HERMITIAN K-THEORY, DERIVED EQUIVALENCES AND KAROUBI'S FUNDAMENTAL THEOREM

MARCO SCHLICHTING

This is a draft which is not (yet) intended for publication!

ABSTRACT. We prove that hermitian K-theory is invariant under derived equivalences when "2 is invertible" or, more generally, when "1 is split". This implies a localization theorem extending an exact sequence of Walter to the left. As corollary we obtain a new, algebraic, and more general proof of Karoubi's fundamental theorem in hermitian K-theory. The localization theorem also implies various (Zariski, Nisnevich) descent theorems for hermitian K and G-theory of (possibly singular) schemes. We also explain the relation of hermitian K-theory to Ranicki's $\mathbb{L}$-groups and Balmer's triangular Witt-grous.

INTRODUCTION

By a result of Thomason [TT90], algebraic K-theory is "invariant under derived equivalences". This is a tremedously powerful fact. Here is the bad news. We show in Appendix C.5 that the corresponding statement is false for higher Grothendieck-Witt groups, alias hermitian K-groups, introduced in [Sch10]. In fact, the examples in Proposition C.5 show that this already fails in degree zero.

The good news is that as long as we restrict ourselves to schemes over $\mathbb{Z}[\frac{1}{2}]$ (or to $\mathbb{Z}[\frac{1}{2}]$-complicial exact categories, or, more generally, to complicial exact categories in which "1 is split"; see condition $(*)$), hermitian K-theory is invariant under derived equivalences. This is proved in Theorem 2.10. Combined with the results of [Sch10], many statements about K-theory, Witt and Grothendieck-Witt groups [TT90], [Gil02], [Wal03a] immediately yield the corresponding results regarding the higher Grothendieck-Witt groups of schemes.

The following is Theorem 3.5 in the main body of the text.

Theorem (Localization). *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying 2.1 $(*)$. Assume that the associated sequence of triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B} \to \mathcal{T}\mathcal{C}$ is exact. Then the sequence of spectra*

$$GW(\mathcal{A}) \to GW(\mathcal{B}) \to GW(\mathcal{C})$$

is a homotopy fibration.

An immediate consequence of our Localization Theorem is a version of Karoubi's fundamental theorem in hermitian K-theory though in the text, we will prove this result before the Localization Theorem and deduce the latter theorem from it. A complicial exact category with weak equivalences and duality $(\mathcal{E}, w, \sharp, \eta)$ comes with a family of dualities $\sharp_i$, $i \in \mathbb{Z}$. We obtain in Theorem 2.7 the following.

Theorem (Periodicity Fibration). *Let $(\mathcal{E}, w, \sharp, \eta)$ be a complicial exact category with duality satisfying 2.1 $(*)$. Then there is a homotopy fibration of spectra*

$$GW(\mathcal{E}, w, \sharp_n, \eta) \xrightarrow{F} K(\mathcal{E}, w) \xrightarrow{H} GW(\mathcal{E}, w, \sharp_{n+1}, \eta),$$

where F and H are the "forgetful" and "hyperbolic" functors, respectively.

With the easy identifications of Karoubi's U and V-theories as

$$_\varepsilon U(\mathcal{E}) \simeq {}_\varepsilon GW(\mathcal{E}, \sharp_{-1}), \;\; _\varepsilon V(\mathcal{E}) \simeq \Omega \; _\varepsilon GW(\mathcal{E}, \sharp_1) \; \text{and} \; _\varepsilon GW(\mathcal{E}, \sharp_{-1}) \simeq {}_{-\varepsilon} GW(\mathcal{E}, \sharp_1),$$

we obtain a new proof and a generalization of the more familiar version of Karoubi's fundamental theorem

$$\Omega \; _\varepsilon U(\mathcal{E}) \simeq {}_{-\varepsilon} V(\mathcal{E});$$

see 3.6 and also Corollary A.3.

Under the assumption $(*)$, we identify in section 5 the colimit of a sequence of spectra $GW(\sharp) \to \Omega^{-1} GW(\sharp_{-1}) \to \Omega^{-2} GW(\sharp_{-2}) \to \dots$ with Ranicki's $\mathbb{L}$-theory spectrum [Ran92] whose homotopy groups $\pi_i \mathbb{L}$ coincide with Balmer's triangular Witt-groups W^{-i}. In particular, our main theorem also holds for $\mathbb{L}$-theory in place of higher Grothendieck-Witt theory – a fact proved some time ago in [Bal00] (though not in the form of spectra) and in [Ran92, ??].

For a complicial exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$, the K-theory spectrum $K(\mathcal{E}, w)$ carries a canonical action of the two element group $\mathbb{Z}/2$. We denote by $K(\mathcal{E}, w)_{h\mathbb{Z}/2}$ the homotopy orbit spectrum. Generalizing a result of Kobal [Kob99], we obtain the following as a consequence of our version of Karoubi's fundamental theorem; see Theorem 5.5.

Theorem. *Let $(\mathcal{E}, w, \sharp, \eta)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$ (see 2.1). Then there is a homotopy fibration of spectra*

$$K(\mathcal{E}, w)_{h\mathbb{Z}/2} \longrightarrow GW(\mathcal{E}, w, \sharp, \eta) \longrightarrow \mathbb{L}(\mathcal{E}, w, \sharp, \eta).$$

An analogous statement for the homotopy fixed point spectrum generalizing [Wil05] and [Kob99] is also given in Theorem 5.5.

