THE HOMOTOPY FIXED POINT THEOREM AND THE QUILLEN-LICHTENBAUM CONJECTURE IN HERMITIAN K-THEORY

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Abstract. Let X be a noetherian scheme of finite Krull dimension, having 2 invertible in its ring of regular functions, an ample family of line bundles, and a global bound on the virtual mod-2 cohomological dimensions of its residue fields. We prove that the comparison map from the hermitian K-theory of X to the homotopy fixed points of K-theory under the natural $\mathbb{Z}/2$-action is a 2-adic equivalence in general, and an integral equivalence when X has no formally real residue field. We also show that the comparison map between the higher Grothendieck-Witt (hermitian K-) theory of X and its étale version is an isomorphism on homotopy groups in the same range as for the Quillen-Lichtenbaum conjecture in K-theory. Applications compute higher Grothendieck-Witt groups of complex algebraic varieties and rings of 2-integers in number fields, and hence values of Dedekind zeta-functions.

1. Introduction

In this paper we settle two central conjectures for computing higher Grothendieck-Witt groups (also known as hermitian K-groups) of commutative rings and more generally of schemes under some mild finiteness assumptions. [1]

The first well-known conjecture is a so-called homotopy limit problem formulated by Thomason in [40] and conjectured by Williams in [46, p. 667]. It expresses higher Grothendieck-Witt groups as homotopy groups of the homotopy fixed point spectrum for the $\mathbb{Z}/2$-action on K-theory given by the duality functor. We prove this conjecture in Theorems 1.1 and 1.3 for noetherian schemes (under certain suitable finiteness assumptions). In this way, one can deduce general theorems for higher Grothendieck-Witt groups from their K-theory counterparts. As an example of application, we give a conceptual computation of the higher Grothendieck-Witt groups of rings of 2-integers in certain totally real number fields [6] and relate these groups to values of Dedekind zeta-functions; see Theorems 4.6 and 4.9.

The second conjecture we prove – inextricably linked with Williams' conjecture on homotopy fixed points – is the counterpart, for hermitian K-theory, of the Quillen-Lichtenbaum conjecture in K-theory. The main goal is to compare the higher Grothendieck-Witt groups to their étale analogues, as has been done successfully for K-theory, based on works by Voevodsky, Rost, Suslin and others. In Theorem 1.6, we show that the étale comparison map for hermitian K-theory is an isomorphism on homotopy groups in the same range and under the same hypotheses as it is for K-theory. As applications, we compute the higher Grothendieck-Witt groups of complex algebraic varieties and totally imaginary number fields in terms of topological data and étale cohomology, respectively; see Theorems 4.1 and 4.8 for precise statements of these results.

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[1]The results of this paper were found independently by the third author in the general case [35] and the other authors in the case of schemes of characteristic 0.

Williams' conjecture on homotopy fixed points was verified first for finite fields [10], then for the real and complex numbers [9], and somewhat more recently for rings of 2-integers (including $\mathbb{Z}[\frac{1}{2}]$) in certain number fields [5], [6]. It was recently proved in cases of fields of characteristic 0 by Hu-Kriz-Ormsby in [13]. The second conjecture was explored in [7], where the étale comparison map was shown to be split surjective in sufficiently high degrees. Its K-theoretical counterpart is proved using a motivic cohomology to K-theory spectral sequence and Voevodsky's solution of Milnor's conjecture on Galois cohomology [42]. As yet, no satisfactory hermitian analogues are known for the motivic spectral sequence. This forced us to employ a different route in the proof of the hermitian Quillen-Lichtenbaum conjecture. In fact, we shall see that the two conjectures under consideration here are equivalent. After extending the result of Hu-Kriz-Ormsby to fields of positive characteristic, this allows us to bootstrap the result for fields to (possibly singular) schemes.

Here is a more detailed description of the results in this paper. Our arguments take place in the setting of spectra associated to a noetherian scheme X of finite Krull dimension. We assume that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$, a condition that is needed for our main theorems; see Remark 3.8. We also require the following geometric finiteness assumption on X. Let $\mathrm{vcd}_2(X)$ be shorthand for $\sup\{\mathrm{vcd}_2(k(x) \mid x \in X\}$, where, for any field k, the virtual mod-2 cohomological dimension $\mathrm{vcd}_2(k)$ is the mod-2 cohomological dimension of $k(\sqrt{-1})$. With this definition, $\mathrm{vcd}_2(X) < \infty$ if X is of finite type over $\mathbb{Z}[\frac{1}{2}]$ or $\mathrm{Spec}(k)$, where k is a field for which $\mathrm{vcd}_2(k) < \infty$. Another assumption we make is that X has an ample family of line bundles, *i.e.*, X is the finite union of open affine subsets of the form $\{f_i \neq 0\}$ with f_i a section of a line bundle $\mathcal{L}_i$ on X. Examples include all affine schemes, separated regular noetherian schemes, and quasi-projective schemes over a scheme with an ample family of line bundles.

For a fixed line bundle $\mathcal{L}$ on X, let $GW^{[n]}(X, \mathcal{L})$ denote the Grothendieck-Witt spectrum of X with coefficients in the nth shifted chain complex $\mathcal{L}[n]$. This is the Grothendieck-Witt spectrum of the category of bounded chain complexes of vector bundles over X equipped with the duality functor $E \mapsto \mathrm{Hom}(E, \mathcal{L}[n])$ and quasi-isomorphisms as the weak equivalences [37, § 8]. If $X = \mathrm{Spec}(R)$ is affine, $n = 0$ or 2 and $\mathcal{L} = \mathcal{O}_X$, its nonnegative homotopy groups coincide with Karoubi's hermitian K-groups of R [17], with the sign of symmetry $\varepsilon = \pm 1$. If $n = 1$ or 3 we recover the so-called U-groups [17]. For $GW^{[n]}(X, \mathcal{L})$, we employ the delooping constructed in [34, Theorem 5.5 and Proposition 5.6], whose ith homotopy group $GW^{[n]}_i(X, \mathcal{L})$ is naturally isomorphic to Balmer's triangular Witt group $W^{n-i}(X, \mathcal{L})$ when $i < 0$; see [4], [34, Proposition 6.3].

We write $K^{[n]}(X, \mathcal{L})$ for the connective K-theory spectrum $K(X)$ of X equipped with the $C_2 = \mathbb{Z}/2$-action induced by the duality functor $\mathrm{Hom}(-, \mathcal{L}[n])$. Recall from [23], [34, § 7.2] the natural map

$$GW^{[n]}(X, \mathcal{L}) \longrightarrow K^{[n]}(X, \mathcal{L})^{hC_2} \tag{1-a}$$

between hermitian K-theory and the homotopy fixed points of K-theory.

Theorem 1.1 (Homotopy Fixed Point Theorem)**.** *Let X be a noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(X,\mathcal{O}_X)$. Assume that X has an ample family of line bundles and $\mathrm{vcd}_2(X) < \infty$. Then for all $\nu \geq 1$, the map (1-a) induces an equivalence[2] of spectra mod 2^ν:*

$$GW^{[n]}(X, \mathcal{L};\, \mathbb{Z}/2^\nu) \xrightarrow{\simeq} K^{[n]}(X, \mathcal{L};\, \mathbb{Z}/2^\nu)^{hC_2}.$$

[2]Throughout the paper we use the term "equivalence" as shorthand for a "map that induces isomorphisms on all homotopy groups."

