

Hermitian K -theory of exact categories

by

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Abstract

We study the theory of higher Grothendieck-Witt groups, alias algebraic hermitian K -theory, of symmetric bilinear forms in exact categories, and prove additivity, cofinality, dévissage and localization theorems – preparing the ground for the theory of higher Grothendieck-Witt groups of schemes as developed in [Sch08a] and [Sch08b]. No assumption on the characteristic is being made.

Key Words: hermitian K -theory, Quillen K -theory, Q -construction, exact categories with duality

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1. Introduction

In his foundational paper [Qui73], Quillen introduced the higher algebraic K -theory of an exact category and proved theorems which go under the names of “localization in abelian categories” [Qui73, §5 Theorem 5], “dévissage” [Qui73, §5 Theorem 4], and “additivity” [Qui73, §3 Theorem 2 and Corollary 1]. These theorems provide the technical background for many results regarding the higher algebraic K -theory of rings and schemes. In this article, we generalize these and some other abstract fundamental theorems from algebraic K -theory to (algebraic) hermitian K -theory. The main point here is that nowhere in the present paper we will have to assume “2 to be invertible” (an unfortunate but common assumption in hermitian K -theory). All our results hold for arbitrary \mathbb{Z} -linear exact categories with duality.

We consider groups $GW_i(\mathcal{E})$, $i \geq 0$, associated with an exact category with duality \mathcal{E} , generalizing the usual Grothendieck-Witt group $GW_0(\mathcal{E})$ of non-degenerate symmetric bilinear forms in \mathcal{E} as defined in [Qui71, §5.1], [Kne77, §4], [QSS79, pp. 280, 281]. As an example, let R be a (commutative) ring and \mathcal{E} the category of finitely generated projective R -modules equipped with an appropriate duality. Then the group $GW_0(\mathcal{E})$ is the Grothendieck group of (that is, the universal group associated with) the abelian monoid of regular symmetric bilinear spaces over R , see [Sch85, §1.6] where it is denoted $\widehat{W}(R)$. For the category $\mathcal{E} = \text{Vect}(X)$

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of vector bundles on a scheme X equipped with the usual duality $E \mapsto E^\vee$, $E \in \text{Vect}(X)$, the group $GW_0(\mathcal{E})$ is Knebusch’s Grothendieck-Witt group $L(X)$ of X [Kne77, §4].

The higher Grothendieck-Witt groups $GW_i(\mathcal{E})$, $i \geq 0$, are defined as the homotopy groups $\pi_i GW(\mathcal{E})$ of a space $GW(\mathcal{E})$ which is the homotopy fibre of a map on classifying spaces of a functor

$$Q^h\mathcal{E} \rightarrow Q\mathcal{E}$$

of categories associated with the exact category with duality \mathcal{E} (definition 4.12). Here $Q\mathcal{E}$ is Quillen’s Q -construction, and $Q^h\mathcal{E}$ its hermitian analog (definition 4.1) introduced by Giffen and Karoubi. The definition of the space $GW(\mathcal{E})$ generalizes Karoubi’s hermitian K -theory of a ring with involution (see remark 4.13) and already appeared in [Hor02, Definitions 3.2 and 3.6] for precisely that reason. The higher Grothendieck-Witt (or hermitian K -) groups also generalize higher algebraic K -theory in the sense that one can associate to any exact category \mathcal{E} an exact category with duality $H\mathcal{E}$ such that there is an isomorphism $GW_i(H\mathcal{E}) \cong K_i(\mathcal{E})$, $i \in \mathbb{N}$ (proposition 4.7).

In this article we prove “dévissage” (theorem 6.1) and “additivity” (theorems 7.1 and 7.2) for the hermitian K -theory of exact categories with duality, generalizing Quillen’s results mentioned above. The hermitian version of Quillen’s localization theorem for noetherian abelian categories (the categories of interest in algebraic geometry and number theory) follows from dévissage since noetherian abelian categories with duality are artinian (see example 6.2). A version of Quillen’s resolution theorem also holds in the hermitian setting but is more conveniently proved in [Sch08a].

We also prove cofinality (theorem 5.1 and corollary 5.2), generalizing [Gra79, theorem 1.1], and a localization theorem for an “s-filtering inclusion of exact categories” (theorem 8.2), generalizing [Sch04a, theorem 2.1], see also remark 8.4. We obtain an explicit non-connective delooping of the hermitian K -theory of an exact category with duality (see theorem 9.11 and remark 9.12), generalizing [Sch04a, theorem 3.4] and [Wag72], [Kar80b, 1.11]. If one wishes, one can use the non-connective delooping to extend Karoubi’s fundamental theorem of [Kar80a] from rings with involution to exact categories with duality following the line of arguments in [Kar80a] (we will, however, refrain from giving the details here since a different approach not relying on [Kar80a] is taken in [Sch08b]).

Unfortunately, the theorems proved in this article for exact categories with duality don’t immediately yield results for the hermitian K -theory of rings and schemes contrary to the K -theory situation in Quillen’s work. This is because there are basically no interesting noetherian (hence artinian) abelian categories with

duality, as mentioned above. The results of this article, however, *are essential* in the theory of higher Grothendieck-Witt groups of schemes and derived categories and provide the foundational background for [Sch08a] and [Sch08b].

We give a short outline of the article. In §2 we recall the definitions of Witt and Grothendieck-Witt groups of exact categories with duality. The expert will find nothing new here. In §3 we introduce “form functors”. These are functors between categories with duality together with the extra structure that compares the dualities of the two categories. This is classical. However, a little less classical is our interpretation of them as “symmetric spaces in a functor category with duality” (see remark 3.3). Though not very deep, this is useful. In §4, we recall the hermitian Q -construction and introduce the Grothendieck-Witt space $GW(\mathcal{E})$ of an exact category with duality. We also identify the low dimensional homotopy groups $\pi_0|Q^h(\mathcal{E})|$, $\pi_1|Q^h(\mathcal{E})|$ and $\pi_0GW(\mathcal{E})$ as the Witt group of \mathcal{E} , the Grothendieck-Witt group of formations in \mathcal{E} , and the Grothendieck-Witt group of symmetric spaces in \mathcal{E} . As the corresponding titles indicate, in §5, §6, §7 and §8 we prove the generalizations to hermitian K -theory of the classical cofinality, dévissage, additivity and filtering localization theorems. In §9, we construct for a certain embedding of (idempotent complete) exact categories with duality $\mathcal{A} \subset \mathcal{U}$, a “cone” exact category with duality $\mathcal{C}(\mathcal{U}, \mathcal{A})$, which contains \mathcal{U} , such that the sequence $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ induces a homotopy fibration in hermitian K -theory. This will be used in [Sch08a].

Prerequisites

The article should be accessible to anyone with a basic knowledge in algebraic topology and who has read §1 of Quillen’s paper [Qui73]. The only results we need from [Qui73] are Quillen’s theorems A and B (for an alternative proof of theorem B, see [GJ99, theorem 5.6]).

Warning to the expert

I sometimes use non-standard notation and terminology, for instance, Q^h instead of \mathcal{W} , GW instead of K^h , and “form functor” instead of “duality preserving functor”. They are all explained at the appropriate place in the article.

2. Witt and Grothendieck-Witt groups of exact categories

We start with fixing standard terminology by recalling the definition of an exact category.

2.1. Exact categories, exact functors and fully exact inclusions

Recall from [Qui73] (see also [Kel96, §4]) that an *exact category* is an additive category \mathcal{E} equipped with a family of sequences of maps in \mathcal{E} , called *conflations* (or *admissible short exact sequences*, or simply *exact sequences*),

$$X \xrightarrow{i} Y \xrightarrow{p} Z \quad (1)$$

satisfying the properties (a) – (f) below. In a conflation (1), the map i is called *inflation* (or admissible monomorphism) and may be depicted as \rightrightarrows , and the map p is called *deflation* (or admissible epimorphism) and may be depicted as \rightarrowtail in diagrams.

- (a) In a conflation (1), the map i is a kernel of p , and p is a cokernel of i .
- (b) Conflations are closed under isomorphisms.
- (c) Inflations are closed under compositions, and deflations are closed under compositions.
- (d) Any diagram $Z \leftarrow X \rightrightarrows Y$ with i an inflation can be completed to a cocartesian square

$$\begin{array}{ccc} X & \rightrightarrows & Y \\ \downarrow & & \downarrow \\ Z & \rightrightarrows & W \end{array}$$

with j an inflation.

- (e) Dually, any diagram $X \rightarrow Z \leftarrowtail Y$ with p a deflation can be completed to a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ q \downarrow & & \downarrow p \\ X & \longrightarrow & Z \end{array}$$

with q a deflation.

- (f) The following sequence is a conflation

$$X \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} Y. \quad (2)$$

Quillen lists another axiom and its dual which, however, follow from the above (cf. [Kel90, appendix]). An additive functor between exact categories is called *exact* if it sends conflations to conflations. Unless otherwise stated, all exact categories in this article will be (essentially) small.

Let \mathcal{A}, \mathcal{B} be exact categories such that $\mathcal{B} \subset \mathcal{A}$ is a full subcategory. We say that \mathcal{B} is a *fully exact subcategory* of \mathcal{A} , or that $\mathcal{B} \subset \mathcal{A}$ is a *fully exact inclusion*, if \mathcal{B} is closed under extensions in \mathcal{A} (that is, if in a conflation (1) in \mathcal{A} , X and Z are isomorphic to objects in \mathcal{B} then Y is isomorphic to an object in \mathcal{B}), and if the inclusion $\mathcal{B} \subset \mathcal{A}$ preserves and detects conflations.

A conflation is called *split* if it is isomorphic to a sequence of the form (2) above. An exact category is called *split exact* if every conflation is split.

Definition 2.1 An *exact category with duality* is a triple $(\mathcal{E}, *, \eta)$, where \mathcal{E} is an exact category, $*$: $\mathcal{E}^{op} \rightarrow \mathcal{E}$ is an exact functor, and $\eta_A : A \xrightarrow{\sim} A^{**}$ a natural isomorphism such that $1_{A^*} = \eta_A^* \circ \eta_{A^*}$ for all objects A of \mathcal{E} . We may write \mathcal{E} in place of $(\mathcal{E}, *, \eta)$ if the duality functor $*$ and the double dual identification η are understood.

Example 2.2 Let R be a ring with involution $R \rightarrow R^{op} : a \mapsto \bar{a}$ (a ring isomorphism satisfying $\bar{\bar{a}} = a$), and let $\mathcal{P}(R)$ be the category of finitely generated right R -modules. For a right R -module M , the R -bimodule structure of R makes the set $\text{Hom}_R(M, R)$ of right R -module maps into a left R -module which we consider as a right R -module via the ring isomorphism given by the involution. Thus, we have a functor

$$* : \mathcal{P}(R)^{op} \rightarrow \mathcal{P}(R) : P \mapsto \text{Hom}_R(P, R).$$

For a finitely generated projective R -module P , there is a natural isomorphism of right R -modules

$$\text{can}_P : P \xrightarrow{\cong} P^{**}$$

defined by $\text{can}_P(x)(f) = \overline{f(x)}$ which makes the triple

$$(\mathcal{P}(R), *, \text{can})$$

into a split exact category with duality.

Example 2.3 Let X be a scheme, and let \mathcal{L} be a line-bundle on X . Then the triple

$$(\text{Vect}(X), \text{Hom}_{O_X}(\cdot, \mathcal{L}), \text{can})$$

is an exact category with duality, where $\text{Vect}(X)$ is the exact category of locally free O_X -modules of finite rank, $\text{Hom}_{O_X}(\cdot, \cdot)$ is the sheaf of O_X -module homomorphisms, and the double dual identification $\text{can} : V \rightarrow \text{Hom}_{O_X}(\text{Hom}_{O_X}(V, \mathcal{L}), \mathcal{L})$ is the map defined by $\text{can}(x)(f) = f(x)$ on sections.

Definition 2.4 A *symmetric form* (on an object X) in an exact category with duality $(\mathcal{E}, *, \eta)$ is a pair (X, φ) where $\varphi : X \rightarrow X^*$ is a morphism in \mathcal{E} satisfying $\varphi^* \eta_X = \varphi$. If φ is an isomorphism, the form is called *non-singular*, or *non-degenerate*; otherwise it is called *singular* or *degenerate*. A non-singular form is also called a *symmetric space*.

Let (X, φ) be a symmetric form in \mathcal{E} , and $f : Y \rightarrow X$ a morphism. Then $(Y, \varphi|_Y)$ is a symmetric form on Y , the restriction (via f) of φ , where $\varphi|_Y = f^* \varphi f$. A *map of symmetric forms* $f : (Y, \psi) \rightarrow (X, \varphi)$ is a morphism $f : Y \rightarrow X$ in \mathcal{E} such that $\psi = \varphi|_Y$. It is an *isometry* if f is an isomorphism. Composition of morphisms in \mathcal{E} defines the composition of maps between symmetric forms.

The *orthogonal sum* $(X, \varphi) \perp (Y, \psi)$ of two symmetric forms (X, φ) and (Y, ψ) is the symmetric form $(X \oplus Y, \varphi \oplus \psi)$. If (X, φ) and (Y, ψ) are non-singular, then so is their orthogonal sum. If \mathcal{E} is essentially small, then the set of isometry classes of symmetric spaces is an abelian monoid where the monoid addition is given by the orthogonal sum.

Definition 2.5 Let (X, φ) be a symmetric space in \mathcal{E} . A *totally isotropic subspace* of X is an inflation $i : L \twoheadrightarrow X$ such that $0 = \varphi|_L = i^* \varphi i$, and such that the induced map $L \rightarrow L^\perp \subset X$ from L to its *orthogonal* $L^\perp = \ker(i^* \varphi)$ is also an inflation.

The totally isotropic subspace $L \subset X$ is called *Lagrangian* of (X, φ) if $L = L^\perp$, that is, $i : L \twoheadrightarrow X$ is a Lagrangian if and only if the following sequence is a conflation

$$L \twoheadrightarrow X \twoheadrightarrow L^*. \quad (3)$$

A symmetric space (X, φ) is called *metabolic* if it has a Lagrangian. If L is any object of \mathcal{E} , then $H(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix})$ is a symmetric space called *hyperbolic space* of L . It is metabolic with Lagrangian $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : L \twoheadrightarrow L \oplus L^*$. Note that the symmetric spaces $H(L \oplus M)$ and $H(L) \perp H(M)$ are isometric.

Before we recall the definition of Witt and Grothendieck-Witt groups, we first record the following well-known lemma, cf. [QSS79, Lemma 5.2 (3)].

Lemma 2.6 *Let (X, φ) be a symmetric space in an exact category with duality $(\mathcal{E}, *, \eta)$, and let $i : L \twoheadrightarrow X$ be a totally isotropic subspace of X . Then there is a unique non-singular form $\bar{\varphi} : L^\perp/L \rightarrow (L^\perp/L)^*$ on the quotient L^\perp/L such that $\varphi|_{L^\perp} = \bar{\varphi}|_{L^\perp}$, where $\varphi|_{L^\perp}$ is the restriction of φ via i , and $\bar{\varphi}|_{L^\perp}$ is the restriction to L^\perp of $\bar{\varphi}$ via the quotient map $L^\perp \twoheadrightarrow L^\perp/L$. Moreover, the symmetric space $(X, \varphi) \oplus (L^\perp/L, -\bar{\varphi})$ is metabolic with Lagrangian L^\perp .*

Proof: The inflation i factors as a composition of inflations $L \xrightarrow{i_0} L^\perp \xrightarrow{i_1} X$. The map $\varphi|_{L^\perp} = i_1^* \varphi i_0$ satisfies $(\varphi|_{L^\perp})^* \circ \eta = \varphi|_{L^\perp}$. If we denote by $p : L^\perp \twoheadrightarrow L^\perp/L$ the quotient map, there is a unique map $\bar{\varphi} : L^\perp/L \rightarrow (L^\perp/L)^*$ such that $\varphi|_{L^\perp} =$

$p^* \bar{\varphi} p$ since $\varphi_{|L^\perp} \circ i_0 = 0$ and $i_0^* \circ \varphi_{|L^\perp} = 0$. The map $\bar{\varphi}$ is symmetric, that is, $\bar{\varphi}^* \eta = \bar{\varphi}$, because $p^*(\bar{\varphi}^* \eta) p = p^* \bar{\varphi}^* p^* \eta = (p^* \bar{\varphi} p)^* \eta = (\varphi_{|L^\perp})^* \eta = \varphi_{|L^\perp} = p^* \bar{\varphi} p$. By construction, i_1 induces an isomorphism $\ker(p) \rightarrow \ker(i_1^* \varphi)$, both kernels being L (as a subobject of X). Therefore, we have an admissible exact sequence

$$L^\perp \xrightarrow{\begin{pmatrix} i_1 \\ p \end{pmatrix}} X \oplus L^\perp / L \xrightarrow{(-i_1^* \varphi \quad p^* \bar{\varphi})} (L^\perp)^*. \quad (4)$$

The triple $(\eta, -\varphi \oplus \bar{\varphi}, 1)$ defines a map from the conflation (4) to its dual. By the five-lemma, $-\varphi \oplus \bar{\varphi}$ and thus $\bar{\varphi}$ are isomorphisms. Hence, $(L^\perp / L, \bar{\varphi})$ is a symmetric space. The last assertion follows from the conflation (4). \square

2.2. Witt and Grothendieck-Witt groups

We recall the classical definitions of Witt and Grothendieck-Witt groups of an exact category with duality (for Grothendieck-Witt groups, see [Qui71, §5.1], [Kne77, §4], [QSS79, pp. 280, 281]; and for Witt groups, see furthermore [Knu91, VIII §1], [Bal01]). Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. The *Witt group* $W_0(\mathcal{E})$ of \mathcal{E} is the abelian monoid of isometry classes $[X, \varphi]$ of symmetric spaces (X, φ) in \mathcal{E} modulo the sub-monoid of metabolic spaces. The quotient monoid $W_0(\mathcal{E})$ is a group because $(X, \varphi) \oplus (X, -\varphi)$ is metabolic with Lagrangian $\begin{pmatrix} 1 \\ 1 \end{pmatrix} : X \rightarrow X \oplus X$, so that we have $-[X, \varphi] = [X, -\varphi]$.

The *Grothendieck-Witt group* $GW_0(\mathcal{E})$ of \mathcal{E} is the Grothendieck group of the abelian monoid of isometry classes $[X, \varphi]$ of symmetric spaces (X, φ) in \mathcal{E} modulo the following relation: if M is metabolic with Lagrangian L , then $[M] = [H(L)]$ in $GW_0(\mathcal{E})$.

Remark 2.7 Let \mathcal{E} be an exact category with duality. An element $[X] \in W_0(\mathcal{E})$ is 0 if and only if there is a metabolic space M such that $X \perp M$ is metabolic. Such a symmetric space X is called *stably metabolic*.

An element $[X] - [Y] \in GW_0(\mathcal{E})$ is 0 if and only if there are metabolic spaces M_1, M_2 with associated Lagrangians L_1, L_2 and an isometry

$$X \perp M_1 \perp H(L_2) \cong Y \perp M_2 \perp H(L_1).$$

These statements follow immediately from the definition of Witt and Grothendieck-Witt groups, details are left to the reader.

Lemma 2.8 *For any exact category with duality \mathcal{E} , we have the following identities in $GW_0(\mathcal{E})$.*

$$(a) \quad [X, \varphi] + [X, -\varphi] = [H(X)]$$

(b) If $X \xrightarrow{i} Y \xrightarrow{p} Z$ is a conflation in \mathcal{E} then $[H(X)] + [H(Z)] = [H(Y)]$.

(c) If (X, φ) is a symmetric space with a totally isotropic subspace $L \subset X$ then $[X, \varphi] = [L^\perp/L, \bar{\varphi}] + [H(L)]$.

Moreover, we have an exact sequence

$$K_0(\mathcal{E}) \xrightarrow{H} GW_0(\mathcal{E}) \rightarrow W_0(\mathcal{E}) \rightarrow 0. \quad (5)$$

Proof: Part (a) holds since $(X, \varphi) \oplus (X, -\varphi)$ has Lagrangian $\begin{pmatrix} 1 \\ 1 \end{pmatrix} : X \rightarrow X \oplus X$.

Part (b) holds since $i \oplus p^* : X \oplus Z^* \rightarrow Y \oplus Y^*$ is a Lagrangian in $H(Y)$ and $\begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} : Z \oplus Z^* \rightarrow Z^* \oplus Z^{**}$ defines an isometry $H(Z) \cong H(Z^*)$.

Part (c) holds because $[X, \varphi] + H(L^\perp/L) = [X, \varphi] + [L^\perp/L, -\bar{\varphi}] + [L^\perp/L, \bar{\varphi}] = H(L^\perp) + [L^\perp/L, \bar{\varphi}] = H(L^\perp/L) + H(L) + [L^\perp/L, \bar{\varphi}]$, where the equations follow from (a), Lemma 2.6, and (b).

Finally, (b) shows that the first map in the sequence (5) is well-defined. It follows from the definitions of Witt and Grothendieck-Witt groups that the sequence is exact, see also Remark 2.7. \square

For completeness' sake, we include the following well-known lemma (cf. [QSS79, Lemma 5.4]) and its corollary which identifies the Grothendieck-Witt group of a ring with involution R (that is, of the split exact category with duality $(\mathcal{P}(R), *, \text{can})$ of example 2.2) with the universal group associated with the abelian monoid of isometry classes of symmetric spaces over R .

Lemma 2.9 *Let $(\mathcal{A}, *, \eta)$ be a split exact category with duality, and let (X, φ) be a metabolic space in \mathcal{A} with Lagrangian $i : L \rightarrow X$. Then there is a metabolic space M and an isometry $H(L) \perp M \cong (X, \varphi) \perp M$.*

Proof: If $\gamma : L^* \rightarrow L^{**}$ is a symmetric form on L^* , we write $M_\gamma(L)$ for the metabolic space $(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \eta & \gamma \end{pmatrix})$ with Lagrangian $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : L \rightarrow L \oplus L^*$.

A retract $r : X \rightarrow L$ of $i : L \rightarrow X$ (so we have $ri = 1$) induces an isometry $(i \ \varphi^{-1}r^*) : M_\gamma(L) \rightarrow (X, \varphi)$ with $\gamma = \eta r \varphi^{-1}r^*$, under which the Lagrangian $i : L \rightarrow X$ corresponds to $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : L \rightarrow L \oplus L^*$. So we can assume $(X, \varphi) = M_\gamma(L)$ and $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since the map

$$\begin{pmatrix} 1 & 0 & -1 & \eta^{-1}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (L \oplus L^*) \oplus (L \oplus L^*) \longrightarrow (L \oplus L^*) \oplus (L \oplus L^*)$$

defines an isometry $H(L) \perp M_{-\gamma}(L) \xrightarrow{\cong} M_\gamma(L) \perp M_{-\gamma}(L)$, we are done. \square

Corollary 2.10 *Let \mathcal{A} be a split exact category with duality. Then the Grothendieck-Witt group $GW_0(\mathcal{A})$ of \mathcal{A} (as defined in 2.2) is the Grothendieck group of the abelian monoid of isometry classes of symmetric spaces in \mathcal{A} .*

3. Form functors and functor categories

In this section we assemble various definitions mostly related to the functoriality behavior of (higher) Grothendieck-Witt groups. In order to do so, it will be convenient to have a more general notion at our disposal than that of an exact category with duality. This is the purpose of the following definitions.

Definition 3.1 A *category with duality* is a triple $(\mathcal{C}, *, \eta)$ with \mathcal{C} a category, $*$: $\mathcal{C}^{op} \rightarrow \mathcal{C}$ a functor, $\eta : 1 \rightarrow **$ a natural transformation such that $1_{A^*} = \eta_A^* \circ \eta_{A^*}$ for all objects A in \mathcal{C} . Note that we don't require η to be an isomorphism. If η is a natural isomorphism, we say that the duality is *strong*. In case η is the identity (in which case $** = id$), we call the duality *strict*. In this article exact categories with duality will always have a strong duality.

The concept of a symmetric form makes sense in any category with duality. More precisely, a *symmetric form* in a category with duality $(\mathcal{C}, *, \eta)$ is a pair (X, φ) where $\varphi : X \rightarrow X^*$ is a morphism in \mathcal{C} satisfying $\varphi^* \eta_X = \varphi$. A map of symmetric forms $(X, \varphi) \rightarrow (Y, \psi)$ is a map $f : X \rightarrow Y$ in \mathcal{C} such that $\varphi = f^* \circ \psi \circ f$. Composition of such maps is composition in \mathcal{C} . For a category with duality $(\mathcal{C}, *, \eta)$, we denote by \mathcal{C}_h the category of symmetric forms in \mathcal{C} and maps between them, and call it the *hermitian category associated with* $(\mathcal{C}, *, \eta)$.

Next, we introduce what some authors call “duality preserving functors”. We will call them “form functors” (this is short for “functors equipped with a form”, see remark 3.3 below), and we will reserve the term “duality preserving functor” for a special kind of (form) functor, see below.

Definition 3.2 A *form functor* from a category with duality $(\mathcal{A}, *, \alpha)$ to $(\mathcal{B}, *, \beta)$ is a pair (F, φ) with $F : \mathcal{A} \rightarrow \mathcal{B}$ a functor and $\varphi : F^* \rightarrow *F$ a natural transformation making diagram (6) commute.

$$\begin{array}{ccc}
 FA & \xrightarrow{\beta_{FA}} & (FA)^{**} \\
 \downarrow F(\alpha_A) & & \downarrow \varphi_A^* \\
 F(A^{**}) & \xrightarrow{\varphi_{A^*}} & F(A^*)^*
 \end{array} \tag{6}$$

The natural transformation φ is called *duality compatibility morphism*. When the duality compatibility morphism φ is understood, we may simply write F for the form functor (F, φ) .

Form functors are composed as follows. Let $(F, \varphi) : (\mathcal{A}, *, \alpha) \rightarrow (\mathcal{B}, *, \beta)$ and $(G, \psi) : (\mathcal{B}, *, \beta) \rightarrow (\mathcal{C}, *, \gamma)$ be form functors. Their composition is the form functor $(G \circ F, \psi \star \varphi)$, where

$$(\psi \star \varphi)_A = \psi_{FA} \circ G(\varphi_A).$$

Composition of form functors is associative and unital, where the unit, that is, the identity form functor of a category with duality $(\mathcal{A}, *, \alpha)$, is $(id_{\mathcal{A}}, 1)$.

