, all inclusions $U_{i_0} \cap \ldots \cap U_{i_k} \subset X$ are affine maps because X is separated. By Sect. A.3.5 and the remark above, for a complex E of quasi-coherent O_U-modules, we have an isomorphism

$$i^* R j_* E = i^* j_* \check{C}(\mathscr{U}, E) \overset{\cong}{\to} j_{V*} i_U^* \check{C}(\mathscr{U}, E) = j_{V*} \check{C}(\mathscr{U} \cap V, i_U^* E)$$

where $\mathscr{U} \cap V$ is the cover $\{U_0 \cap V, \ldots, U_n \cap V\}$ of V. Pull-backs of affine maps are affine maps. Hence, all inclusions $(U_{i_0} \cap V) \cap \ldots \cap (U_{i_k} \cap V) \subset V$ are affine maps. By Sect. A.3.5, the functor $j_{V*} \check{C}(\mathscr{U} \cap V)$ represents $R j_{V*}$. So, the isomorphism above represents an isomorphism of functors $i^* \circ R j_* \overset{\cong}{\longrightarrow} R j_{V*} \circ i_U^*$. □

A.3.8 Lemma

Let $X = V_1 \cup V_2$ be a quasi-compact and separated scheme which is covered by two quasi-compact open subschemes $V_1, V_2 \subset X$. Denote by $j_i : V_i \hookrightarrow X$ and $j_{12} : V_1 \cap V_2 \hookrightarrow X$ the corresponding open immersions. Then for $E \in D\,\mathrm{Qcoh}(X)$, there is a distinguished triangle in $D\,\mathrm{Qcoh}(X)$

$$E \longrightarrow R j_{1*}(j_1^* E) \oplus R j_{2*}(j_2^* E) \longrightarrow R j_{12*}(j_{12}^* E) \longrightarrow E[1].$$

Proof

Consider the commutative square in $D\,\mathrm{Qcoh}(X)$

$$\begin{array}{ccc} E & \longrightarrow & R j_{1*}(j_1^* E) \\ \downarrow & & \downarrow \\ R j_{2*}(j_2^* E) & \longrightarrow & R j_{12*}(j_{12}^* E) \end{array}$$

in which the maps are induced by the unit of adjunction maps $1 \to R j_* \circ j^*$ and the base-change isomorphism in Lemma A.3.7. By Sect. A.2.2, we can complete this square to the right into a good map of distinguished triangles. The map on horizontal cones is an isomorphism when restricted to U_1 (since both cones are zero, by Base-Change A.3.7) and when restricted to U_2 (by the Five Lemma and Base-Change A.3.7). Therefore, the map on horizontal cones is an isomorphism in $D\,\mathrm{Qcoh}(X)$. Finally, the sequence $E \to R j_{1*}(j_1^* E) \oplus R j_{2*}(j_2^* E) \to R j_{12*}(j_{12}^* E)$ can be completed to a distinguished triangle by the last paragraph in Sect. A.2.2. □

A.4 Proof of Compact Generation of $D_Z \operatorname{Qcoh}(X)$

In this appendix, we prove Proposition 3.4.6, first for schemes with an ample family of line bundles and then, by a formal induction argument, for general quasi-compact and separated schemes. To summarize, in Lemma A.4.10 we show that $D_Z \operatorname{Qcoh}(X)$ is compactly generated and in Lemmas A.4.8 and A.4.9 we show that the compact objects in $D_Z \operatorname{Qcoh}(X)$ are precisely those complexes which are isomorphic (in the derived category) to bounded complexes of vector bundles when restricted to the open subsets of an affine open cover of X. Part of the exposition is taken from [83]. When $Z = X$, the reader may also find proofs in [65, Corollary 2.3 and Proposition 2.5] and [18, Theorems 3.1.1 and 3.1.3].

We first recall the usual technique of extending a section of a quasi-coherent sheaf from an open subset cut out by a divisor to the scheme itself. For a proof, see [37, Théorème 9.3.1], [41, Lemma II.5.14].

A.4.1 Lemma

Let X be a quasi-compact and separated scheme, $s \in \Gamma(X,L)$ be a global section of a line bundle L on X, and $X_s = \{x \in X \mid s(x) \neq 0 \in L_x/m_xL_x\}$ be the non-vanishing locus of s. Let F be a quasi-coherent sheaf on X. Then the following hold.