Remark 1.2. Williams conjectured this theorem in [46, p. 627] for affine X (and noncommutative rings), but with no restriction on the cohomological dimension. In that generality, however, there are counterexamples; see [13] for fields of infinite virtual mod-2 cohomological dimension and [6, Appendix C] for noncommutative rings.

Most of the results of this paper deal with 2-primary coefficients. For p-primary coefficients with p an odd prime see Remark 1.8 below. One exception is the following result, which Proposition 3.7 shows is the best that we may expect integrally.

Theorem 1.3 (Integral Homotopy Fixed Point Theorem)**.** *Let X be a noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(X,\mathcal{O}_X)$. Assume that X has an ample family of line bundles and $\mathrm{vcd}_2(X) < \infty$. If -1 is a sum of squares in all residue fields of X, then the map (1-a) is an equivalence of spectra*

$$GW^{[n]}(X,\mathcal{L}) \xrightarrow{\simeq} K^{[n]}(X,\mathcal{L})^{hC_2}.$$

For example, the map (1-a) is an equivalence when X is of finite type over the Gaussian 2-integers $\mathbb{Z}[\frac{1}{2},\sqrt{-1}]$ or when X is defined over a field that is not formally real, *e.g.*, an algebraically closed field of characteristic $\neq 2$ or a field of odd characteristic. If $\mathcal{L} = \mathcal{O}_X$ then the converse holds; see Proposition 3.7. For example, the map (1-a) is not an integral equivalence for $X = \mathrm{Spec}(R)$ where $R = \mathbb{Z}[\frac{1}{2}]$, $\mathbb{Q}$ or $\mathbb{R}$.

Recall from [34, Definition 7.1] (for affine X; see also [30]) the L-theory spectrum $L(X,\mathcal{L})$ of a $\mathbb{Z}[\frac{1}{2}]$-scheme X with coefficients in the line-bundle $\mathcal{L}$. By [34, Proposition 7.2], its homotopy groups $\pi_i L(X,\mathcal{L})$ are naturally isomorphic to the higher Witt-groups $W^{-i}(X,\mathcal{L})$ of Balmer [4]. Further, denote by $\widehat{H}(C_2,\ F)$ the Tate-spectrum of a spectrum F with C_2-action. Then we have the following corollary.

Corollary 1.4. *Under the hypotheses of Theorem 1.1, the map*

$$L(X,\mathcal{L}) \longrightarrow \widehat{H}(C_2,\ K(X,\mathcal{L}))$$

is a 2-adic equivalence. It is an integral equivalence under the further hypothesis of Theorem 1.3.

In the formulation of the next theorem we employ the "non-connective" versions $\mathbb{G}W$ of GW [37, p. 430, Definition 8], [34, Definition 8.6 and Remark 8.8] and $\mathbb{K}$ of K [41, p. 360, Definition 6.4], [36, p. 123, Definition 12.1]. We note the following consequence of our previous results.

Corollary 1.5. *Theorems 1.1 and 1.3 remain valid if one replaces GW and K with $\mathbb{G}W$ and $\mathbb{K}$, respectively.*

We write $GW^{[n]}(X_{\text{ét}},\mathcal{L})$ for the value at X of a globally fibrant replacement of $GW^{[n]}(\ \ ,\mathcal{L})$ on the small étale site $X_{\text{ét}}$ of X; see [7], [14], or 2.3 below. Also, recall that a map is said to be m-connected when its homotopy fibre is; equivalently, on the i th homotopy groups the map induces an isomorphism whenever $i > m$ and a monomorphism when $i = m$.

Theorem 1.6 (Hermitian Quillen-Lichtenbaum)**.** *Let X be a noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(X,\mathcal{O}_X)$. Assume that X has an ample family of line bundles and $\mathrm{vcd}_2(X) < \infty$. Then for all $\nu \geq 1$ the natural map*

$$GW^{[n]}(X,\mathcal{L};\ \mathbb{Z}/2^\nu) \longrightarrow GW^{[n]}(X_{\text{ét}},\mathcal{L};\ \mathbb{Z}/2^\nu) \tag{1-b}$$

is $(\mathrm{vcd}_2(X) - 2)$-connected.

Theorem 1.6 is the evident analogue for hermitian K-theory of the K-theoretic Quillen-Lichtenbaum conjecture solved by Voevodsky, Rost, Suslin and others. In [32], the K-theory analogue was proved in essentially the same generality as in Theorem 1.6 above; see also Theorem 2.7 below. The statement of Theorem 1.6 was conjectured in [7], where the map was shown to be split surjective in sufficiently high degrees.

Remark 1.7. The theorems above remain valid if one drops the "ample family of line bundles" assumption and replaces GW- and K-theories defined via vector bundles by their versions based on perfect complexes [34].

Remark 1.8. The odd-primary analogue of the Hermitian Quillen-Lichtenbaum Theorem 1.6 can be read off from the isomorphisms [34, Remark 7.8]

$$GW_i^{[n]}(X,\mathcal{L})\otimes\mathbb{Z}[1/2]\cong\left[K_i^{[n]}(X,\mathcal{L})^{C_2}\otimes\mathbb{Z}[1/2]\right]\oplus\left[W^{n-i}(X)\otimes\mathbb{Z}[1/2]\right].$$

Here, the K-summand may be computed by étale techniques thanks to the solution of the Bloch-Kato conjecture by Voevodsky, Rost, Suslin and others. Also, W^r denotes Balmer's Witt groups, which coincide up to 2-torsion with the higher Witt groups defined in [17] (for affine schemes). On the other hand, the odd-primary analogue of the Homotopy Fixed Point Theorem is false in general, even with our standard assumptions: see Proposition 3.7.

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2. Preliminaries

In this section, we collect a few well-known facts; no originality is claimed.

For a given scheme X, fix a line bundle $\mathcal{L}$ on X and set $\ell=2^\nu$. For legibility we often write $GW(X)$ for the spectrum $GW^{[n]}(X,\mathcal{L})$, $GW/\ell(X)$ for $GW^{[n]}(X,\mathcal{L};\mathbb{Z}/\ell)$, $K(X)$ for $K^{[n]}(X,\mathcal{L})$, etc. We also sometimes drop the parameter scheme X.

2.1. The C_2-action on K-theory and homotopy fibrations.

Although it will not be needed in this paper, we note that the C_2-action on $K^{[0]}(X,\mathcal{O}_X)$ (resp. $K^{[2]}(X,\mathcal{O}_X)$) coincides up to homotopy with the C_2-action defined in [6] and [7] (in the affine case) with the sign of symmetry $\varepsilon=1$ (resp. $\varepsilon=-1$).

Now suppose that the scheme X has an ample family of line bundles, and $\frac{1}{2}\in\Gamma(X,\mathcal{O}_X)$. In [34, Theorem 7.6], the following are shown to hold.

(1) There is a homotopy fibration of spectra

$$K^{[n]}(X,\mathcal{L})_{hC_2}\longrightarrow GW^{[n]}(X,\mathcal{L})\longrightarrow L^{[n]}(X,\mathcal{L}).$$

The first term is the homotopy orbit spectrum for the C_2-action on the K-theory spectrum $K^{[n]}(X,\mathcal{L})$. The homotopy groups $\pi_jL^{[n]}(X,\mathcal{L})$ are naturally isomorphic to Balmer's Witt groups $W^{n-j}(X,\mathcal{L})$ for all $n,j\in\mathbb{Z}$. Recall from [4] that the groups W^i are 4-periodic in i and coincide with the classical Witt groups in degrees $\equiv 0 \pmod 4$. For a local ring R with $\frac{1}{2}\in R$ we have $W^i(R)=0$ for $i\not\equiv 0 \pmod 4$.