In this article and its sequels, a *duality preserving functor* is a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ which commutes with (“preserves”) dualities and double dual identifications, that is, we have $F* = *F$ and $F(\alpha) = \beta_F$. In this case, (F, id) is a form functor. We will consider duality preserving functors F as form functors (F, id) .

3.1. Non-singular exact form functors

A form functor between exact categories with duality $(F, \varphi) : (\mathcal{A}, *, \alpha) \rightarrow (\mathcal{B}, *, \beta)$ is called *exact* if F is an exact functor. The form functor is called *non-singular* if φ is a natural isomorphism. Note that the composition of non-singular form functors is non-singular. A non-singular exact form functor $(F, \varphi) : (\mathcal{A}, *, \alpha) \rightarrow (\mathcal{B}, *, \beta)$ between exact categories with duality sends symmetric spaces (A, a) in \mathcal{A} to symmetric spaces $(FA, \varphi_A \circ Fa)$ in \mathcal{B} , and it preserves metabolic spaces, Lagrangians, totally isotropic subspaces etc. In particular, it induces maps on (Witt and) Grothendieck-Witt groups

$$GW_0(F) : GW_0(\mathcal{A}) \rightarrow GW_0(\mathcal{B}) : (A, a) \mapsto (FA, \varphi_A \circ Fa).$$

It is sometimes convenient to adopt a more conceptual point of view which we will explain next.

3.2. Functor categories

Let $(\mathcal{A}, *, \alpha)$ and $(\mathcal{B}, *, \beta)$ be (small) categories with duality. Consider the category $\text{Fun}(\mathcal{A}, \mathcal{B})$ of functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and natural transformations as maps. We define a duality \sharp on $\text{Fun}(\mathcal{A}, \mathcal{B})$ by declaring the dual of a functor F to be the functor $F^\sharp = *F* : \mathcal{A} \mapsto F(A^*)^*$. The dual of a natural transformation $g : F \rightarrow G$ of functors is the natural transformation g^\sharp defined by $g_A^\sharp = g_{A^*}^* : G(A^*)^* \rightarrow F(A^*)^*$. The double dual identification $\eta_F : F \rightarrow F^{\sharp\sharp}$ is the map $\eta_F(A) = \beta_{F(A^{**})} \circ F(\alpha_A) = F(\alpha_A)^{**} \circ \beta_{FA} : F(A) \rightarrow F(A^{**})^{**}$. One verifies the equality $\eta_F^\sharp \eta_{F^\sharp} = id_{F^\sharp}$ so that the triple

$$(\text{Fun}(\mathcal{A}, \mathcal{B}), \sharp, \eta)$$

is indeed a category with duality.

Remark 3.3 (Important remark) We will make frequent use of the following observation. There is a natural bijection between form functors $(F, \varphi) : (\mathcal{A}, *, \alpha) \rightarrow (\mathcal{B}, *, \beta)$ and symmetric forms $(F, \hat{\varphi})$ in the category with duality $(\text{Fun}(\mathcal{A}, \mathcal{B}), \sharp, \eta)$. The bijection is given by the formulas $\varphi_A = F(\alpha_A)^* \circ \hat{\varphi}_{A^*}$ and $\hat{\varphi}_A = \varphi_{A^*} \circ F(\alpha_A)$.

Note that if α and β are natural isomorphisms, then (F, φ) is a non-singular form functor if and only if $(F, \hat{\varphi})$ is a symmetric space in $(\text{Fun}(\mathcal{A}, \mathcal{B}), \sharp, \eta)$.

3.3. Natural transformations

Using remark 3.3, we define a *natural transformation* $(F, \varphi) \rightarrow (G, \psi)$ of form functors $(\mathcal{A}, *, \alpha) \rightarrow (\mathcal{B}, *, \beta)$ to be a map of associated symmetric forms in $(\text{Fun}(\mathcal{A}, \mathcal{B}), \sharp, \eta)$ (in the sense of 2.4).

3.4. Functor categories with exact target

If \mathcal{E} is an exact category, and \mathcal{A} an arbitrary (small) category, then the category of functors $\text{Fun}(\mathcal{A}, \mathcal{E})$ is an exact category by declaring a sequence $F \rightarrow G \rightarrow H$ of functors to be admissible exact if $FA \twoheadrightarrow GA \twoheadrightarrow HA$ is a conflation in \mathcal{E} for all objects A of \mathcal{A} . If, furthermore, \mathcal{E} is an exact category with duality, and \mathcal{A} a category with strong duality, then $\text{Fun}(\mathcal{A}, \mathcal{E})$ is an exact category with duality (see 3.2). If \mathcal{A} is also an exact category with duality, then the full subcategory $\text{ExFun}(\mathcal{A}, \mathcal{E}) \subset \text{Fun}(\mathcal{A}, \mathcal{E})$ of exact functors is closed under extensions and the duality functor, and thus inherits a structure of an exact category with duality from $\text{Fun}(\mathcal{A}, \mathcal{E})$. Note that the bijection in remark 3.3 yields a bijection between non-singular exact form functors from \mathcal{A} to \mathcal{E} and symmetric spaces in $\text{ExFun}(\mathcal{A}, \mathcal{E})$.

Remark/Definition 3.4 The interpretation of non-singular form functors between exact categories with duality in terms of symmetric spaces in the category of functors now allows us to transfer to functors the terminology applying to objects in categories with duality. In this sense, metabolic (form) functors, totally isotropic subfunctors, Lagrangians of a metabolic form functor, the hyperbolic (form) functor HF associated with an exact functor $F : \mathcal{A} \rightarrow \mathcal{E}$, and natural isometries between form functors are to be understood.

We finish this section by introducing “hyperbolic categories”. They allow us to consider algebraic K -theory as a special case of Grothendieck-Witt theory (see proposition 4.7).

3.5. Hyperbolic categories

To any exact category \mathcal{E} we can associate an exact category with duality $H\mathcal{E}$, called *hyperbolic category* of \mathcal{E} . As a category, $H\mathcal{E}$ is $\mathcal{E} \times \mathcal{E}^{op}$. Thus, its objects are pairs (A, B) of objects A, B in \mathcal{E} . A morphism $(A, B) \rightarrow (A', B')$ is a pair (f, g) of morphisms $f : A \rightarrow A'$ and $g : B' \rightarrow B$. The dual $(A, B)^*$ is (B, A) , and $(f, g)^* = (g, f)$. The double dual identification $1 \rightarrow **$ is the identity. So $H\mathcal{E}$ is an exact category with strict duality.

If $(\mathcal{E}, *, \eta)$ is an exact category with duality, we have a non-singular exact form functor

$$H : H\mathcal{E} \rightarrow (\mathcal{E}, *, \eta) : (X, Y) \mapsto X \oplus Y^*,$$

called *hyperbolic (form) functor*, with duality compatibility $H(Y, X) \rightarrow H(X, Y)^*$ the map $\begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} : Y \oplus X^* \rightarrow X^* \oplus Y^{**}$.

4. The Grothendieck-Witt space of an exact category

In this section, we introduce the main object of study of the article. This is the Grothendieck-Witt space of an exact category with duality (see definition 4.4). For the convenience of the reader, we recall Quillen's Q -construction.

4.1. Quillen's Q -construction

Let \mathcal{E} be an exact category. Quillen defined in [Qui73] a new category $Q\mathcal{E}$. Its objects are the objects of \mathcal{E} . A map $X \rightarrow Y$ in $Q\mathcal{E}$ is an equivalence class of diagrams

$$X \xleftarrow{p} U \xrightarrow{i} Y \tag{7}$$

with p a deflation and i an inflation. The datum (U, i, p) is equivalent to (U', i', p') if there is an isomorphism $g : U \rightarrow U'$ in \mathcal{E} such that $p = p' \circ g$ and $i = i' \circ g$. The composition in $Q\mathcal{E}$ of maps $X \rightarrow Y$ and $Y \rightarrow Z$ represented by the data (U, i, p) and (V, j, q) is given by the datum $(U \times_Y V, p\bar{q}, j\bar{i})$ where $\bar{q} : U \times_Y V \rightarrow U$ and $\bar{i} : U \times_Y V \rightarrow V$ are the canonical projections to U and V , respectively.

Now we introduce the hermitian analog of Quillen's Q -construction. This is the "hermitian Q -construction" $Q^h(\mathcal{E})$ due to Giffen and Karoubi.

Definition 4.1 Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. We define a category $Q^h(\mathcal{E}, *, \eta)$ as follows. The objects are the symmetric spaces (X, φ) in \mathcal{E} . A map $(X, \varphi) \rightarrow (Y, \psi)$ is an equivalence class of diagrams (7) as in the Q -construction with p a deflation and i and inflation such that $\varphi|_U = \psi|_U$ and i induces an isomorphism $\ker(p) \rightarrow \ker(i^*\psi)$. The last two conditions are equivalent to the conditions that $\ker(p) \subset Y$ is totally isotropic with orthogonal U , and p induces an isometry $(U/\ker(p), \bar{\varphi}) \cong (X, \varphi)$. They are also equivalent to the diagram

$$\begin{array}{ccc} U & \xrightarrow{i} & Y \\ p \downarrow & & \downarrow i^*\phi \\ X & \xrightarrow{p^*\varphi} & U^* \end{array}$$

being (commutative and) bicartesian. As in Quillen's Q -construction, the datum (U, i, p) is equivalent to the datum (U', i', p') if there is an isomorphism $g : U \rightarrow U'$ such that $p = p'g$ and $i = i'g$. Composition is also as in Quillen's Q -construction 4.1. Sometimes, we may write $Q^h\mathcal{E}$ in place of $Q^h(\mathcal{E}, *, \eta)$ if $*$ and η are understood.

Remark 4.2 The category $Q^h\mathcal{E}$ appeared at various places in the literature. In [CL86], [Uri90], [Hor02], [HS04] it was denoted $\mathcal{W}(\mathcal{E})$, presumably because $\pi_0\mathcal{W}(\mathcal{E})$ is the usual Witt group $W_0(\mathcal{E})$ of \mathcal{E} , see proposition 4.8 below. I use the notation $Q^h\mathcal{E}$ because I would like to emphasize the close relationship to Quillen's Q -construction and because I would like to reserve the letter W (or \mathcal{W}) for a space whose homotopy groups are the Witt groups of Balmer (or the \mathbb{L} -groups of Ranicki) not only in degree 0 but in all degrees.

Remark 4.3 For any exact category \mathcal{E} , the functor

$$Q^h H\mathcal{E} \rightarrow Q\mathcal{E} : ((A, B), (f, g)) \mapsto A$$

is an equivalence of categories. This is an easy exercise in unraveling the definitions, see [Hor02, Lemma 3.1].

4.2. Classifying spaces of groups and categories

For a small category \mathcal{C} we denote by $|\mathcal{C}|$ its classifying space, that is, the topological realization of the nerve simplicial set of \mathcal{C} , see for instance [GJ99, Example I.1.4].

As an example and to fix notation, let G be a group, and consider the category \underline{G} which has one object e and morphisms the elements of G . Composition is multiplication in G . It is well-known [GJ99, Example I.1.5], that the map $G \rightarrow \pi_1|\underline{G}| : g \mapsto g$ is an isomorphism, and that $\pi_i|\underline{G}| = 0$ for $i \neq 0$.

To simplify notation, we will often apply to a category \mathcal{C} a terminology (or notation) from topology when the terminology (or notation) strictly speaking applies to the classifying space $|\mathcal{C}|$ of \mathcal{C} . For instance, we may write $\pi_i\mathcal{C}$ instead of $\pi_i|\mathcal{C}|$ for the homotopy groups of $|\mathcal{C}|$, and we may call a sequence of categories a homotopy fibration when the associated sequence of classifying spaces is a homotopy fibration etc.

Now we come to the definition of the Grothendieck-Witt space. Recall that the algebraic K -theory space is $K(\mathcal{E}) = \Omega|Q\mathcal{E}|$.

Definition 4.4 Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. The forgetful functor $Q^h\mathcal{E} \rightarrow Q\mathcal{E} : (X, \varphi) \mapsto X$ defines a map

$$|Q^h\mathcal{E}| \rightarrow |Q\mathcal{E}| \tag{8}$$

on classifying spaces whose homotopy fibre (with respect to a zero object of \mathcal{E} as base point of $Q\mathcal{E}$) is defined to be the *Grothendieck-Witt space* $GW(\mathcal{E}, *, \eta)$ of \mathcal{E} . Thus, we have a homotopy fibration

$$GW(\mathcal{E}) \rightarrow |Q^h\mathcal{E}| \rightarrow |Q\mathcal{E}|.$$

Here, as always, we write $GW(\mathcal{E})$ instead of $GW(\mathcal{E}, *, \eta)$ if $*$ and η are understood.

Remark 4.5 The space $GW(\mathcal{E})$ already appeared in [Hor02, Definition 3.2] under the name $\mathcal{F}(\mathcal{E})$. The idea of considering this space probably goes back to Giffen and Karoubi.

Definition 4.6 For a ring with involution R and $\varepsilon \in \{-1, +1\}$, we define its Grothendieck-Witt (or hermitian K -theory) space as

$${}_{\varepsilon}GW(R) = GW(\mathcal{P}(R), *, \varepsilon \text{can})$$

where the category with duality on the right hand side is defined in example 2.2.

Similarly, for a scheme X we define its Grothendieck-Witt space with coefficients in a line bundle \mathcal{L} as

$${}_{\varepsilon}GW(X, \mathcal{L}) = GW(\text{Vect}(X), \text{Hom}_{O_X}(\cdot, \mathcal{L}), \varepsilon \text{can})$$

where the category with duality on the right hand side is defined in example 2.3. When $\mathcal{L} = O_X$, we may simply write ${}_{\varepsilon}GW(X)$ for this space.

The next proposition shows that higher algebraic K -theory is a particular case of the theory of higher Grothendieck-Witt groups. A proof can also be found in [Hor05, Proposition 3.3].

Proposition 4.7 *For any exact category \mathcal{E} , there is a natural homotopy equivalence*

$$GW(H\mathcal{E}) \simeq K(\mathcal{E}).$$

Proof: The space $GW(H\mathcal{E})$ is the homotopy fibre of the map $Q^h(H\mathcal{E}) \rightarrow Q(H\mathcal{E})$ which, under the equivalence in 4.3 and the isomorphism $Q(H\mathcal{E}) = Q\mathcal{E} \times Q(\mathcal{E}^{op}) \cong Q\mathcal{E} \times Q\mathcal{E}$, corresponds to the diagonal map $Q\mathcal{E} \rightarrow Q\mathcal{E} \times Q\mathcal{E} : X \mapsto (X, X)$. This implies the claim since for any pointed topological space Y , the homotopy fibre of the diagonal map $\Delta : Y \rightarrow Y \times Y$ is ΩY . In detail, the commutative triangle consisting of $\Delta : Y \rightarrow Y \times Y$, the projection on to the first component $p : Y \times Y \rightarrow Y$, and their composition $id_Y = p \circ \Delta$ induces a homotopy fibration $F(\Delta) \rightarrow F(id_Y) \rightarrow F(p)$ with fibre, total space and base the homotopy fibres of the maps Δ , $id_Y = p \circ \Delta$ and p . Since $F(id_Y)$ is contractible, and since $F(p) \simeq Y$, we are done. \square

In the remainder of the section we calculate some low dimensional homotopy groups of $Q^h\mathcal{E}$ and of $GW(\mathcal{E})$. The next proposition which identifies $\pi_0 Q^h\mathcal{E}$ with the usual Witt group of \mathcal{E} is due to Uridia [Uri90] and is included here for completeness' sake. In its statement, we denote by $i\mathcal{C}$ the subcategory $i\mathcal{C} \subset \mathcal{C}$ of a category \mathcal{C} whose morphism set is the set of isomorphisms of \mathcal{C} .

Proposition 4.8 *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Then the map $(i\mathcal{E})_h = iQ^h\mathcal{E} \hookrightarrow Q^h\mathcal{E}$ induces an isomorphism of abelian groups*

$$W_0(\mathcal{E}) \rightarrow \pi_0|Q^h\mathcal{E}|.$$

Proof: Orthogonal sum operation makes the classifying spaces of $(i\mathcal{E})_h$ and $Q^h\mathcal{E}$ into commutative H -spaces, and we obtain a map $\pi_0|(i\mathcal{E})_h| \rightarrow \pi_0|Q^h\mathcal{E}|$ of abelian monoids. Metabolic objects are trivial in $\pi_0|Q^h\mathcal{E}|$ since a Lagrangian defines a path to 0 in $|Q^h\mathcal{E}|$. We therefore obtain a well-defined map $W_0(\mathcal{E}) \rightarrow \pi_0|Q^h\mathcal{E}|$ which is clearly surjective. As a set, $\pi_0|Q^h\mathcal{E}|$ is generated by the objects (X, φ) of $Q^h\mathcal{E}$ modulo the relation $(X, \varphi) \sim (Y, \varphi)$ whenever there is a map $(X, \varphi) \rightarrow (Y, \varphi)$ in $Q^h\mathcal{E}$. This relation also holds in $W_0(\mathcal{E})$, by Lemma 2.8 (c). Hence, the surjective map $\pi_0 Q^h\mathcal{E} \rightarrow W_0(\mathcal{E}) : (X, \varphi) \mapsto [X, \varphi]$ is well-defined and thus inverse to the map $W_0(\mathcal{E}) \rightarrow \pi_0|Q^h\mathcal{E}|$. \square

Our next goal is to “calculate” $\pi_1|Q^h\mathcal{E}|$. We will identify this group as the “Grothendieck-Witt group of formations” the definition of which we recall now.

4.3. Formations

Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. A *formation* in \mathcal{E} is a quadruple (X, φ, L_1, L_2) with (X, φ) a metabolic space in \mathcal{E} and L_1, L_2 two Lagrangians of (X, φ) . Two formations (X, φ, L_1, L_2) and $(X', \varphi', L'_1, L'_2)$ are isometric if there is an isometry $(X, \varphi) \cong (X', \varphi')$ carrying L'_1 and L'_2 into L_1 and L_2 . The orthogonal sum of two formations is defined by

$$(X, \varphi, L_1, L_2) \perp (X', \varphi', L'_1, L'_2) = (X \oplus X', \varphi \oplus \varphi', L_1 \oplus L'_1, L_2 \oplus L'_2).$$

The *Grothendieck-Witt group of formations*

$$GW_{form}(\mathcal{E})$$

is the free abelian group generated by isometry classes $[X, \varphi, L_1, L_2]$ of formations subject to the following three relations (see [Kar74, p. 370], where, in special cases, the group was denoted $U(\mathcal{E})$, [Wal03]):

$$(a) \quad [(X, \varphi, L_1, L_2) \perp (X', \varphi', L'_1, L'_2)] = [X, \varphi, L_1, L_2] + [X', \varphi', L'_1, L'_2]$$

$$(b) [X, \varphi, L_1, L_2] + [X, \varphi, L_2, L_3] = [X, \varphi, L_1, L_3]$$

(c) If $L \subset X$ is totally isotropic and a common admissible subspace of L_1 and L_2 , then one has $[L^\perp/L, \bar{\varphi}, L_1/L, L_2/L] = [X, \varphi, L_1, L_2]$.

A Lagrangian $L \hookrightarrow X$ of a metabolic space (X, φ) defines a path $[L] : 0 \rightarrow X$ in $Q^h\mathcal{E}$ given by the datum $0 \leftarrow L \hookrightarrow X$. Therefore, a formation (X, φ, L_1, L_2) defines a loop $[L_2]^{-1}[L_1]$ in $Q^h\mathcal{E}$ based at zero. For such loops, the three relations above hold in $\pi_1 Q^h(\mathcal{E})$. The first relation holds because the group structure on $\pi_1 Q^h\mathcal{E}$ is determined by the H-space structure on $Q^h\mathcal{E}$ (as is the case for any H-group). The second relation holds because concatenation of loops also defines addition in $\pi_1 Q^h\mathcal{E}$. And the third relation holds because backtracking the path $(L^\perp/L, \bar{\varphi}) \rightarrow (X, \varphi)$ in $Q^h\mathcal{E}$ doesn't change the homotopy class of the loop. Therefore, the map in the following proposition is well-defined.

Proposition 4.9 *For any exact category with duality \mathcal{E} , the following map is an isomorphism*

$$GW_{form}(\mathcal{E}) \xrightarrow{\cong} \pi_1 |Q^h\mathcal{E}| : [X, \varphi, L_1, L_2] \mapsto [L_2]^{-1}[L_1].$$

Proof: A symmetric space X is called *stably metabolic* if there is a metabolic space M such that $X \perp M$ is metabolic. Write $Q_{met}^h\mathcal{E} \subset Q_{smet}^h\mathcal{E} \subset Q^h\mathcal{E}$ for the full subcategories of metabolic and stably metabolic spaces. The subcategory $Q_{smet}^h\mathcal{E}$ is the connected component of 0 in $Q^h\mathcal{E}$ (see remark 2.7). By strong cofinality (see lemma 4.15 below), $Q_{met}^h\mathcal{E} \subset Q_{smet}^h\mathcal{E}$ is a homotopy equivalence. Thus, it suffices to show that $GW_{form}(\mathcal{E}) \rightarrow \pi_1 Q_{met}^h(\mathcal{E}) : [X, \varphi, L_1, L_2] \mapsto [L_2]^{-1}[L_1]$ is an isomorphism.

Fix for every metabolic object (X, φ) a Lagrangian $L_X \hookrightarrow X$. Every loop in $Q_{met}^h(\mathcal{E})$ is homotopic to a loop of the form $a_n^\pm a_{n-1}^\pm \cdots a_2^\pm a_1^\pm : 0 \rightarrow 0$ with a_i maps in $Q_{met}^h(\mathcal{E})$. The loop is homotopic to $a_n^\pm [L_{X_{n-1}}][L_{X_{n-1}}]^{-1} a_{n-1}^\pm \cdots a_2^\pm [L_{X_1}][L_{X_1}]^{-1} a_1^\pm$, where X_i is the target of a_i^\pm and the source of a_{i+1}^\pm . This shows that the group $\pi_1 Q_{met}^h(\mathcal{E})$ is generated by loops of the form $[L_Y]^{-1} a [L_X]$ with $a : X \rightarrow Y$ a map in $Q_{met}^h(\mathcal{E})$. The composition $a \circ [L_X]$ defines a Lagrangian L in Y so that the generating loops are homotopic to loops of the form $[L_Y]^{-1}[L]$. It follows that $GW_{form}(\mathcal{E}) \rightarrow \pi_1 Q_{met}^h(\mathcal{E})$ is surjective.

To show the injectivity of the map, we consider the category $\underline{GW}_{form}(\mathcal{E})$ associated with the abelian group $GW_{form}(\mathcal{E})$, see 4.2. Let $Q_{met}^h(\mathcal{E}) \rightarrow \underline{GW}_{form}(\mathcal{E})$ be the functor defined as follows. Every object goes to the unique object of $\underline{GW}_{form}(\mathcal{E})$. A morphism $(X, \varphi) \rightarrow (Y, \phi)$ given by the datum $X \xleftarrow{p} U \xrightarrow{i} Y$ is sent to $[Y, \phi, L, L_Y]$ where $L \hookrightarrow Y$ is $p^{-1}(L_X)$, that is, L is the pull-back of $L_X \hookrightarrow X$ along p . Using the three relations defining GW_{form} in 4.3,

one checks that this is compatible with composition. In particular, we obtain a map $\pi_1 Q_{met}^h(\mathcal{E}) \rightarrow \pi_1 \underline{GW}_{form}(\mathcal{E}) = GW_{form}(\mathcal{E})$. Since the composition $GW_{form}(\mathcal{E}) \rightarrow \pi_1 Q_{met}^h(\mathcal{E}) \rightarrow GW_{form}(\mathcal{E})$ is the identity, and since the first map is surjective, we are done. \square

Finally, we will identify $\pi_0 GW(\mathcal{E})$ with the Grothendieck-Witt group of \mathcal{E} . For that, we need the following lemma.

Lemma 4.10 *The map $F : GW_{form}(\mathcal{E}) \rightarrow K_0(\mathcal{E}) : [X, \varphi, L_1, L_2] \mapsto [L_1] - [L_2]$ extends the exact sequence (5) to an exact sequence*

$$GW_{form}(\mathcal{E}) \xrightarrow{F} K_0(\mathcal{E}) \xrightarrow{H} GW_0(\mathcal{E}) \rightarrow W_0(\mathcal{E}) \rightarrow 0.$$

Proof: The only thing that needs to be checked is exactness at K_0 . Clearly, the composition $H \circ F$ is 0. Given $[X] - [Y] \in K_0(\mathcal{E})$ such that $[H(X)] = [H(Y)] \in GW_0(\mathcal{E})$, there are metabolic objects M, N with Lagrangians L_M, L_N , respectively, and an isometry $f : H(X) \oplus M \oplus H(L_N) \cong Z := H(Y) \oplus N \oplus H(L_M)$, see remark 2.7. The formation $[Z, f(X) \oplus f(L_M) \oplus f(L_N), Y \oplus L_N \oplus L_M]$ has image $[X] - [Y]$ in $K_0(\mathcal{E})$. \square

Proposition 4.11 *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Then there is a natural isomorphism*

$$GW_0(\mathcal{E}) \xrightarrow{\cong} \pi_0 GW(\mathcal{E}).$$

Proof: The hyperbolic functor $K_0(\mathcal{E}) \rightarrow GW_0(\mathcal{E}) : [X] \mapsto [HX]$ induces a functor $H : \underline{K}_0(\mathcal{E}) \rightarrow \underline{GW}_0(\mathcal{E})$ on associated categories, see 4.2. Every morphism in $\underline{GW}_0(\mathcal{E})$ is an isomorphism. Therefore, Quillen's theorem B trivially applies to the functor H , and we have a homotopy fibration

$$(e \downarrow H) \rightarrow \underline{K}_0(\mathcal{E}) \rightarrow \underline{GW}_0(\mathcal{E}). \quad (9)$$

The category $(e \downarrow H)$ has objects the elements x of $GW_0(\mathcal{E})$. A map from y to x in $(e \downarrow H)$ is an element $\xi \in K_0(\mathcal{E})$ such that $y + H(\xi) = x$. The functor $(e \downarrow H) \rightarrow \underline{K}_0(\mathcal{E})$ sends the map $\xi \in K_0(\mathcal{E})$ above to $\xi \in \underline{K}_0(\mathcal{E})$. By the long exact sequence associated with the homotopy fibration (9), we calculate the homotopy groups $\pi_0(e \downarrow H) = \text{coker } H = W_0(\mathcal{E})$, $\pi_1(e \downarrow H) = \text{ker } H$, and $\pi_i(e \downarrow H) = 0$ for $i \geq 2$.