(a) *For every $f \in \Gamma(X_s, F)$, there is an $n \in \mathbb{N}$ such that $f \otimes s^n$ extends to a global section of $F \otimes L^{\otimes n}$.*
(b) *For every $f \in \Gamma(X,F)$ such that $f_{|X_s} = 0$, there is an $n \in \mathbb{N}$ such that $f \otimes s^n = 0$.*

A.4.2 Schemes with an Ample Family of Line Bundles

A scheme X has an *ample family of line bundles* if there is a finite set $L_1, \ldots, L_n$ of line bundles on X and if there are global sections $s_i \in \Gamma(X, L_i)$ such that the non-vanishing loci $X_{s_i} = \{x \in X \mid s_i(x) \neq 0 \in L_x/m_xL_x\}$ form an open affine cover of X; see [94, Definition 2.1], [85, II 2.2.4]. Note that such an X is necessarily quasi-compact.

Recall that if $f \in \Gamma(X,L)$ is a global section of a line bundle L on a scheme X, then the open inclusion $X_f \subset X$ is an affine map (as can be seen by choosing an open affine cover of X trivializing the line bundle L). As a special case, the open subscheme X_f is affine whenever X is affine. Thus, for the affine cover $X = \bigcup X_{s_i}$ associated with an ample family of line-bundles as above, all finite intersections of the X_{s_i}'s are affine.

Let X be a scheme which has an ample family of line bundles. Then there is a set $\{L_i \mid i \in I\}$ of line bundles on X together with global sections $s_i \in \Gamma(X, L_i)$ such that the set $\{X_{s_i} \mid i \in I\}$ of non-vanishing loci forms an open affine basis for the topology of X [94, 2.1.1 (b)]. If X is affine, this follows from the definition of the Zariski topology. For a general X (with an ample family of line bundles), the sections which give rise to a basis of topology on an open affine X_s can be extended (up to a power of s) to global sections, by Lemma A.4.1. Therefore, every open subset of a basis for X_s is also the non-vanishing locus of a global section of some line bundle on X.

Let X be a scheme which has an ample family of line bundles $L_1, \ldots, L_n$. Then for every quasi-coherent sheaf F on X, there is a surjective map $M \twoheadrightarrow F$ of quasi-coherent sheaves where M is a (possibly infinite) direct sum of line bundles of the form L_i^k for $i = 1, \ldots, n$ and $k < 0$. This follows from the definition of an ample family of line bundles and Lemma A.4.1.

A.4.3 Truncated Koszul Complexes

Let X be a quasi-compact and separated scheme, and let L_i, $i = 1,...,l$ be a finite set of line bundles together with global sections $s_i \in \Gamma(X, L_i)$. Let $U = \bigcup_{i=1}^{l} X_{s_i}$ be the union of the non-vanishing loci X_{s_i} of the s_i's, and $j : U \subset X$ be the corresponding open immersion. The global sections s_i define maps $s_i : O_X \to L_i$ of line-bundles whose O_X-duals are denoted by s_i^{-1} : $L_i^{-1} \to O_X$. We consider the maps s_i^{-1} as (cohomologically graded) chain-complexes with O_X placed in degree 0. For an l-tuple $n = (n_1,...,n_l)$ of negative integers, the Koszul complex

$$\bigotimes_{i=1}^{l} (L_i^{n_i} \xrightarrow{s^{n_i}} O_X) \tag{32}$$

is acyclic over U. This is because the map $s^{n_i} = (s_i^{-1})^{\otimes |n_i|} : L_i^{n_i} \to O_X$ is an isomorphism when restricted to X_{s_i}, hence the Koszul complex (32) is acyclic (even contractible) over each X_{s_i}. Let $K(s^n)$ denote the bounded complex which is obtained from the Koszul complex (32) by deleting the degree zero part O_X and placing the remaining non-zero part in degrees $-l+1,...,0$. The last differential d^{-1} of the Koszul complex defines a map

$$K(s^n) = \left[\bigotimes_{i=1}^{l} (L_i^{n_i} \xrightarrow{s^{n_i}} O_X) \right]^{\leq -1} [-1] \xrightarrow{\varepsilon} O_X$$

of complexes of vector bundles. This map of complexes is a quasi-isomorphism over U, since its cone, the Koszul complex, is acyclic over U. For a complex M of quasi-coherent O_X-modules, we write ε_M for the tensor product map $\varepsilon_M = 1_M \otimes \varepsilon : M \otimes K(s^n) \to M \otimes O_X \cong M$.

The following proposition is a generalization of Lemma A.4.1. It is implicit in the proof of [94, Proposition 5.4.2]. We omit the proof (which is not very difficult, but not very enlightening either). Details can be found in *loc.cit.* and in [83, Lemma 9.6].