(2) There is a homotopy cartesian square of spectra

$$\begin{array}{ccc} GW^{[n]}(X,\mathcal{L}) & \longrightarrow & L^{[n]}(X,\mathcal{L}) \\ \downarrow & & \downarrow \\ K^{[n]}(X,\mathcal{L})^{hC_2} & \longrightarrow & \hat{H}(C_2,\, K^{[n]}(X,\mathcal{L})) \end{array}$$

where for a spectrum Y with C_2-action, the Tate spectrum $\hat{H}(C_2, Y)$ is the cofiber of the hypernorm map $Y_{hC_2} \to Y^{hC_2}$; see [15, Ch. 3].

(3) Let $\eta \in GW^{[-1]}_{-1}(\mathbb{Z}[\frac{1}{2}]) \cong W^0(\mathbb{Z}[\frac{1}{2}])$ correspond to the unit $1 \in W^0(\mathbb{Z}[\frac{1}{2}])$. Then the horizontal maps in (2) induce equivalences of spectra

$$GW^{[n]}[\eta^{-1}] \xrightarrow{\cong} L^{[n]} \quad \text{and} \quad (K^{[n]})^{hC_2}[\eta^{-1}] \xrightarrow{\cong} \hat{H}(C_2,\, K^{[n]}).$$

Both spectra $L^{[n]}(X,\mathcal{L})$ and $\hat{H}(C_2,\, K^{[n]}(X,\mathcal{L}))$ are 4-periodic and the map $L^{[n]}(X,\mathcal{L}) \to \hat{H}(C_2,\, K^{[n]}(X,\mathcal{L}))$ commutes with the periodicity maps by [34, Remark 7.7], [45]. Hence the homotopy fibre $\mathcal{F}$ of $GW^{[n]}(X,\mathcal{L}) \to K^{[n]}(X,\mathcal{L})^{hC_2}$ satisfies

$$\pi_i\mathcal{F} \cong \pi_{i+4}\mathcal{F} \quad \text{for all } i \in \mathbb{Z}.$$

Remarks 2.2. For the affine non-connective analogues of the previous statements, see also [23] and [46, Theorem 13]. A possible generalization to schemes goes via the Mayer-Vietoris principle [37]. An alternate approach is developed in [34].

2.3. Presheaves of spectra

Let X be a scheme. Its small étale site $X_{\text{ét}}$ is comprised of finite type étale X-schemes $U \to X$ and maps between X-schemes, along with étale coverings. If $\text{vcd}_2(X) = n$, then $\text{vcd}_2(U) \le n$ for all $U \in X_{\text{ét}}$.

We denote by $\text{PSp}(X_{\text{ét}})$ the model category of presheaves of spectra on $X_{\text{ét}}$ [14]. Its objects are contravariant functors from $X_{\text{ét}}$ to spectra and maps are natural transformations of such functors. We are mainly interested in the presheaves of spectra $GW^{[n]}(\mathcal{L})$ sending $p : U \to X$ to $GW^{[n]}(U, p^*\mathcal{L})$ and its K-theory analogue. We shall often suppress $\mathcal{L}$ and $[n]$ in the notation.

A map of presheaves of spectra $\mathcal{F} \to \mathcal{G}$ on $X_{\text{ét}}$ is:
(1) a *pointwise weak equivalence* if for all $U \in X_{\text{ét}}$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of spectra;
(2) a *local weak equivalence* if for all points $x \in X$, $\mathcal{F}_x \to \mathcal{G}_x$ is a weak equivalence of spectra, where $\mathcal{F}_x$ is the filtered colimit $\mathcal{F}_x = \text{colim}_{U \to X}\, \mathcal{F}(U)$ over all étale neighbourhoods U of x;
(3) a *cofibration* if it is pointwise a cofibration, that is, if for all $U \in X_{\text{ét}}$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a cofibration of spectra in the sense of [8]; and
(4) a *local fibration* if it has the right lifting property with respect to all cofibrations which are also local weak equivalences.

It is proved in [14] that the category $\text{PSp}(X_{\text{ét}})$ together with the local weak equivalences, cofibrations and local fibrations is a proper closed (simplicial) model category.

Note that by Theorem 2.6 below, $(GW^{[n]}/\ell)_x \simeq GW^{[n]}/\ell(k)$, where k is a separable closure of the residue field of x. However, there is, *a priori*, no evident equivalence between $(K^{hC_2}/\ell)_x$ and $K/\ell(k)^{hC_2}$ since the homotopy fixed point functor $(\ \ )^{hC_2}$ does not commute with filtered colimits, in general. Compare Lemma 3.5 below.

From the theory of model categories, there exists a globally fibrant replacement functor

$$\text{PSp}(X_{\text{ét}}) \longrightarrow \text{PSp}(X_{\text{ét}}) : \mathcal{F} \longmapsto \mathcal{F}^{\text{ét}}. \tag{2-a}$$

By definition, this is a functor equipped with a natural local weak equivalence $\mathcal{F} \to \mathcal{F}^{\text{ét}}$ for which the map from $\mathcal{F}^{\text{ét}}$ to the final object is a local fibration. The Hermitian Quillen-Lichtenbaum Theorem 1.6 is a statement about the map $\mathcal{F} \to \mathcal{F}^{\text{ét}}$ when $\mathcal{F}$ is the hermitian K-theory presheaf.

Call a square of presheaves of spectra *pointwise homotopy cartesian* if it becomes a homotopy cartesian square of spectra when evaluated at all finite type étale X-schemes. We need the following observations.

Lemma 2.4. (1) *The globally fibrant replacement functor (2-a) sends pointwise homotopy cartesian squares to pointwise homotopy cartesian squares.*
(2) *Let n be an integer. If a presheaf of spectra $\mathcal{F}$ satisfies $\pi_i(\mathcal{F}_x) = 0$ for all $i \geq n$ and all points $x \in X$ then $\pi_i(\mathcal{F}^{\text{ét}}(U)) = 0$ for all $i \geq n$ and $U \in X_{\text{ét}}$.*

Proof. Both statements are true for any small Grothendieck site (with enough points so that we can formulate the second part of the lemma). For the first part, recall that in the category of spectra, homotopy cartesian is the same as homotopy co-cartesian. Since cofibrations are pointwise cofibrations and pointwise weak equivalences are local weak equivalences, it is clear that the globally fibrant replacement functor preserves pointwise homotopy co-cartesian squares. The second part is explicitly stated in [15, Proposition 6.12]. □

There are evident analogs for the Nisnevich topology X_{Nis} on X. Details are *mutatis mutandis* the same.

For later reference we include the following results. Recall that $\ell = 2^\nu$.