We consider the map of homotopy fibrations

$$\begin{array}{ccccc} GW(\mathcal{E}) & \longrightarrow & Q^h \mathcal{E} & \longrightarrow & Q\mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ GW_0(\mathcal{E}) & \longrightarrow & (e \downarrow H) & \longrightarrow & \underline{K}_0(\mathcal{E}), \end{array}$$

where the left vertical map is induced by the commutativity of the right square in which the vertical functors are defined as follows. The functor $Q^h\mathcal{E} \rightarrow (e \downarrow H)$ sends (X, φ) to $[X, \varphi] \in GW_0(\mathcal{E})$, and a map $(Y, \psi) \rightarrow (X, \varphi)$ given by the datum $Y \xleftarrow{p} U \rightarrow X$ to $[\ker(p)] \in K_0(\mathcal{E})$. Similarly, the functor $Q\mathcal{E} \rightarrow \underline{K}_0(\mathcal{E})$ sends an object X to the unique object of $\underline{K}_0(\mathcal{E})$, and a map $Y \rightarrow X$ given by the datum $Y \xleftarrow{p} U \rightarrow X$ to $[\ker(p)] \in K_0(\mathcal{E})$. The lower left corner is the (realization of) the discrete category whose objects are the elements of $GW_0(\mathcal{E})$ and all morphisms are identity morphisms; the lower homotopy fibration being induced from (9).

Taking homotopy groups, we obtain a map of exact sequences

$$\begin{array}{ccccccccc} \pi_1 Q^h\mathcal{E} & \longrightarrow & \pi_1 Q\mathcal{E} & \longrightarrow & \pi_0 GW(\mathcal{E}) & \longrightarrow & \pi_0 Q^h\mathcal{E} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ker(H) & \longrightarrow & K_0(\mathcal{E}) & \xrightarrow{H} & GW_0(\mathcal{E}) & \longrightarrow & W_0(\mathcal{E}) & \longrightarrow & 0, \end{array}$$

in which the left vertical map is surjective, by lemma 4.10. The second and the fourth vertical maps are isomorphisms. By the five lemma, $\pi_0 GW(\mathcal{E}) \rightarrow GW_0(\mathcal{E})$ is an isomorphism. \square

Definition 4.12 The higher Grothendieck-Witt (or hermitian K -) groups of an exact category with duality $(\mathcal{E}, *, \eta)$ are the homotopy groups

$$GW_i(\mathcal{E}, *, \eta) = \pi_i GW(\mathcal{E}, *, \eta)$$

of the Grothendieck-Witt space $GW(\mathcal{E})$ of \mathcal{E} , $i \geq 0$. By proposition 4.11, this is compatible with the definition of $GW_0(\mathcal{E})$ given in 2.2.

Remark 4.13 For a ring with involution R , Karoubi defined in [Kar80a, p. 261] its hermitian K -theory space ${}_\varepsilon\mathcal{L}(R) \sim {}_\varepsilon\mathcal{L}_0(R) \times B_\varepsilon O(R)^+$ where ${}_\varepsilon\mathcal{L}_0(R)$ is the Grothendieck group of the abelian monoid of isometry classes of finitely generated projective R -modules equipped with a non-degenerate ε -quadratic form and where ${}_\varepsilon O(R)$ is the “infinite ε -orthogonal group of R ”. It follows from [Sch04b, Theorem 4.2] that the two spaces ${}_\varepsilon\mathcal{L}(R)$ and ${}_\varepsilon GW(R)$ are homotopy equivalent when $\frac{1}{2} \in R$ (another proof of which not relying on Karoubi’s fundamental theorem [Kar80a] will appear in [Sch08b]). Of course, the two spaces have different homotopy types, in general, since in the general case, symmetric and quadratic forms are not the same.

As alluded to above, the definition of the groups $GW_i(\mathcal{E})$ as the homotopy groups of the space $GW(\mathcal{E})$ (under the restriction “2 invertible”) already appeared in [Hor02, Definition 3.6] where they were called hermitian K -groups and denoted by $K_i^h(\mathcal{E})$. However, for general exact categories with duality, propositions 4.9 and 4.11 seem to be new, even in the case “2 invertible”.

Remark 4.14 In [Wal03], Walter defines a sequence $GW^i(\mathcal{T})$ of Grothendieck-Witt groups, $i \in \mathbb{Z}$, associated with a triangulated category with duality in which “2 is invertible”. Applied to the bounded derived category $D^b(\mathcal{E})$ of an exact category with duality $(\mathcal{E}, *, \eta)$, this yields a sequence $GW^i(\mathcal{E}, *, \eta)$, $i \in \mathbb{Z}$, of Grothendieck-Witt groups of \mathcal{E} in case \mathcal{E} is a $\mathbb{Z}[\frac{1}{2}]$ -linear category. His Grothendieck-Witt groups should not be confused with our higher Grothendieck-Witt groups!

Essentially by definition, Walter’s Grothendieck-Witt groups $GW^i(\mathcal{E}, *, \eta)$ are 4-periodic, $GW^i(\mathcal{E}) \cong GW^{i+4}(\mathcal{E})$, and it is shown in [Wal03], that they are the usual Grothendieck-Witt groups of symmetric spaces in $(\mathcal{E}, *, \eta)$, of formations in $(\mathcal{E}, *, -\eta)$, of symmetric spaces in $(\mathcal{E}, *, -\eta)$, and of formations in $(\mathcal{E}, *, \eta)$ for $i = 0, 1, 2, 3 \pmod{4}$. With our calculation of low-dimensional homotopy groups above, we obtain the following table for any $\mathbb{Z}[\frac{1}{2}]$ -linear exact category with duality $(\mathcal{E}, *, \eta)$:

$$\begin{aligned} GW^{4i}(\mathcal{E}, *, \eta) &= \pi_0 GW(\mathcal{E}, *, \eta) = GW_0(\mathcal{E}, *, \eta) \\ GW^{4i+1}(\mathcal{E}, *, \eta) &= \pi_1 Q^h(\mathcal{E}, *, -\eta) = GW_{form}(\mathcal{E}, *, -\eta) \\ GW^{4i+2}(\mathcal{E}, *, \eta) &= \pi_0 GW(\mathcal{E}, *, -\eta) = GW_0(\mathcal{E}, *, -\eta) \\ GW^{4i+3}(\mathcal{E}, *, \eta) &= \pi_1 Q^h(\mathcal{E}, *, \eta) = GW_{form}(\mathcal{E}, *, \eta). \end{aligned}$$

We conclude this section with the (well-known) “strong cofinality” lemma which was used in the proof of proposition 4.9.

Lemma 4.15 (Strong cofinality) *Let $(\mathcal{B}, \oplus, 0)$ be a unital symmetric monoidal category, and $\mathcal{A} \subset \mathcal{B}$ a full subcategory which contains the unit 0 and is closed under the monoidal operation \oplus . If the abelian monoids $\pi_0|\mathcal{A}|$ and $\pi_0|\mathcal{B}|$ are groups, and if for every object X of \mathcal{B} there is an object A of \mathcal{A} such that $X \oplus A$ is isomorphic to an object of \mathcal{A} , then the inclusion $\mathcal{A} \subset \mathcal{B}$ induces a homotopy equivalence on classifying spaces*

$$|\mathcal{A}| \xrightarrow{\sim} |\mathcal{B}|.$$

Proof: We can assume that an object in \mathcal{B} which is isomorphic to an object of \mathcal{A} is actually in \mathcal{A} , since equivalences of categories induce homotopy equivalences of classifying spaces. Let $\mathcal{C} \subset \mathcal{B}$ be a full subcategory with finitely many objects. By our assumption, there is an object A of \mathcal{A} such that $A \oplus \mathcal{C}$ is in \mathcal{A} . Since $\pi_0\mathcal{A}$ is a group, there is an object \bar{A} and a zigzag in \mathcal{A} of maps between $\bar{A} \oplus A$ and 0. Thus, the given inclusion $\mathcal{C} \subset \mathcal{B}$ is homotopic to the functor $\mathcal{C} \rightarrow \mathcal{A} : X \mapsto \bar{A} \oplus A \oplus X$ with image in \mathcal{A} . Moreover, for any subcategory $\mathcal{C}' \subset \mathcal{C}$ which lies in \mathcal{A} , the above homotopy, restricted to \mathcal{C}' , stays entirely in \mathcal{A} . Thus,

any map of pairs $(K, K') \rightarrow (|\mathcal{B}|, |\mathcal{A}|)$, with K, K' compact spaces, is homotopic (within such maps of pairs) to a map with image in $(|\mathcal{A}|, |\mathcal{A}|)$. This implies that the relative integral homology groups $H_*(|\mathcal{B}|, |\mathcal{A}|)$ are trivial, as every relative homology class is supported on a compact pair. In detail, the natural map from the filtered colimit $\operatorname{colim}_{(K, K')} H_*(K, K')$ to $H_*(|\mathcal{B}|, |\mathcal{A}|)$ is an isomorphism where $(K, K') \subset (|\mathcal{B}|, |\mathcal{A}|)$ ranges over the pairs of finite subcomplexes of $(|\mathcal{B}|, |\mathcal{A}|)$. Every map $H_*(K, K') \rightarrow H_*(|\mathcal{B}|, |\mathcal{A}|)$ factors through $H_*(|\mathcal{A}|, |\mathcal{A}|) = 0$, by the argument above, so that the natural map (an isomorphism) from the colimit to the relative homology group is in fact the zero map, hence $H_*(|\mathcal{B}|, |\mathcal{A}|) = 0$. Therefore, the inclusion $|\mathcal{A}| \subset |\mathcal{B}|$ induces an isomorphism on integral homology. The spaces $|\mathcal{A}|$ and $|\mathcal{B}|$ are homotopy commutative H -groups (hence nilpotent spaces), so that the homology isomorphism implies that $|\mathcal{A}| \subset |\mathcal{B}|$ is a homotopy equivalence. \square

5. Cofinality

5.1. Cofinal functors and idempotent completion

A fully exact inclusion $\mathcal{A} \subset \mathcal{B}$ of exact categories (see 2.1 for the definition of “fully exact”!) is called *cofinal* if for every object B of \mathcal{B} there is an object \bar{B} of \mathcal{B} such that $B \oplus \bar{B}$ is isomorphic to an object of \mathcal{A} . An exact category \mathcal{E} is *idempotent complete* if every idempotent $p = p^2 : A \rightarrow A$ defines a direct sum decomposition $A \cong \operatorname{im}(p) \oplus \operatorname{im}(1 - p)$ in \mathcal{E} under which the idempotent p corresponds to the map $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \operatorname{im}(p) \oplus \operatorname{im}(1 - p) \rightarrow \operatorname{im}(p) \oplus \operatorname{im}(1 - p)$. In an idempotent complete exact category, a map $A \rightarrow B$ for which there is a map $B \rightarrow C$ such that the composition $A \rightarrow C$ is an inflation is itself an inflation (this is an easy direct exercise, it also follows from [TT90, Theorem A.7.16 (b)]).

Let \mathcal{E} be an exact category, the *idempotent completion* $\tilde{\mathcal{E}}$ of \mathcal{E} has objects pairs (A, p) with $p = p^2 : A \rightarrow A$ an idempotent in \mathcal{E} . A map $(A, p) \rightarrow (B, q)$ is a map $f : A \rightarrow B$ in \mathcal{E} such that $f = fp = qf$. Composition in \mathcal{E} defines composition in $\tilde{\mathcal{E}}$. The functor $\mathcal{E} \rightarrow \tilde{\mathcal{E}} : A \mapsto (A, 1)$ makes \mathcal{E} into a full subcategory of $\tilde{\mathcal{E}}$. The idempotent completion of an exact category has a canonical structure of an exact category such that $\mathcal{E} \rightarrow \tilde{\mathcal{E}}$ is exact, closed under extensions, and reflects exactness (see for instance [TT90, Theorem A.9.1], where “idempotent complete” is called “Karoubian”). Of course, the idempotent completion is idempotent complete, and the inclusion $\mathcal{E} \subset \tilde{\mathcal{E}}$ is cofinal. If $(\mathcal{E}, *, \eta)$ is an exact category with duality, its idempotent completion $\tilde{\mathcal{E}}$ is an exact category with duality where the dual $(A, p)^*$ of an object (A, p) is (A^*, p^*) with double dual identification $\eta_{(A, p)} = \eta_A$. The inclusion $\mathcal{E} \subset \tilde{\mathcal{E}}$ is a (non-singular, exact) form functor with duality compatibility isomorphism the identity map.

5.2. The groups $GW_0(\mathcal{B}, \mathcal{A})$ and $K_0(\mathcal{B}, \mathcal{A})$

Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a duality preserving cofinal inclusion of exact categories with duality. Write $GW_0(\mathcal{B}, \mathcal{A})$ for the quotient monoid of isometry classes of symmetric spaces in \mathcal{B} modulo the submonoid of symmetric spaces in \mathcal{A} . The quotient monoid is actually a group. To see this, let (X, φ) be a symmetric space in \mathcal{B} , and choose an \bar{X} in \mathcal{B} with $X \oplus \bar{X}$ in \mathcal{A} . Then $X \oplus \bar{X}^* \cong (X \oplus \bar{X})^*$ is also in \mathcal{A} , so that $(X, \varphi) \perp (X, \varphi) \perp H(\bar{X})$ is a symmetric space in \mathcal{A} , and thus

$$-[X, \varphi] = [X, \varphi] + [H(\bar{X})]$$

in $GW_0(\mathcal{B}, \mathcal{A})$. The map $GW_0(\mathcal{B}) \rightarrow GW_0(\mathcal{B}, \mathcal{A}) : [X, \varphi] \mapsto [X, \varphi]$ is well-defined. This is because for a metabolic space (M, φ) in \mathcal{B} with Lagrangian L , we can choose an object \bar{L} of \mathcal{B} such that $L \oplus \bar{L}$ is in \mathcal{A} . Then we have an admissible exact sequence

$$L \oplus \bar{L} \twoheadrightarrow M \oplus H(\bar{L}) \twoheadrightarrow L^* \oplus \bar{L}^*$$

in \mathcal{B} with outer two terms in \mathcal{A} . Therefore, the middle term $M \oplus H(\bar{L})$ is in \mathcal{A} , and we have $[M, \varphi] = -[H\bar{L}] = [H(L \oplus \bar{L})] - [H\bar{L}] = [HL]$ in $GW_0(\mathcal{B}, \mathcal{A})$. This shows that the map $GW_0(\mathcal{B}) \rightarrow GW_0(\mathcal{B}, \mathcal{A})$ is well-defined and induces an isomorphism between $GW_0(\mathcal{B}, \mathcal{A})$ and the cokernel of the map $GW_0(\mathcal{A}) \rightarrow GW_0(\mathcal{B})$.

Similarly, let $K_0(\mathcal{B}, \mathcal{A})$ be the quotient of the monoid of isomorphism classes of objects in \mathcal{B} by the submonoid of isomorphism classes of objects in \mathcal{A} . As above, $K_0(\mathcal{B}, \mathcal{A})$ is actually a group, and $K_0(\mathcal{B}) \rightarrow K_0(\mathcal{B}, \mathcal{A}) : [X] \mapsto [X]$ is well-defined and induces an isomorphism between $K_0(\mathcal{B}, \mathcal{A})$ and the cokernel of the map $K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$.

5.3. The category $\mathcal{H}(\mathcal{B}, \mathcal{A})$

The hyperbolic functor induces a map

$$H : K_0(\mathcal{B}, \mathcal{A}) \rightarrow GW_0(\mathcal{B}, \mathcal{A}) : [X] \mapsto [HX]$$

of abelian groups which yields a functor $H : \underline{K}_0(\mathcal{B}, \mathcal{A}) \rightarrow \underline{GW}_0(\mathcal{B}, \mathcal{A})$ between associated categories. Let $\mathcal{H}(\mathcal{B}, \mathcal{A})$ be the comma category $(e \downarrow H)$ (a variant of which has already appeared in the proof of proposition 4.11), where e denotes the unique object of $\underline{GW}_0(\mathcal{B}, \mathcal{A})$. In detail, an object of $\mathcal{H}(\mathcal{B}, \mathcal{A})$ is an element $[X, \varphi] \in GW_0(\mathcal{B}, \mathcal{A})$. A map $[X, \varphi] \rightarrow [Y, \psi]$ in $\mathcal{H}(\mathcal{B}, \mathcal{A})$ is an element $[T] \in K_0(\mathcal{B}, \mathcal{A})$ such that $[Y, \psi] = [HT] + [X, \varphi]$ in $GW_0(\mathcal{B}, \mathcal{A})$. Composition is addition in $K_0(\mathcal{B}, \mathcal{A})$. Since every morphism in $\underline{GW}_0(\mathcal{B}, \mathcal{A})$ is an isomorphism, theorem B

of Quillen trivially applies and gives a homotopy fibration $\mathcal{H}(\mathcal{B}, \mathcal{A}) \rightarrow \underline{K}_0(\mathcal{B}, \mathcal{A}) \rightarrow \underline{GW}_0(\mathcal{B}, \mathcal{A})$ which implies the following calculation of homotopy groups

$$\pi_i \mathcal{H}(\mathcal{B}, \mathcal{A}) = \begin{cases} \text{coker}(H : K_0(\mathcal{B}, \mathcal{A}) \rightarrow GW_0(\mathcal{B}, \mathcal{A})), & i = 0 \\ \ker(H : K_0(\mathcal{B}, \mathcal{A}) \rightarrow GW_0(\mathcal{B}, \mathcal{A})), & i = 1 \\ 0, & i \geq 2. \end{cases}$$

There is a functor $F : Q^h \mathcal{B} \rightarrow \mathcal{H}(\mathcal{B}, \mathcal{A})$ which sends on object (X, φ) of $Q^h \mathcal{B}$ to $[X, \varphi]$, and a map $(X, \varphi) \rightarrow (Y, \psi)$ represented by the datum $X \xleftarrow{p} U \rightarrow Y$ to the map $[\ker(p)] : [X, \varphi] \rightarrow [Y, \psi]$ in $\mathcal{H}(\mathcal{B}, \mathcal{A})$.

For the next theorem, recall that a ‘‘cofinal functor’’ is a functor which (among other) is fully faithful and extension closed (see 5.1 and 2.1).

Theorem 5.1 (Cofinality for Q^h) *Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a duality preserving cofinal inclusion of exact categories with duality. Then the sequence of categories*

$$Q^h \mathcal{A} \longrightarrow Q^h \mathcal{B} \xrightarrow{F} \mathcal{H}(\mathcal{B}, \mathcal{A})$$

induces a homotopy fibration on classifying spaces.

Proof: Since the category $\mathcal{H}(\mathcal{B}, \mathcal{A})$ is a groupoid, Quillen’s theorem B trivially applies to the functor $F : Q^h \mathcal{B} \rightarrow \mathcal{H}(\mathcal{B}, \mathcal{A})$, so that we have a homotopy fibration

$$\mathcal{C} \rightarrow Q^h \mathcal{B} \rightarrow \mathcal{H}(\mathcal{B}, \mathcal{A}),$$

where \mathcal{C} is the comma category $(0 \downarrow F)$. The objects of \mathcal{C} are pairs $(X, \varphi, [S])$, where (X, φ) is a symmetric space in \mathcal{B} , and $[S] \in K_0(\mathcal{B}, \mathcal{A})$ such that $[X, \varphi] = [HS]$ in $GW_0(\mathcal{B}, \mathcal{A})$. A map $(X, \varphi, [S]) \rightarrow (Y, \psi, [T])$ in \mathcal{C} is a map $(X, \varphi) \rightarrow (Y, \psi)$ in $Q^h \mathcal{B}$ such that, if the map is represented by the datum $X \xleftarrow{p} U \rightarrow Y$, we have $[T] = [S] + [\ker(p)]$ in $K_0(\mathcal{B}, \mathcal{A})$. Composition is composition of maps in $Q^h \mathcal{B}$.

Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of those objects $(X, \varphi, [S])$ with $X \in \mathcal{A}$ and $[S] = 0$. Note that the functor $\mathcal{C} \rightarrow Q^h \mathcal{B}$ maps the subcategory \mathcal{C}' isomorphically onto the subcategory of $Q^h \mathcal{B}$ whose objects are the symmetric spaces in \mathcal{A} , and whose morphisms are represented by diagrams $A \xleftarrow{p} U \rightarrow B$ with $[\ker(p)] = 0$ in $K_0(\mathcal{B}, \mathcal{A})$. To prove the theorem, it suffices to show that the inclusions

$$Q^h \mathcal{A} \subset \mathcal{C}' \subset \mathcal{C}$$

are homotopy equivalences.

The inclusion $I : \mathcal{C}' \subset \mathcal{C}$ is a homotopy equivalence by the following argument. Let $\mathcal{X} = (X, \varphi, [S])$ be an object of \mathcal{C} , by Quillen’s theorem A it suffices to show that the comma category $(\mathcal{X} \downarrow I)$ is contractible. By definition of the category \mathcal{C} ,

we have $[X, \varphi] = [HS]$ in $GW_0(\mathcal{B}, \mathcal{A})$. Let T be an object in \mathcal{B} such that $S \oplus T$ is in \mathcal{A} . Then $(X, \varphi) \perp HT$ is zero in the quotient monoid $GW_0(\mathcal{B}, \mathcal{A})$ which implies the existence of a symmetric space (A, α) in \mathcal{A} such that $(X, \varphi) \perp HT \perp (A, \alpha)$ is a symmetric space in \mathcal{A} . The space $M = (A, \alpha) \perp (A, -\alpha)$ is metabolic in \mathcal{A} with Lagrangian A . The Lagrangians $L_0 = A \oplus T$ and $L = A \oplus S \oplus T$ in $N_0 = M \perp HT$ and $N = M \perp HS \perp HT$ define maps $l_0 : 0 \rightarrow N_0$ and $l : 0 \rightarrow N$ in $Q^h\mathcal{B}$ and $Q^h\mathcal{A}$, respectively, such that $a \circ l_0 = l$ in $Q^h\mathcal{B}$, where $a : N_0 \rightarrow N$ is the orthogonal sum of the identity on N_0 and the map $0 \rightarrow HS$ given by the Lagrangian S . Note that $(X, \varphi) \perp N_0$ is a symmetric space in \mathcal{A} . The identity functor on $(\mathcal{X} \downarrow I)$ is homotopic to the functor $\square \perp l : (\mathcal{X} \downarrow I) \rightarrow (\mathcal{X} \downarrow I)$ sending the object $b : \mathcal{X} \rightarrow (B, \beta, 0)$ to $b \perp l : \mathcal{X} \rightarrow ((B, \beta) \perp N, 0)$ via the homotopy given by the natural transformation $id \perp l : id \rightarrow \square \perp l$. The functor $\square \perp l$ is homotopic to the constant functor pt with image $id_X \perp l_0 : \mathcal{X} \rightarrow ((X, \varphi) \perp N_0, 0)$ via the homotopy given by the natural transformation $id \perp a : pt \rightarrow \square \perp l$. Therefore, the identity functor on $(\mathcal{X} \downarrow I)$ is homotopic to a constant map, so that the classifying space of $(\mathcal{X} \downarrow I)$ is contractible.

The inclusion $J : Q^h\mathcal{A} \subset \mathcal{C}'$ is a homotopy equivalence by Quillen's theorem A once we prove that for every object (A, α) of \mathcal{C}' the comma category $(A, \alpha \downarrow J)$ is contractible. Recall that $Q^h\mathcal{A}$ and \mathcal{C}' have the same objects, but \mathcal{C}' may have more maps. In particular, the categories $(A, \alpha \downarrow J)$ are non-empty. Let $\mathcal{C}'' \subset (A, \alpha \downarrow J)$ be a full subcategory with a non-empty, finite set of objects, and denote by G the inclusion functor. It suffices to show that G is null-homotopic as this implies that the homotopy groups of $(A, \alpha \downarrow J)$ are trivial. This is because every of its homotopy classes is supported on some \mathcal{C}'' , and every based map from the n -sphere S^n which is homotopic to the constant base point preserving map is also null-homotopic through base point preserving maps.

The finitely many objects $b_i : (A, \alpha) \rightarrow (B_i, \beta_i)$ in \mathcal{C}'' are represented by diagrams $A \xleftarrow{p_i} U_i \xrightarrow{\quad} B_i$ with $[\ker(p_i)] = 0$ in $K_0(\mathcal{B}, \mathcal{A})$. This implies that $\ker(p_i)$ is stably in \mathcal{A} so that there are objects A_i of \mathcal{A} with $\ker(p_i) \oplus A_i$ in \mathcal{A} . The Lagrangian $N = \bigoplus_i A_i$ defines a map $a : 0 \rightarrow HN$ in $Q^h\mathcal{A}$. The functor G is homotopic to the functor $\square \perp a : \mathcal{C}'' \rightarrow (A, \alpha \downarrow J) : b_i \mapsto b_i \perp a$ via the homotopy given by the natural transformation $id \perp a : G \rightarrow \square \perp a$. Since the maps $b_i \perp a$ are in $Q^h\mathcal{A}$, they define a natural transformation from the constant functor with image $id_A : (A, \alpha) \rightarrow (A, \alpha)$ to the functor $\square \perp a$. Hence, the inclusion G is null-homotopic. This finishes the proof of the theorem. \square

Corollary 5.2 (Cofinality for GW) *Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a duality preserving cofinal functor between exact categories with duality. Then the maps $GW_i(\mathcal{A}) \rightarrow GW_i(\mathcal{B})$ are isomorphisms for $i \geq 1$ and a monomorphism for $i = 0$.*

Proof: This follows by taking vertical homotopy fibres in the diagram

$$\begin{array}{ccccc} Q^h \mathcal{A} & \longrightarrow & Q^h \mathcal{B} & \longrightarrow & \mathcal{H}(\mathcal{B}, \mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ Q \mathcal{A} & \longrightarrow & Q \mathcal{B} & \longrightarrow & \underline{K}_0(\mathcal{B}, \mathcal{A}) \end{array}$$

of categories and functors in which the rows are homotopy fibrations on classifying spaces by theorem 5.1 and cofinality for K -theory (see [Gra79, theorem 1.1], or theorem 5.1 applied to $H\mathcal{A} \subset H\mathcal{B}$). \square

6. Dévissage

The following theorem generalizes [QSS79, Theorem 6.7] and [Qui73, Dévissage].