A.4.4 Proposition

Let X be a quasi-compact and quasi-separated scheme, and $L_1,...L_n$ be a finite set of line bundles together with global sections $s_i \in \Gamma(X, L_i)$ for $i = 1,...,n$. Let $U = \bigcup_{i=1}^{n} X_{s_i}$ be the union of the non-vanishing loci X_{s_i} of the s_i's, and $j : U \subset X$ be the corresponding open immersion. Let M be a complex of quasi-coherent O_X-modules and let A be a bounded complex of vector bundles on X. Then the following hold.

(a) For every map $f : j^*A \to j^*M$ of complexes of O_U-modules between the restrictions of A and M to U, there is an l-tuple of negative integers $n = (n_1,...,n_l)$ and a map $\tilde{f} : A \otimes K(s^n) \to M$ of complexes of O_X-modules such that $f \circ j^*(\varepsilon_A) = j^*(\tilde{f})$.

(b) For every map $f : A \to M$ of complexes of O_X-modules such that $j^*(f) = 0$, there is an l-tuple of negative integers $n = (n_1,...,n_l)$ such that $f \circ \varepsilon_A = 0$.

A.4.5 Lemma

Let X be a quasi-compact scheme, and $s \in \Gamma(X, L)$ be a global section of a line-bundle L such that X_s is affine. Let $N \to E$ be a map of complexes of quasi-coherent O_X-modules such that its restriction to X_s is a quasi-isomorphism. If E is a bounded complex of vector bundles on X, then there is an integer $k > 0$ and a map of complexes $E \otimes L^{-k} \to N$ whose restriction to X_s is a quasi-isomorphism.

Proof:

Write $j : X_s \subset X$ for the open inclusion. Since X_s is affine, say $X_s = \operatorname{Spec} A$, we have an equivalence of categories between quasi-coherent O_{X_s}-modules and A-modules under which the map $j^*N \to j^*E$ becomes a quasi-isomorphism of complexes of A-modules with j^*E a bounded complex of projectives. Any quasi-isomorphism of complexes of A-modules with target a bounded complex of projectives has a retraction up to homotopy. Therefore, the choice of a homotopy right inverse $f : j^*E \to j^*N$ yields a quasi-isomorphisms. By Lemma A.4.1, there is a map of complexes $\tilde{f} : E \otimes L^k \to N$ such that $j^*\tilde{f} = f \cdot s^k$ for some $k < 0$. In particular, $\tilde{f}$ is a quasi-isomorphism when restricted to X_s. □

Lemma A.4.5 has the following generalization.

A.4.6 Lemma

Let X be a quasi-compact and separated scheme. Let $U = \bigcup_{i=1}^n X_{s_i}$ be the union of affine non-vanishing loci X_{s_i} associated with global sections $s_i \in \Gamma(X, L_i)$ of line bundles L_i on X where $i = 1, \ldots, n$. Denote by $j : U \subset X$ the open immersion. Let $b : M \to B$ be a map of complexes of quasi-coherent O_X-modules such that its restriction j^*b to U is a quasi-isomorphism. If B is a bounded complex of vector bundles on X, then there is a map of complexes $a : A \to M$ such that its restriction to U is a quasi-isomorphism and A is a bounded complex of vector bundles on X.

Proof:

We prove the lemma by induction on n. For $n = 1$, this is Lemma A.4.5. Let $U_0 = \bigcup_{i=1}^{n-1} X_{s_i} \subset X$. By the induction hypothesis, there is map $a_0 : A_0 \to M$ from a bounded complex of vector bundles A_0 such that a_0 is a quasi-isomorphism when restricted to U_0. The induced map $b_0 : C(a_0) \to C(ba_0)$ on mapping cones is a quasi-isomorphism when restricted to U (hence when restricted to X_{s_n}) since b is. Moreover, both cones are acyclic on U_0. Note that we have a distinguished triangle $A_0 \to M \to C(a_0) \to A_0[1]$ in $\mathscr{K}\operatorname{Qcoh}(X)$. Since X_{s_n} is affine and $C(ba_0)$ a bounded complex of vector bundles, Lemma A.4.5 implies the existence of a map $a_1 : A_1 \to C(a_0)$ of complexes with $A_1 = C(ba_0) \otimes L_n^k$ a bounded complex of vector bundles on X which is acyclic when restricted to U_0, and a_1 is a quasi-isomorphism when restricted to X_{s_n}. It follows that a_1 is a quasi-isomorphism when restricted to U. Let A be a complex such that $A \to A_1 \to A_0[1]$ extends to a distinguished triangle in $\mathscr{K}\operatorname{Qcoh}(X)$ where the last map is $A_1 \to C(a_0) \to A_0[1]$. We can choose A to be a bounded complex of vector bundles because A_0 and A_1 are also of this form. Let $a : A \to M$ be a map such that $(a, a_1, 1_{A_0[1]})$ is a map of triangles. By the Five-lemma, the map $a : A \to M$ is a quasi-isomorphism when restricted to U. □