Lemma 2.5. *Let $F \subset L$ be a purely inseparable algebraic extension of fields of characteristic $\neq 2$. Then the inclusion $F \subset L$ induces equivalences of spectra*

$$K/\ell(F) \xrightarrow{\simeq} K/\ell(L), \qquad GW^{[n]}/\ell(F) \xrightarrow{\simeq} GW^{[n]}/\ell(L)$$

and an isomorphism of Witt groups

$$W(F) \xrightarrow{\cong} W(L).$$

Proof. The K-theory statement is due to Quillen [28, Proposition 4.8]. For Witt-groups, see [3, p. 456, §2]. The result for $GW^{[n]}/\ell$ now follows from the homotopy fibration (2.1 (1)) and the vanishing of $W^i(k)$ for k a field and $i \not\equiv 0 \pmod 4$. □

Theorem 2.6 (Rigidity)**.** *Let R be a henselian local ring with residue field k and $\frac{1}{2} \in R$. Then the map $R \to k$ induces equivalences of spectra*

$$K/\ell(R) \xrightarrow{\simeq} K/\ell(k), \qquad GW^{[n]}/\ell(R) \xrightarrow{\simeq} GW^{[n]}/\ell(k)$$

and an isomorphism of Witt groups

$$W(R) \xrightarrow{\cong} W(k).$$

Proof. The K-theory (resp. Witt-theory) result is due to Gabber [11] (resp. Knebusch [21, Satz 3.3]). The claim for Grothendieck-Witt theory then follows from the homotopy fibration (2.1 (1)). □

The following theorem is implicit in [32] but was formulated only as an equivalence on $(\mathrm{vcd}_2(X) - 2)$-connected covers.

Theorem 2.7 (K-theoretic Quillen-Lichtenbaum)**.** *Let X be a noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Assume that X has an ample family of line bundles and $\mathrm{vcd}_2(X) < \infty$. Then for all $\nu \geq 1$ the natural map*

$$K(X; \mathbb{Z}/2^\nu) \longrightarrow K^{\text{ét}}(X; \mathbb{Z}/2^\nu)$$

is $(\mathrm{vcd}_2(X) - 2)$-connected.

Proof. The map in the theorem factors as

$$K/\ell(X) \to (K/\ell)^{\mathrm{Nis}}(X) \to (K/\ell)^{\text{ét}}(X) \tag{2-b}$$

where $(K/\ell)^{\mathrm{Nis}}$ denotes a globally fibrant model for the Nisnevich topology. We first show that the second map is $(\mathrm{vcd}_2(X)-2)$-connected. For fields, this is [32, (11), §5]. The case of henselian rings reduces to the case of fields, by Rigidity for K-theory and its étale version (see *e.g.* the proof of [26, Lemma 4.14] and [27, Proposition 6.1]). For general X as in the theorem, the result follows from the henselian case in view of the strongly convergent Nisnevich descent spectral sequence applied to the homotopy fibre of the second map in (2-b).

To finish the proof, we note that the first map in (2-b) is always 0-connected, so the theorem follows as soon as $\mathrm{vcd}_2(X) \geq 2$. Since $\dim X \leq \mathrm{vcd}_2 X$, we are left with the cases $\dim X = 0, 1$. If $\dim X = 0$ then the first map in (2-b) is an equivalence. If $\dim X = 1$ then this map is (-1)-connected. This assertion follows from the fact that K_{-1} is torsion free for noetherian schemes of Krull dimension ≤ 1; see [44, Lemma 2.5 (2)]. Therefore, the maps $K/\ell \to \mathbb{K}/\ell$ and hence $(K/\ell)^{\mathrm{Nis}} \to (\mathbb{K}/\ell)^{\mathrm{Nis}}$ are (-1)-connected for such schemes. Moreover, $\mathbb{K}/\ell \to (\mathbb{K}/\ell)^{\mathrm{Nis}}$ is a pointwise weak equivalence, by [41]. □

Lemma 2.8. *Suppose that X is a quasi-compact scheme with an ample family of line bundles. Then the following are equivalent.*

(1) *There exists an integer $n > 0$ such that $2^n W(X) = 0$.*
(2) *-1 is a sum of squares in all the residue fields of X.*

Proof. In [22, Theorem 3, p. 189] it is shown that (1) is equivalent to the statement that all the residue fields of X have 2-primary torsion Witt groups. The latter is equivalent to (2); see for instance [33, Theorem II.7.1]. □

Remark 2.9. If X is affine, the condition that -1 is a sum of squares in all the residue fields of X is equivalent to -1 being a sum of squares in $\Gamma(X, \mathcal{O}_X)$; see for instance [22, Proposition 4, p. 190]. If X is non-affine, then -1 might be a sum of squares in all residue fields without being a sum of squares in $\Gamma(X, \mathcal{O}_X)$. Indeed, every smooth projective real curve X with function field of level > 1 has the property that -1 is a sum of squares in all of its residue fields, but not in $\Gamma(X, \mathcal{O}_X) = \mathbb{R}$. For example, take the closed subscheme of $\mathbb{P}^2_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[X,Y,Z])$ cut out by the equation $X^2 + Y^2 + Z^2 = 0$.

3. Proofs

Lemma 3.1. *Theorem 1.1 holds for fields k with $\mathrm{vcd}_2(k) < \infty$ and $\mathrm{char}(k) \neq 2$.*

Proof. We claim that the homotopy fibre $\mathcal{F}$ of $GW/\ell \to K/\ell^{hC_2}$ is contractible. If $\mathrm{char}(k) = 0$, this holds by [13]. If $\mathrm{char}(k) > 0$, we reduce the claim to the case of characteristic 0 by means of "Teichmüller lifting". In effect, we may assume that k is perfect, since K/ℓ and GW/ℓ are invariant under purely inseparable algebraic field extensions (Lemma 2.5). Then the ring V of Witt-vectors over k is a complete (hence henselian) dvr with residue field k and fraction field F of characteristic 0. Furthermore,

$$\mathrm{vcd}_2(F) \leq \mathrm{cd}_2(F) = \mathrm{cd}_2(k) + 1 = \mathrm{vcd}_2(k) + 1 < \infty,$$

by [1, Exposé X, Théorème 2.2 (ii)] and [38, II §4.1].

Let $\pi \in V$ be a uniformizer. We claim that $\alpha\colon V[t,t^{-1}] \to F :\ t \mapsto \pi$, induces an equivalence $\mathcal{F}(\alpha) : \mathcal{F}(V[t,t^{-1}]) \xrightarrow{\simeq} \mathcal{F}(F)$. It is known that $K/\ell(\alpha)$ is an equivalence (one may use the same argument as for Witt groups below), and hence $K/\ell^{hC_2}(\alpha)$ and $K/\ell_{hC_2}(\alpha)$ are equivalences. To show that $GW/\ell(\alpha)$ is an equivalence, we consider the spectrum L defined by the homotopy fibration $K_{hC_2} \to GW \to L$ (2.1 (1)). The groups $\pi_i L$ are 4-periodic, trivial for local rings in degrees $\not\equiv 0 \pmod 4$ and homotopy invariant for regular rings. Using the localization exact sequence for $V[t] \to V[t,t^{-1}]$, we get $W^i(V[t,t^{-1}]) = 0$ for $i \not\equiv 0 \pmod 4$. Thus, it remains to check that $W^0(V[t,t^{-1}]) \to W^0(F)$ is an isomorphism. This follows by a comparison of the localization exact sequences for $V[t] \to V[t,t^{-1}]$ and $V \to F$, which reduces to a map between short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & W^0(V[t]) & \longrightarrow & W^0(V\left[t,t^{-1}\right]) & \longrightarrow & W^0(V) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & W^0(V) & \longrightarrow & W^0(F) & \longrightarrow & W^0(k) & \to & 0. \end{array}$$

Theorem 2.6 shows that the right vertical map, induced by the reduction map modulo π, is an isomorphism. The left vertical map is an isomorphism by homotopy invariance of Witt-theory. It follows that $W^0(\alpha)$ is an isomorphism, as claimed.