Theorem 6.1 (Dévissage) *Let $\mathcal{A} \subset \mathcal{B}$ be a duality preserving full inclusion of abelian categories with duality. Assume that \mathcal{A} is closed under taking subobjects and quotients in \mathcal{B} . Assume furthermore that*

- (a) *every object X in \mathcal{B} has a finite filtration $0 = X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X$ such that the successive quotients X_{i+1}/X_i are in \mathcal{A} for $i = 0, \dots, n-1$, and that*
- (b) *for every symmetric space (X, φ) in \mathcal{B} there is a totally isotropic subspace $N \subset X$ such that N^\perp/N is in \mathcal{A} .*

Then the inclusion $\mathcal{A} \rightarrow \mathcal{B}$ induces a homotopy equivalence

$$Q^h \mathcal{A} \xrightarrow{\sim} Q^h \mathcal{B}.$$

Proof: Let (X, φ) be a symmetric space in \mathcal{B} . The filtration in (a) and the totally isotropic subspace in (b) induce a finite filtration

$$0 \subset X_1 \cap N \subset \dots \subset X_{n-1} \cap N \subset N \subset N^\perp \subset (X_{n-1} \cap N)^\perp \subset \dots \subset (X_1 \cap N)^\perp \subset X$$

with successive quotients in \mathcal{A} . We write F for the inclusion $Q^h \mathcal{A} \rightarrow Q^h \mathcal{B}$, and note that F is fully faithful. By Quillen's theorem A it suffices to show that $(F \downarrow X)$ is non-empty and contractible. The category $(F \downarrow X)$ is equivalent to the category whose objects are totally isotropic subspaces $N \subset X$ such that $N^\perp/N \in \mathcal{A}$. Morphisms are inclusions of totally isotropic subspaces.

By condition (b), the category $(F \downarrow X)$ is non-empty. The filtration above with successive quotients in \mathcal{A} yields a string of maps

$$N^\perp/N \rightarrow \dots \rightarrow (X_1 \cap N)^\perp/(X_1 \cap N) \rightarrow X \tag{10}$$

in $Q^h(\mathcal{B})$. We will show that base-change w.r.t. these maps induces weak equivalences on comma categories $(F \downarrow \quad)$. Since $(F \downarrow N^\perp/N)$ has a final object, namely $N^\perp/N \in \mathcal{A}$, this will prove the claim.

Let (Y, ψ) be a symmetric space in \mathcal{B} . Given a totally isotropic subspace $L \subset Y$ with $L \in \mathcal{A}$ (hence $Y/L^\perp \cong L^* \in \mathcal{A}$), we have to show that the map $L^\perp/L \rightarrow Y$ in $Q^h\mathcal{B}$ induces a homotopy equivalence $G : (F \downarrow L^\perp/L) \rightarrow (F \downarrow Y)$, since all maps in (10) are of the form $L^\perp/L \rightarrow Y$. Write p for the quotient map $L^\perp \rightarrow L^\perp/L$. Then the functor G sends the totally isotropic subspace $M \subset L^\perp/L$ to the totally isotropic subspace $p^{-1}(M) \subset Y$. There is a functor $H : (F \downarrow Y) \rightarrow (F \downarrow L^\perp/L)$ which sends a totally isotropic subspace $N \subset Y$ with $N^\perp/N \in \mathcal{A}$ to the totally isotropic subspace $M = (N \cap L^\perp)/(N \cap L) \subset L^\perp/L$. The orthogonal of M in L^\perp/L is $M^\perp = (N^\perp \cap L^\perp)/(N^\perp \cap L)$ so that the quotient M^\perp/M is itself a quotient of the object $(N^\perp \cap L^\perp)/(N \cap L^\perp) \subset N^\perp/N$ of \mathcal{A} , hence $M^\perp/M \in \mathcal{A}$. We have $HG = id$, and we have a well-defined zigzag of functors and natural transformations

$$GH(N) = p^{-1} \left(\frac{N \cap L^\perp}{N \cap L} \right) \leftrightarrow N \cap L^\perp \hookrightarrow N$$

between GH and id since the object $(N \cap L^\perp)^\perp/(N \cap L^\perp) = (N^\perp \cup L)/(N \cap L^\perp)$ is in \mathcal{A} . This is because it is a quotient of $(N^\perp/(N \cap L^\perp)) \oplus L$ which itself is a subobject of the object $(Y/L^\perp) \oplus (N^\perp/N) \oplus L$ in \mathcal{A} . We conclude that G and H induce inverse homotopy equivalences. \square

For the following example, recall (e.g. from the definitions in [Pop73, §5.7, p. 366, p. 370]) that an abelian category \mathcal{B} is noetherian if and only if its dual \mathcal{B}^{op} is artinian, so that if \mathcal{B} is a noetherian abelian category with duality, it is also artinian, as $*$: $\mathcal{B} \cong \mathcal{B}^{op}$ is an equivalence of exact categories.

Example 6.2 The dévissage theorem applies when \mathcal{B} is a noetherian (hence artinian) abelian category with duality, and $\mathcal{A} \subset \mathcal{B}$ its full subcategory of semi-simple objects. Note that in this case \mathcal{A} is a split exact category with duality.

We obtain a more explicit description of the higher Grothendieck-Witt groups of \mathcal{B} as follows. For a simple object A of \mathcal{B} , either A is isomorphic to A^* or it is not. In the first case, let $\alpha : A \rightarrow A^*$ be an isomorphism. Then $\alpha + \alpha^*\eta$ is either zero in which case $\alpha = -\alpha^*\eta$, or it is an isomorphism, in which case we can replace α with $\alpha + \alpha^*\eta$. In either case, we can assume $\alpha = \varepsilon_A \alpha^*\eta$, where $\varepsilon_A \in \{-1, 1\}$, so that the map $\text{End}(A) \rightarrow \text{End}(A)^{op} : a \mapsto \bar{a} = \alpha^{-1} \circ a^* \circ \alpha$ defines an involution on the ring $\text{End}(A)$. We have a form functor

$$(\text{End}(A), \bar{\quad}, \varepsilon_A) \rightarrow (\mathcal{A}, *, \eta) : \text{End}(A) \mapsto A, a \mapsto a$$

with duality compatibility morphism $\alpha : A \rightarrow A^*$ which extends to an equivalence of split exact categories with duality between $(\mathcal{P}\text{End}(A), *, \varepsilon_A \text{ can})$ (as defined in example 2.2) and the semi-simple subcategory with duality of \mathcal{A} generated by A .

In the second case, when $A \not\cong A^*$, the semi-simple subcategory with duality of \mathcal{A} generated by A and A^* is equivalent to the hyperbolic category $H\mathcal{P}\text{End}(A)$ associated with the category $\mathcal{P}\text{End}(A)$. By dévissage, we therefore obtain isomorphisms of higher Grothendieck-Witt groups

$$GW_i(\mathcal{B}) \cong GW_i(\mathcal{A}) \cong \bigoplus_{\varepsilon_A} GW_i(\text{End}(A)) \oplus \bigoplus K_i(\text{End}(A))$$

where the first sum is taken over the isomorphism classes $[A]$ of simple objects A of \mathcal{B} such that $A \cong A^*$, and the second sum is taken over the set

$$\{\{[A], [A^*]\} \mid A \not\cong A^*, A \text{ simple object of } \mathcal{B}\}.$$

7. Additivity

The aim of this section is to prove the following theorem and its (equivalent) version in theorem 7.2 below. Recall from remark 3.3 that a non-singular exact form functor $(F, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ between two exact categories with duality \mathcal{A} and \mathcal{B} is the same as a symmetric space $(F, \hat{\varphi})$ in the exact category with duality $\text{ExFun}(\mathcal{A}, \mathcal{B})$ of exact functors $\mathcal{A} \rightarrow \mathcal{B}$. With this in mind, the meaning of notions such as “totally isotropic subfunctor” $G \succrightarrow F$, “orthogonal G^\perp of a functor” $G \subset F$, “induced form” $(G^\perp/G, \bar{\varphi})$, see lemma 2.6, and “hyperbolic functor HG associated with G ” are to be interpreted as taking place in $\text{ExFun}(\mathcal{A}, \mathcal{B})$.

Theorem 7.1 (Additivity for functors) *Let $(F, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ be a non-singular exact form functor between exact categories with duality \mathcal{A} and \mathcal{B} . Let $G \succrightarrow F$ be a totally isotropic subfunctor. Then the form functors (F, φ) and $(G^\perp/G, \bar{\varphi}) \perp H(G)$ induce homotopic maps $Q^h\mathcal{A} \rightarrow Q^h\mathcal{B}$ on hermitian Q -constructions and homotopic maps $GW(\mathcal{A}) \rightarrow GW(\mathcal{B})$ on Grothendieck-Witt spaces.*

The proof is delayed until after the proof of theorem 7.2 below. To formulate theorem 7.2, we need the notion of “admissible short complexes”.

7.1. Short complexes

In this article, a *short complex* in an exact category \mathcal{E} is a complex

$$A_\bullet : \quad 0 \rightarrow A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \rightarrow 0, \quad (d_0 \circ d_1 = 0) \quad (11)$$

in \mathcal{E} which is concentrated in degrees $-1, 0, 1$. In displaying short complexes we may sometimes omit the two outer zero objects. We call the short complex *admissible*

if d_1 and d_0 are inflation and deflation, respectively, and the maps $A_1 \rightarrow \ker(d_0)$ and $\text{coker}(d_1) \rightarrow A_{-1}$ are inflation and deflation, respectively. “Admissibility” can be checked as follows. Let $\mathcal{E} \subset \mathcal{A}$ be a fully exact embedding of \mathcal{E} into an abelian category [TT90, Appendix A], [Kel90, Appendix A]. Then a short complex A_\bullet in \mathcal{E} is admissible if and only if the homology $H_*(A_\bullet)$ computed in the ambient abelian category \mathcal{A} satisfies $H_i(A_\bullet) = 0$, $i \neq 0$, and $H_0(A_\bullet)$ is isomorphic to an object in \mathcal{A} (exercise!).

The homology $H_0(A_\bullet)$ of an admissible short complex is defined independently of an embedding $\mathcal{E} \subset \mathcal{A}$ into an abelian category. It can be computed for instance as $H_0(A_\bullet) = \text{im}(\varepsilon_{A_\bullet})$, where the map $\varepsilon_{A_\bullet} : \ker d_0 \rightarrow \text{coker} d_1$ is the composition of the canonical maps $\ker d_0 \twoheadrightarrow A_0 \twoheadrightarrow \text{coker} d_1$. Here (and elsewhere in the paper), we only use the notation $\text{im}(f)$ for a map $f : A \rightarrow B$ in an exact category \mathcal{E} which admits a factorization $A \twoheadrightarrow \text{im}(f) \twoheadrightarrow B$ into a deflation followed by an inflation. Any two choices of such a factorization are canonically isomorphic.

A map $f : A_\bullet \rightarrow B_\bullet$ of short complexes is a map of chain complexes, that is, a sequence of maps $f_i : A_i \rightarrow B_i$ commuting with differentials. We call a sequence $A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet$ of admissible short complexes *exact* if $A_i \rightarrow B_i \rightarrow C_i$ is exact in \mathcal{E} , $i = -1, 0, 1$. The exact category of admissible short complexes in \mathcal{E} is denoted by $s\text{Cx}(\mathcal{E})$.

Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Duality and double dual identification of \mathcal{E} induce a duality and double dual identification on the category of (admissible) short complexes, where the dual of the complex (11) is the (admissible) short complex

$$A_{-1}^* \xrightarrow{d_0^*} A_0^* \xrightarrow{d_1^*} A_1^*,$$

and the double dual identification $\eta_{A_\bullet} : A_\bullet \rightarrow (A_\bullet)^{**}$ is η_{A_i} in degree $i = -1, 0, 1$. To give a symmetric space (A_\bullet, φ) in $s\text{Cx}(\mathcal{E})$ is the same (up to isometry) as to give a symmetric space (A_0, φ_0) in \mathcal{E} together with a totally isotropic subspace $i : A_1 \twoheadrightarrow A_0$. The associated admissible short complex is

$$A_1 \xrightarrow{i} A_0 \xrightarrow{i^* \varphi_0} A_1^* \tag{12}$$

equipped with the form $(\eta, \varphi_0, 1)$. The admissible short complex (12) with its form $(\eta, \varphi_0, 1)$ is called symmetric space *in standard form*. Every symmetric space in $s\text{Cx}(\mathcal{E})$ is isometric to a symmetric space in standard form.

7.2. Homology of short complexes

The (exact) degree-zero homology functor

$$H_0 : s\text{Cx}(\mathcal{E}) \rightarrow \mathcal{E}$$

is a (non-singular exact) form functor with duality compatibility isomorphism $H_0^* \rightarrow {}^*H_0$ the unique map making the diagram

$$\begin{array}{ccccc} \ker d_1^* & \longrightarrow & H_0^* & \longrightarrow & \operatorname{coker} d_0^* \\ \downarrow & & \downarrow & & \downarrow \\ (\operatorname{coker} d_1)^* & \longrightarrow & {}^*H_0 & \longrightarrow & (\ker d_0)^* \end{array}$$

commute, where $\ker d_1^* \rightarrow (\operatorname{coker} d_1)^*$ and $\operatorname{coker} d_0^* \rightarrow (\ker d_0)^*$ are the canonical isomorphisms induced by the exactness of the duality functor, and the compositions of the horizontal maps are $\varepsilon_{(A_\bullet)^*}$ and $\varepsilon_{A_\bullet}^*$. In particular, a symmetric space (A_\bullet, φ) in $s\text{Cx}(\mathcal{E})$ defines a symmetric space $H_0(A_\bullet, \varphi)$ in \mathcal{E} . This space is canonically isometric to $(A_1^\perp/A_1, \bar{\varphi}_0)$ in the notation of lemma 2.6. More generally, the homology form functor H_0 sends a symmetric quasi-isomorphism $\varphi : A_\bullet \rightarrow (A_\bullet)^*$ in $s\text{Cx}(\mathcal{E})$, to a symmetric space $H_0(A_\bullet, \varphi)$ in \mathcal{E} .

Theorem 7.2 (Additivity for admissible short complexes) *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Then the exact functor $ev_1 : s\text{Cx}(\mathcal{E}) \rightarrow \mathcal{E} : A_\bullet \mapsto A_1$ and the non-singular exact form functor $H_0 : s\text{Cx}(\mathcal{E}) \rightarrow \mathcal{E}$ induce a homotopy equivalence*

$$(H_0, ev_1) : Q^h s\text{Cx}(\mathcal{E}) \xrightarrow{\sim} Q^h \mathcal{E} \times Q\mathcal{E}.$$

Remark 7.3 One version of Quillen’s additivity theorem [Qui73, §3 Theorem 2] uses the category of exact sequences in an exact category \mathcal{E} . An analogous statement also exists in hermitian K -theory, see exercise 7.8 and remark 7.9. In [Wal85], Waldhausen considers generalizations, denoted by $S_n\mathcal{E}$, of the category of exact sequences, which itself is denoted by $S_2\mathcal{E}$ in this context. The category of admissible short complexes $s\text{Cx}(\mathcal{E})$ is equivalent, as an exact category with duality, to Waldhausen’s category $S_3\mathcal{E}$ but it carries less (for our purpose) redundant information than $S_3\mathcal{E}$.

The proof of theorem 7.2 uses “admissible short bicomplexes”. Here are their definition and relevant properties.

7.3. Short bicomplexes

A short bicomplex in \mathcal{E} is a bicomplex in \mathcal{E} which is concentrated in bidegrees $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0, -1)$. Leaving out all the zeros, we can display a short

bicomplex as

$$\begin{array}{ccccc}
 X_{\bullet\bullet} : & & X_{0,1} & & \\
 & & \downarrow m & & \\
 X_{-1,0} & \xrightarrow{n} & X_{0,0} & \xrightarrow{n'} & X_{1,0} \\
 & & \downarrow m' & & \\
 & & X_{0,-1} & &
 \end{array}$$

with $m'm = n'm = m'n = n'n = 0$.

If row and column of the bicomplex are admissible short complexes, then horizontal and vertical homologies $H_{i,j}^h$ and $H_{i,j}^v$ are defined. In this case, the following assertions are equivalent.

- (a) $H_{*,0}^v(X_{\bullet\bullet})$ is an admissible short complex,
- (b) $H_{0,*}^h(X_{\bullet\bullet})$ is an admissible short complex,
- (c) the total complex $\text{Tot}(X_{\bullet\bullet})$ is an admissible short complex.

This is because there is a zigzag $H_0^v X_{\bullet\bullet} \leftarrow E_{\bullet}^X \rightarrow \text{Tot}(X_{\bullet\bullet})$ of quasi-isomorphisms of short complexes as in the diagram

$$\begin{array}{ccccc}
 H_0^v X_{\bullet\bullet} : & X_{-1,0} & \longrightarrow & H_{0,0}^v X_{\bullet\bullet} & \longrightarrow & X_{1,0} & (13) \\
 \uparrow & (1 \ 0) \uparrow & & \uparrow & & \uparrow 1 & \\
 E_{\bullet}^X : & X_{-1,0} \oplus X_{0,1} & \longrightarrow & \ker(m') & \longrightarrow & X_{1,0} & \\
 \downarrow & \downarrow 1 & & \downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \\
 \text{Tot}(X_{\bullet\bullet}) : & X_{-1,0} \oplus X_{0,1} & \xrightarrow{(n \ m)} & X_{0,0} & \xrightarrow{\begin{pmatrix} n' \\ m' \end{pmatrix}} & X_{1,0} \oplus X_{0,-1} &
 \end{array}$$

in which upper left and lower right squares are bicartesian. Then $H_0^v X_{\bullet\bullet}$ is admissible if and only if E_{\bullet}^X is admissible if and only if $\text{Tot}(X_{\bullet\bullet})$ is admissible since all three complexes have isomorphic homology computed in some ambient abelian category. For $H_0^h X_{\bullet\bullet}$ in place of $H_0^v X_{\bullet\bullet}$, the argument is similar. If $X_{\bullet\bullet}$ satisfies either of the three equivalent conditions (a) - (c), we call the short bicomplex $X_{\bullet\bullet}$ *admissible*.

An admissible short bicomplex defines an object in the exact category with duality $s\text{Cx}s\text{Cx}(\mathcal{E})$ of admissible short complexes in $s\text{Cx}(\mathcal{E})$. If $X_{\bullet\bullet}$ is equipped with a non-singular symmetric form φ , then $H_0^v(X_{\bullet\bullet}, \varphi)$, $H_0^h(X_{\bullet\bullet}, \varphi)$, and $\text{Tot}(X_{\bullet\bullet}, \varphi)$ are

symmetric spaces in $s\text{Cx}(\mathcal{E})$. In this case, the zigzag of quasi-isomorphisms (13) induces isometries

$$H_0 H_0^v(X_{\bullet\bullet}, \varphi) \xleftarrow{\cong} H_0(E_{\bullet}^X, \varphi|_{E_{\bullet}^X}) \xrightarrow{\cong} H_0(\text{Tot}(X_{\bullet\bullet}), \varphi)$$

of symmetric spaces in \mathcal{E} because the forms on $H_0^v X_{\bullet\bullet}$ and on $\text{Tot}(X_{\bullet\bullet})$ restrict to the same form on E_{\bullet}^X . Agreement of the two forms restricted to E_{\bullet}^X is clear in degree 0, by definition of $H_{0,0}^v$, and it also holds in degrees $-1, 1$ since a form on an admissible short complex is determined by its value in degree 0. Summarizing, if $(X_{\bullet\bullet}, \varphi)$ is an admissible short bicomplex equipped with a (non-singular) symmetric form, then we have canonical isometries

$$H_0 H_0^v(X_{\bullet\bullet}, \varphi) \cong H_0 \text{Tot}(X_{\bullet\bullet}, \varphi) \cong H_0 H_0^h(X_{\bullet\bullet}, \varphi).$$

Proof of theorem 7.2

Write F for the functor $(H_0, ev_1) : Q^h s\text{Cx}(\mathcal{E}) \rightarrow Q^h \mathcal{E} \times Q\mathcal{E}$. By Quillen's theorem A, it suffices to show that for any pair of objects $(R, \rho; E)$ with $(R, \rho) \in Q^h \mathcal{E}$ and $E \in Q\mathcal{E}$, the comma category $\mathcal{C} = (F \downarrow R, \rho; E)$ is (non-empty and) contractible.

An object of \mathcal{C} is a triple (X, r_X, e_X) where X is a symmetric space $X = (X_{\bullet}, \varphi_X)$ in $s\text{Cx}(\mathcal{E})$, which we can assume to be in standard form

$$X_1 \xrightarrow{i_X} X_0 \xrightarrow{i_X^* \circ \varphi_{X_0}} X_1^*, \quad \varphi_X = (\eta, \varphi_{X_0}, 1), \quad (14)$$

together with maps $r_X : H_0(X_{\bullet}, \varphi_X) \rightarrow (R, \rho)$ and $e_X : X_1 \rightarrow E$ in $Q^h \mathcal{E}$ and $Q\mathcal{E}$, respectively.

Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of those objects (X, r_X, e_X) for which, in the notation of 4.1, the map e_X is $e_X = (E, 1, q_X)$ for a deflation q_X of \mathcal{E} . Further, let $\mathcal{C}'' \subset \mathcal{C}'$ be the full subcategory of those objects (X, r_X, e_X) for which $X_1 = 0$. We will show that the inclusions $\mathcal{C}'' \subset \mathcal{C}' \subset \mathcal{C}$ are homotopy equivalences, and that \mathcal{C}'' is contractible. This proves that \mathcal{C} is contractible, and thus that F is a homotopy equivalence.

Lemma 7.4 *The inclusion $\mathcal{C}' \subset \mathcal{C}$ is a homotopy equivalence.*

Proof: Denote $J : \mathcal{C}' \subset \mathcal{C}$ the inclusion. We will construct a functor $G : \mathcal{C} \rightarrow \mathcal{C}'$, a natural isomorphism $GJ \cong id_{\mathcal{C}'}$ and a natural transformation $1_{\mathcal{C}} \rightarrow JG$. This implies that the inclusion $J : \mathcal{C}' \subset \mathcal{C}$ is a homotopy equivalence (with inverse G).

Let (X, r_X, e_X) be an object of \mathcal{C} . The map e_X can be written as the composition $e_X = (E, 1, q_X) \circ (X_1, j_X, 1)$ where $j_X : X_1 \rightarrow X_2$ and $q_X : E \rightarrow X_2$ are inflation and deflation in \mathcal{E} , respectively. The object X_2 , together with the maps j_X and q_X

in \mathcal{E} , is uniquely determined, up to canonical isomorphism, by the map e_X in $Q\mathcal{E}$. Consider the following short bicomplex $B_{\bullet\bullet}^X$ in \mathcal{E}

$$B_{\bullet\bullet}^X : \begin{array}{ccccc} & & X_1 & & \\ & & \downarrow m & & \\ X_2 & \xrightarrow{n} & X_0 \oplus HX_2 & \xrightarrow{n^*\psi} & X_2^* \\ & & \downarrow m^*\psi & & \\ & & X_1^* & & \end{array}$$

where $m = \begin{pmatrix} i_X \\ j_X \\ 0 \end{pmatrix}$, $n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and where ψ is the form on $(X_0, \varphi_{X_0}) \perp HX_2$. The bicomplex is admissible because $H_0^h(B_{\bullet\bullet}^X)$ is an admissible short complex. Together with the form ψ_X which is of standard form in vertical and horizontal direction and which is ψ in bidegree $(0,0)$, the bicomplex $B_{\bullet\bullet}^X$ defines a symmetric space $(B_{\bullet\bullet}^X, \psi_X)$ in $sCxsCx(\mathcal{E})$. It follows that $H_0^h H_0^v(B_{\bullet\bullet}^X, \psi_X) = H_0^v H_0^h(B_{\bullet\bullet}^X, \psi_X) = H_0(X_\bullet, \varphi_X)$. The functor G we wish to define sends the object (X, r_X, e_X) of \mathcal{C} to the object (GX, r_X, q_X^1) of \mathcal{C}' , where the symmetric space GX in $sCx(\mathcal{E})$ is $H_0^v(B_{\bullet\bullet}^X, \psi_X)$, and where $q_X^1 = (E, 1, q_X)$.