The following proposition is a more precise version of Proposition 3.4.6 in case X has an ample line bundle. For a closed subscheme $Z \subset X$ of a scheme X, denote by $D^b_Z\operatorname{Vect}(X) \subset D^b\operatorname{Vect}(X)$ the full triangulated subcategory of those complexes of vector bundles which are acyclic over $X - Z$. Recall from Sect. 3.4.4 the definition of a "compact object" and of a "compactly generated triangulated category".

A.4.7 Proposition

Let X be a quasi-compact and separated scheme which has an ample family of line bundles $L_1, \ldots, L_n$. Let $Z \subset X$ be a closed subset with quasi-compact open complement $j : U = X - Z \subset X$. Then the following hold.

(a) $D\,\mathrm{Qcoh}(X)$ *is a compactly generated triangulated category with generating set of compact objects the set*

$$\mathscr{L} = \{\, L_i^{k_i}[l] \mid \quad i = 1,\dots,n,\ \ k_i < 0,\ \ k_i, l \in \mathbb{Z} \,\}.$$

The inclusion $\mathrm{Vect}(X) \subset \mathrm{Qcoh}(X)$ *yields a triangle functor* $D^b\,\mathrm{Vect}(X) \subset D\,\mathrm{Qcoh}(X)$ *which is fully faithful and induces an equivalence of* $D^b\,\mathrm{Vect}(X)$ *with the triangulated subcategory of compact objects in* $D\,\mathrm{Qcoh}(X)$*. In particular,* $D^b\,\mathrm{Vect}(X)$ *is generated – as an idempotent complete triangulated category – by the set of line bundles* $L_i^{k_i}$ *where* $i = 1,\dots,n$*, where* $k_i < 0$ *and* $k_i \in \mathbb{Z}$*.*

(b) *The following sequence of triangulated categories is exact up to factors*

$$D^b_Z\,\mathrm{Vect}(X) \to D^b\,\mathrm{Vect}(X) \to D^b\,\mathrm{Vect}(U).$$

(c) *The triangulated category* $D_Z\mathrm{Qcoh}(X)$ *is compactly generated, and the inclusion* $\mathrm{Vect}(X) \subset \mathrm{Qcoh}(X)$ *of vector bundles into quasi-coherent sheaves yields a fully faithful triangle functor* $D^b_Z\,\mathrm{Vect}(X) \subset D_Z\,\mathrm{Qcoh}(X)$ *which induces an equivalence of* $D^b_Z\,\mathrm{Vect}(X)$ *with the triangulated subcategory of compact objects in* $D_Z\,\mathrm{Qcoh}(X)$*.*

Proof:

For (a), we first note that a vector bundle A on X is compact in $D\,\mathrm{Qcoh}(X)$. This is because the functor $E \mapsto \mathrm{Hom}(A,E)$ is, in the notation of Sect. A.3.2, the same as the functor $E \mapsto H^0(Rg_*(E \otimes A^\vee))$. The latter functor commutes with infinite direct sums since its component functors $E \mapsto E \otimes A^\vee$, Rg_* and $H^0 : D(\mathbb{Z}\text{-Mod}) \to \mathbb{Z}\text{-Mod}$ have this property. Secondly, recall that the compact objects form a triangulated subcategory. Therefore, every complex of vector bundles is compact in $D\,\mathrm{Qcoh}(X)$. Next, we will check that the set $\mathscr{L}$ which consists of compact objects generates $D\,\mathrm{Qcoh}(X)$. For that, let E be a complex such that every map $L \to E$ is zero in $D\,\mathrm{Qcoh}(X)$ when $L \in \mathscr{L}$. We have to show that $E = 0$ in $D\,\mathrm{Qcoh}(X)$, that is, that $H^*E = 0$. Since $\mathscr{L}$ is closed under shifts, it suffices to show that the cohomology sheaf $H^0E = \ker(d^0)/\mathrm{im}\,(d^{-1})$ is zero where d^i is the i-th differential of E. By ampleness of the family $L_1,\dots,L_n$, we can choose a surjection $M \twoheadrightarrow \ker(d^0)$ of quasi-coherent O_X-modules with M a (possibly infinite) direct sum of line bundles of the form $L_i^{k_i}$ where $i = 1,\dots,n$ and $k_i < 0$. Composing the inclusion of complexes $\ker(d^0) \to E$ with this surjection defines a map of complexes $M \twoheadrightarrow \ker(d^0) \to E$ which induces a surjective map $M = H^0M \twoheadrightarrow \ker(d^0) \twoheadrightarrow H^0E$ of cohomology sheaves. Since every map $L_i^{k_i} \to E$ is zero in $D\,\mathrm{Qcoh}(X)$, the induced surjective map $M \twoheadrightarrow H^0E$ is the zero map, hence $H^0E = 0$. Altogether, the arguments above show that $D\,\mathrm{Qcoh}(X)$ is compactly generated by the set $\mathscr{L}$. Finally, the triangle functors $D^b\,\mathrm{Vect}(X) \subset D^b\mathrm{Qcoh}(X) \subset D\,\mathrm{Qcoh}(X)$ are fully faithful. The first by the dual of Sect. 3.1.7 (b) and the second by Sect. A.2.10. The remaining statements in (a) follow directly from Neeman's Theorem 3.4.5 (a) and from Sect. 3.1.7 (c).

For part (b), denote by U-quis the set of maps of complexes of vector bundles on X which are quasi-isomorphisms when restricted to $U = X - Z$. By construction, the following sequence of triangulated categories is exact

$$D^b_Z\,\mathrm{Vect}(X) \to D^b\,\mathrm{Vect}(X) \to \mathscr{T}(\mathrm{Ch}^b\,\mathrm{Vect}(X), U\text{-quis}),$$

and the triangle functor $\mathscr{T}(\mathrm{Ch}^b\,\mathrm{Vect}(X), U\text{-quis}) \to D^b\,\mathrm{Vect}(U)$ is conservative. Using Proposition A.4.4 and Lemma A.4.6 we see that the last triangle functor is full. Any conservative and full triangle functor is fully faithful, by Sect. A.2.4. Hence, the last triangle functor is fully faithful. The restriction to U of an ample family of line bundles on X is an ample family of line bundles on U. Therefore, part (a) shows that the triangle functor is also cofinal. It follows that the sequence in part (b) of the Proposition is exact up to factors.

For part (c), we already know that the functor $D^b_Z\operatorname{Vect}(X) \to D_Z\operatorname{Qcoh}(X)$ is fully faithful since both categories are full subcategories of $D\operatorname{Qcoh}(X)$. By part (a), every object in $D^b_Z\operatorname{Vect}(X)$ is compact in $D\operatorname{Qcoh}(X)$. Since the inclusion $D_Z\operatorname{Qcoh}(X) \subset D\operatorname{Qcoh}(X)$ commutes with infinite sums, the objects of $D^b_Z\operatorname{Vect}(X)$ are also compact in $D_Z\operatorname{Qcoh}(X)$. Let $\mathscr{S} \subset D_Z\operatorname{Qcoh}(X)$ be the smallest full triangulated subcategory closed under arbitrary coproducts in $D_Z\operatorname{Qcoh}(X)$ which contains $D^b_Z\operatorname{Vect}(X)$. Then $\mathscr{S}$ is compactly generated with category of compact objects $D^b_Z\operatorname{Vect}(X)$. By Neeman's Theorem 3.4.5 (b), the triangulated category $D\operatorname{Qcoh}(X)/\mathscr{S}$ is compactly generated. It has as category of compact objects the idempotent completion of $D^b\operatorname{Vect}(X)/D^b_Z\operatorname{Vect}(X)$. By part (b), this category is $D^b\operatorname{Vect}(U)$. The functor $D\operatorname{Qcoh}(X)/\mathscr{S} \to D\operatorname{Qcoh}(U)$ preserves coproducts and compact objects, and it induces an equivalence of categories of compact objects. Any triangle functor between compactly generated triangulated categories which commutes with coproducts and which induces an equivalence on compact objects is an equivalence. Therefore, the triangle functor $D\operatorname{Qcoh}(X)/\mathscr{S} \to D\operatorname{Qcoh}(U)$ is an equivalence. It follows that $\mathscr{S} = D_Z\operatorname{Qcoh}(X)$. □

For the remainder of the subsection, write $D_X(A,F)$ for maps in $D\operatorname{Qcoh}(X)$ from A to F, and similarly for U, V and $U \cap V$ in place of X.