With all these abstract results at hand, we can now transport to the theory of higher Grothendieck-Witt groups results known for algebraic K-theory and triangular Witt-theory among them Zariski-excision, Nisnevich excision (both for possibly singular schemes), homotopy invariance for regular noetherian separated schemes, projective bundle formulas, generalizations to quasi-separated and quasi-compact schemes using perfect complexes... One can develop a theory of coherent higher Grothendieck-Witt groups with its localization sequence, Poincare duality, homotopy invariance, Brown-Gersten spectral sequences etc. The proofs are now mutatis mutandis the same as in K-theory and triangular Witt theory. We will describe some of these applications in §9 and §10.

0.1. **Avertissement.** We freely use terminology and results from [Sch10] with the exception that we write $GW_{\geq 0}$ for the (-1)-connected specrtrum whose Ω^∞-space was denoted by GW in [Sch10].

0.2. **Terminology.** We say that a map of $(n-1)$-connected spectra is an *equivalence up to π_n* if its cofibre has non-trivial homotopy groups only in degree n. We say that a commutative square of $(n-1)$-connected spectra is *homotopy cartesian up to π_n* if the total cofibre of the square has only non-trivial homotopy groups in degree n. This is equivalent to the map on vertical (or horizontal) cofibres of the square to be an equivalence up to π_n.

CONTENTS

1. COMPLICIAL EXACT CATEGORIES

All known proofs of the analog in algebraic K-theory of our abstract localization theorem 3.5 need more structure than just that of an "exact category with weak equivalences". Usually one demands the existence of a "cylinder functor" ([Wal85], [TT90]), which, in the cases of interest to us, amounts to giving an action of the category $\mathrm{Ch}^b(\mathbb{Z})$ of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules.

In this section, we introduce the hermitian analog: complicial exact categories with weak equivalences and duality. These are the exact categories with weak equivalences and duality which carry an action of $\mathrm{Ch}^b(\mathbb{Z})$ compatible with involutions.

Many categories with duality carry an action by a closed symmetric monoidal category. We start by reviewing the necessary concepts regarding such categories.

1.1. **Closed symmetric monoidal categories.** Recall that a *symmetric monoidal category* is a category $\mathcal{A}$ equipped with a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, an (unit) object $\mathbb{1} \in \mathcal{A}$, and functorial isomorphisms $a : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$, $l : \mathbb{1} \otimes X \to X$, $r : X \otimes \mathbb{1} \to X$, and $c : X \otimes Y \cong Y \otimes X$ making certain "coherence" diagrams commute []. In order to simplify formulas, we may assume that a, l, and r are the identity morphisms. This is justified by the fact that any symmetric monoidal category is equivalent to one where a, l, and r are the identity morphisms [May74].

A *closed symmetric monoidal category* is a symmetric monoidal category $\mathcal{A}$ together with a functor $[\,,\,] : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$ and functorial evaluation $e : [X, Y] \otimes X \to$

Y and coevaluation $d : X \to [Y, X \otimes Y]$ such that the compositions

$$[A, X] \xrightarrow{d} [A, [A, X] \otimes A] \xrightarrow{[1,e]} [A, X] \quad \text{and} \quad X \otimes A \xrightarrow{d \otimes 1} [A, X \otimes A] \otimes A \xrightarrow{e} X \otimes A$$

are the identity maps. A natural transformation which is built out of iterated application of $\otimes, [\ ,\ ], a, c, d, e, l, r$ is called admissible. There is a coherence theorem which roughly says, that two admissible natural transformations coincide provided they have the same graph (defined in [KML71]) and no term $[\ , \mathbb{1}]$ occurs in the formula. For details, the reader is referred to [KML71].

For every object A of $\mathcal{A}$, we have a functor $\sharp_A : \mathcal{A}^{op} \to \mathcal{A} : X \mapsto [X, A]$. The composition $X \xrightarrow{d} [[X, A], X \otimes [X, A]] \xrightarrow{[1,c]} [[X, A], [X, A] \otimes A] \xrightarrow{[1,e]} [[X, A], A]$ defines a natural transformation

$$\tag{1} \eta_X^A : X \to X^{\sharp_A \sharp_A} = [[X, A], A]$$

which satisfies $(\eta_X^A)^{\sharp_A} \eta_{X^{\sharp_A}}^A = 1_{X^{\sharp_A}}$ [KML71] so that $(\mathcal{A}, \sharp_A, \eta^A)$ is a category with duality. We may write $\sharp$ and η for $\sharp_{\mathbb{1}}$ and $\eta^{\mathbb{1}}$.