A map in \mathcal{C} from (X, r_X, e_X) to (Y, r_Y, e_Y) is a map $f : (X_\bullet, \varphi_X) \rightarrow (Y_\bullet, \varphi_Y)$ in $Q^h sCx(\mathcal{E})$ such that $r_Y \circ H_0(f) = r_X$, and $e_Y \circ f_1 = e_X$. The map f in $Q^h sCx(\mathcal{E})$ is given by a datum

$$X_\bullet \xleftarrow{p} U_\bullet \xrightarrow{i} Y_\bullet$$

such that the square of maps $p, i, p^*\varphi_X, i^*\varphi_Y$ is bicartesian. The equality $e_Y \circ f_1 = e_X$ means that there are maps $j : U_{-1}^* \rightarrow X_2$ and $q : Y_2 \rightarrow X_2$ in \mathcal{E} such that the square of maps $j_Y, q, i_{-1}^* \eta_{Y_1}, j$ is bicartesian, and such that $q_X = q \circ q_Y$ and $j \circ p_{-1}^* \circ \eta_{X_1} = j_X$. The maps j and q are uniquely determined by these properties. The image $G(f)$ of f under the functor $G : \mathcal{C} \rightarrow \mathcal{C}'$ we wish to define is the map $H_0^v(\tilde{f})$ obtained by taking vertical homology of the map \tilde{f} in $Q^h sCxsCx(\mathcal{E})$ given by the datum

$$B_{\bullet\bullet}^X \xleftarrow{v} \bar{B}_{\bullet\bullet}^U \xrightarrow{u} B_{\bullet\bullet}^Y$$

with $\bar{B}_{\bullet\bullet}^U$ the short bicomplex

$$\bar{B}_{\bullet\bullet}^U : \begin{array}{ccc} & & U_1 \\ & & \downarrow m \\ Y_2 & \xrightarrow{n} & U_0 \oplus Y_2 \oplus X_2^* \xrightarrow{n'} X_2^* \\ & & \downarrow m' \\ & & U_{-1} \end{array}$$

where $m = \begin{pmatrix} d_1^U \\ j_Y i_1 \\ 0 \end{pmatrix}$, $n = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $m' = (d_0^U \ 0 \ \eta^{-1} j^*)$, and $n' = (0 \ 0 \ 1)$. The bicomplex is admissible since its horizontal homology is an admissible short complex. In bidegrees $(-1,0)$, $(0,0)$, $(1,0)$, $(0,1)$, $(0,-1)$, the maps $v : \bar{B}_{\bullet\bullet}^U \rightarrow B_{\bullet\bullet}^X$ and $u : \bar{B}_{\bullet\bullet}^U \rightarrow B_{\bullet\bullet}^Y$ are the maps q , $p_0 \oplus q \oplus 1$, 1 , p_1 , p_{-1} , and 1 , $i_0 \oplus 1 \oplus q^*$, q^* , i_1 , i_{-1} , respectively. To see that the datum (u, v) does indeed define a map $\tilde{f} : (B_{\bullet\bullet}^X, \psi_X) \rightarrow (B_{\bullet\bullet}^Y, \psi_Y)$ in $Q^h sCxsCx(\mathcal{E})$ we have to verify that $\psi_X|_{\bar{B}_{\bullet\bullet}^U} = \psi_Y|_{\bar{B}_{\bullet\bullet}^U}$ and that the square of maps u , v , $u^* \psi_Y$, $v^* \psi_X$ is bicartesian in $sCxsCx(\mathcal{E})$. Also, to see that $G : \mathcal{C} \rightarrow \mathcal{C}'$ defines a functor, we have to verify $G(fg) = G(f) \circ G(g)$. These verifications are checked for the horizontal and vertical parts of the bicomplexes separately. We omit the details.

In order to define the natural transformation $1_{\mathcal{C}} \rightarrow JG$, we will first construct a non-singular exact form functor $sCxsCx(\mathcal{E}) \rightarrow sCxsCx(\mathcal{E}) : X_{\bullet} \mapsto A_{\bullet\bullet}^X$ together with a natural isometry of form functors $X_{\bullet} \rightarrow H_0^v A_{\bullet\bullet}^X$. The natural transformation $1_{\mathcal{C}} \rightarrow JG$ will then be H_0^v of a certain natural transformation of bicomplexes.

The functor $A_{\bullet\bullet}$ sends an admissible short complex $X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1}$ to the bicomplex

$$A_{\bullet\bullet}^X : \begin{array}{ccc} & & X_1 \\ & & \downarrow m \\ X_1 & \xrightarrow{n} & X_0 \oplus X_1 \oplus X_{-1} \xrightarrow{n'} X_{-1} \\ & & \downarrow m' \\ & & X_{-1} \end{array}$$

where $m = \begin{pmatrix} d_1 \\ 1 \\ 0 \end{pmatrix}$, $n = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $m' = (d_0 \ 0 \ 1)$, and $n' = (0 \ 0 \ 1)$. The duality compatibility isomorphism $A_{\bullet\bullet}^{X^*} \rightarrow (A_{\bullet\bullet}^X)^*$ is the identity in bidegrees $(-1,0)$, $(1,0)$, $(0,-1)$, $(0,1)$, and in bidegree $(0,0)$ it is the map $X_0^* \oplus X_{-1}^* \oplus X_1^* \rightarrow X_0^* \oplus X_1^* \oplus X_{-1}^*$ which switches the two factors X_1^* and X_{-1}^* .

The natural isometry of form functors $X_\bullet \rightarrow H_0^v A_{\bullet\bullet}^X$ is induced by a zig-zag $X_{\bullet\bullet} \leftarrow C_{\bullet\bullet}^X \rightarrow A_{\bullet\bullet}^X$ of vertical quasi-isomorphisms (that is, H_*^v -isomorphisms) of form functors $sCx(\mathcal{E}) \rightarrow sCx_sCx(\mathcal{E})$, which yields natural isometries of form functors $X_\bullet = H_0^v(X_{\bullet\bullet}) \cong H_0^v(C_{\bullet\bullet}^X) \cong H_0^v(A_{\bullet\bullet}^X)$. Here $X_{\bullet\bullet}$ and $C_{\bullet\bullet}^X$ are the bicomplexes

$$\begin{array}{ccc}
 X_{\bullet\bullet} : & & C_{\bullet\bullet}^X : \\
 & \begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 X_1 & \xrightarrow{d_1} & X_0 & \xrightarrow{d_0} & X_{-1} \\
 & & \downarrow & & \\
 & & 0 & &
 \end{array} & & \begin{array}{ccccc}
 & & X_1 & & \\
 & & \downarrow \begin{pmatrix} d_1 \\ 1 \end{pmatrix} & & \\
 X_1 & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & X_0 \oplus X_1 & \xrightarrow{\begin{pmatrix} d_0 & 0 \end{pmatrix}} & X_{-1} \\
 & & \downarrow & & \\
 & & 0 & &
 \end{array}
 \end{array}$$

and in bidegrees $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0, -1)$, the maps $C_{\bullet\bullet}^X \rightarrow X_{\bullet\bullet}$ and $C_{\bullet\bullet}^X \rightarrow A_{\bullet\bullet}^X$ are the maps -1_{X_1} , $(1 - d_1)$, $1_{X_{-1}}$, 0 , 0 , and 1_{X_1} , $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $-1_{X_{-1}}$, 1_{X_1} , 0 . Obviously, $X_\bullet = H_0^v(X_{\bullet\bullet})$. One checks that the two maps of bicomplexes are vertical quasi-isomorphisms, and that the forms on the functors $X_\bullet \mapsto X_{\bullet\bullet}$ and on $X_\bullet \mapsto A_{\bullet\bullet}^X$ restrict to the same form on $X_\bullet \mapsto C_{\bullet\bullet}^X$.

In view of the natural isometry $(X_\bullet, \varphi_X) \cong H_0^v(A_{\bullet\bullet}^X, \varphi_X)$, the natural transformation $1_C \rightarrow JG$ for an object (X, r_X, e_X) of \mathcal{C} , with X as in (14), is determined by the $Q^h sCx(\mathcal{E})$ -map $H_0^v(g)$ induced by the map $g : (A_{\bullet\bullet}^X, \varphi_X) \rightarrow (B_{\bullet\bullet}^X, \psi_X)$ in $Q^h sCx_sCx(\mathcal{E})$ given by the datum $A_{\bullet\bullet}^X \leftarrow D_{\bullet\bullet}^X \rightarrow B_{\bullet\bullet}^X$, where $D_{\bullet\bullet}^X$ is the bicomplex

$$\begin{array}{ccc}
 D_{\bullet\bullet}^X : & & \\
 & \begin{array}{ccccc}
 & & X_1 & & \\
 & & \downarrow m & & \\
 X_1 & \xrightarrow{n} & X_0 \oplus X_1 \oplus X_2^* & \xrightarrow{n'} & X_2^* \\
 & & \downarrow m' & & \\
 & & X_1^* & &
 \end{array}
 \end{array}$$

with differentials $m = \begin{pmatrix} i_X \\ 1 \\ 0 \end{pmatrix}$, $n = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $m' = (i_X^* \varphi_{X_0} \ 0 \ j_X^*)$, and $n' = (0 \ 0 \ 1)$. The maps $D_{\bullet\bullet}^X \rightarrow A_{\bullet\bullet}^X$ and $D_{\bullet\bullet}^X \rightarrow B_{\bullet\bullet}^X$ are 1_{X_1} , $1_{X_0} \oplus 1_{X_1} \oplus j_X^*$, j_X^* , 1_{X_1} , $1_{X_1^*}$ and j_X , $1_{X_0} \oplus j_X \oplus 1_{X_2^*}$, $1_{X_2^*}$, 1_{X_1} , $1_{X_1^*}$ in bidegrees $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0, -1)$, respectively. We omit the details of the verification that $H_0^v(g)$ defines a natural transformation $1_C \rightarrow JG$. Finally, we note that the natural isometry $(X_\bullet, \varphi) \cong H_0^v(A_{\bullet\bullet}^X, \varphi_X)$ defines a natural isomorphism $id_{C'} \cong GJ$ since for an object (X, r_X, e_X) of \mathcal{C}' , we have $(B_{\bullet\bullet}^X, \psi_X) = (A_{\bullet\bullet}^X, \varphi_X)$. \square

Lemma 7.5 *The inclusion $\mathcal{C}'' \subset \mathcal{C}'$ is a homotopy equivalence.*

Proof: Let $J' : \mathcal{C}'' \subset \mathcal{C}'$ be the inclusion functor. We will construct a functor $G' : \mathcal{C}' \rightarrow \mathcal{C}''$ and a natural transformation $J'G' \rightarrow 1_{\mathcal{C}'}$ such that $G'J' = id_{\mathcal{C}''}$. The functor G' sends the object (X, r_X, e_X) to $(G'X, r_X, 0^!)$, where $G'X$ is the symmetric space

$$0 \rightarrow H_0 X_{\bullet} \rightarrow 0$$

in $s\text{Cx}(\mathcal{E})$ equipped with the form $(0, H_0(\varphi_X), 0)$, and where $0^! = (E, 1, 0)$ and $e_X = (E, 1, q_X)$. Clearly, we have $G'J' = id_{\mathcal{C}''}$.

For (X, r_X, e_X) in \mathcal{C}' , the natural transformation $J'G' \rightarrow 1_{\mathcal{C}'}$ is determined by the map $(G'X, \varphi_X) \rightarrow (X_{\bullet}, \varphi_X)$ in $Q^h s\text{Cx}(\mathcal{E})$ given by the datum $G'X \leftarrow U_{\bullet} \rightarrow X_{\bullet}$ as in the diagram

$$\begin{array}{ccccc}
 X_{\bullet} : & X_1 & \xrightarrow{i_X} & X_0 & \xrightarrow{i_X^* \varphi_X} & X_1^* \\
 \uparrow & \uparrow & & \uparrow & & \uparrow \\
 U_{\bullet} : & X_1 & \longrightarrow & X_1^{\perp} & \longrightarrow & 0 \\
 \downarrow & \downarrow & & \downarrow & & \downarrow \\
 G'X : & 0 & \longrightarrow & H_0 X_{\bullet} & \longrightarrow & 0
 \end{array}$$

where X_1^{\perp} is the orthogonal $\ker(i_X^* \varphi_{X_0})$ of X_1 in (X_0, φ_{X_0}) , and $X_1 \rightarrow X_1^{\perp} \rightarrow X_0$ are the canonical inclusions. \square

Lemma 7.6 *The category \mathcal{C}'' is contractible.*

Proof: The category \mathcal{C}'' has a final object, namely $(R, 1_R, 0)$, where we consider R as the complex $0 \rightarrow R \rightarrow 0$ equipped with the form $(0, \rho, 0)$. It follows that the category \mathcal{C}'' is contractible. \square

Corollary 7.7 *For an exact category with duality \mathcal{E} , the non-singular exact form functor $s\text{Cx}(\mathcal{E}) \rightarrow \mathcal{E} \times H\mathcal{E} : A_{\bullet} \mapsto A_0, (A_1, A_{-1}^*)$ induces a homotopy equivalence of Grothendieck-Witt spaces. In particular, we have a homotopy equivalence*

$$GW(s\text{Cx}(\mathcal{E})) \simeq GW(\mathcal{E}) \times K(\mathcal{E}).$$

Proof: This follows immediately from the additivity theorem for Q^h (theorem 7.2), the definition of the Grothendieck-Witt space, and additivity in K -theory ([Qui73], or theorem 7.2 for hyperbolic categories). \square

Proof of theorem 7.1

A non-singular exact form functor $(F, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ together with a totally isotropic subfunctor $i : G \rightarrow F$ induces a non-singular exact form functor $F_\bullet : \mathcal{A} \rightarrow s\text{Cx}(\mathcal{B})$ into the category of admissible short complexes given by

$$F_\bullet : \quad G \xrightarrow{i} F \xrightarrow{i^\# \circ \hat{\varphi}} G^\#,$$

where $\#$ denotes the duality on the functor category $\text{ExFun}(\mathcal{A}, \mathcal{B})$, see 3.3.

The additivity form functor $s\text{Cx}(\mathcal{B}) \rightarrow \mathcal{B} \times H\mathcal{B}$ which gives rise to the homotopy equivalence in theorem 7.2 has a section $\mathcal{B} \times H\mathcal{B} \rightarrow s\text{Cx}(\mathcal{B})$ which maps $B_0 \in \mathcal{B}$ and $(B_1, B_{-1}) \in H\mathcal{B}$ to the short complex

$$B_1 \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} B_0 \oplus B_1 \oplus B_{-1}^* \xrightarrow{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}} B_{-1}^*.$$

Consider the composition of form functors $\mathcal{A} \rightarrow s\text{Cx}(\mathcal{B}) \rightarrow \mathcal{B} \times H\mathcal{B} \rightarrow s\text{Cx}(\mathcal{B}) \rightarrow \mathcal{B}$ of F_\bullet , the additivity form functor, its section, and evaluation in degree 0. It induces a sequence of functors on hermitian Q -constructions

$$Q^h \mathcal{A} \longrightarrow Q^h s\text{Cx}(\mathcal{B}) \xrightarrow{\sim} Q^h \mathcal{B} \times Q\mathcal{B} \xrightarrow{\sim} Q^h s\text{Cx}(\mathcal{B}) \longrightarrow Q^h \mathcal{B}$$

in which the middle two arrows are inverse to each other (up to homotopy), by the additivity theorem 7.2, so that the composition of the middle two arrows is homotopic to the identity. The composition of the sequence is $(G^\perp/G, \bar{\varphi}) \oplus H(G)$, whereas the composition of the first and the last arrow $Q^h \mathcal{A} \rightarrow Q^h s\text{Cx}(\mathcal{B}) \rightarrow Q^h \mathcal{B}$ is $(F, \varphi) : Q^h \mathcal{A} \rightarrow Q^h \mathcal{B}$. Since these two compositions are homotopic, we are done.

The same argument works for Grothendieck-Witt spaces in place of the hermitian Q -constructions; we omit the details. \square

We conclude the section with an exercise and a remark about the statement of additivity in hermitian K -theory.

Exercise 7.8 Let \mathcal{E} be an exact category, and let $E(\mathcal{E})$ be the exact category with duality of exact sequences in \mathcal{E} , that is, the fully exact subcategory with duality of $s\text{Cx}(\mathcal{E})$ of those admissible short complexes which have zero homology. Show that the following functor induces a homotopy equivalence of classifying spaces

$$Q^h E(\mathcal{E}) \xrightarrow{\sim} Q\mathcal{E} : (E_\bullet, \varphi) \mapsto E_{-1}. \tag{15}$$

In particular, there is a homotopy equivalence $GW(E(\mathcal{E})) \xrightarrow{\sim} K(\mathcal{E})$.

Remark 7.9 In view of Quillen’s additivity theorem [Qui73, §3 Theorem 2] in terms of the category $E(\mathcal{E})$, the homotopy equivalence (15) seems the more natural generalization of additivity to hermitian K -theory. Indeed, one can deduce theorems 7.1 and 7.2 from the homotopy equivalence (15), but proving (15) directly doesn’t seem to be much simpler than proving theorem 7.2. In any case, the formulation in theorem 7.1 is more useful in applications than the homotopy equivalence (15).

8. Filtering Localization

In this section we prove the “filtering localization theorem” (theorem 8.2) which will be used in §9 to construct cone categories and non-connective deloopings.

8.1. The s -filtering condition

Recall from [Sch04a] that a fully exact inclusion $\mathcal{A} \subset \mathcal{U}$ of exact categories is called *s -filtering* if

- (a) every map $A \rightarrow U$ with $A \in \mathcal{A}$ factors through an admissible subobject of U belonging to \mathcal{A} ,
- (b) every map $U \rightarrow A$ with A in \mathcal{A} factors through an admissible quotient object of U belonging to \mathcal{A} ,
- (c) for every deflation $U \twoheadrightarrow A$ in \mathcal{U} with A in \mathcal{A} , there is an inflation $B \twoheadrightarrow U$ with $B \in \mathcal{A}$ such that the composition $B \rightarrow A$ is a deflation in \mathcal{A} , and
- (d) for every inflation $A \twoheadrightarrow U$ in \mathcal{U} with A in \mathcal{A} , there is a deflation $U \twoheadrightarrow B$ with $B \in \mathcal{A}$ such that the composition $A \rightarrow B$ is an inflation in \mathcal{A} .

Items (a) and (b) imply that \mathcal{A} is closed under admissible sub-objects and quotient-objects in \mathcal{U} .

Remark 8.1 A fully exact inclusion $\mathcal{A} \subset \mathcal{U}$ satisfying only (a) and (b) is called filtering. For instance, a Serre inclusion of abelian categories is filtering. An inflation $A \twoheadrightarrow U$ satisfying (d) is called special, as well as a deflation $U \twoheadrightarrow A$ satisfying (c) is called special. The reason to introduce special inflations and deflations is to ensure that the inclusion $\mathcal{A} \subset \mathcal{U}$ induces a fully faithful functor on bounded derived categories (a fact we won’t need in this article). For this, (a) and (b) don’t suffice. The letter “ s ” in “ s -filtering” refers to the special inflations and deflations in conditions (c) and (d).

8.2. The exact quotient \mathcal{U}/\mathcal{A}

Let $\mathcal{A} \subset \mathcal{U}$ be an s-filtering inclusion of exact categories. Call a composition of deflations with kernel in \mathcal{A} and inflations with cokernel in \mathcal{A} a *weak isomorphism*. By definition, weak isomorphisms are closed under composition. They satisfy a calculus of fractions [Sch04a, Lemma 1.13]. In diagrams, weak isomorphisms are displayed as $\xrightarrow{\sim}$, inflations with cokernel in \mathcal{A} as $\xrightarrow{\succsim}$, and deflations with kernel in \mathcal{A} as $\xrightarrow{\sim\sim}$. By [Sch04a, Proposition 1.16], the exact quotient category \mathcal{U}/\mathcal{A} exists and is obtained from \mathcal{U} by formally inverting the set of weak isomorphisms. A sequence of \mathcal{U}/\mathcal{A} is a conflation in \mathcal{U}/\mathcal{A} if and only if it is isomorphic to the image of a conflation of \mathcal{U} .

Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving s-filtering inclusion of exact categories with duality. The duality on \mathcal{U} induces an exact duality and a natural double dual identification on the quotient \mathcal{U}/\mathcal{A} , and we consider \mathcal{U}/\mathcal{A} equipped with this structure of an exact category with duality. The sequence of functors

$$\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$$

is a sequence of duality preserving exact (form) functors.

The rest of the section is devoted to the proof of the following theorem.

Theorem 8.2 *Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving s-filtering inclusion of exact categories with duality. Assume that \mathcal{A} is idempotent complete. Then the sequence $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$ induces a homotopy fibration on classifying spaces*

$$Q^h(\mathcal{A}) \longrightarrow Q^h(\mathcal{U}) \longrightarrow Q^h(\mathcal{U}/\mathcal{A}).$$

Remark 8.3 In theorem 8.2, the map $\pi_0 Q^h(\mathcal{U}) \rightarrow \pi_0 Q^h(\mathcal{U}/\mathcal{A})$, that is, the map $W_0(\mathcal{U}) \rightarrow W_0(\mathcal{U}/\mathcal{A})$ need not be surjective.

Remark 8.4 Special cases of theorem 8.2 were proved in [Sch04a, theorem 2.1] (the K -theory version), [PW89, theorem 5.3] (the K -theory version in the split exact case, see also [CP97] for an alternative proof) and [HS04, theorem 3.5] (the hermitian K -theory version for split exact categories in which 2 is invertible). Note that the main theorem in [PW89] is implied by [PW89, theorem 5.3], so that, replacing [PW89, theorem 5.3] with theorem 8.2, we obtain the hermitian analog of the main theorem of [PW89] (see also [Ran92] for an L -theory version).

For the rest of this section, $\mathcal{A} \subset \mathcal{U}$ will be an s-filtering inclusion of exact categories, and \mathcal{A} will be idempotent complete. Before proving theorem 8.2, we collect some useful facts about s-filtering inclusions of exact categories.

Lemma 8.5 (a) *For any weak isomorphism $f : X \xrightarrow{\sim} Y$ in \mathcal{U} there is an inflation $s : U \xrightarrow{\succsim} X$ with cokernel in \mathcal{A} such that fs is an inflation with cokernel in \mathcal{A} .*

- (b) Every weak isomorphism f in \mathcal{U} is $f = pt$ with t an inflation with cokernel in \mathcal{A} and p a deflation with kernel in \mathcal{A} .
- (c) A map in \mathcal{U} is an isomorphism in \mathcal{U}/\mathcal{A} iff it is a weak isomorphism.
- (d) An object X of \mathcal{U} is zero in \mathcal{U}/\mathcal{A} iff X is in \mathcal{A} .
- (e) For any diagram $X \xrightarrow{u} Y \xleftarrow{s} Z$ of inflations in \mathcal{U} with $\text{coker}(s)$ in \mathcal{A} , there is a diagram of inflations $X \xleftarrow{t} W \xrightarrow{v} Z$ with $\text{coker}(t)$ in \mathcal{A} such that $ut = sv$.
- (f) A map $X \rightarrow Y$ in \mathcal{U} is an inflation in \mathcal{U}/\mathcal{A} iff there is an inflation $X_0 \xrightarrow{\sim} X$ in \mathcal{U} with cokernel in \mathcal{A} such that the composition $X_0 \rightarrow Y$ is an inflation in \mathcal{U} .
- (g) Let $f : X \rightarrow Y$ be a map in \mathcal{U} such that there is a map $g : Y \rightarrow Z$ with gf an inflation in \mathcal{U} . If either X is in \mathcal{A} or f is a weak isomorphism, then f is an inflation.

Proof: Parts (a) and (c) are proved in [Sch04a, Lemma 1.17 (3), (6)].

For part (b), choose an inflation s with cokernel in \mathcal{A} as in (a) such that fs is an inflation with cokernel in \mathcal{A} . The map f induces a map from the push-out of fs along s to the target of f . This map is a deflation whose kernel is $\text{coker}(s)$. Then f is the composition of the pushout of fs and this deflation.

For part (d), the map $0 \rightarrow X$ is an isomorphism in \mathcal{U}/\mathcal{A} , hence by part (c) a weak isomorphism. Then part (a) implies that the cokernel of $0 \rightarrow X$ is in \mathcal{A} .

Part (e) is [Sch04a, Lemma 1.17 (5)] (up to the typographical errors in [Sch04a, Lemma 1.17 (5)], see the proof given there).

We prove part (f). The map $f : X \rightarrow Y$ in \mathcal{U} becomes an inflation in \mathcal{U}/\mathcal{A} and only if it is isomorphic in \mathcal{U}/\mathcal{A} to the image of an inflation $f_1 : X_1 \rightarrow Y_1$ of \mathcal{U} . By part (e) and the calculus of fractions w.r.t inflations with cokernel in \mathcal{A} , we can find the following commutative diagram in \mathcal{U}

$$\begin{array}{ccccc}
 X & \xleftarrow{\sim} & X_0 & \xrightarrow{\sim} & X_1 \\
 f \downarrow & & \downarrow & & \downarrow f_1 \\
 Y & \xleftarrow{\sim} & Y_0 & \xrightarrow{\sim} & Y_1.
 \end{array}$$

For part (g), assume first that f is a weak isomorphism. By assumption, the map f is an inflation in the idempotent completion $\widetilde{\mathcal{U}}$ of \mathcal{U} with cokernel, say N in $\widetilde{\mathcal{U}}$ (see 5.1). By part (b), we have $f = pt$ with p a deflation with kernel A in \mathcal{A} and t an inflation with cokernel B in \mathcal{A} . Then we have a conflation $A \xrightarrow{i} B \twoheadrightarrow N$ in $\widetilde{\mathcal{U}}$. Choose an object \bar{N} of $\widetilde{\mathcal{U}}$ such that $N \oplus \bar{N}$ is in \mathcal{U} . Then $B \oplus \bar{N}$ is (isomorphic to)

an object of \mathcal{U} . By definition of s -filtering, there is a deflation $q : B \oplus \bar{N} \twoheadrightarrow C$ in \mathcal{U} with target in \mathcal{A} such that the composition $q \circ \binom{j}{0}$ is an inflation. On quotients, we obtain a map $N = B/A \rightarrow C/A$ such that the square involving $B, C, N, C/A$ is bicartesian. This induces a conflation $B \twoheadrightarrow C \oplus N \twoheadrightarrow C/A$ in \mathcal{U} . The objects B and C/A are in \mathcal{A} (the latter because $A \twoheadrightarrow C$ is an inflation in \mathcal{U} and \mathcal{A} is closed under quotient objects in \mathcal{U}). Since the inclusions $\mathcal{A} \subset \mathcal{U} \subset \tilde{\mathcal{U}}$ are extension closed, $C \oplus N$ is in \mathcal{A} , and since \mathcal{A} is idempotent complete, N is in \mathcal{A} .

Finally, assume X in \mathcal{A} , then, by the filtering condition, the map f is $f = ia$ with a a map in \mathcal{A} and i an inflation. By the dual of [Sch04a, Lemma 1.17 (1)], a is an inflation. As a composition of inflations, f is an inflation itself. \square

Let \mathcal{E} be an exact category. We write $\mathcal{I}(\mathcal{E})$ for the exact category of inflations in \mathcal{E} . Objects are inflations $X \xrightarrow{j} Y$ in \mathcal{E} , and maps $(X_0 \xrightarrow{j_0} Y_0) \rightarrow (X_1 \xrightarrow{j_1} Y_1)$ are commutative squares in \mathcal{E} , that is, they are pairs (f, g) of maps $f : X_0 \rightarrow X_1, g : Y_0 \rightarrow Y_1$ in \mathcal{E} such that $j_1 f = g j_0$. A sequence of inflations is exact if the corresponding sequences of source objects and target objects are exact in \mathcal{E} . This means that a map $(f, g) : j_0 \rightarrow j_1$ as above is an inflation if and only if f and g are inflations in \mathcal{E} and the canonical map from the push-out of f along j_0 to Y_1 is an inflation. Note that $\mathcal{I}(\mathcal{E})$ is equivalent to the exact category $E(\mathcal{E})$ of conflations in \mathcal{E} .