A.4.8 Lemma

Let $X = U \cup V$ be a quasi-compact and separated scheme covered by two quasi-compact open subschemes U, V. Then a complex $A \in D\operatorname{Qcoh}(X)$ is compact iff $A_{|U} \in D\operatorname{Qcoh}(U)$ and $A_{|V} \in D\operatorname{Qcoh}(V)$ are compact.

Proof:

Write $j : U \hookrightarrow X$ for the open immersion. Let $A \in D\operatorname{Qcoh}(X)$ be a compact object, and let $F_i \in D\operatorname{Qcoh}(U)$ be a set of complexes on U where $i \in I$. In the sequence of equations

$$\begin{aligned} D_U(j^*A, \textstyle\bigoplus_I F_i) &= D_X(A, Rj_* \textstyle\bigoplus_I F_i) \\ &= D_X(A, \textstyle\bigoplus_I Rj_* F_i) \\ &= \textstyle\bigoplus_I D_X(A, Rj_* F_i) \\ &= \textstyle\bigoplus_I D_U(j^*A, F_i), \end{aligned}$$

the first and last are justified by adjointness of j^* and Rj_*, the second by Sect. A.3.6, and the third by compactness of A. This shows that $A_{|U}$ is compact. The same argument also shows that $A_{|V}$ is compact.

For the other direction, assume that $A_{|U}$ and $A_{|V}$ are compact. Then $A_{|U\cap V}$ is also compact, by the argument above. Let $F_i \in D\operatorname{Qcoh}(X)$ be a set of complexes of quasi-coherent sheaves on X where $i \in I$. We have

$$D_X(A, Rj_* j^* \textstyle\bigoplus_I F_i) = D_U(j^*A, j^* \textstyle\bigoplus_I F_i) = D_U(j^*A, \textstyle\bigoplus_I j^* F_i), \qquad (*)$$

by adjointness of j^* and Rj_* and the fact that j^* commutes with infinite sums (as it is a left adjoint). Similarly, for V and $U \cap V$ in place of U. For every $i \in I$, Lemma A.3.8 provides us with a distinguished triangle

$$F_i \longrightarrow Rj_{1*}(j_1^* F_i) \oplus Rj_{2*}(j_2^* F_i) \longrightarrow Rj_{12*}(j_{12}^* F_i) \longrightarrow F_i[1] \qquad (**)$$

where $j_1 : U \subset X$, $j_2 : V \subset X$ and $j_{12} : U \cap V \subset X$ are the corresponding open immersions. Taking direct sum, we obtain a distinguished triangle

$$\bigoplus_I F_i \to \bigoplus_I Rj_{1*}(j_1^* F_i) \oplus \bigoplus_I Rj_{2*}(j_2^* F_i) \to \bigoplus_I Rj_{12*}(j_{12}^* F_i) \to \bigoplus_I F_i[1]$$

which receives a canonical map from (**). Using Lemma A.3.6, we have a canonical isomorphism $\bigoplus_I Rg_*(g^* F_i) \xrightarrow{\cong} Rg_*(g^* \bigoplus_I F_i)$ for $g = j_1, j_2, j_{12}$, and the last distinguished triangle becomes

$$\bigoplus_I F_i \to Rj_{1*}(j_1^* \bigoplus_I F_i) \oplus Rj_{2*}(j_2^* \bigoplus_I F_i) \to Rj_{12*}(j_{12}^* \bigoplus_I F_i) \to \bigoplus_I F_i[1].$$

Applying the functor $D_X(A, \ )$ to the last triangle, the triangles (**) and the natural map from (**) to the last triangle, we obtain a map of long exact sequences of abelian groups. In view of the identification (*) above, this is the commutative diagram

$$\begin{array}{ccccccc}
\cdots \to \bigoplus_I D_X(A, F_i) & \to & \bigoplus_I D_U(A, F_i) \oplus \bigoplus_I D_V(A, F_i) & \to & \bigoplus_I D_{U\cap V}(A, F_i) \to \cdots \\
\downarrow & & \downarrow \cong & & \downarrow \cong \\
\cdots \to D_X(A, \bigoplus_I F_i) & \to & D_U(A, \bigoplus_I F_i) \oplus D_V(A, \bigoplus_I F_i) & \to & D_{U\cap V}(A, \bigoplus_I F_i) \to \cdots
\end{array}$$

where we wrote $D_U(A, F_i)$ in place of $D_U(A_{|U}, F_{i|U})$, similarily for V and $U \cap V$. All but every third vertical map in the diagram is an isomorphism, by compactness of $A_{|U}$, $A_{|V}$ and $A_{|U\cap V}$. By the Five Lemma, the remaining vertical maps are also isomorphisms. Hence, A is compact. □