Because augmentation makes $\mathcal{F}(V)$ a retract of $\mathcal{F}(V\left[t,t^{-1}\right])$, combining with the equivalence $\mathcal{F}(\alpha) : \mathcal{F}(V[t,t^{-1}]) \xrightarrow{\simeq} \mathcal{F}(F)$ makes $\mathcal{F}(V)$ also a retract of $\mathcal{F}(F)$. However, since $\mathrm{char}(F) = 0$ and $\mathrm{vcd}_2(F) < \infty$, we have $\mathcal{F}(F) \simeq *$, by [13]; this now implies that $\mathcal{F}(V) \simeq *$. Finally, by rigidity, $\mathcal{F}(V) \to \mathcal{F}(k)$ is an equivalence. Hence, $\mathcal{F}(k) \simeq *$ as sought. □

3.2. Bott elements. Let $p = 8m$ for $m \geq 1$ an integer, and let ℓ be the highest power of 2 that divides $3^{4m} - 1$; that is, $\ell = \max\{2^k \mid 2^k \text{ divides } 3^{4m} - 1\}$. An element $\beta \in \pi_p(S^0; \mathbb{Z}/\ell)$ in the mod ℓ stable stems is called a *(positive) Bott element* if it maps to the reduction mod ℓ of a generator of $KO_p \cong \mathbb{Z}$ under the unit map $S^0 \to KO$, where KO denotes the Bott-periodic real topological K-theory spectrum. We require Bott elements in our proof (Lemma 3.5) of the étale version of the Homotopy Fixed Point Theorem 1.1. Bott elements for higher Grothendieck-Witt theory and arbitrary coefficients were first constructed in [7]. In what follows we give a simpler construction that suffices for our purposes.

Lemma 3.3. *Let ℓ be as in Section 3.2. Then there is an element $\beta \in \pi_p(S^0/\ell)$ in the mod ℓ stable stem whose image in $\pi_p(KO/\ell)$ under the unit map $S^0 \to KO$ is the reduction mod ℓ of a generator of $KO_p = \mathbb{Z}$.*

Proof. The construction of β, essentially due to Quillen, is based on Adams' work on the image $J(\pi_*O) \subseteq \pi_*(S^0)$ of the J-homomorphism [2]. Recall that the 2-primary part $J(\pi_{p-1}O)_{(2)}$ of $J(\pi_{p-1}O)$ is cyclic of order ℓ. For a spectrum F, write F_{tor} for the homotopy fibre of the rationalization map $F \to F_{\mathbb{Q}}$, and note that the natural map $F_{tor}/\ell \to F/\ell$ is an equivalence. Further, recall that $\pi_*(S^0_{tor}) \to \pi_*(S^0)$ is an isomorphism for $* > 0$. So, the J-homomorphism has image in $\pi_*(S^0_{tor})$. Consider the commutative diagram

$$\begin{tikzcd} & \pi_p(S^0_{tor}/\ell) \arrow[r] \arrow[d, "\delta"] & \pi_p(KO_{tor}/\ell) \arrow[d, "\delta"] \\ J(\pi_{p-1}O)_{(2)} \cong \mathbb{Z}/\ell \arrow[ur] \arrow[r, hook] & \pi_{p-1}(S^0_{tor}) \arrow[r] & \pi_{p-1}(KO_{tor}) \end{tikzcd}$$

in which the diagonal map exists because of the long exact sequence of homotopy groups associated with the fibration $S^0_{tor} \xrightarrow{\cdot\ell} S^0_{tor} \to S^0_{tor}/\ell$. In [29, p. 183, §2], Quillen shows that the composition of the lower two maps is injective. It follows that the composition of the diagonal map with the upper horizontal map in the previous and in the following commutative diagram is injective

$$\begin{tikzcd} & \pi_p(S^0_{tor}/\ell) \arrow[r] \arrow[d, "\cong"] & \pi_p(KO_{tor}/\ell) \arrow[d, "\cong"] \\ J(\pi_{p-1}O)_{(2)} \cong \mathbb{Z}/\ell \arrow[r] \arrow[ur] & \pi_p(S^0/\ell) \arrow[r] & \pi_p(KO/\ell) \cong \mathbb{Z}/\ell. \end{tikzcd}$$

Therefore, the composition of the two lower horizontal maps in the last diagram is an injection of finite groups of the same order. Hence, this composition is an isomorphism. In particular, the map $\pi_p(S^0/\ell) \to \pi_p(KO/\ell)$ is surjective, and we can lift the generator mod ℓ of KO_p to an element $\beta \in \pi_p(S^0/\ell)$. □

Lemma 3.4. *Let k be a separably closed field of characteristic $\mathrm{char}(k) \neq 2$, and let $\eta \in GW^{[-1]}_{-1}(k) \cong W^0(k)$ correspond to the unit of the ring $W^0(k)$. Then $\beta\eta^p = 0$ in $GW(k; \mathbb{Z}/\ell)$. In particular, $L/\ell(k)[\beta^{-1}] \simeq *$.*

Proof. By [18], [19] the ring spectrum $GW/\ell(k)[\beta^{-1}]$ is equivalent to KO/ℓ and the natural map $GW(k)/\ell \to GW/\ell(k)[\beta^{-1}]$ is an equivalence on connective covers. Thus, it suffices to check that $\beta\eta^p = 0$ in $\pi_0(KO/\ell)$. In KO/ℓ, the elements β and η are reductions of integral classes. More precisely, β is the reduction mod ℓ of b^m, where $b \in KO_8 = \mathbb{Z}$ is a generator, and $\eta^4 \in GW^{[-4]}_{-4}(\mathbb{C})$. However, the map

$$\mathbb{Z}/2 \cong GW^{[-4]}_{-4}(\mathbb{C}) = GW^{[0]}_{-4}(\mathbb{C}) \longrightarrow KO_{-4} = \mathbb{Z}$$

is trivial. Consequently, $\eta^p = 0$ in KO_{-p} and hence $\beta\eta^p = 0$.

For the L-theory statement, recall that $L = GW[\eta^{-1}]$; see (2.1 (3)). Therefore, $\beta\eta^p = 0$ implies $L/\ell(k)[\beta^{-1}] \simeq *$. □

Lemma 3.5. *Let $\nu > 0$ be an integer and $\ell = 2^\nu$. Let X be a noetherian scheme with an ample family of line bundles, $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$ and $\mathrm{vcd}_2(X) < \infty$. Then the map*

$$GW^{\text{ét}}/\ell(X) \longrightarrow (K^{hC_2})^{\text{ét}}/\ell(X)$$

is an equivalence.

Proof. For an integer $\nu > 0$, a map of spectra is an equivalence mod 2^ν if and only if it is an equivalence mod 2. Therefore, we can assume ℓ to be as in Section 3.2, and we have Bott elements at our disposal. Consider the commutative diagram

$$\begin{array}{ccccc} (GW/\ell)^{\text{ét}}(X) & \longrightarrow & (GW/\ell[\beta^{-1}])^{\text{ét}}(X) & \longrightarrow & (L/\ell[\beta^{-1}])^{\text{ét}}(X) \\ \downarrow & & \downarrow & & \downarrow \\ (K/\ell^{hC_2})^{\text{ét}}(X) & \longrightarrow & (K/\ell^{hC_2}[\beta^{-1}])^{\text{ét}}(X) & \longrightarrow & (\hat{H}/\ell[\beta^{-1}])^{\text{ét}}(X) \end{array}$$

in which the right-hand square is obtained from the homotopy cartesian square (2.1 (2)) by reduction mod ℓ, inverting the positive Bott element constructed in Lemma 3.3, and taking étale globally fibrant replacements. All these operations preserve (pointwise) homotopy cartesian squares. Thus, the right-hand square in the diagram is homotopy cartesian. By Lemma 3.4, the upper right corner of the diagram is trivial. Since $\hat{H}/\ell[\beta^{-1}]$ is a module spectrum over $L/\ell[\beta^{-1}]$, the lower right corner of the diagram is trivial as well. Hence, the middle vertical arrow is an equivalence. In view of Lemma 2.4, the upper left horizontal arrow is an equivalence on connective covers since $GW/\ell(F) \to GW/\ell(F)[\beta^{-1}]$ has this property for separably closed fields F. By the same lemma, the lower left horizontal map is an equivalence on some connected cover, because by the solution of the K-theoretic Quillen-Lichtenbaum conjecture [32] $K/\ell^{hC_2} \to K/\ell^{hC_2}[\beta^{-1}]$ is a pointwise (hence local) weak equivalence on $(\mathrm{vcd}_2(X) - 2)$-connected covers. Hence, the fibre of the left vertical map has trivial homotopy groups in high degrees. By periodicity (2.1 (3)), we are done. □