Lemma 8.6 (a) $\mathcal{I}(\mathcal{A}) \subset \mathcal{I}(\mathcal{U})$ is s -filtering

(b) A commutative diagram in \mathcal{U}

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ x \downarrow \wr & & \wr \downarrow y \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

with x and y weak isomorphisms and f_0, f_1 inflations in \mathcal{U} defines a weak isomorphism in $\mathcal{I}(\mathcal{U})$ from f_0 to f_1 .

(c) The canonical functor $\mathcal{I}(\mathcal{U})/\mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{U}/\mathcal{A})$ is an equivalence of exact categories.

Proof: Part (a) was proved in [Sch04a, Lemma 1.17 (4)] in view of the equivalence $\mathcal{I}(\mathcal{A})^{op} \cong \mathcal{I}(\mathcal{A}^{op})$.

For part (b), let P be the push-out of f_0 along x , the induced maps being $\bar{f}_0 : X_1 \twoheadrightarrow P, \bar{x} : Y_0 \twoheadrightarrow P$ and $\bar{y} : P \rightarrow Y_1$. It is clear that (x, \bar{x}) is a weak isomorphism from f_0 to \bar{f}_0 in $\mathcal{I}(\mathcal{U})$. By lemma 8.5 (c), \bar{y} is a weak isomorphism, that is, $\bar{y} = y_0 \circ y_1 \circ \cdots \circ y_n$ with y_i an inflation with cokernel in \mathcal{A} or a deflation with kernel

in \mathcal{A} . Then $(1_{X_1}, y_i)$ defines a map in $\mathcal{I}(\mathcal{U})$ which is an inflation with cokernel in $\mathcal{I}(\mathcal{A})$ or a deflation with kernel in $\mathcal{I}(\mathcal{A})$ so that (x, \bar{y}) is a weak isomorphism in $\mathcal{I}(\mathcal{U})$. This shows part (b).

Finally, by Lemma 8.5 together with part (b), the functor $\mathcal{I}(\mathcal{U})/\mathcal{I}(\mathcal{A}) \rightarrow \mathcal{I}(\mathcal{U}/\mathcal{A})$ is essentially surjective. Fully faithfulness follows from the calculus of fractions and part (b). We omit the details. \square

Lemma 8.7 *Let $j : X \twoheadrightarrow Y$ and $s_1, s_2 : Y \xrightarrow{\sim} Z$ be inflations in \mathcal{U} with $\text{coker}(s_i)$ in \mathcal{A} , $i = 1, 2$. Assume that $s_1 j = s_2 j$ in \mathcal{U} and $s_1 = s_2$ in \mathcal{U}/\mathcal{A} . Then there are inflations $l : X \twoheadrightarrow U$ and $s : U \xrightarrow{\sim} Y$ such that $\text{coker}(s)$ is in \mathcal{A} , $j = sl$ and $s_1 s = s_2 s$ in \mathcal{U} .*

Proof: Since $s_1 = s_2$ in \mathcal{U}/\mathcal{A} , there is an inflation $y_0 : Y_0 \xrightarrow{\sim} Y$ with cokernel in \mathcal{A} such that $s_1 y_0 = s_2 y_0$ in \mathcal{U} . By lemma 8.5 (e) there is an inflation $x_0 : X_0 \xrightarrow{\sim} X$ with cokernel in \mathcal{A} and an inflation $j_0 : X_0 \twoheadrightarrow Y_0$ in \mathcal{U} such that $y_0 j_0 = j x_0$. By lemma 8.6 (b), the map (x_0, y_0) is a weak isomorphism in $\mathcal{I}(\mathcal{U})$. By lemmas 8.5 (a) and 8.6 (b), there is an inflation (x_1, y_1) in $\mathcal{I}(\mathcal{U})$ with cokernel in $\mathcal{I}(\mathcal{A})$ from $j_1 : X_1 \twoheadrightarrow Y_1$ to j_0 such that the composition $(x_0 x_1, y_0 y_1)$ is an inflation with cokernel in $\mathcal{I}(\mathcal{A})$. Let U be the push-out of j_1 along $x_0 x_1$, and $s : U \rightarrow Y, l : X \twoheadrightarrow U$ the induced maps. Then s is an inflation in \mathcal{U} since $(x_0 x_1, y_0 y_1)$ is an inflation in $\mathcal{I}(\mathcal{U})$, and $\text{coker}(s)$ is 0 in \mathcal{U}/\mathcal{A} , hence $\text{coker}(s)$ in \mathcal{A} , by lemma 8.5 (d). \square

From now on until the end of the section $\mathcal{A} \subset \mathcal{U}$ will be a duality preserving s -filtering inclusion of exact categories with duality.

Lemma 8.8 (a) *Every symmetric space in \mathcal{U}/\mathcal{A} is isometric to a (possibly singular) symmetric form (X, φ) in \mathcal{U} with φ a weak isomorphism.*

(b) *A symmetric form (X, φ) in \mathcal{U} with φ a weak isomorphism is metabolic in \mathcal{U}/\mathcal{A} with Lagrangian $L_0 \subset X$ if and only if there is a weak isomorphism of conflations in \mathcal{U} of the form*

$$\begin{array}{ccccc} L & \xrightarrow{i} & X & \xrightarrow{p} & X/L \\ l \downarrow \wr & & \wr \downarrow \varphi & & \wr \downarrow l^* \eta \\ (X/L)^* & \xrightarrow{p^*} & X^* & \xrightarrow{i^*} & L^* \end{array}$$

such that L is isomorphic to L_0 as subobjects of X in \mathcal{U}/\mathcal{A} .

Proof: For part (a), let Y be an object of \mathcal{U}/\mathcal{A} (that is, an object of \mathcal{U}), together with a non-singular symmetric form represented by the fraction ψs^{-1} where $s : Z \xrightarrow{\sim} Y$ and $\psi : Z \xrightarrow{\sim} Y^*$ are weak isomorphisms in \mathcal{U} . The equality $\psi s^{-1} = (\psi s^{-1})^* \eta$ in \mathcal{U}/\mathcal{A} implies the existence of a weak isomorphism $t : X \xrightarrow{\sim} Z$ such

that $\psi^* \eta st = s^* \psi t$. Let $\varphi = t^* s^* \psi t$, then (X, φ) is a symmetric form in \mathcal{U} with φ a weak isomorphism, and $st : (X, \varphi) \rightarrow (Y, \psi s^{-1})$ defines an isometry in \mathcal{U}/\mathcal{A} .

For part (b), it is clear that a symmetric form (X, φ) in \mathcal{U} for which there is a diagram as in (b) defines a metabolic space in \mathcal{U}/\mathcal{A} with Lagrangian L_0 . For the other implication, let (X, φ) be a symmetric form in \mathcal{U} which is a metabolic space in \mathcal{U}/\mathcal{A} with Lagrangian $L_0 \subset X$. The map $L_0 \twoheadrightarrow X$ in \mathcal{U}/\mathcal{A} is represented by a fraction is^{-1} with $i : L \rightarrow X$ and $s : L \xrightarrow{\sim} L_0$ maps in \mathcal{U} and s a weak isomorphism. By lemma 8.5 (f), we can assume i to be an inflation in \mathcal{U} . The map $i^* \varphi i$ is zero in \mathcal{U}/\mathcal{A} so that there is an inflation $j : L' \xrightarrow{\sim} L$ with cokernel in \mathcal{A} such that $i^* \varphi i j = 0$ in \mathcal{U} . Replacing L with L' we can assume $i : L \twoheadrightarrow X$ to be an inflation in \mathcal{U} such that $i^* \varphi i = 0$ in \mathcal{U} . Let p be the quotient map $X \twoheadrightarrow X/L$. Then the equation $i^* \varphi i = 0$ implies that the map φi factors as $p^* l$. \square

Lemma 8.9 *The following sequence of abelian groups is exact*

$$W_0(\mathcal{A}) \rightarrow W_0(\mathcal{U}) \rightarrow W_0(\mathcal{U}/\mathcal{A}).$$

Proof: Let (X, φ) be a symmetric space in \mathcal{U} which is stably metabolic in \mathcal{U}/\mathcal{A} . Then there is a metabolic space (M, μ) in \mathcal{U}/\mathcal{A} such that $(X, \varphi) \perp (M, \mu)$ is metabolic in \mathcal{U}/\mathcal{A} . By lemma 8.8 (a), we can assume (M, μ) to be a symmetric form in \mathcal{U} with μ a weak isomorphism. The symmetric space $(M, -\mu)$ is also metabolic with same Lagrangian as (M, μ) , so that the sum of metabolic spaces $(X, \varphi) \perp (M, \mu) \perp (M, -\mu)$ is metabolic in \mathcal{U}/\mathcal{A} .

We have the classical isometry in \mathcal{U}/\mathcal{A}

$$\begin{pmatrix} -1 & 1 \\ \mu & 0 \end{pmatrix} : (M, -\mu) \perp (M, \mu) \xrightarrow{\cong} H^\mu(M) = (M \oplus M^*, \begin{pmatrix} \mu & 1 \\ \eta & 0 \end{pmatrix})$$

which shows that the symmetric space $(Y, \psi) = (X, \varphi) \perp H^\mu(M)$ in \mathcal{U} is metabolic in \mathcal{U}/\mathcal{A} . By lemma 8.8 (b), there is an inflation $i : L \twoheadrightarrow Y$ in \mathcal{U} such that $i^* \psi i = 0$ in \mathcal{U} and such that the canonical map $L \rightarrow (Y/L)^* \cong L^\perp$ is a weak isomorphism. By lemma 8.5 (g), $L \rightarrow L^\perp$ is an inflation with cokernel A in \mathcal{A} . Since $H^\mu(M)$ is metabolic in \mathcal{U} , we have in $W_0(\mathcal{U})$ the equalities $[X, \varphi] = [Y, \psi] = [L^\perp/L, \bar{\psi}]$ which is in the image of the map $W_0(\mathcal{A}) \rightarrow W_0(\mathcal{U})$ because L^\perp/L is in \mathcal{A} . \square

For an exact category with duality \mathcal{E} , let $Q_m^h \mathcal{E} \subset Q^h \mathcal{E}$ be the full subcategory of metabolic spaces in \mathcal{E} . For an s-filtering duality preserving inclusion $\mathcal{A} \subset \mathcal{U}$ of exact categories with duality, let $Q_{\mathcal{A}}^h \mathcal{U} \subset Q^h \mathcal{U}$ be the full subcategory of those symmetric spaces (X, φ) in \mathcal{U} for which there is a map $(A, \alpha) \rightarrow (X, \varphi)$ in $Q^h \mathcal{U}$ with A in \mathcal{A} . Note that we have inclusions $Q_m^h \mathcal{U} \subset Q_{\mathcal{A}}^h \mathcal{U} \subset Q^h \mathcal{U}$.

Lemma 8.10 *On homotopy groups, the inclusions $Q_m^h \mathcal{U} \subset Q_{\mathcal{A}}^h \mathcal{U} \subset Q^h \mathcal{U}$ and $Q_m^h(\mathcal{U}/\mathcal{A}) \rightarrow Q^h(\mathcal{U}/\mathcal{A})$ induce monomorphisms on π_0 and isomorphisms on π_i , $i \geq 1$.*

Proof: Consider the abelian monoid of isomorphism classes of those symmetric spaces in \mathcal{U} which are the objects of $Q_{\mathcal{A}}^h \mathcal{U}$. We denote by $W_{\mathcal{A}}(\mathcal{U})$ the quotient monoid of this monoid by the submonoid of metabolic spaces. As for $W_0(\mathcal{U})$, the monoid $W_{\mathcal{A}}(\mathcal{U})$ is actually a group, and an element $[X, \varphi]$ in $W_{\mathcal{A}}(\mathcal{U})$ is zero if and only if it is stably metabolic. The map $W_{\mathcal{A}}(\mathcal{U}) \rightarrow W_0(\mathcal{U}) : [X, \varphi] \mapsto [X, \varphi]$ is thus injective. As in proposition 4.8, one shows that the map $W_{\mathcal{A}}(\mathcal{U}) \rightarrow \pi_0 Q_{\mathcal{A}}^h \mathcal{U} : [X, \varphi] \mapsto [X, \varphi]$ is well-defined and an isomorphism. It follows that the maps $\pi_0 Q_m^h \mathcal{U} \rightarrow \pi_0 Q_{\mathcal{A}}^h \mathcal{U} \rightarrow \pi_0 Q^h \mathcal{U}$ are injective (the first is because $\pi_0 Q_m^h \mathcal{U} = 0$).

Let $Q_{sm}^h \mathcal{U} \subset Q^h \mathcal{U}$ and $Q_{sm\mathcal{A}}^h \mathcal{U} \subset Q_{\mathcal{A}}^h \mathcal{U}$ be the full subcategories of stably metabolic objects. They are the connected components of $Q^h \mathcal{U}$ and $Q_{\mathcal{A}}^h \mathcal{U}$, so that the inclusions $Q_{sm}^h \mathcal{U} \subset Q^h \mathcal{U}$ and $Q_{sm\mathcal{A}}^h \mathcal{U} \subset Q_{\mathcal{A}}^h \mathcal{U}$ induce isomorphisms in π_i , $i \geq 1$. The full inclusions $Q_m^h \mathcal{U} \subset Q_{sm\mathcal{A}}^h \mathcal{U}$ and $Q_m^h \mathcal{U} \subset Q_{sm}^h \mathcal{U}$ induce isomorphisms on π_i , $i \geq 1$ by the strong cofinality lemma 4.15. We conclude that the sequence of inclusions $Q_m^h \mathcal{U} \subset Q_{\mathcal{A}}^h \mathcal{U} \subset Q^h \mathcal{U}$ induces isomorphisms on π_i for $i \geq 1$. Similarly, $Q_m^h(\mathcal{U}/\mathcal{A}) \rightarrow Q^h(\mathcal{U}/\mathcal{A})$ is a monomorphism on π_0 and an isomorphism on π_i , $i \geq 1$. \square

The quotient functor $\mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$ maps $Q_{\mathcal{A}}^h \mathcal{U}$ into the subcategory $Q_m^h(\mathcal{U}/\mathcal{A})$ of metabolic objects of $Q^h(\mathcal{U}/\mathcal{A})$. In fact, $Q_{\mathcal{A}}^h \mathcal{U}$ is precisely the full subcategory of $Q^h \mathcal{U}$ of those objects which have image in $Q_m^h(\mathcal{U}/\mathcal{A})$ (in view of lemmas 8.8 (b) and 8.5 (g)).

Theorem 8.2 follows from lemmas 8.9, 8.10 and the following proposition the proof of which will occupy the rest of the section.

Proposition 8.11 *Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving s -filtering inclusion of exact categories with duality. Assume that \mathcal{A} is idempotent complete. Then the sequence $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$ induces a homotopy fibration on classifying spaces*

$$Q^h(\mathcal{A}) \longrightarrow Q_{\mathcal{A}}^h(\mathcal{U}) \longrightarrow Q_m^h(\mathcal{U}/\mathcal{A}).$$

Proof: Let ρ be the functor $Q_{\mathcal{A}}^h(\mathcal{U}) \rightarrow Q_m^h(\mathcal{U}/\mathcal{A})$ induced by the localization functor $\mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$. In order to identify the homotopy fibre of ρ , we will use theorem B of Quillen. We will show that for a map $g : M \rightarrow N$ in $Q_m^h(\mathcal{U}/\mathcal{A})$, the induced functor $g^* : (N \downarrow \rho) \rightarrow (M \downarrow \rho)$ is a homotopy equivalence on classifying spaces. Since every object in $Q_m^h(\mathcal{U}/\mathcal{A})$ is metabolic, every object is the target of a map from 0. Thus it suffices to show that g^* is a homotopy equivalence for $g : 0 \rightarrow N$. Recall that the category $(N \downarrow \rho)$ has objects pairs (X, α) with X an object of $Q_{\mathcal{A}}^h(\mathcal{U})$ and $\alpha : N \rightarrow \rho X$ a map in $Q_m^h(\mathcal{U}/\mathcal{A})$. Maps from (X, α) to (Y, β) correspond to maps $s : X \rightarrow Y$ in $Q_{\mathcal{A}}^h(\mathcal{U})$ such that $\beta = \rho(s) \circ \alpha$. Composition is composition of the maps s in $Q_{\mathcal{A}}^h(\mathcal{U})$.

Consider the following diagram of categories and functors

$$\begin{array}{ccc}
 E_g & \xrightarrow{p} & (N \downarrow \rho) \xrightarrow{i_N} (N \downarrow \rho) \\
 \gamma \downarrow & & \downarrow g^* \\
 Q^h(\mathcal{A}) & \xlongequal{\quad} & (0 \downarrow \rho) \xrightarrow{i_0} (0 \downarrow \rho),
 \end{array} \tag{16}$$

where the categories and functors are as follows.

- The functor i_N is the inclusion of the full subcategory $(N \downarrow \rho)$ of $(N \downarrow \rho)$ whose objects are those pairs (X, α) for which α is an isomorphism. Note that $Q^h(\mathcal{A}) = (0 \downarrow \rho)$ since $Q^h(\mathcal{A}) \subset Q_A^h(\mathcal{U})$ is the full subcategory of objects which are isomorphic to 0 in $Q^h(\mathcal{U}/\mathcal{A})$.
- The category E_g has objects the pairs (a, α) with $a : A \rightarrow X$ a map in $Q_A^h(\mathcal{U})$, $A \in Q^h(\mathcal{A})$, and $\alpha : N \rightarrow \rho X$ an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$ such that $g = \alpha^{-1}\rho(a)$ where ρA is identified with 0 via the unique isomorphism $0 \rightarrow \rho A$ in $Q^h(\mathcal{U}/\mathcal{A})$. A map $(a, \alpha) \rightarrow (b : B \rightarrow Y, \beta)$ is a pair of maps $c : A \rightarrow B$ and $s : X \rightarrow Y$ in $Q_A^h(\mathcal{U})$ such that $sa = bc$ and $\beta = \rho(s)\alpha$. Composition is composition of the individual maps of the pairs.
- The functor $p : E_g \rightarrow (N \downarrow \rho)$ sends $(a : A \rightarrow X, \alpha)$ to (X, α) .
- The functor $\gamma : E_g \rightarrow Q^h(\mathcal{A})$ sends $(a : A \rightarrow X, \alpha)$ to A .

There is a natural transformation of functors $i_0 \circ \gamma \rightarrow g^* \circ i_N \circ p$ given by

$$i_0 \circ \gamma(a : A \rightarrow X, \alpha) = (A, 0) \xrightarrow{a} g^* \circ i_N \circ p(a, \alpha) = (X, \alpha \circ g).$$

Thus, on classifying spaces, the diagram commutes up to homotopy. The proof of proposition 8.11 is complete once we show that i_N (hence i_0), p and γ induce homotopy equivalences on classifying spaces. \square

Lemma 8.12 *For every object N of $Q_m^h(\mathcal{U}/\mathcal{A})$, the full inclusion*

$$i_N : (N \downarrow \rho) \rightarrow (N \downarrow \rho)$$

is a homotopy equivalence on classifying spaces.

Proof: The lemma follows from theorem A of Quillen once we show that for all objects (X, α) of $(N \downarrow \rho)$, the comma category $\mathcal{C} = (i_N \downarrow (X, \alpha))$ is contractible. The map $\alpha : N \rightarrow \rho X$ is represented by a totally isotropic subspace $L_0 \subset X$ of X in \mathcal{U}/\mathcal{A} together with an isometry $L_0^\perp/L_0 \cong N$ in \mathcal{U}/\mathcal{A} . By the calculus of

fractions, lemmas 8.5 (a) and (f), we can assume that $L_0 \subset X$ is actually a totally isotropic subspace of X in \mathcal{U} . This shows that \mathcal{C} is non-empty, as L_0^\perp/L_0 together with the canonical map in $Q^h\mathcal{U}$ to X and the isometry $L_0^\perp/L_0 \cong N$ in \mathcal{U}/\mathcal{A} defines an object of \mathcal{C} .

The comma category \mathcal{C} is equivalent to the category with objects the totally isotropic subspaces $L \subset X$ in \mathcal{U} such that L is isomorphic, as a subobject of X in \mathcal{U}/\mathcal{A} , to L_0 . There is a (unique) map $L \rightarrow L'$ in \mathcal{C} if $L' \subset L$. Using the calculus of fractions for the set of weak isomorphisms and lemma 8.5 (a), we see that \mathcal{C} is a filtering category so that the classifying space of \mathcal{C} is contractible [Qui73, §1 Corollary 2]. \square

Lemma 8.13 *The functor $p : E_g \rightarrow (N \downarrow \rho)$ is a homotopy equivalence on classifying spaces.*

Proof: In view of Quillen's theorem A it suffices to show that for any object (X, α) of $(N \downarrow \rho)$, the comma category $\mathcal{C} = (p \downarrow (X, \alpha))$ is contractible. An object of \mathcal{C} is a sequence $A \xrightarrow{a} U \xrightarrow{s} X$ of composable maps a, s in $Q^h_{\mathcal{A}}(\mathcal{U})$ with $A \in Q^h(\mathcal{A})$ and s an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$ such that $\rho(sa) = \alpha g$. A map from (a, s) to $B \xrightarrow{b} V \xrightarrow{t} X$ is a pair of maps $c : A \rightarrow B, r : U \rightarrow V$ in $Q^h_{\mathcal{A}}(\mathcal{U})$ such that $ra = bc$ and $s = tr$. Composition is composition of commutative diagrams in $Q^h_{\mathcal{A}}(\mathcal{U})$.

Let $J : \mathcal{C}' \subset \mathcal{C}$ be the full subcategory of those sequences which are of the form $A \xrightarrow{a} X \xrightarrow{1} X$. The functor $G : \mathcal{C} \rightarrow \mathcal{C}'$ which sends an object (a, s) to $(s \circ a, id)$ and a map (c, r) to (c, id_X) is a homotopy inverse to J because $GJ = id$ and because $(id, s) : (a, s) \rightarrow (s \circ a, id)$ defines a natural transformation $id \rightarrow JG$.

The category \mathcal{C}' is non-empty and filtered, and thus contractible, by exactly the same arguments as in the proof of lemma 8.12. In detail, the map $\alpha g : 0 \rightarrow X$ in $Q^h(\mathcal{U}/\mathcal{A})$ is represented by a Lagrangian L_0 of X in \mathcal{U}/\mathcal{A} which, by lemmas 8.5 (a) and (f), we can assume to be a totally isotropic subspace L_0 of X in \mathcal{U} such that the quotient L_0^\perp/L_0 is in \mathcal{A} . The category \mathcal{C}' is equivalent to the category whose objects are totally isotropic subspaces $L \subset X$ in \mathcal{U} such that L is isomorphic to L_0 as subobjects of X in \mathcal{U}/\mathcal{A} . Maps are inclusions of totally isotropic subspaces. This category is filtered by the calculus of fractions and lemma 8.5 (a). \square

The rest of this section is devoted to proving the following lemma which completes the proof of proposition 8.11 and thus of theorem 8.2.

Lemma 8.14 *The functor $\gamma : E_g \rightarrow Q^h(\mathcal{A})$ is a homotopy equivalence on classifying spaces.*

Proof: Again, the lemma follows from theorem A of Quillen, once we show that for all objects A of $Q^h(\mathcal{A})$, the comma category $\mathcal{C} = (A \downarrow \gamma)$ is contractible. The category \mathcal{C} has objects $(A \xrightarrow{b} B \xrightarrow{u} X, N \xrightarrow{s} \rho X)$ with b, u maps in $Q^h_{\mathcal{A}}(\mathcal{U})$, $B \in$

$Q^h(\mathcal{A})$ and s an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$ such that $\rho(u) = sg$. A map from (b, u, s) to $(A \xrightarrow{c} C \xrightarrow{v} Y, t)$ is a pair of maps $a : B \rightarrow C$ and $w : X \rightarrow Y$ in $Q^h(\mathcal{U})$ such that $c = ab, va = wu$ and $t = \rho(w)s$. Composition is composition of commutative diagrams in $Q^h(\mathcal{U})$.

Let $J : \mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects (b, u, s) with $b = id_A$. The functor $G : \mathcal{C} \rightarrow \mathcal{C}'$ which sends an object (b, u, s) to (id, ub, s) and a map (a, w) to (id_A, w) is a homotopy inverse of J because $GJ = id$ and because the map $(b, id_X) : (id_A, ub, s) \rightarrow (b, u, s)$ defines a natural transformation of functors $JG \rightarrow id$. We will show that \mathcal{C}' is a non-empty filtered (projective) system, so that \mathcal{C}' and hence \mathcal{C} will be contractible.