Let $j : U \subset X$ be an open immersion of quasi-compact and separated schemes with closed complement $Z = X - U$. Recall that $j^* : D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(U)$ has a right adjoint Rj_* such that the counit of adjunction $j^* Rj_* \to 1$ is an isomorphism. By Exercise A.2.8 (a), the inclusion $J : D_Z\mathrm{Qcoh}(X) \subset D\,\mathrm{Qcoh}(X)$ has a right adjoint which we denote by $R : D\,\mathrm{Qcoh}(X) \to D_Z\mathrm{Qcoh}(X)$. It is part of a functorial distinguished triangle

$$JR(E) \to E \to Rj_* j^* E \to JR(E)[1]. \qquad (33)$$

A.4.9 Lemma

Let X be a quasi-compact and separated scheme, $Z \subset X$ a closed subset with quasi-compact open complement $j : U = X - Z \subset X$. Then an object $A \in D_Z\mathrm{Qcoh}(X)$ is compact in $D_Z\mathrm{Qcoh}(X)$ iff it is compact in $D\,\mathrm{Qcoh}(X)$.

Proof:

Let A be an object of $D_Z\mathrm{Qcoh}(X)$. If A is compact in $D\,\mathrm{Qcoh}(X)$ then A is also compact in $D_Z\mathrm{Qcoh}(X)$ because the inclusion $D_Z\mathrm{Qcoh}(X) \subset D\,\mathrm{Qcoh}(X)$ commutes with infinite coproducts.

For an object $B \in D_Z \operatorname{Qcoh}(X)$, we have

$$D_X(B, Rj_* j^* E) = D_U(j^* B, j^* E) = 0.$$

Therefore, the long exact sequence of hom-sets associated with the distinguished triangle (33) yields an isomorphism $D_X(B, JRE) \cong D_X(B, E)$. Since j^* and Rj_* commute with infinite coproducts, the distinguished triangle (33) shows that $IR : D\operatorname{Qcoh}(X) \to D\operatorname{Qcoh}(X)$ also commutes with infinite coproducts. Let A be a compact object of $D_Z \operatorname{Qcoh}(X)$, and let F_i be a set of objects in $D\operatorname{Qcoh}(X)$ where $i \in I$. Then

$$\begin{aligned} D_X(JA, \textstyle\bigoplus_I F_i) &= D_X(JA, JR \textstyle\bigoplus_I F_i) = D_X(JA, \textstyle\bigoplus_I JRF_i) = D_X(JA, J\textstyle\bigoplus_I RF_i) \\ &= D_Z \operatorname{Qcoh}(X)(A, \textstyle\bigoplus_I RF_i) = \textstyle\bigoplus_I D_Z \operatorname{Qcoh}(X)(A, RF_i) \\ &= \textstyle\bigoplus_I D_X(JA, F_i). \end{aligned}$$

Thus, A is also compact in $D\operatorname{Qcoh}(X)$. □

A.4.10 Lemma

Let X be a quasi-compact and separated scheme, $Z \subset X$ a closed subset with quasi-compact open complement $X - Z \subset X$. Then the triangulated category $D_Z \operatorname{Qcoh}(X)$ is compactly generated.

Proof:

The lemma is true when X has an ample family of line bundles, by Proposition A.4.7. In particular, it is true for affine schemes and their quasi-compact open subschemes. The proof for general quasi-compact and separated X is by induction on the number of elements in a finite cover of X by open subschemes which have an ample family of line bundles. We only need to prove the induction step. Assume $X = U \cup V$ is covered by two open subschemes U and V such that the lemma holds for U, V and $U \cap V$ in place of X. Denote by i, $\bar{i}$, j, and $\bar{j}$ the open immersions $V \hookrightarrow X$, $U \cap V \hookrightarrow U$, $U \hookrightarrow X$, and $U \cap V \hookrightarrow V$, respectively.