There is a unique admissible natural transformation

$$\tag{2} \text{can} : [A, B] \otimes [X, E] \to [A \otimes X, B \otimes E].$$

It is associative and unital in the obvious sense. Moreover, the diagram

$$\tag{3}
\begin{array}{ccc}
E \otimes X & \xrightarrow{\eta_{E \otimes X}^{A \otimes B}} & [[E \otimes X, A \otimes B], A \otimes B] \\
{\scriptstyle \eta_X^A \otimes \eta_X^B} \downarrow & & \downarrow {\scriptstyle [\text{can},1]} \\
[[E, A], A] \otimes [[X, B], B] & \xrightarrow{\text{can}} & [[E, A] \otimes [X, B], A \otimes B]
\end{array}$$

commutes, since, by the coherence theorem, there is a unique admissible natural transformation from the upper left corner to the lower right corner. In summary,

$$\tag{4} (\mathcal{A}, \sharp_A) \times (\mathcal{A}, \sharp_B) \xrightarrow{(\otimes, \text{can})} (\mathcal{A}, \sharp_{A \otimes B})$$

defines a functor of categories with duality which is associative and unital (with unit $(\mathbb{1}, 1) \to (\mathcal{A}, \sharp)$).

1.2. **Remark.** A map $\mu : A \otimes B \to E$ in a closed symmetric monoidal category defines, by adjunction, a map

$$\varphi_\mu : A \xrightarrow{d} [B, A \otimes B] \xrightarrow{[1, \mu]} [B, E] = B^{\sharp_E}$$

which satisfies $\varphi_\mu^{\sharp_E} \circ \eta_B^E = \varphi_{\mu \circ c}$ where $\mu \circ c : B \otimes A \xrightarrow{c} A \otimes B \xrightarrow{\mu} C$. In particular, a map $\mu : A \otimes A \to E$ with $\mu \circ c = \mu$ defines a symmetric form $\varphi_\mu : A \to A^{\sharp_E}$.

1.3. **The category of dg R-modules.** Here is a well-known example of a closed symmetric monoidal category. Let R be a commutative ring with unit, and let dgR-Mod be the category of differential graded R-modules (and maps the closed degree 0 morphisms) [Kel94]. This is the same as the category of complexes of R-modules and chain maps as morphisms. Our convention is that differentials raise the degree (by 1). Usual tensor product $\otimes = \otimes_R$ and internal hom functors $[\ ,\ ]$ of complexes make dgR-Mod into a closed symmetric monoidal category with unit $\mathbb{1} = R$, the chain complex which is R concentrated in degree 0.

More precisely, tensor product and internal homs are defined by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j$$

with differential $d_{A \otimes B}(x \otimes y) = d_A x \otimes y + (-1)^{|x|} x \otimes d_B y$, and

$$[A, B]^n = \Pi_{i+j=n}[A^{-i}, B^j]$$

with differential $d_{[A,B]}(f) = d_B \circ f - (-1)^{|f|} f \circ d_A$. Note that the differentials are defined in such a way that evaluation $e : [A, B] \otimes A \to B : f \otimes x \mapsto f(x)$, coevaluation $d : A \to [B, A \otimes B] : a \mapsto (b \mapsto a \otimes b)$ and commutativity $c : A \otimes B \to B \otimes A : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ are maps of chain-complexes.

The double dual identification $\eta_X^A : X \to X^{\sharp_A \sharp_A} = [[X, A], A] : x \mapsto \eta_X(x)$ of 1.1 (1) is the map satisfying $\eta_X^A(x)(f) = (-1)^{|x||f|} f(x)$ for $f \in X^{\sharp_A}$. The map can $: [X, A] \otimes [Y, B] \to [X \otimes Y, A \otimes B]$ of 1.1 (2) is given by the formula $[\operatorname{can}(f \otimes g)](x \otimes y) = (-1)^{|g||m|} f(x) \otimes g(y)$.

As usual, a sequence $(A, d) \to (B, d) \to (C, d)$ of dg R-modules is exact if $A \to B \to C$ is an exact sequence of (graded) R-modules (forgetting the differentials). A map $(A, d) \to (B, d)$ of dg R-modules is called *weak equivalence* or *quasi-isomorphism* if it induces an isomorphism of cohomology modules.

1.4. The category $\operatorname{Ch}^b(R)$ of finite, semi-free dg R-modules.
A dg R-module (A, d) is *finite and semi-free* if A is a finitely generated free graded R-module, that is, A has a finite homogeneous R-basis. We denote by $\operatorname{Ch}^b(R) \subset dgR$-Mod the full subcategory of finite and semi-free dg R-modules. This is the same as the category of bounded chain complexes of finitely generated free R-modules. The closed symmetric monoidal structure of dgR-Mod respects finite, semi-free dg R-modules so that it makes $\operatorname{Ch}^b(R)$ into a closed symmetric monoidal category. The category $\operatorname{Ch}^b(R)$ inherits the notion of exact sequences and that of weak equivalences from dgR-Mod.

1.5. Complicial exact categories.
Fix a commutative ring with unit R, and write $\otimes$ for $\otimes_R$. An exact category $\mathcal{E}$ is called R-*complicial* if it is equipped with a bi-exact tensor product

$$\otimes : \operatorname{Ch}^b(R) \times \mathcal{E} \to \mathcal{E}$$

which defines an associative and unital left action of the monoidal category $\operatorname{Ch}^b(R)$ on $\mathcal{E}$. The words "associative" and "unital" mean that there are natural isomorphisms $A \otimes (B \otimes X) \cong (A \otimes B) \otimes X$ and $\mathbb{1} \otimes X \cong X$ making the usual pentagonal and triangular diagrams [ML98, VII §1] commute (see also [Gra76] for the definition of an action of a monoidal category on another category).