Omitting the information $b = id_A$, the objects of \mathcal{C}' are pairs $(A \xrightarrow{u} X, N \xrightarrow{s} \rho X)$ with u a map in $Q^h(\mathcal{U})$ and s an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$ such that $\rho(u) = sg$. The category \mathcal{C}' is non-empty by the following argument. By lemma 8.8 (a), there is a weak isomorphism of confections

$$\begin{array}{ccccc}
 L & \xrightarrow{-i} & N & \xrightarrow{-p} & N/L \\
 \downarrow -l \wr & & \downarrow \wr v & & \downarrow \wr -l^* \eta \\
 (N/L)^* & \xrightarrow{p^*} & N^* & \xrightarrow{i^*} & L^*
 \end{array} \tag{17}$$

such that $i : L \subset N$ defines a Lagrangian in \mathcal{U}/\mathcal{A} representing the map $g : 0 \rightarrow N$. Consider the admissible short complex (centered in degree 0)

$$N_\bullet : \quad L \xrightarrow{\begin{pmatrix} -i \\ l \end{pmatrix}} N \oplus (N/L)^* \rightarrow 0$$

equipped with the symmetric form ψ which in degree 0 is $\begin{pmatrix} v & p^* \\ \eta p & 0 \end{pmatrix}$. The form $\psi : N_\bullet \rightarrow N_\bullet^*$ is a quasi-isomorphism since its cone is isomorphic to the cone of (17) which is acyclic. We therefore obtain a symmetric space $H_0(N_\bullet, \psi)$ in \mathcal{U} which turns out to be metabolic with Lagrangian $(N/L)^*$ and thus defines a map $h : 0 \rightarrow H_0(N_\bullet, \psi)$ in $Q^h\mathcal{U}$. More precisely, the inclusion of the Lagrangian $(N/L)^*$ into $H_0(N_\bullet, \psi)$ is H_0 of the map of short complexes $(N/L)^* \rightarrow N_\bullet$ where $(N/L)^*$ denotes the complex concentrated in degree 0 where it is $(N/L)^*$. The map of complexes $(N/L)^* \rightarrow N_\bullet$ is the canonical inclusion of $(N/L)^*$ into $N \oplus (N/L)^*$ in degree 0. Furthermore, the canonical inclusion of N into $N \oplus (N/L)^*$ induces a map of complexes $N \rightarrow N_\bullet$, where, again, N is concentrated in degree 0. This map of complexes is a quasi-isomorphism in \mathcal{U}/\mathcal{A} . It induces an isometry $s : (N, v) \cong H_0(N_\bullet, \psi)$ in \mathcal{U}/\mathcal{A} under which the Lagrangian $(N/L)^*$ corresponds to $L \subset N$. Since $\rho A = 0$, the pair

$$(A \xrightarrow{1_A \perp h} A \perp H_0(N_\bullet, \psi), N \xrightarrow{s} \rho H_0(N_\bullet, \psi))$$

defines an object of \mathcal{C}' . This shows that the category \mathcal{C}' is non-empty. The fact that \mathcal{C}' is a filtered projective system is shown in lemmas 8.16 and 8.17 below. \square

In order to reduce the proof of lemma 8.16 to $A = 0$, we need the following.

Lemma 8.15 *For any map $a : A \rightarrow X$ in $Q^h\mathcal{U}$ with $A \in Q^h\mathcal{A}$ there are maps $l : 0 \rightarrow M$, $t : A \perp M \rightarrow X$ in $Q^h\mathcal{U}$ such that $a = t \circ (id_A \perp l)$ and $\rho(t)$ is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$.*

Proof: Let α and φ be the forms on A and X , respectively. The map $A \xrightarrow{a} X$ in $Q^h\mathcal{U}$ is represented by a datum $A \xleftarrow{p} U^\perp \rightrightarrows X$ such that $\ker(p) = U$ and $\alpha|_{U^\perp} = \varphi|_{U^\perp}$. By the s-filtering property, there is an admissible subobject $B \in \mathcal{A}$ of U^\perp such that the composition $B \rightrightarrows U^\perp \rightarrow A$ is a deflation. Let $C \in \mathcal{A}$ be the kernel of the composition $B \rightarrow A$. It is an admissible subobject of U , and thus totally isotropic in X and defines a map $t : (C^\perp/C, \gamma) \rightarrow (X, \varphi)$ in $Q^h\mathcal{U}$, where γ is the (unique non-singular) form induced by φ via C^\perp . Since C is an object of \mathcal{A} , the map t is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$.

The two chains of inclusions $C \subset B, U \subset U^\perp \subset C^\perp$ from C to C^\perp induce a commutative diagram

$$\begin{array}{ccccc} & & B/C & & \\ & \cong \swarrow & \downarrow i_0 & \searrow i & \\ U^\perp/U & \longleftarrow & U^\perp/C & \longrightarrow & C^\perp/C \end{array}$$

in \mathcal{U} in which all maps are compatible with forms (since all of them are induced by φ). The lower row defines a map $(A, \alpha) \rightarrow (C^\perp/C, \gamma)$ in $Q^h\mathcal{U}$. Since the form α on $B/C = A$ is non-singular, the maps i_0 and i have retractions $r_0 = \alpha^{-1}i^*\gamma_0$ and $r = \alpha^{-1}i^*\gamma$, where γ_0 is the restriction of γ to U^\perp/C . Let $q_0 = i_0r_0$ and $q = ir$ be the corresponding projectors. Then the lower row decomposes as the direct sum of $A = \text{im}q_0 = \text{im}q$ and the map $l : 0 \rightarrow M$ in $Q\mathcal{U}$ represented by the datum

$$0 \leftarrow \text{im}(-q_0 + 1_{U^\perp/C}) \rightrightarrows M := \text{im}(-q + 1_{C^\perp/C})$$

(which is isomorphic to $0 \leftarrow U^\perp/B \rightrightarrows C^\perp/B$). Under this decomposition, the form γ corresponds to the orthogonal sum of α and $\mu := j^*\gamma j$, where $j : M \rightarrow \text{im}(-q + 1_{C^\perp/C}) \rightrightarrows C^\perp/C$ is the canonical inclusion. The form (M, μ) is non-singular since $(A, \alpha) \perp (M, \mu) \cong (C^\perp/C, \gamma)$ is non-singular. The map $l : 0 \rightarrow (M, \mu)$ in $Q^h\mathcal{U}$ satisfies $a = t \circ (id_A \perp l)$, by construction. \square

Lemma 8.16 *Let $u_i : A \rightarrow X_i$, $i = 1, 2$, be two maps in $Q^h(\mathcal{U})$ with A an object of $Q^h\mathcal{A}$. Assume that there is an isomorphism $s : \rho X_1 \xrightarrow{\cong} \rho X_2$ in $Q^h(\mathcal{U}/\mathcal{A})$ such that $s\rho(u_1) = \rho(u_2)$ in $Q^h(\mathcal{U}/\mathcal{A})$. Then there are maps $u : A \rightarrow V$ and $s_i : V \rightarrow X_i$*

in $Q^h\mathcal{U}$ such that $s_i u = u_i$, $sp(s_1) = \rho(s_2)$, and such that s_i is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$, $i = 1, 2$.

Proof: By lemma 8.15, we can assume $A = 0$. Let u_i be represented by the Lagrangians $a_i : L_i \twoheadrightarrow X_i$. By the calculus of fractions, there are inflations $b_i : L \twoheadrightarrow L_i$ with cokernel in \mathcal{A} such that $s \circ a_1 b_1 = a_2 b_2$ in \mathcal{U}/\mathcal{A} .

By the following argument, we can assume $b_i : L \twoheadrightarrow L_i$ to be split injective, that is, to have a retraction. By the s -filtering condition, there are admissible subobjects $B_i \subset L_i$ with B_i in \mathcal{A} such that $B_i \twoheadrightarrow L_i/L$ is a deflation in \mathcal{A} . Its kernel $C_i = \ker(B_i \twoheadrightarrow L_i/L)$ is an object of \mathcal{A} and a subobjects of L . By the filtering condition, we can choose an admissible subobject C of L in \mathcal{A} such that $C_1, C_2 \subset C \subset L$. Let $C^\perp \subset X_i$ be the orthogonal of C in X_i . Then u_i factors as $0 \rightarrow C^\perp/C \rightarrow X_i$, and $C^\perp/C \rightarrow X_i$ is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$. Moreover, the map $0 \rightarrow C^\perp/C$ is represented by the Lagrangian $L_i/C \subset C^\perp/C$. The inflation $L/C \twoheadrightarrow L_i/C$ in \mathcal{U} is split injective because the quotient map $L_i/C \twoheadrightarrow L_i/L$ has a section since the composition $B_i/C_i \twoheadrightarrow L_i/C_i \twoheadrightarrow L_i/C \twoheadrightarrow L_i/L$ is an isomorphism. Replacing X_i, L_i and L by $C^\perp/C, L_i/C, L/C$, we can assume $L \twoheadrightarrow L_i$ to be split injective.

Next, we will show that we can assume $b_i : L \twoheadrightarrow L_i$ to be the identity map. For that, let $A_i \in \mathcal{A}$ be a complement of L in L_i so that we have isomorphisms $L_i \cong L \oplus A_i$ under which the inclusion $L \subset L_i$ corresponds to the canonical inclusion $L \twoheadrightarrow L \oplus A_i$ of L into the first factor. We consider A_i embedded into $L_i = L \oplus A_i$ via the canonical inclusion into the second factor. Let $A_i^\perp \subset X_i$ be the orthogonal of A_i in X_i , and let Y_i be the symmetric space A_i^\perp/A_i with form induced by X_i . Then the map u_i in $Q^h\mathcal{U}$ factors as $0 \rightarrow Y_i \rightarrow X_i$ where $0 \rightarrow Y_i$ is given by the Lagrangian $L_i/A_i = L \subset Y_i$, and where the map $Y_i \rightarrow X_i$ is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$. Thus, in order to prove the lemma, we can assume $u_i : 0 \rightarrow X_i$ to be given by Lagrangians $a_i : L_i \twoheadrightarrow X_i$ with $L_1 = L_2 = L$.

By assumption, the map $s : \rho(X_1) \rightarrow \rho(X_2)$ is an isometry in \mathcal{U}/\mathcal{A} which fixes L . By the calculus of fractions, we can find weak isomorphisms $t_i : Y \xrightarrow{\sim} X_i$ in \mathcal{U} such that the forms on X_1 and X_2 restrict to the same form on Y , the two compositions $Y \rightarrow X_i \rightarrow X_i^* \rightarrow L^*$ agree, and $s = t_2 t_1^{-1}$ in \mathcal{U}/\mathcal{A} . Since $sa_1 = a_2$, hence $t_1^{-1}a_1 = t_2^{-1}a_2$ in \mathcal{U}/\mathcal{A} , lemma 8.5 allows us to find inflations $j : N \twoheadrightarrow L$ and $a : N \twoheadrightarrow Y$ in \mathcal{U} such that $a_i j = t_i a$ and such that the cokernel of j is in \mathcal{A} . By lemma 8.6 (b), the pairs (j, t_i) define weak isomorphisms in $\mathcal{I}(\mathcal{U})$, so that by lemma 8.5 (a) we can precompose them with a weak isomorphism so that the compositions are inflations in $\mathcal{I}(\mathcal{U})$ with cokernel in $\mathcal{I}(\mathcal{A})$. Thus, we can assume (j, t_i) to be inflations in $\mathcal{I}(\mathcal{U})$ with cokernel in $\mathcal{I}(\mathcal{A})$, so that the natural maps $Z := Y \sqcup_N L \rightarrow X_i$ are inflations in \mathcal{U} (with cokernels in \mathcal{A}). The forms on X_i , $i = 1, 2$ restrict to the same form on Z since the two compositions $Y \twoheadrightarrow X_i \rightarrow X_i^* \rightarrow L^*$, $i = 1, 2$, agree, and the forms on X_i restrict to the same form in Y . It follows that the orthogonal of

Z in X_i is independent of $i = 1, 2$ as it is the kernel of the form $Z \rightarrow Z^*$. The forms on X_i induce a non-singular symmetric form on $V = Z/Z^\perp$ which is independent of $i = 1, 2$, since their restrictions to Z agree. The Lagrangian $L/Z^\perp \twoheadrightarrow Z/Z^\perp$ defines a map $u : 0 \rightarrow V$ in $Q^h\mathcal{U}$, and we have maps $s_i : V \rightarrow X_i$ in $Q^h\mathcal{U}$ given by the datum $V = Z/Z^\perp \leftarrow Z \twoheadrightarrow X_i$ such that $s_i u = u_i$, and $s \circ \rho(s_1) = \rho(s_2)$. Moreover, s_i is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$. \square

Lemma 8.17 *Let $x : A \rightarrow X$, $u_i : X \rightarrow Y$, $i = 1, 2$ be maps in $Q^h(\mathcal{U})$ with A in $Q^h\mathcal{A}$ and $\rho(u_i)$ isomorphisms in $Q^h(\mathcal{U}/\mathcal{A})$ satisfying $u_1 x = u_2 x$ and $\rho(u_1) = \rho(u_2)$. Then there are maps $z : A \rightarrow Z$, $v : Z \rightarrow X$ in $Q^h(\mathcal{U})$ such that $\rho(v)$ is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$, $u_1 v = u_2 v$, and $x = vz$.*

Proof: Let α , φ and ψ be the forms on A , X and Y . The maps $x : A \rightarrow X$ and $u_i : X \rightarrow Y$ in $Q^h\mathcal{U}$ are represented by diagrams $A \xleftarrow{p} V \xrightarrow{j} X$ and $X \xleftarrow{q_i} U_i \xrightarrow{s_i} Y$.

The compositions $u_i x$ are represented by diagrams $A \xleftarrow{p\bar{q}_i} W_i \xrightarrow{s_i\bar{j}_i} Y$, where W_i is the pull-back of j along the deflation q_i with $\bar{q}_i : W_i \twoheadrightarrow V$ and $\bar{j}_i : W_i \twoheadrightarrow U_i$ the induced maps. The assumption that $u_1 x = u_2 x$ in $Q^h\mathcal{U}$ implies the existence of an isomorphism $g : W_1 \rightarrow W_2$ such that $s_1\bar{j}_1 = s_2\bar{j}_2 g$ and $p\bar{q}_1 = p\bar{q}_2 g$. Replacing \bar{j}_2 and \bar{q}_2 with $\bar{j}_2 g$ and $\bar{q}_2 g$, we can assume $W_1 = W_2 =: W$ and $g = id_W$, so that $s_1\bar{j}_1 = s_2\bar{j}_2$ and $p\bar{q}_1 = p\bar{q}_2$. The maps x , u_i and their compositions are depicted in the following diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{\bar{j}_i} & U_i & \xrightarrow{\sim s_i} & Y \\
 \bar{q}_i \downarrow \wr & & \downarrow \wr q_i & & \\
 V & \xrightarrow{j} & X & & \\
 p \downarrow & & & & \\
 A & & & &
 \end{array}$$

In \mathcal{U}/\mathcal{A} , we have the two equations $j\bar{q}_i = (q_i s_i^{-1}) \circ (s_i \bar{j}_i)$, $i = 1, 2$, in which the right-hand side is independent of i (in \mathcal{U}/\mathcal{A} , we have $q_1 s_1^{-1} = q_2 s_2^{-1}$ since $u_1 = u_2$). Since j is an inflation in \mathcal{U}/\mathcal{A} , this implies that $\bar{q}_1 = \bar{q}_2$ in \mathcal{U}/\mathcal{A} . By the dual of Lemma 8.7, we can therefore write p as a composition $p_2 \circ p_1$ with $p_1 : V \twoheadrightarrow \bar{V}$ and $p_2 : \bar{V} \twoheadrightarrow A$ deflations, $\ker(p_1)$ in \mathcal{A} and $p_1\bar{q}_i$ independent of $i = 1, 2$. Since $\ker(p_1) \subset \ker(p)$, we obtain a symmetric space $\bar{Z} = \ker(p_1)^\perp / \ker(p_1)$ equipped with the form induced by the form φ on X . In $Q^h\mathcal{U}$, the map $x : A \rightarrow X$ factors as $A \rightarrow \bar{Z} \rightarrow X$, where the individual maps are given by the data $A \leftarrow \bar{V} = V/\ker(p_1) \twoheadrightarrow \ker(p_1)^\perp / \ker(p_1) = \bar{Z}$ and $\bar{Z} \leftarrow \ker(p_1)^\perp \twoheadrightarrow X$. Note that $\bar{Z} \rightarrow X$ is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$, so that (for the purpose of the lemma), replacing X by \bar{Z} allows us to assume $\bar{q}_1 = \bar{q}_2$ in the diagram above. Since $s_1\bar{j}_1 = s_2\bar{j}_2$, and

since U_i is the orthogonal of $\ker(\bar{q}_1) = \ker(\bar{q}_2)$ in Y , we can also assume $U_1 = U_2$, $\bar{j}_1 = \bar{j}_2$ and $s_1 = s_2$ in the diagram above.

The equality $u_1 = u_2$ of isomorphisms in $Q^h(\mathcal{U}/\mathcal{A})$ implies that $q_1 \circ s_1^{-1} = q_2 \circ s_2^{-1}$ in \mathcal{U}/\mathcal{A} . Since we assume $s_1 = s_2$, this implies $q_1 = q_2$ in \mathcal{U}/\mathcal{A} . In \mathcal{U} , the compositions $j^* \varphi q_i$ are independent of $i = 1, 2$ since their composition with the inflation $\bar{q}_1^* = \bar{q}_2^*$ is $\bar{j}_i^* \psi s_i^*$ which is independent of i . Another application of the dual of Lemma 8.7 yields a factorization of $j^* \varphi$ into $r_2 \circ r_1$ where $r_1 : X \twoheadrightarrow \bar{X}$ and $r_2 : \bar{X} \twoheadrightarrow V^*$ are deflations, $\ker(r_1)$ is in \mathcal{A} and $r_1 q_i$ is independent of i . As above $\ker(r_1) \subset \ker(p) = \ker(j^* \varphi)$ so that the map $x : A \rightarrow X$ in $Q^h \mathcal{U}$ factors as $A \xrightarrow{z} Z = \ker(r_1)^\perp / \ker(r_1) \xrightarrow{v} X$. Since $\ker(r_1)$ is in \mathcal{A} , the map v is an isomorphism in $Q^h(\mathcal{U}/\mathcal{A})$. By construction, we have $u_1 v = u_2 v$. \square

Corollary 8.18 *Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving s -filtering inclusion of exact categories with duality. Assume that \mathcal{A} is idempotent complete. Then the sequence $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of Grothendieck-Witt spaces*

$$GW(\mathcal{A}) \longrightarrow GW(\mathcal{U}) \longrightarrow GW(\mathcal{U}/\mathcal{A}).$$

Proof: In view of the definition of the Grothendieck-Witt space, this follows from theorem 8.2 and its K -theory analog [Sch04a, Theorem 2.1] which is theorem 8.2 applied to the sequence of hyperbolic categories $H\mathcal{A} \rightarrow H\mathcal{U} \rightarrow H(\mathcal{U}/\mathcal{A})$. \square

9. Cone and Suspension in Hermitian K -theory

In §8, we constructed an exact category with duality \mathcal{U}/\mathcal{A} associated with a duality preserving s -filtering inclusion $\mathcal{A} \subset \mathcal{U}$, and we obtained a homotopy fibration of Grothendieck-Witt spaces induced by the sequence $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{U}/\mathcal{A}$ (when \mathcal{A} is idempotent complete). Unfortunately, the s -filtering condition is rather artificial, and it is rarely present in applications. The aim of this section is to construct an exact category with duality $\mathcal{C}(\mathcal{U}, \mathcal{A})$ which plays the role of a quotient \mathcal{U}/\mathcal{A} in hermitian K -theory and which is attached to any duality preserving fully exact inclusion $\mathcal{A} \subset \mathcal{U}$ of idempotent complete exact categories with duality. The category $\mathcal{C}(\mathcal{U}, \mathcal{A})$, called cone of $\mathcal{A} \subset \mathcal{U}$, contains \mathcal{U} , and the sequence $\mathcal{A} \rightarrow \mathcal{U} \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ induces a homotopy fibration in hermitian K -theory (see theorem 9.7).

We begin with constructing the category $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ a localization of which will be the cone category $\mathcal{C}(\mathcal{U}, \mathcal{A})$.

9.1. The category $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$

Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving fully exact inclusion of exact categories with duality (2.1 and 3.2). As in §8, we will depict inflations in \mathcal{U} with cokernel in \mathcal{A} as

\succsim , deflations in \mathcal{U} with kernel in \mathcal{A} as \rightsquigarrow , and compositions of these two types of maps as $\overset{\sim}{\rightarrow}$ in diagrams.

We define a category $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Its objects are commutative diagrams in \mathcal{U} of the form

$$\begin{array}{ccccccc} U_0 & \overset{\sim}{\rightarrow} & U_1 & \overset{\sim}{\rightarrow} & U_2 & \overset{\sim}{\rightarrow} & U_3 & \overset{\sim}{\rightarrow} & \dots \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ U^0 & \overset{\sim}{\leftarrow} & U^1 & \overset{\sim}{\leftarrow} & U^2 & \overset{\sim}{\leftarrow} & U^3 & \overset{\sim}{\leftarrow} & \dots \end{array} \quad (18)$$

such that

- (a) for all $n \leq m \in \mathbb{N}$, the maps $U_n \overset{\sim}{\rightarrow} U_m$ and $U^m \overset{\sim}{\rightarrow} U^n$ are inflations with cokernel in \mathcal{A} and deflations with kernel in \mathcal{A} , respectively, and
- (b) there is a $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the maps $U_n \overset{\sim}{\rightarrow} U^{n+k}$ are inflations with cokernel in \mathcal{A} , and the maps $U_{n+k} \overset{\sim}{\rightarrow} U^n$ are deflations with kernel in \mathcal{A} .

Maps in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ are natural transformations of diagrams. We will abbreviate such a diagram as $(U_\bullet \xrightarrow{u} U^\bullet)$, where $U_\bullet : \mathbb{N} \rightarrow \mathcal{U} : n \mapsto U_n$ is the functor representing the first row, $U^\bullet : \mathbb{N}^{op} \rightarrow \mathcal{U} : n \mapsto U^n$ is the functor representing the second row, and u is the matrix (u_{ji}) of maps $u_{ji} : U_i \rightarrow U^j$ in \mathcal{U} , $i, j \in \mathbb{N}$, such that $U_i \rightarrow U_j \rightarrow U^k \rightarrow U^l$ is $U_i \rightarrow U^l$ whenever $i \leq j$, and $l \leq k$.

It is clear that $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is an additive category. Declare a sequence in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ to be exact if it is exact at each of the U_i and U^j spots. To see that this makes $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ into an exact category, we consider it as a full subcategory of the functor category $\text{Fun}(\mathcal{N}, \mathcal{U})$ (which itself is an exact category, see 3.4) where \mathcal{N} denotes the totally ordered set $\mathbb{N} \sqcup \mathbb{N}^{op}$ in which every element of \mathbb{N}^{op} is greater than every element of \mathbb{N} . Since inflations with cokernel in \mathcal{A} and deflations with kernel in \mathcal{A} are closed under extensions in the exact category of morphisms in \mathcal{U} , the category $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is closed under extensions in $\text{Fun}(\mathcal{N}, \mathcal{U})$. This shows that the choice of exact sequences above makes $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ into an exact category.

The category \mathcal{N} is a category with duality where the duality interchanges \mathbb{N} and \mathbb{N}^{op} so that the double dual identification is the identity. Therefore, the category of functors $\text{Fun}(\mathcal{N}, \mathcal{U})$ is canonically equipped with an exact duality (see 3.4). The full subcategory $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is closed under this duality, so that, by restriction of the duality, we equip $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ with the induced structure of an exact category with duality. This means that the dual of an object $U = (U_\bullet \xrightarrow{u} U^\bullet)$ of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is $U^* = ((U^\bullet)^* \xrightarrow{u^*} (U_\bullet)^*)$.

The constant diagram functor $C : \mathcal{U} \rightarrow \mathcal{C}_0(\mathcal{U}, \mathcal{A})$ which sends an object U of \mathcal{U} to the diagram $CU = (U \xrightarrow{1} U)$ of the form (18) in which all maps are the identity on U , is fully faithful, exact, reflects exactness and is duality preserving.

9.2. The hermitian cone $\mathcal{C}(\mathcal{U}, \mathcal{A})$

For an object $U = (U_\bullet \xrightarrow{u} U^\bullet)$ of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ given by diagram (18), we can forget the upper left corner U_0 . This gives us a new object $U_{[1]} = (U_{\bullet+1} \xrightarrow{u_{[1]}} U^\bullet)$ where $(U_{\bullet+1})_n = U_{n+1}$ and $(u_{[1]})_{ji} = u_{j,i+1}$. We have a canonical map $e_{[1]} : U \rightarrow U_{[1]}$ induced by the structure maps of U_\bullet and the identity on U^\bullet . Similarly, forgetting the lower left corner U^0 in diagram (18) defines a new object $U^{[1]} = (U_\bullet \xrightarrow{u^{[1]}} U^{\bullet+1})$ and a canonical map $e^{[1]} : U^{[1]} \rightarrow U$, where $(U^{\bullet+1})^n = U^{n+1}$. We define the objects $U_{[k]}$ and $U^{[k]}$ iteratively by the formulas $U_{[k]} = (U_{[k-1]})_{[1]}$ and $U^{[k]} = (U^{[k-1]})^{[1]}$. For $i = 1, \dots, k$, composition of the canonical maps $e_{[i]} : U_{[i-1]} \rightarrow U_{[i]}$ yields a map $e_{[k]} : U \rightarrow U_{[k]}$, and similarly the maps $e^{[i]} : U^{[i]} \rightarrow U^{[i-1]}$ define a map $e^{[k]} : U^{[k]} \rightarrow U$ for $k \in \mathbb{N}$. Note that $(U^{[i]})_{[j]} = (U_{[j]})^{[i]}$ and that the diagram

$$\begin{array}{ccc} U^{[i]} & \xrightarrow{e^{[i]}} & U \\ e_{[j]} \downarrow & & \downarrow e_{[j]} \\ U_{[j]}^{[i]} & \xrightarrow{e^{[i]}} & U_{[j]} \end{array}$$

commutes. Clearly, the assignments $U \mapsto U^{[k]}$ and $U \mapsto U_{[j]}$ are functorial for objects in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, and the canonical maps $e_{[k]} : U \rightarrow U_{[k]}$ and $e^{[k]} : U^{[k]} \rightarrow U$ are natural transformations.