Lemma 3.6. *If Theorem 1.6 holds for the residue fields of X, then the map (1-b) is n-connected for some integer n.*

Proof. By definition, $GW^{\text{ét}}/\ell$ satisfies Nisnevich descent. In positive degrees the same holds for GW/ℓ [34]. More precisely, $\mathbb{G}W$ satisfies Nisnevich descent [34, Theorem 9.7] and the map $GW \to \mathbb{G}W$ is an equivalence on connective covers [34, Proposition 8.7 or Theorem 8.14]. The map between the E^2-pages of the corresponding Nisnevich descent spectral sequences takes the form

$$H^p_{\mathrm{Nis}}(X;\, \tilde{\pi}_q(GW/\ell)) \longrightarrow H^p_{\mathrm{Nis}}(X;\, \tilde{\pi}_q(GW^{\text{ét}}/\ell)).$$

By rigidity, see Theorem 2.6, the assumption shows that the canonically induced map

$$\tilde{\pi}_q(GW/\ell) \longrightarrow \tilde{\pi}_q(GW^{\text{ét}}/\ell)$$

of Nisnevich sheaves is an isomorphism for $q \geq \mathrm{vcd}_2(X) - 1$. The result follows from the fact that $H^p_{\mathrm{Nis}}(X, A) = 0$ for $p > \dim X$ and $p < 0$ and strong convergence of the descent spectral sequences [15, Theorem 7.58]. □

Proofs of Theorems 1.1 and 1.6. Consider the commutative diagram:

$$\begin{array}{ccc} GW/\ell(X) & \longrightarrow & GW^{\text{ét}}/\ell(X) \\ \downarrow & & \downarrow \\ \left[K/\ell^{hC_2}\right](X) & \longrightarrow & \left[K/\ell^{hC_2}\right]^{\text{ét}}(X). \end{array} \tag{3-a}$$

By the solution of the K-theoretic Quillen-Lichtenbaum conjecture (Theorem 2.7), the homotopy fibre of the lower horizontal map is $(\mathrm{vcd}_2(X)-2)$-connected. Lemma 3.5 shows the right vertical map is an equivalence, while Lemma 3.1 shows the left vertical map is an equivalence for fields. This implies the Hermitian Quillen-Lichtenbaum Theorem 1.6 for fields. Using Lemma 3.6, we have that the upper horizontal map is an isomorphism in high degrees. It follows that the homotopy fibre of the left vertical map in (3-a) is trivial in high degrees. By periodicity (2.1 (3)), the homotopy fibre has trivial homotopy in all degrees. Thus the left vertical map in (3-a) is an equivalence. This proves the Homotopy Fixed Point Theorem 1.1. Since both vertical maps are equivalences, the homotopy fibre of the upper horizontal map has trivial homotopy groups in the same range as the homotopy fibre of the lower horizontal map. This proves the Hermitian Quillen-Lichtenbaum Theorem 1.6 for schemes X. □

Proof of Theorem 1.3. By Lemma 2.8, we have $2^m W^0(X) = 0$ for some $m > 0$. The homotopy groups of $L^{[n]}(X)$ and the Tate spectrum $\hat{H}(C_2,\, K^{[n]}(X))$ acquire compatible actions by W^0; see [45], [34, Remark 7.7]. It follows that the homotopy groups of the fibre $\mathcal{F}(X)$ of $L^{[n]}(X) \to \hat{H}(C_2,\, K^{[n]}(X))$ also admit such an action, and are therefore annihilated by 2^m. However, by the Homotopy Fixed Point Theorem 1.1, the homotopy cofiber of multiplication by $2^m\colon \mathcal{F}(X) \to \mathcal{F}(X)$ is trivial; that is, multiplication by 2^m is an isomorphism on the homotopy groups of $\mathcal{F}(X)$. As we have just noticed, this is the zero map. Hence, $\mathcal{F}(X) \simeq *$, and the map (1-a) is an equivalence. □

Proof of Corollary 1.4. This follows from Theorems 1.1 and 1.3 in view of the homotopy cartesian square 2.1 (2). □

Proof of Corollary 1.5. This follows from the fact that the diagram

$$\begin{array}{ccc} GW(X) & \longrightarrow & \mathbb{G}W(X) \\ \downarrow & & \downarrow \\ K(X)^{hC_2} & \longrightarrow & \mathbb{K}(X)^{hC_2} \end{array}$$

is a homotopy cartesian square; see [34, Theorem 8.14]. □

If $\mathcal{L} = \mathcal{O}_X$ then the converse of the Integral Homotopy Fixed Point Theorem 1.3 holds.

Proposition 3.7. *Let X be a scheme with an ample family of line bundles and $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. If the map (1-a) is an equivalence for $\mathcal{L} = \mathcal{O}_X$ then no residue field of X is formally real. More generally, this conclusion holds if we assume that the map (1-a) is only an equivalence modulo some odd prime.*

Proof. For any prime q, the map $GW^{[n]}/q(X) \to (K^{[n]}/q(X))^{hC_2}$ is an equivalence if the map (1-a) is an integral equivalence. If q is odd, then by (2.1 (2)), the map $L^{[n]}/q(X) \to \hat{H}(C_2, K^{[n]}/q(X))$ is an equivalence. Now multiplication by 2 is an equivalence on $K^{[n]}/q$, which implies that $\hat{H}(C_2, K^{[n]}/q(X)) \simeq *$, and hence $L^{[n]}/q(X) \simeq *$. Therefore, the Witt ring $W(X)$ is a $\mathbb{Z}[\frac{1}{q}]$-algebra since multiplication by q on $W(X) = L_n^{[n]}(X)$ is an isomorphism. If X has a formally real residue field k, then we obtain ring maps $\mathbb{Z}[\frac{1}{q}] \to W(X) \to W(k) \to W(\bar{k}) = \mathbb{Z}$, where $\bar{k}$ is a real closure of k, which leads to a contradiction. ∎

Remark 3.8. We should point out the necessity of our standing assumption that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$, although the cited results in [37] are proved in greater generality. Without this assumption, the Homotopy Fixed Point Theorem 1.1 cannot hold for the following reason. As proved in [34, § 2], the fundamental theorem in hermitian K-theory [17] fails for the $GW^{[n]}$-spectrum whenever X has a residue field of characteristic 2, whereas it does hold for the $(K^{[n]})^{hC_2}$-spectrum; see the proof of [34, Theorem 6.2]. In particular, if X has a residue field of characteristic 2 then (1-a) is not an integral equivalence, in general, even if $\mathrm{vcd}_2(X) < \infty$. Moreover, it is not a 2-adic equivalence for fields of characteristic 2 (in this case the fibre of (1-a) is 2-adically complete).