To simplify formulas, we may sometimes suppress the tensor product from the notation, so that, for instance, the associativity isomorphism above will be written as $A(BX) \cong (AB)X$.

1.6. The fundamental exact sequence.
Let C be the dg-R-algebra freely generated in degree 0 by the unit ε and in degree -1 by an element ν subject to the relations $d\nu = \varepsilon$, $\nu^2 = 0$. Note that C is a commutative dg-R-algebra. We denote by $p : C \to T$ the cokernel of the dg-R-module map $i : \mathbb{1} \to C : 1 \mapsto \varepsilon$. It is the

dg-R-module T freely generated in degree -1 by the image ν_T of ν. We denote by Γ the *fundamental exact sequence* in $\mathrm{Ch}^b(R)$

$$(5) \qquad\qquad \Gamma \qquad \mathbb{1} \rightarrowtail^{\ i\ } C \xrightarrow{\ p\ }\!\!\!\!\twoheadrightarrow T.$$

1.7. The Frobenius structure of a complicial exact category. Let $\mathcal{E}$ be a complicial exact category, and keep in mind that we agreed to write AX instead of $A \otimes X$ for $A \in \mathrm{Ch}^b$ and $X \in \mathcal{E}$. A map $i : X \to Y$ in $\mathcal{E}$ which has a cokernel in $\mathcal{E}$ is called *Frobenius inflation* if for all maps $f : X \to CU$ in $\mathcal{E}$ there is a map $f' : Y \to CU$ such that $f'i = f$. Dually, a map $p : Y \to Z$ in $\mathcal{E}$ which has a kernel in $\mathcal{E}$ is called *Frobenius deflation* if for all maps $f : CU \to Z$ in $\mathcal{E}$ there is a map $f' : CU \to Y$ such that $pf' = f$.

1.8. Lemma. *Let $\mathcal{E}$ be a complicial exact category. Then*

    (1) *$X \to CX$ is a Frobenius inflation, $C^\sharp X \to X$ is a Frobenius deflation.*
    (2) *Frobenius inflations (deflations) are inflations (deflations) in $\mathcal{E}$.*
    (3) *Frobenius inflations (deflations) are closed under composition.*
    (4) *Frobenius inflations (deflations) are preserved under push-outs (pull-backs)*
    (5) *Split injections (surjections) are Frobenius inflations (deflations).*

Proof. Recall that C is a commutative dg R-algebra with multiplication $C \otimes C \to C$ and unit map $\varepsilon : \mathbb{1} \to C$.

Given a map $f : X \to CU$, let $f' : CX \to CU$ be the composition $CX \xrightarrow{1 \otimes f} CCU \xrightarrow{\mu \otimes 1} CU$. We have $f'i = f$ since the composition $C \xrightarrow{1 \otimes \varepsilon} C \otimes C \xrightarrow{\mu} C$ is the identity. Since the map $X \to CX$ has a cokernel, namely TX, it follows that it is a Frobenius inflation. The proof that $C^\sharp X \to X$ is a Frobenius deflation is similar, using the fact that $C^\sharp = [C, \mathbb{1}]$ is a co-algebra.

Let $X \to Y$ be a Frobenius inflation, then there is a map $Y \to CX$ such that the composition $X \to Y \to CX$ is the Frobenius inflation $X \to CX$. Since $X \to CX$ is an inflation in $\mathcal{E}$, and $X \to Y$ has a cokernel, the map $X \to Y$ is an inflation, too. The proof that Frobenius deflations are deflations is dual.

The rest follows easily from the definition of Frobenius inflations and deflations. $\qquad\square$

1.9. Lemma/Definition. *Let $X \to Y \to Z$ be a conflation in $\mathcal{E}$. Then $X \to Y$ is a Frobenius inflation iff $Y \to Z$ is a Frobenius deflation. Such conflations are called* Frobenius conflations.

Proof. The map $CX \to TX$ is a Frobenius deflation because it is isomorphic to $C^\sharp TX \to TX$ via the isomorphism $C \to [C, T] = C^\sharp T$ which is adjoint to $C \otimes C \xrightarrow{\mu} C \to T$.

Let $X \to Y$ be a Frobenius inflation, then there is a map $Y \to CX$ such that the composition $X \to Y \to CX$ is the canonical Frobenius inflation $X \to CX$. Passing to quotients, we see that $Y \to Z$ is a pull-back of $CX \to TX$. Since the latter is a Frobenius deflation, so is $Y \to Z$.

The other direction is dual. $\qquad\square$

Recall that a Frobenius category is an exact category with enough projectives and injectives, and the class of projectives and injectives coincide.

1.10. **Corollary.** *The category $\mathcal{E}$ equipped with the Frobenius conflations is a Frobenius exact category. An object is injective/projective iff it is a direct factor of an object of the form CX.* $\square$

The stable category $\underline{\mathcal{E}}$ of an R-linear Frobenius exact category $\mathcal{E}$ is the category which has the same objects as $\mathcal{E}$, and whose morphisms are the morphisms in $\mathcal{E}$ modulo those which factor through some injective projective object. The stable category $\underline{\mathcal{E}}$ has a canonical structure of an R-linear triangulated category [**?**], [Kel96]. In the case of the Frobenius structure associated with an R-complicial exact category $\mathcal{E}$, a sequence in the stable category $\underline{\mathcal{E}}$ is a distinguished triangle iff it is isomorphic in $\underline{\mathcal{E}}$ to a sequence of the form

$$X \xrightarrow{f} Y \to C(f) \to TX$$

where $Y \to C(f)$ is the pushout of $X \to CX$ along f, and $C(f) \to TX$ is the induced map which is zero on Y and on CX it is the quotient map $CX \to TX$.