Definition 9.1 Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving fully exact inclusion of exact categories with duality. The category $\mathcal{C}(\mathcal{U}, \mathcal{A})$, called *cone* of $\mathcal{A} \subset \mathcal{U}$, is the localization of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ with respect to the set of maps $e_{[i]}(U) : U \rightarrow U_{[i]}$ and $e^{[i]}(U) : U^{[i]} \rightarrow U$, for $U \in \mathcal{C}_0(\mathcal{U}, \mathcal{A})$ and $i \in \mathbb{N}$. In detail, the category $\mathcal{C}(\mathcal{U}, \mathcal{A})$ has objects the objects of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, and morphism sets

$$\text{hom}_{\mathcal{C}(\mathcal{U}, \mathcal{A})}(U, V) = \text{colim}_{i, j \in \mathbb{N}} \text{hom}_{\mathcal{C}_0(\mathcal{U}, \mathcal{A})}(U^{[i]}, V_{[j]})$$

for objects U, V of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, where the colimit is taken over the projective system $\dots U^{[3]} \rightarrow U^{[2]} \rightarrow U^{[1]} \rightarrow U$ and over the inductive system $V \rightarrow V_{[1]} \rightarrow V_{[2]} \rightarrow V_{[3]} \rightarrow \dots$. Composition of maps is defined as follows. If the maps $[f] : U \rightarrow V$ and $[g] : V \rightarrow W$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ are represented by the maps $f : U^{[i]} \rightarrow V_{[j]}$ and $g : V^{[k]} \rightarrow W_{[l]}$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, their composition $[g] \circ [f]$ is represented by the map

$g_{[j]} \circ f^{[k]} : U^{[i+k]} \rightarrow V_{[j]}^{[k]} \rightarrow W_{[i+j]}$. Note that the identity map $U \rightarrow U$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ is represented by any of the maps $e_{[i]} : U \rightarrow U_{[i]}$ and $e^{[j]} : U^{[j]} \rightarrow U$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Therefore, the maps $U \rightarrow U_{[i]}$ and $U^{[j]} \rightarrow U$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ represented by $e_{[i]}$ and $e^{[j]}$ are isomorphisms in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ with inverses $U_{[i]} \rightarrow U$ and $U \rightarrow U^{[j]}$ represented by $1 : U_{[i]} \rightarrow U_{[i]}$ and $1 : U^{[j]} \rightarrow U^{[j]}$. The identity on objects and morphisms defines a functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ which satisfies the universal property of the localization of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ with respect to the set of maps $U \rightarrow U_{[i]}$ and $U^{[j]} \rightarrow U$, where $i, j \in \mathbb{N}$ and U runs through the objects of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Note that, since the maps $e^{[i]}$ and $e_{[j]}$ are epic and monic, respectively, the functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is faithful.

We define a sequence in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ to be a conflation if and only if it is isomorphic in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ to the image under the localization functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ of a conflation of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Moreover, the duality on $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ defines a duality on $\mathcal{C}(\mathcal{U}, \mathcal{A})$ such that the functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is duality preserving.

Lemma 9.2 *Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving fully exact inclusion of exact categories with duality. Then the cone category $\mathcal{C}(\mathcal{U}, \mathcal{A})$ is an exact category with duality.*

Proof: We first show that the functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ sends push-outs of arbitrary maps along inflations to cocartesian squares and pull-backs of arbitrary maps along deflations to cartesian squares. We will only prove the first statement, the second being dual. Consider a cocartesian square in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ b \downarrow & & \downarrow \bar{b} \\ W & \xrightarrow{\bar{a}} & X \end{array}$$

with a an inflation. Let $[f] : V \rightarrow Y$ and $[g] : W \rightarrow Y$ be maps in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ represented by the maps $f : V^{[i]} \rightarrow Y_{[j]}$ and $g : W^{[k]} \rightarrow Y_{[l]}$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ such that $[g] \circ [b] = [f] \circ [a]$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$. Composing and pre-composing f and g with the appropriate $e_{[n]}$ and $e^{[n]}$ (this doesn't change the classes $[f]$ and $[g]$), we can assume $i = k$ and $j = l$. Since $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is faithful, we have a commutative diagram in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$

$$\begin{array}{ccc} U^{[i]} & \xrightarrow{a^{[i]}} & V^{[i]} \\ b^{[i]} \downarrow & & \downarrow \bar{b}^{[i]} \\ W^{[i]} & \xrightarrow{\bar{a}^{[i]}} & X^{[i]} \end{array} \begin{array}{c} \searrow f \\ \searrow g \\ \searrow \end{array} \begin{array}{c} \\ \\ Y_{[i]} \end{array}$$

The upper left square is cocartesian in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ because the functor $U \mapsto U^{[i]}$ is exact. The universal property of push-out squares yields a map $h : X^{[i]} \rightarrow Y_{[j]}$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ such that $f = h \circ \bar{b}^{[i]}$ and $g = h \circ \bar{a}^{[i]}$. The map h represents a map $[h] : X \rightarrow Y$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ such that $[f] = [h] \circ [\bar{b}]$ and $[g] = [h] \circ [\bar{a}]$. To show uniqueness of the map $[h]$, consider a diagram in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ of the form

$$\begin{array}{ccc}
 U^{[i]} & \xrightarrow{a^{[i]}} & V^{[i]} \\
 \downarrow b^{[i]} & & \downarrow \bar{b}^{[i]} \\
 W^{[i]} & \xrightarrow{\bar{a}^{[i]}} & X^{[i]} \\
 & \searrow 0 & \searrow h \\
 & & Y_{[j]}
 \end{array}$$

which commutes in $\mathcal{C}(\mathcal{U}, \mathcal{A})$. Since $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is faithful, the diagram also commutes in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. But then $h = 0$ because the upper left square is cocartesian in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$.

Now we are ready to check the axioms 2.1 (a) – (f) for $\mathcal{C}(\mathcal{U}, \mathcal{A})$. Axiom (a) is satisfied because in a conflation (1), Z is the push-out of $X \rightarrow 0$ along i , and X is the pull-back of $0 \rightarrow Z$ along p , and – as we have seen – this property is preserved under the localization functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$.

Axiom (b) holds by definition. Axiom (c) can be proved as follows. Given inflations $a : U \twoheadrightarrow V$ and $b : W \twoheadrightarrow X$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ and an isomorphism $[f] : V \rightarrow W$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ with inverse $[g] : W \rightarrow V$ represented by $g : W^{[i]} \rightarrow V_{[j]}$. Let Y be the push-out in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ of g along $b^{[i]}$. As we have seen, $X^{[i]} \rightarrow Y$ is an isomorphism in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ since $[g]$ is. We have a commutative diagram in $\mathcal{C}(\mathcal{U}, \mathcal{A})$

$$\begin{array}{ccccccc}
 U & \xrightarrow{a} & V & \xrightarrow{[f]} & W & \xrightarrow{b} & X \\
 \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
 U_{[j]} & \xrightarrow{a_{[j]}} & V_{[j]} & \xleftarrow{g} & W^{[i]} & \xrightarrow{b^{[i]}} & X^{[i]} \\
 & & & \searrow & & \searrow & \\
 & & & & & & Y
 \end{array}$$

in which the solid arrows are maps in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, solid squares commute in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, arrows labeled with \sim are isomorphisms in $\mathcal{C}(\mathcal{U}, \mathcal{A})$, the dashed arrow is a map in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ and the non-labeled vertical maps are the maps $e^{[i]}$ and $e_{[j]}$. This shows that $[b] \circ [f] \circ [a]$ is an inflation in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ because the diagram displays it as being isomorphic in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ to the image of the inflation $U_{[j]} \twoheadrightarrow Y$. By a dual argument, deflations are closed under composition in $\mathcal{C}(\mathcal{U}, \mathcal{A})$.

Axiom (d) can be proved as follows. Let $[f] : U \rightarrow W$ and $[g] : U \rightarrow V$ be maps in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ with $[f]$ an inflation. Changing $[f]$ up to isomorphism, we can assume $[f]$ to be represented by an inflation $f : U \twoheadrightarrow W$ of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Let $[g]$ be represented by $g : U^{[i]} \rightarrow V_{[j]}$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Consider the commutative diagram in $\mathcal{C}(\mathcal{U}, \mathcal{A})$

$$\begin{array}{ccccc} V & \xleftarrow{[g]} & U & \xrightarrow{f} & W \\ \wr \downarrow & & \wr \uparrow & & \wr \uparrow \\ V_{[j]} & \xleftarrow{g} & U^{[i]} & \xrightarrow{f^{[i]}} & W^{[i]}. \end{array}$$

As we have seen, the push-out of $f^{[i]}$ along g in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ induces a cocartesian diagram in $\mathcal{C}(\mathcal{U}, \mathcal{A})$. Therefore, the push-out of $[f]$ along $[g]$ exists in $\mathcal{C}(\mathcal{U}, \mathcal{A})$. It is an inflation in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ as it is isomorphic to the push-out of $f^{[i]}$ along g in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. A dual argument shows axiom (e). Axiom (f) holds in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ since it holds in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Summarizing, we have shown that $\mathcal{C}(\mathcal{U}, \mathcal{A})$ is an exact category.

The exact duality on $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ satisfies $(U^{[i]})^* = (U^*)_{[i]}$, $(U_{[i]})^* = (U^*)^{[i]}$, $(e_{[i]}(U))^* = e^{[i]}(U^*)$ and $(e^{[i]}(U))^* = e_{[i]}(U^*)$. Therefore, it induces an exact duality functor on $\mathcal{C}(\mathcal{U}, \mathcal{A})$ such that the localization functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is duality preserving. \square

Lemma 9.3 *The constant diagram functor $\mathcal{C} : \mathcal{U} \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is fully faithful, preserves and reflects exactness, and is duality preserving. It makes \mathcal{U} into an s -filtering subcategory of $\mathcal{C}(\mathcal{U}, \mathcal{A})$.*

Proof: From the definition of $\mathcal{C}(\mathcal{U}, \mathcal{A})$ it is clear that the functor $\mathcal{U} \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is fully faithful, exact and duality preserving. We claim that if an object X of $\mathcal{C}(\mathcal{U}, \mathcal{A})$ is isomorphic in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ to a constant diagram CU , for some U in \mathcal{U} , then there are $i, j \in \mathbb{N}$ such that $X_{[j]}^{[i]}$ is isomorphic to CU in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, that is, all structure maps for $X_{[j]}^{[i]}$ are isomorphisms. To prove the claim, let $f : CU^{[i]} \rightarrow X_{[j]}$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ represent an isomorphism $[f] : CU \rightarrow X$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$, and let $[g] : X \rightarrow CU$ be an inverse of $[f]$ represented by $g : X^{[k]} \rightarrow CU_{[l]}$. The composition $g_{[j]} \circ f^{[k]} : CU^{[i+k]} \rightarrow CU_{[j+l]}$ is $e_{[j+l]} \circ e^{[i+k]}$ since $g_{[j]} \circ f^{[k]}$ represents the identity $1 = [g][f]$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$, and $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is faithful. But $e_{[j+l]} \circ e^{[i+k]} : CU^{[i+k]} \rightarrow CU_{[j+l]}$, and thus $g_{[j]} \circ f^{[k]}$ is an isomorphism in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Therefore, $f^{[k]}$ has a retraction r with $rf^{[k]} = 1$. The projector $1 - f^{[k]}r$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is zero in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ because $f^{[k]}$ is an isomorphism in $\mathcal{C}(\mathcal{U}, \mathcal{A})$. Since the localization functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ is faithful, $1 - f^{[k]}r$ is already zero in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, so that $f^{[k]}$ has to be an isomorphism in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. It follows that \mathcal{U} is extension closed in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ since it is extension closed in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, and the inclusion $\mathcal{U} \subset \mathcal{C}(\mathcal{U}, \mathcal{A})$ preserves and detects conflations.

Now we check the s-filtering conditions 8.1 (a) – (d). For every object X of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, the structure maps of X define a map $C(X_0) \rightarrow X$ such that every map $C(U) \rightarrow X$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ factors (uniquely) through $C(X_0) \rightarrow X$. By property 9.1 (b), there is a $k \in \mathbb{N}$ such that the map $C(X_0) \rightarrow X^{[k]}$ is an inflation in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. Therefore, a map $[f] : C(U) \rightarrow X$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ represented by $f : C(U)^{[l]} \rightarrow X_{[l]}$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ factors through the inflation $C(X_0) \rightarrow X_{[l]}^{[k]}$, for some k , representing the inflation $C(X_0) \rightarrow X$ in $\mathcal{C}(\mathcal{U}, \mathcal{A})$. This shows 8.1 (a). Part (b) is dual, and we omit the proof.

For part (c), let $X \twoheadrightarrow C(U)$ be a deflation in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ with constant target. Changing X and $C(U)$ up to isomorphism in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ (and truncating the diagram X if necessary), we can assume $X \twoheadrightarrow C(U)$ to be a deflation in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$. But then the composition $C(X_0) \rightarrow X \rightarrow C(U)$ is a deflation in \mathcal{U} , and the map $C(X_0) \rightarrow X$ is an inflation in $\mathcal{C}(\mathcal{U}, \mathcal{A})$, as we have already seen. This shows (c). Part (d) is dual and we omit the proof. \square

Lemma 9.4 *Let $\mathcal{A} \subset \mathcal{U}$ be a fully exact inclusion of exact categories. Then the induced functor $\mathcal{C}(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ yields an equivalence of quotient exact categories*

$$\mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A} \xrightarrow{\sim} \mathcal{C}(\mathcal{U}, \mathcal{A})/\mathcal{U}.$$

Proof: We define the inverse of the functor in the lemma in several steps. First, we construct a functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$ with image in the full subcategory $\mathcal{C}(\mathcal{A}, \mathcal{A}) \subset \mathcal{C}(\mathcal{U}, \mathcal{A})$. To do so, recall that for an object U of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, there is an integer $k_U \in \mathbb{N}$ such that the canonical map $C(U_0) \twoheadrightarrow U^{[k]}$ is an inflation with cokernel in $\mathcal{C}_0(\mathcal{A}, \mathcal{A})$ for all $k \geq k_U$. If $l \geq k$ then the canonical map $\varepsilon^{[k-l]} : U^{[l]}/C(U_0) \rightarrow U^{[k]}/C(U_0)$, induced by $e^{[k-l]}$, is an isomorphism in $\mathcal{C}(\mathcal{U}, \mathcal{A})$, and hence in $\mathcal{C}(\mathcal{A}, \mathcal{A})$, since in the map of exact sequences in $\mathcal{C}(\mathcal{U}, \mathcal{A})$

$$\begin{array}{ccccc} C(U_0) & \twoheadrightarrow & U^{[l]} & \twoheadrightarrow & U^{[l]}/C(U_0) \\ \parallel & & \downarrow e^{[l-k]} & & \downarrow \varepsilon^{[l-k]} \\ C(U_0) & \twoheadrightarrow & U^{[k]} & \twoheadrightarrow & U^{[k]}/C(U_0) \end{array}$$

the left two vertical maps are isomorphisms. In order to construct the functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}, \mathcal{A})$, choose, for every object U of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, an integer $k_U \in \mathbb{N}$ as above. The functor we wish to construct sends an object U of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ to the object $U^{[k_U]}/C(U_0)$ of $\mathcal{C}(\mathcal{U}, \mathcal{A})$ and a map $f : U \rightarrow V$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ to the composition $\varepsilon^{[m-k_V]} \circ f^{[m]} \circ (\varepsilon^{[m-k_U]})^{-1}$, where m is any integer $m \geq k_U, k_V$. The functor has image in the full subcategory $\mathcal{C}(\mathcal{A}, \mathcal{A}) \subset \mathcal{C}(\mathcal{U}, \mathcal{A})$ so that we obtain a functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A}) : U \mapsto U^{[k_U]}/C(U_0)$. It preserves exact sequences, since a

conflation $U \twoheadrightarrow V \twoheadrightarrow W$ in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is sent (up to isomorphism) to the conflation $U^{[k]}/\mathcal{C}(U_0) \twoheadrightarrow V^{[k]}/\mathcal{C}(V_0) \twoheadrightarrow W^{[k]}/\mathcal{C}(W_0)$ in $\mathcal{C}(\mathcal{A}, \mathcal{A})$, where $k \geq k_U, k_V, k_W$.

Composition with the localization functor $\mathcal{C}(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$ yields a well-defined exact functor $\mathcal{C}_0(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A} : U \mapsto U^{[k_U]}/\mathcal{C}(U_0)$. For every object U of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, this functor sends the canonical maps $e^{[k]}(U)$ and $e_{[k]}(U)$ to isomorphisms in $\mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$. This is clear for the maps $e^{[k]}(U)$. The image of the map $e_{[k]} : U \rightarrow U_{[k]}$ under this functor is, up to isomorphism, the composition $U^{[l]}/\mathcal{C}(U_0) \rightarrow U_{[k]}^{[l]}/\mathcal{C}(U_0) \twoheadrightarrow U_{[k]}^{[l]}/\mathcal{C}(U_k)$ for $l \geq k_U, k_{U_{[k]}}$. In this composition, the first map is an isomorphism in $\mathcal{C}(\mathcal{U}, \mathcal{A})$ (and hence in $\mathcal{C}(\mathcal{A}, \mathcal{A})$) as it is the cokernel of the map of morphisms from $id : \mathcal{C}(U_0) \rightarrow \mathcal{C}(U_0)$ to $e_{[k]} : U^{[l]} \rightarrow U_{[k]}^{[l]}$, and the second map is a deflation with kernel U_k/U_0 in \mathcal{A} , hence an isomorphism in $\mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$. Therefore, the maps $e_{[k]}, e^{[k]}$ are sent to isomorphisms, so that we obtain a well-defined exact functor $\mathcal{C}(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$. Clearly, this functor sends $\mathcal{U} \subset \mathcal{C}(\mathcal{U}, \mathcal{A})$ to zero in $\mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$ so that we have an induced exact functor $\mathcal{C}(\mathcal{U}, \mathcal{A})/\mathcal{U} \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$. This functor is inverse to the functor of the lemma. \square

Lemma 9.5 *Let \mathcal{A} be an exact category with duality. There is a non-singular exact form functor $T : \mathcal{C}(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})$ and an isometry of form functors*

$$id \perp T \cong T.$$

Proof: We define a functor $[-1] : \mathcal{C}_0(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}_0(\mathcal{A}, \mathcal{A})$. It sends $A = (A_\bullet \xrightarrow{a} A^\bullet)$ to $A[-1] = (A_\bullet[-1] \xrightarrow{a[-1]} A^\bullet[-1])$, where for $i, j \geq 1$, we set $(A[-1])_i = A_{i-1}$, $A[-1]^i = A^{i-1}$ and $a[-1]_{i,j} = a_{i-1,j-1}$ and $A[-1]_0 = A[-1]^0 = 0$. Note that the functor $[-1]$ is exact and duality preserving. We write $[-l]$ for the l -th iteration $[-1]^l$ of $[-1]$. Let $T : \mathcal{C}_0(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}_0(\mathcal{A}, \mathcal{A})$ be the duality preserving exact functor

$$T = \bigoplus_{l \geq 0} [-l] : A \mapsto \bigoplus_{l \geq 0} A[-l].$$

The functor makes sense, since for every $i \in \mathbb{N}$, the sums $(\bigoplus_{l \geq 0} A[-l])_i$ and $(\bigoplus_{l \geq 0} A[-l])^i$ are finite. Note that we have an isometry of duality preserving functors $id \perp T[-1] \cong T$.

In order to see that T induces a functor $\mathcal{C}(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})$, we have to check that the maps $T(e^{[1]}(A))$ and $T(e_{[1]}(A))$ are isomorphisms in $\mathcal{C}(\mathcal{A}, \mathcal{A})$ for every object A of $\mathcal{C}_0(\mathcal{A}, \mathcal{A})$. For $l \in \mathbb{N}$, we define a natural transformation $\alpha_A^l : A[-l]^{[1]} \rightarrow A^{[1]}[-l]$, $A \in \mathcal{C}_0(\mathcal{A}, \mathcal{A})$, which is the identity at all $(\)^i$ and $(\)_i$ spots except at the $(\)^{l-1}$ -spot where it is (necessarily) the zero map. One checks that

$$e^{[1]}(A[-l]) = (e^{[1]}(A))[-l] \circ \alpha_A^l, \quad \text{and} \quad e_{[1]}(A^{[1]}[-l]) = \alpha_A^l \circ ((e^{[1]}(A))[-l])^{[1]}.$$

The natural transformations α_A^l assemble to a natural transformation $\alpha_A = \bigoplus_{l \geq 0} \alpha_A^l : (TA)^{[1]} \rightarrow T(A^{[1]})$ such that the following diagram commutes

$$\begin{array}{ccccc}
 TA & \xleftarrow{T(e^{[1]})} & T(A^{[1]}) & & \\
 & \searrow e^{[1]} & \uparrow \alpha_A & \swarrow e^{[1]} & \\
 & & (TA)^{[1]} & \xleftarrow{(T(e^{[1]}))^{[1]}} & (T(A^{[1]}))^{[1]}
 \end{array}$$

In the diagram, the diagonal maps are isomorphisms in $\mathcal{C}(\mathcal{A}, \mathcal{A})$. It follows that $T(e^{[1]}(A))$ is an isomorphism in $\mathcal{C}(\mathcal{A}, \mathcal{A})$ for all objects A of $\mathcal{C}_0(\mathcal{A}, \mathcal{A})$. Similarly, the maps $T(e_{[1]}(A))$ are isomorphisms in $\mathcal{C}(\mathcal{A}, \mathcal{A})$. Therefore, the duality preserving exact functor T induces a duality preserving exact functor $T : \mathcal{C}(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})$.

The diagram of functors and natural transformations in $\mathcal{C}_0(\mathcal{A}, \mathcal{A})$

$$\begin{array}{ccc}
 id = [-1]_{[1]}^{[1]} & \xleftarrow{e^{[1]}} & [-1]^{[1]} \\
 \downarrow e^{[1]} & & \downarrow e^{[1]} \\
 [-1]_{[1]} & \xleftarrow{e^{[1]}} & [-1]
 \end{array}$$

commutes and induces an isometry $[-1] \rightarrow id$ of form functors $\mathcal{C}(\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{A})$. Thus we have an isometry of form functors $T[-1] \cong T$ which transforms the isometry $id \perp T[-1] \cong T$ into an isometry $id \perp T \cong T$. \square

Corollary 9.6 *For any exact category with duality \mathcal{A} , the cone category $\mathcal{C}(\mathcal{A}, \mathcal{A})$ has contractible Q^h - and Grothendieck-Witt spaces:*

$$Q^h\mathcal{C}(\mathcal{A}, \mathcal{A}) \simeq *, \text{ and } \quad GW(\mathcal{C}(\mathcal{A}, \mathcal{A})) \simeq *$$

Proof: The isometry of form functors $id \perp T \cong T$ of lemma 9.5 induces a homotopy between the induced maps of $id \perp T$ and T on Q^h and Grothendieck-Witt spaces. Since $Q^h\mathcal{C}(\mathcal{A}, \mathcal{A})$ and $GW(\mathcal{C}(\mathcal{A}, \mathcal{A}))$ are H -groups under orthogonal sum \perp , this implies that the identity maps of $Q^h\mathcal{C}(\mathcal{A}, \mathcal{A})$ and $GW(\mathcal{C}(\mathcal{A}, \mathcal{A}))$ are null-homotopic. \square

Putting everything together, we obtain the main result of this section.

Theorem 9.7 *Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving fully exact inclusion of idempotent complete exact categories with duality. Then the commutative square*

$$\begin{array}{ccc}
 \mathcal{A} & \longrightarrow & \mathcal{U} \\
 \downarrow & & \downarrow \\
 \mathcal{C}(\mathcal{A}, \mathcal{A}) & \longrightarrow & \mathcal{C}(\mathcal{U}, \mathcal{A})
 \end{array}$$

induces homotopy cartesian squares of Q^h -constructions and Grothendieck-Witt spaces with contractible lower left corner.

Proof: This follows from theorem 8.2 (corollary 8.18 for the case of Grothendieck-Witt spaces), lemma 9.4, and corollary 9.6. \square

Remark 9.8 The maps $Q^h\mathcal{U} \rightarrow Q^h\mathcal{C}(\mathcal{U}, \mathcal{A})$ and $GW(\mathcal{U}) \rightarrow GW(\mathcal{C}(\mathcal{U}, \mathcal{A}))$ need not be surjective on π_0 .

Remark 9.9 If $\mathcal{A} \subset \mathcal{U}$ is s -filtering, functoriality of the cone construction yields a functor $\mathcal{C}(\mathcal{U}, \mathcal{A}) \rightarrow \mathcal{C}(\mathcal{U}/\mathcal{A}, 0) \cong \mathcal{U}/\mathcal{A}$ which induces isomorphisms of higher Grothendieck-Witt groups $GW_i(\mathcal{C}(\mathcal{U}, \mathcal{A})) \cong GW_i(\mathcal{U}/\mathcal{A})$ for $i \geq 1$, by theorems 9.7 and 8.2. I believe that this is also an isomorphism in degree 0, but I haven't checked it.

Definition 9.10 For an idempotent complete exact category with duality \mathcal{A} , we define the exact category with duality

$$S\mathcal{A} = \mathcal{C}(\mathcal{A}, \mathcal{A})/\mathcal{A}$$

and call it the (*hermitian*) *suspension* of \mathcal{A} .

Theorem 9.11 *Let \mathcal{A} be an idempotent complete exact category with duality. Then there are homotopy equivalences*

$$GW(\mathcal{A}) \simeq \Omega GW(S\mathcal{A}) \quad \text{and} \quad |Q^h\mathcal{A}| \simeq \Omega|Q^hS\mathcal{A}|.$$

Proof: This follows from theorem 8.2, corollary 8.18 and corollary 9.6. \square

Remark 9.12 The idempotent completion map $\Omega GW(S\mathcal{A}) \rightarrow \Omega GW(\widetilde{S\mathcal{A}})$ is an equivalence, by cofinality (theorem 5.2), so that we can iterate, and we obtain an Ω -spectrum for \mathcal{A} an idempotent complete exact category with duality

$$\{GW(\mathcal{A}), GW(\widetilde{S\mathcal{A}}), GW(\widetilde{S^2\mathcal{A}}), GW(\widetilde{S^3\mathcal{A}}), \dots\}.$$

The spectrum is non-connective, in general, since for $\mathcal{A} = H\mathcal{E}$, it represents non-connective algebraic K -theory of \mathcal{E} , see [Sch04a], [Sch06].

In contrast, the idempotent completion map $\Omega|Q^h(S\mathcal{A})| \rightarrow \Omega|Q^h(\widetilde{S\mathcal{A}})|$ is not an isomorphism on π_0 , in general, see theorem 5.1. Therefore, the resulting spectrum

$$\{|Q^h(\mathcal{A})|, |Q^h(\widetilde{S\mathcal{A}})|, |Q^h(\widetilde{S^2\mathcal{A}})|, |Q^h(\widetilde{S^3\mathcal{A}})|, \dots\}$$

is not an Ω -spectrum, in general. This happens, for instance, when \mathcal{A} is the hyperbolic category $H\mathcal{E}$ associated with an exact category \mathcal{E} which has non-trivial negative K -groups.

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