If K-theory of symmetric bilinear forms (that is, GW-spectra) is replaced with K-theory of quadratic forms, then (1-a) is not an equivalence either, because the latter is not homotopy invariant for regular rings, whereas K-theory and its homotopy fixed points are. In particular, the quadratic analogue of the map (1-a) is not generally a 2-adic equivalence in characteristic 2.

4. Applications

Theorem 4.1. *Let X be a complex algebraic variety of (complex) dimension d which has an ample family of line bundles. Let $X_{\mathbb{C}}$ be the associated analytic topological space of complex points. Then for $\ell = 2^\nu > 1$ and $n \in \mathbb{Z}$, the canonical map*

$$GW_i^{[n]}(X; \mathbb{Z}/\ell) \longrightarrow KO^{2n-i}(X_{\mathbb{C}}; \mathbb{Z}/\ell)$$

is an isomorphism for $i \geq d-1$ and a monomorphism for $i = d-2$.

Proof. The theories $GW^{[n]}$ have Bott-periodic topological counterparts $GW_{\mathrm{top}}^{[n]}$ first explored in [16] as

$$GW_{\mathrm{top}}^{[0]}(X_{\mathbb{C}}) = {}_1\mathcal{L}(X_{\mathbb{C}}) = KO(X_{\mathbb{C}}), \qquad GW_{\mathrm{top}}^{[-1]}(X_{\mathbb{C}}) = {}_1\mathcal{U}(X_{\mathbb{C}}) = \Omega^2 KO(X_{\mathbb{C}}),$$
$$GW_{\mathrm{top}}^{[-2]}(X_{\mathbb{C}}) = {}_{-1}\mathcal{L}(X_{\mathbb{C}}) = \Omega^4 KO(X_{\mathbb{C}}), \quad GW_{\mathrm{top}}^{[-3]}(X_{\mathbb{C}}) = {}_{-1}\mathcal{U}(X_{\mathbb{C}}) = \Omega^6 KO(X_{\mathbb{C}}),$$

which induce the maps

$$GW^{[-n]}(X; \mathbb{Z}/\ell) \longrightarrow GW_{\mathrm{top}}^{[-n]}(X_{\mathbb{C}}; \mathbb{Z}/\ell) = \Omega^{2n} KO(X_{\mathbb{C}}; \mathbb{Z}/\ell)$$

in the theorem. In the commutative diagram

$$\begin{array}{ccc} GW^{[n]}(X;\,\mathbb{Z}/\ell) & \longrightarrow & GW^{[n]}_{\mathrm{top}}(X_{\mathbb{C}};\,\mathbb{Z}/\ell) \\ \downarrow & & \downarrow \\ \left[K^{[n]}(X;\,\mathbb{Z}/\ell)\right]^{hC_2} & \longrightarrow & \left[KU^{[n]}(X_{\mathbb{C}};\,\mathbb{Z}/\ell)\right]^{hC_2} \end{array},$$

the lower horizontal map is $(d-2)$-connected, by a theorem of Voevodsky [42, Theorem 7.10]. By Theorem 1.3, the left vertical map is an (integral) equivalence. Finally, it is a classical theorem that the right vertical map is also an (integral) equivalence. Indeed, for $n=0$, this is the usual homotopy equivalence $KO \simeq KU^{hC_2}$ (see *e.g.* [20]); and for other $n \in \mathbb{Z}$, it follows from the topological version of the fundamental theorem in hermitian K-theory [16]. □

Remark 4.2. The proof shows that the theorem also holds for odd prime powers. We simply need to remark that the map $K^{[n]}(X;\,\mathbb{Z}/\ell) \to KU^{[n]}(X_{\mathbb{C}};\,\mathbb{Z}/\ell)$ is also $(d-2)$-connected for odd prime powers ℓ due to the solution of the Bloch-Kato conjecture by Voevodsky, Rost, Suslin and others. However, the odd prime analogue of Theorem 4.1 can be more easily proved using Remark 1.8 in place of the Integral Homotopy Fixed Point Theorem. See also [7] for another argument in that case.

From now on, let ℓ again be a power of 2. As a second application, we give new and more conceptual proofs of the main results of [5] and [6]. Let $\mathbb{Z}'$ be short for $\mathbb{Z}[\frac{1}{2}]$. Because $K^{[n]}$ has the same (nonequivariant) homotopy type as K, from [31], [43, Corollary 8] we have the existence of a homotopy cartesian square

$$\begin{array}{ccc} K^{[n]}(\mathbb{Z}')/\ell & \longrightarrow & K^{[n]}_{\mathrm{top}}(\mathbb{R})/\ell \\ \downarrow & & \downarrow \\ K^{[n]}(\mathbb{F}_3)/\ell & \longrightarrow & K^{[n]}_{\mathrm{top}}(\mathbb{C})/\ell \end{array},$$

where K_{top} stands for connective topological K-theory. Now, since the fixed spectrum of $K^{[n]}$ is $GW^{[n]}$, the Homotopy Fixed Point Theorem 1.1 applied to this square yields the following.

Theorem 4.3. *For $\ell = 2^\nu > 1$ and $n \in \mathbb{Z}$, the square*

$$\begin{array}{ccc} GW^{[n]}(\mathbb{Z}')/\ell & \longrightarrow & GW^{[n]}_{\mathrm{top}}(\mathbb{R})/\ell \\ \downarrow & & \downarrow \\ GW^{[n]}(\mathbb{F}_3)/\ell & \longrightarrow & GW^{[n]}_{\mathrm{top}}(\mathbb{C})/\ell \end{array}$$

is homotopy cartesian on connective covers. □

Remark 4.4. According to [39], these results do not depend on whether the fields $\mathbb{R}$ and $\mathbb{C}$ are taken with the discrete or standard Euclidean topology.

This theorem enables complete computation of the groups $GW^{[n]}(\mathbb{Z}')$, up to finite groups of odd order (see [5]). In particular, if $n=0$, the right vertical map in the above square can be identified with the split surjective map $KO \times KO \to KO$ mod ℓ. Therefore, using 2-adic completions we get the following corollary.

Corollary 4.5. *For $i \geq 0$ and any one-point space* pt*, the natural map*

$$GW^{[0]}_i(\mathbb{Z}') \longrightarrow GW^{[0]}_i(\mathbb{F}_3) \oplus KO^{-i}(\mathrm{pt})$$

is an isomorphism modulo finite groups of odd order. □

The groups $KO^{-i}(\mathrm{pt})$ are given by Bott periodicity, and the groups $GW^{[0]}_i(\mathbb{F}_3)$ were computed by Friedlander [10].

Similarly, let F be a number field and $\mathcal{O}'_F = \mathcal{O}_F[\frac{1}{2}]$ be its ring of 2-integers. Assume that F is a 2-regular totally real number field with r real embeddings. Let

q be a prime number such that the elements corresponding to the Adams operations ψ^{-1} and ψ^q in the ring of operations of the periodic complex topological K-theory spectrum generate the Galois group $F(\mu_{2^\infty})$ over F, where $F(\mu_{2^\infty})$ is obtained from F by adjoining all 2-primary roots of unity. From [12] and [27] we have a homotopy cartesian square of connective spectra

$$\begin{array}{ccc} K^{[n]}(\mathcal{O}'_F)/\ell & \longrightarrow & K^{[n]}_{\mathrm{top}}(\mathbb{R})^r/\ell \\ \downarrow & & \downarrow \\ K^{[n]}(\mathbb{F}_q)/\ell & \longrightarrow & K^{[n]}_{\mathrm{top}}(\mathbb{C})^r/\ell. \end{array}$$

After application of the functor $(-)^{hC_2}$ to this homotopy cartesian square, the Homotopy Fixed Point Theorem 1.1 implies the following result, which was first proved in [6] and which allows us to compute completely the groups $GW_i^{[n]}(\mathcal{O}'_F)$ up to finite groups of odd order.