1.11. **Definition.** An exact category with weak equivalences $(\mathcal{E}, w)$ is called R-*complicial* if $\mathcal{E}$ is R-complicial, and the tensor product preserves weak equivalences in both variables.

In this situation, the fully exact subcategory $\mathcal{E}^w \subset \mathcal{E}$ of w-acyclic objects (that is, of those objects X for which $0 \to X$ is a weak equivalence) is an R-complicial exact subcategory of $\mathcal{E}$. In particular, $(\mathcal{E}, \mathcal{E}^w)$ is a Frobenius pair in the sense of [Sch04a]. The associated triangulated or derived category $\mathcal{T}(\mathcal{E}, w)$ of $(\mathcal{E}, w)$ is the category $w^{-1}\mathcal{E}$, obtained from $\mathcal{E}$ by formally inverting the weak equivalences in $\mathcal{E}$. It is equivalent to the Verdier quotient $\underline{\mathcal{E}}/\underline{\mathcal{E}^w}$ of the stable category of $\mathcal{E}$ modulo the stable category of $\mathcal{E}^w$, and thus carries a canonical structure of an R-linear triangulated category.

Now we add dualities to the picture.

1.12. **Categories with dualities in a closed symmetric monoidal category.** Let $\mathcal{A}$ be a closed symmetric monoidal category. A *category with dualities in $\mathcal{A}$* is a category $\mathcal{U}$ together with functors

$$\mathcal{A} \times \mathcal{U} \xrightarrow{\otimes} \mathcal{U} : (A, X) \mapsto A \otimes X \quad \text{and}$$

$$\mathcal{U}^{op} \times \mathcal{A} \xrightarrow{[\ ,\ ]} \mathcal{U} : (X, A) \mapsto [X, A]$$

and natural transformations η and can as in 1.1 (1) and 1.1 (2) with $X \in \mathcal{U}$ and $A, B, E \in \mathcal{A}$. As usual, we write $\sharp_A$ for the functor $\mathcal{U}^{op} \to \mathcal{U} : X \mapsto [X, A]$. We require that $\otimes$ and can are associative and unital (from the left), η satisfies $(\eta_X^A)^{\sharp_A} \circ \eta_{X^{\sharp_A}}^A = 1_{X^{\sharp_A}}$ and 1.1 (3) commutes where X is in $\mathcal{U}$ and A, B, E are in $\mathcal{A}$. In particular,

$$(6) \qquad\qquad (\mathcal{A}, \sharp_A) \times (\mathcal{U}, \sharp_B) \xrightarrow{(\otimes, \text{can})} (\mathcal{U}, \sharp_{A \otimes B})$$

is a functor of categories with duality which defines an associative and unital left action.

Of course, any closed symmetric monoidal category $\mathcal{A}$ defines a category with dualities in $\mathcal{A}$. Also, if $\mathcal{A} \to \mathcal{B}$ is a closed symmetric monoidal functor, then $\mathcal{B}$ is a category with dualities in $\mathcal{A}$ via $\mathcal{A} \to \mathcal{B}$.

1.13. **Example.** *Complexes of O_X-modules.* Let X be a scheme over $\operatorname{Spec} R$. The category $\operatorname{Ch}(O_X\text{-Mod})$ of complexes of O_X-modules is closed symmetric monoidal via the same formulas as in 1.3. The map $R \to \Gamma(X, O_X)$ defines a closed symmetric monoidal functor $\otimes_R O_X : \operatorname{Ch}^b(R) \to \operatorname{Ch}(O_X\text{-Mod})$. Let K be any complex of O_X-modules, then, varying $X \in \operatorname{Ch}(O_X\text{-Mod})$ and $A, B, E \in \operatorname{Ch}^b(R)$, the data $A \otimes_R X$, $[X, A]_K = [X, A \otimes K]$, $\eta^{A \otimes K}$ and can $: [A, B] \otimes [X, E]_K \to [A \otimes X, B \otimes E]_K$ define on $\operatorname{Ch}(O_X\text{-Mod})$ the structure $(\otimes, [\quad, \quad]_K, \eta^K, \operatorname{can})$ of a category with dualities in $\operatorname{Ch}^b(R)$.

As usual, a sequence $(A, d) \to (B, d) \to (C, d)$ of complexes of O_X-modules is exact if $A \to B \to C$ is an exact sequence of O_X-modules (forgetting the differentials), and a map $(A, d) \to (B, d)$ of chain complexes of O_X-modules is a *weak equivalence* or *quasi-isomorphism* if it induces an isomorphism of cohomology sheaves.