Consider the diagram of triangulated categories

$$\begin{array}{ccccc} D_{Z-U}\operatorname{Qcoh}(X) & \xrightarrow{J} & D_Z \operatorname{Qcoh}(X) & \xrightarrow{j^*} & D_{Z\cap U}\operatorname{Qcoh}(U) \\ \downarrow \simeq & & \downarrow i^* & & \downarrow \bar{i}^* \\ D_{Z\cap V - U\cap V}\operatorname{Qcoh}(V) & \longrightarrow & D_{Z\cap V}\operatorname{Qcoh}(V) & \xrightarrow{\bar{j}^*} & D_{Z\cap U\cap V}\operatorname{Qcoh}(U\cap V) \end{array}$$

in which the rows are exact, by (the argument in the proof of) Lemma 3.4.3 (a), and the left vertical map is an equivalence, by Lemma 3.4.3 (b).

Let A be a compact object of $D_{Z\cap U}\operatorname{Qcoh}(U)$. We will show that $E = A \oplus A[1]$ is (up to isomorphism) the image j^*C of a compact object C of $D_Z \operatorname{Qcoh}(X)$. By induction hypothesis, the lower row in the diagram is an exact sequence of compactly generated triangulated in which the functors preserve infinite coproducts and compact objects (Lemmas A.4.8 and A.4.9). By Lemmas A.4.8 and A.4.9, $\bar{i}^*A$ is compact. By Neeman's Theorem 3.4.5 (b) and Remark 3.1.14, there is a compact object B of $D_{Z\cap V}\operatorname{Qcoh}(V)$ and an isomorphism $g : \bar{j}^* B \xrightarrow{\cong} \bar{i}^* E$. Define the object C of $D_Z \operatorname{Qcoh}(X)$ to be the third object in the distinguished triangle in $D_Z \operatorname{Qcoh}(X)$

$$C \longrightarrow Ri_*B \oplus Rj_*E \longrightarrow Ri\bar{j}_*(\bar{i}^*E) \longrightarrow C[1]$$

in which the middle map on the summands Ri_*B and Rj_*E are given by the maps $g : (i\bar{j})^*Ri_*B = \bar{j}^*B \to \bar{j}^*E$ and $id : (i\bar{j})^*Rj_*E = (j\bar{i})^*Rj_*E = \bar{i}^*E \to \bar{i}^*E$, in view of the adjunction between $R(i\bar{j})_*$ and $(i\bar{j})^*$. By the Base-Change Lemma A.3.7, we have $j^*C \cong E$ and $i^*C \cong B$. By Lemmas A.4.8 and A.4.9, C is compact. Summarizing, every compact object of $D_{Z\cap U}\,\mathrm{Qcoh}(U)$ is a direct factor of the image of a compact object of $D_Z\,\mathrm{Qcoh}(X)$.

To finish the proof that $D_Z\,\mathrm{Qcoh}(X)$ is compactly generated, let E be an object of $D_Z\,\mathrm{Qcoh}(X)$ such that every map from a compact object of $D_Z\,\mathrm{Qcoh}(X)$ to E is trivial. We have to show that $E = 0$. Since compact objects of $D_{Z-U}\,\mathrm{Qcoh}(X)$ are also compact objects of $D_Z\,\mathrm{Qcoh}(X)$ (Lemma A.4.9), all maps from compact objects of $D_{Z-U}\,\mathrm{Qcoh}(X)$ to E vanish. The category $D_{Z-U}\,\mathrm{Qcoh}(X)$ is compactly generated. This is because it is equivalent to $D_{Z\cap V\cup(V-U)}\,\mathrm{Qcoh}(V)$ which is compactly generated, by induction hypothesis. Therefore, all maps from all objects of $D_{Z-U}\,\mathrm{Qcoh}(X)$ to E are trivial. For the right adjoint R of J, we therefore have $R(E) = 0$. The distinguished triangle (33) then shows that the unit of adjunction $E \to Rj_*j^*E$ is an isomorphism. I claim that $j^*E = 0$. For that, it suffices to show that $D_U(A, j^*E) = 0$ for all compact $A \in D_{Z\cap U}\,\mathrm{Qcoh}(U)$, since $D_{Z\cap U}\,\mathrm{Qcoh}(U)$ is compactly generated, by induction hypothesis. All such compact A's are direct factors of objects of the form j^*C with $C \in D_Z\,\mathrm{Qcoh}(X)$ compact. Therefore, it suffices to show that $D_U(j^*C, j^*E) = 0$ for all compact $C \in D_Z\,\mathrm{Qcoh}(X)$. But $D_U(j^*C, j^*E) = D_X(C, Rj_*j^*E) = D_X(C,E) = 0$. So $j^*E = 0$. In view of the isomorphism $E \cong Rj_*j^*E$, we have $E = 0$. □

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