Theorem 4.6. *Let $\ell = 2^\nu > 1$ and $n \in \mathbb{Z}$. For a 2-regular totally real number field F with r real embeddings, the square of spectra*

$$\begin{array}{ccc} GW^{[n]}(\mathcal{O}'_F)/\ell & \longrightarrow & GW^{[n]}_{\mathrm{top}}(\mathbb{R})^r/\ell \\ \downarrow & & \downarrow \\ GW^{[n]}(\mathbb{F}_q)/\ell & \longrightarrow & GW^{[n]}_{\mathrm{top}}(\mathbb{C})^r/\ell \end{array}$$

is homotopy cartesian on connective covers. In particular, for $i \geq 0$, $n = 0$ and any one-point space pt*, the natural map*

$$GW_i^{[0]}(\mathcal{O}'_F) \longrightarrow GW_i^{[0]}(\mathbb{F}_q) \oplus KO^{-i}(\mathrm{pt})^r.$$

is an isomorphism modulo finite groups of odd order. □

Remark 4.7. Let X be as in the Homotopy Fixed Point Theorem 1.1. Our results also give the isomorphism

$$GW/\ell(X)[\beta^{-1}] \xrightarrow{\cong} GW^{\text{ét}}/\ell(X)[\beta^{-1}] \tag{4-a}$$

which was first proved in [7]. Note that $GW^{\text{ét}}/\ell(X) \to GW^{\text{ét}}/\ell(X)[\beta^{-1}]$ is an equivalence on connective covers. By the Homotopy Fixed Point Theorem 1.1, cup product with the Bott element is an isomorphism in high degrees, as the same is true for K-theory. However, this is also true for étale hermitian K-theory, since by the Hermitian Quillen-Lichtenbaum Theorem 1.6 it coincides with hermitian K-theory in high degrees. Hence the equivalence (4-a).

Let $A \bullet B$ denote an abelian group extension of B by A, so that there exists a short exact sequence

$$0 \to A \to A \bullet B \to B \to 0.$$

Another convention we follow is that $\mu_{2^\nu}^{\otimes i}$ denotes the i th Tate twist of the sheaf of 2^ν th roots of unity μ_{2^ν} (the kernel of multiplication by 2^ν on the multiplicative group scheme $\mathbb{G}_m$ over $\mathcal{O}'_F$). At one extreme, when $\nu = 1$ this is independent of the Tate twist; at the other, we use finiteness of the étale cohomology groups of $\mathcal{O}'_F$ to write $\mathbb{Z}_2^{\otimes j}$ for $\varprojlim \mu_{2^\nu}^{\otimes j}$.

By combining the Hermitian Quillen-Lichtenbaum Theorem 1.6 with [7, Lemmas 6.12, 6.16] we deduce our third computational application for higher Grothendieck-Witt groups.

Theorem 4.8. *Suppose that F is a totally imaginary number field. The 2-adically completed higher Grothendieck-Witt groups $GW_i^{[n]}(\mathcal{O}'_F)_\#$ of the ring of 2-integers $\mathcal{O}'_F$ of F are computed in terms of étale cohomology groups as follows.*

$i \bmod 8$	$GW_i^{[0]}(\mathcal{O}'_F)_\#$	$GW_i^{[1]}(\mathcal{O}'_F)_\#$
$8k>0$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+1})$
$8k+1$	$H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)$
$8k+2$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+2})\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$	$H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$
$8k+3$	$H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+2})$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+3})\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$
$8k+4$	0	$H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+3})$
$8k+5$	0	0
$8k+6$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+4})$	0
$8k+7$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+4})$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+5})$

$i \bmod 8$	$GW_i^{[2]}(\mathcal{O}'_F)_\#$	$GW_i^{[3]}(\mathcal{O}'_F)_\#$
$8k>0$	0	$H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+1})$
$8k+1$	0	0
$8k+2$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+2})$	0
$8k+3$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+2})$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+3})$
$8k+4$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+3})$
$8k+5$	$H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)$
$8k+6$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+4})\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$	$H^1_{\text{ét}}(\mathcal{O}'_F,\mu_2)\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$
$8k+7$	$H^1_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+4})$	$H^2_{\text{ét}}(\mathcal{O}'_F,\mathbb{Z}_2^{\otimes 4k+5})\bullet H^0_{\text{ét}}(\mathcal{O}'_F,\mu_2)$

In particular, for $0 \le n \le 3$ and $i > 0$, the group $GW_i^{[n]}(\mathcal{O}'_F)_\#$ is trivial when

$$\left\lfloor \frac{i-n}{2} \right\rfloor \equiv \left\lfloor \frac{4+n}{2} \right\rfloor \pmod 4. \qquad \square$$

Comparing the above with [31, Theorem 0.4], one sees that the cohomology terms involving twisted $\mathbb{Z}_2$-coefficients are detected by the K-groups of $\mathcal{O}'_F$.

The Lichtenbaum conjectures relate the orders of K-groups to values of Dedekind zeta-functions of totally real number fields [24], [25]. We exhibit precise formulas relating the orders of higher Grothendieck-Witt groups to values of Dedekind zeta-functions. If m is even, let $w_m = 2^{a_F+\nu_2(m)}$ where $a_F := (|\mu_{2^\infty}(F(\sqrt{-1}))|)_2$ is the 2-adic valuation and $2^{\nu_2(m)}$ is the 2-primary part of m. If $F = \mathbb{Q}(\zeta_{2^b} + \bar{\zeta}_{2^b})$, then $a_F = b$; and when $F = \mathbb{Q}(\zeta_r + \bar{\zeta}_r)$ (with r odd) or $\mathbb{Q}(\sqrt{d})$ with $d > 2$, then $a_F = 2$. The following theorem applies to these examples, where we write G_{tor} for the torsion subgroup of an abelian group G. Its proof combines our results for $GW^{[n]}(\mathcal{O}'_F)$ with [31, Theorem 0.2].

Theorem 4.9. *For every 2-regular totally real abelian number field F with r real embeddings, the Dedekind zeta-function of F takes the values*

$$\zeta_F(-1-4k) = \frac{\#GW^{[0]}_{8k+2}(\mathcal{O}'_F)}{2\#GW^{[0]}_{8k+3}(\mathcal{O}'_F)} = 2^{2r}\frac{\#GW^{[2]}_{8k+2}(\mathcal{O}'_F)}{\#GW^{[2]}_{8k+3}(\mathcal{O}'_F)} = \frac{2^r}{w_{4k+2}}$$

$$\zeta_F(-3-4k) = 2^r\frac{\#GW^{[0]}_{8k+6}(\mathcal{O}'_F)}{\#GW^{[0]}_{8k+7}(\mathcal{O}'_F)} = 2^r\frac{\#GW^{[2]}_{8k+6}(\mathcal{O}'_F)_{\text{tor}}}{\#GW^{[2]}_{8k+7}(\mathcal{O}'_F)} = \frac{2^r}{w_{4k+4}}$$

up to odd multiples. $\square$

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