1.14. **Definition.** Let R be a commutative ring with unit. An *R-complicial exact category with weak equivalences and duality* is an exact category with weak equivalences $(\mathcal{A}, w)$ together with a structure $(\otimes, [\quad, \quad], \eta, \operatorname{can})$ on $\mathcal{A}$ of a category with dualities in the symmetric monoidal category $\operatorname{Ch}^b(R)$ of finite, semi-free dg R-modules such that $\eta^{\mathbb{1}}$ is a natural weak equivalence and such that the functors $\otimes : \operatorname{Ch}^b(R) \times \mathcal{A} \to \mathcal{A}$ and $[\quad, \quad] : \mathcal{A}^{op} \times \operatorname{Ch}^b(R) \to \mathcal{A}$ are bi-exact and preserve weak equivalences.

Its *associated triangulated* or *derived category* $\mathcal{T}\mathcal{A}$ is the category $w^{-1}\mathcal{A}$, obtained from $\mathcal{A}$ by formally inverting weak equivalences. Since $(\mathcal{A}, w, \otimes)$ is an R-complicial exact category with weak equivalences, $\mathcal{T}\mathcal{A}$ has a canonical structure of a R-linear triangulated category, see definition 1.11.

A *complicial exact category with weak equivalences and duality* is simply a $\mathbb{Z}$-complicial exact category with weak equivalences and duality.

1.15. **Remark.** If $(\mathcal{A}, w, \otimes, [\quad, \quad], \eta, \operatorname{can})$ is an R-complicial exact category with duality. Then changing η into $-\eta$ defines another R-complicial exact category with duality $(\mathcal{A}, w, \otimes, [\quad, \quad], -\eta, \operatorname{can})$.

1.16. **Remark.** Definition 1.14 implies that can $: [A, B] \otimes [X, E] \to [A \otimes X, B \otimes E]$ is an isomorphism for $A, B, E \in \operatorname{Ch}^b(R)$, $X \in \mathcal{A}$. We can see this as follows. Using biexactness of $\otimes$ and $[\quad, \quad]$ one reduces the claim to showing that can is an isomorphism when $A = R[i]$ and $B = R[j]$. The fact that can is an isomorphism in this case follows from the requirement that it be associative and unital.

1.17. **Invertible objects.** Recall that an object L in a closed symmetric monoidal category is called *invertible* if the evaluation map $e : [L, \mathbb{1}] \otimes L \to \mathbb{1}$ is an isomorphism. For an invertible object, the map can $: [B, A] \otimes L = [B, A] \otimes [\mathbb{1}, L] \to [B, A \otimes L]$ is an isomorphism. The signature sgn_L of L is the composition of the isomorphisms $\operatorname{sgn}_L : \mathbb{1} \xrightarrow{d} [L, L] \xleftarrow{\operatorname{can}} L^\sharp \otimes L \xrightarrow{e} \mathbb{1}$.

If $e_1, ..., e_n$ is a homogeneous base of an object A of $\operatorname{Ch}^b(R)$, we denote by $e_1^\sharp, ..., e_n^\sharp$ its dual base, the base of $A^\sharp = [A, \mathbb{1}]$ defined by the requirements $e_i^\sharp(e_i) = (-1)^{|e_i|}$ and $e_i^\sharp(e_j) = 0$ for $i \neq j$.

Let L be a dg R-module freely generated by an element ν of degree $|\nu|$. Since evaluation $e : L^\sharp \otimes L \to \mathbb{1}$ sends the generator $\nu^\sharp \otimes \nu$ to $\nu^\sharp(\nu) = (-1)^{|e_i|}$, the object L is invertible. Moreover, the sequence of maps $\operatorname{sgn}_L : \mathbb{1} \xrightarrow{d} [L, L] \xleftarrow{\operatorname{can}} L^\sharp \otimes L \xrightarrow{e} \mathbb{1}$ is $1 \mapsto id \mapsto \nu^\sharp \otimes \nu \mapsto (-1)^\nu$, so that the signature of L is $(-1)^{|\nu|}$.

1.18. **Remark.** The requirement in definition 1.14 that $\eta^{\mathbb{1}}$ be a weak equivalence implies that for any invertible object $L \in \mathrm{Ch}^b(R)$, the natural transformation $\eta^L : X \to [[X, L], L]$ is a weak equivalence. This follows from diagram 1.1 (3) with $A = L$, $B = E = \mathbb{1}$. and the fact that can is an isomorphism (1.16). The same argument shows that if $\eta^{\mathbb{1}}$ is an isomorphism then so is η^L for any invertible object L of $\mathrm{Ch}^b(R)$.

1.19. **Remark.** Let $\mathcal{A}$ be an R-complicial exact category, and A, B invertible objects in $\mathrm{Ch}^b(R)$. Let's write as usual $\sharp_A$ for the functor $[\ , A]$. Let ε and δ be 1 or -1. Then we have a bi-exact functor

$$(7) \qquad (\mathrm{Ch}^b R, \mathrm{quis}, \sharp_A, \varepsilon\eta^A) \times (\mathcal{A}, w, \sharp_B, \delta\eta^B) \overset{(\otimes,\mathrm{can})}{\longrightarrow} (\mathcal{A}, w, \sharp_{A \otimes B}, \varepsilon\delta\eta^{A \otimes B})$$

of categories with weak equivalences and duality. Moreover, the tensor product $(\otimes, \mathrm{can})$ is associative and unital.

1.20. **Remark.** Let A and B be invertible objects of $\mathrm{Ch}^b(R)$. Any symmetric space (E, φ) in the (exact) category with duality $(\mathrm{Ch}^b R, \sharp_A, \varepsilon\eta^A)$ defines a functor $(E, \varphi) : \mathrm{pt} \to (\mathrm{Ch}^b R, \sharp_A, \varepsilon\eta^A) : \mathrm{pt} \mapsto E$ of categories with (strong) duality, where pt denotes the category with duality which has one object pt and one morphism 1_{pt}. Composition with (7) defines an exact functor

$$(E, \varphi) : (\mathcal{A}, w, \sharp_B, \delta\eta^B) \to (\mathcal{A}, w, \sharp_{A \otimes B}, \varepsilon\delta\eta^{A \otimes B})$$

of categories with weak equivalences and duality which is "tensor product with the symmetric space (E, φ)". Composition of such functors corresponds to tensor product of symmetric spaces.

Now, let L be an invertible object of $\mathrm{Ch}^b(R)$. As in 1.2, the identity map $id_{L^2} : L \otimes L \to L^2$ defines, by adjunction, a map $\rho_L = \varphi_{id_{L^2}} : L \to [L, L^2] = L^{\sharp_{L^2}}$ which is easily seen to be an isomorphism. Since $id_{L^2} \circ c = \mathrm{sgn}_L id_{L^2}$, the map satisfies $\rho_L^{\sharp_{L^2}} \eta = \mathrm{sgn}_L \rho_L$. In other words, the pair (L, ρ_L) defines a symmetric space in $(\mathrm{Ch}^b(R), \sharp_{L^2}, \mathrm{sgn}_L \eta^{L^2})$.

1.21. **Lemma.** *Let $\mathcal{A}$ be an R-complicial exact category with weak equivalences and duality, and let L, M be invertible objects in $\mathrm{Ch}^b(R)$. Then tensor product with the symmetric space (L, ρ_L) defined in 1.20 defines an equivalence of exact categories with weak equivalences and duality*

$$(L, \rho_L) : (\mathcal{A}, w, \sharp_M, \eta^M) \to (\mathcal{A}, w, \sharp_{L^2 \otimes M}, \mathrm{sgn}_L \eta^{L^2 \otimes M}).$$

Proof. The map $\lambda : (L^\sharp)^2 \otimes L^2 \overset{\mathrm{can} \otimes 1}{\to} (L^2)^\sharp \otimes L^2 \overset{e}{\to} \mathbb{1}$ defines an isomorphism $(id, \lambda) : (\sharp_{(L^\sharp)^2 \otimes L^2 \otimes M}, \eta) \to (\sharp_M, \eta)$ of exact categories with weak equivalences and duality. One checks that the composition $(id, \lambda) \circ (L^\sharp, \rho_{L^\sharp}) \circ (L, \rho_L)$ is naturally isometric to the identity functor of $(\sharp_M, \eta)$. Similarly, there is an isomorphism $(id, \bar{\lambda})$ of categories with duality such that $(id, \bar{\lambda}) \circ (L, \rho) \circ (L^\sharp, \rho)$ is naturally isometric to the identity functor. This implies the claim. $\qquad\square$

1.22. **Definition.** A *(complicial) functor of R-complicial exact categories with duality* from $\mathcal{A}$ to $\mathcal{B}$, is a triple (F, φ, ρ) with $F : \mathcal{A} \to \mathcal{B}$ an exact functor preserving weak equivalences, $\rho : A \otimes F(X) \to F(A \otimes X)$ a natural transformation, and $\varphi : F[X, A] \to [FX, A]$ a natural weak equivalence. We require that ρ be associative and unital from the left with respect to $\otimes$ (this implies that ρ is a natural isomorphism as in 1.16), and that the following two diagrams commute

$$(8) \qquad \begin{array}{ccccc} [A, B] \otimes F[X, E] & \xrightarrow[\cong]{\rho} & F([A, B] \otimes [X, E]) & \xrightarrow[\cong]{F(\mathrm{can})} & F[A \otimes X, B \otimes E] \\ {\scriptstyle 1 \otimes \varphi} \downarrow & & & & \downarrow {\scriptstyle \varphi} \\ [A, B] \otimes [FX, E] & \xrightarrow[\cong]{\mathrm{can}} & [A \otimes FX, B \otimes E] & \xleftarrow[\cong]{[\rho, 1]} & [F(A \otimes X), B \otimes E] \end{array}$$

$$(9) \qquad \begin{array}{ccc} FX & \xrightarrow{F\eta^{\mathcal{A}}} & F[[X, A], A] \\ {\scriptstyle \eta^{\mathcal{B}}} \downarrow & & \downarrow {\scriptstyle \varphi} \\ [[FX, A], A] & \xrightarrow{[\varphi, 1]} & [F[X, A], A] \end{array}$$

Composition of such functors is composition of the F's, ρ's and φ's, respectively,

1.23. **An equivalent definition.** By remark 1.19 with $A = B = \mathbb{1}$ and $\varepsilon = \delta = 1$, an R-complicial exact category with weak equivalences and duality $(\mathcal{A}, w, \otimes, [\ ,\ ], \eta, \mathrm{can})$ induces an associative and unital biexact left action

$$(10) \qquad\qquad (\mathrm{Ch}^b(R), \mathrm{quis}, \sharp, \eta) \times (\mathcal{A}, w, \sharp, \eta) \overset{(\otimes, \mathrm{can})}{\to} (\mathcal{A}, w, \sharp, \eta),$$

where $\sharp = [\ , \mathbb{1}]$ and $\eta = \eta^{\mathbb{1}}$.

In fact, a left action of $(\mathrm{Ch}^b(R), \mathrm{quis}, \sharp, \eta)$ on $(\mathcal{A}, w, \sharp, \eta)$ already defines an R-complicial exact category $(\mathcal{A}, \otimes, [\ ,\ ]', \eta', \mathrm{can}')$ which is isomorphic to $(\mathcal{A}, \otimes, [\ ,\ ], \eta, \mathrm{can})$. Given such a left action (with $\mathrm{can} : A^\sharp \otimes X^\sharp \to (A \otimes X)^\sharp$ a natural isomorphism, and $\eta : X \to X^{\sharp\sharp}$ a natural weak equivalence, $A \in \mathrm{Ch}^b(R)$ and $X \in \mathcal{A}$), we define $[X, A]' = A \otimes X^\sharp$, and η' and can' as the compositions

$$X = \mathbb{1} \otimes X \overset{d \otimes \eta}{\to} [A, A] \otimes X^{\sharp\sharp} \overset{\mathrm{can} \otimes 1}{\leftarrow} A \otimes A^\sharp \otimes X^{\sharp\sharp} \overset{1 \otimes \mathrm{can}}{\to} A \otimes (A \otimes X^\sharp)^\sharp = [[X, A]', A]',$$

$$[A, B][X, E]' = [A, B] E X^\sharp \overset{\mathrm{can}\, 1}{\leftarrow} B A^\sharp E X^\sharp \overset{1 \mathrm{c} 1}{\to} B E A^\sharp X^\sharp \overset{1\, \mathrm{can}}{\to} B E (AX)^\sharp = [AX, BE]',$$

where on the second line we suppressed the tensor product $\otimes$ in the notation. We leave it as an exercise for the reader to check that with these definitions, $(\mathcal{A}, \otimes, [\ ,\ ]', \eta', \mathrm{can}')$ is an R-complicial exact category with duality, and the functor $(id, id, \mathrm{can}) : (\mathcal{A}, \otimes, [\ ,\ ]', \eta', \mathrm{can}') \to (\mathcal{A}, \otimes, [\ ,\ ], \eta, \mathrm{can})$ is an isomorphism of complicial exact categories with duality.

2. Karoubi-periodicity and invariance under derived equivalences

Let $(\mathcal{A}, w, \otimes, [\ ,\ ], \eta, \mathrm{can})$ be a complicial exact category with duality. We write $\sharp_n$ and η^n for $[\ , T^n]$ and η^{T^n}. Then $(\mathcal{A}, w, \sharp_n, \eta^n)$ is an exact category with weak equivalences and duality carrying a left action by $(\mathrm{Ch}^b \mathbb{Z}, \mathrm{quis}, \sharp, \eta)$. Note that by lemma 1.21 (with $L = T$, and noting that $\mathrm{sgn}\, T = -1$), the two categories with weak equivalences and duality $(\mathcal{A}, w, \sharp_n, \eta^n)$ and $(\mathcal{A}, w, \sharp_{n+2}, -\eta^{n+2})$ are equivalent, so that $GW_{\geq 0}(\mathcal{A}, w, \sharp_n, \eta^n) \cong GW_{\geq 0}(\mathcal{A}, w, \sharp_{n+2}, -\eta^{n+2})$ only depends on n modulo 4.

2.1. **Standing Assumption.** Let $(\mathcal{E}, *)$ be a complicial exact category with weak equivalences and duality. We say that $\mathcal{E}$ satisfies the condition $(*)$ if

> $(*)$ there is a natural transformation $\lambda : id_{\mathcal{E}} \to id_{\mathcal{E}}$ such that
> > (1) for every object X of $\mathcal{E}$ we have $\lambda_{X^*} + \lambda_X{}^* = 1_{X^*}$, and
> > (2) for $A \in \mathrm{Ch}^b \mathbb{Z}$, $X \in \mathcal{E}$ we have $\lambda_{A \otimes X} = 1_A \otimes \lambda_X$.

Recall that $\mathcal{E}$ is equipped with a family $\sharp_L$ of dualities with $* = \sharp_{\mathbb{1}}$, where L ranges over the invertible objects of $\mathrm{Ch}^b \mathbb{Z}$. Since $X^{\sharp_L} = [X, L]$ is naturally isomorphic to $L \otimes X^*$, we have $\lambda_{X^{\sharp_L}} + (\lambda_X)^{\sharp_L} = \lambda_{L \otimes X^*} + L \otimes \lambda_X^* = 1_L \otimes (\lambda_{X^*} + \lambda_X^*) = 1_{X^{\sharp_L}}$. In other words, $(\mathcal{E}, w, \sharp_L)$ also satisfies $(*)$.

2.2. **Examples.** Here are some examples of complicial categories satisfying $(*)$. If $\mathcal{E}$ is a $\mathbb{Z}[\frac{1}{2}]$-linear category (in which case we say "2 is invertible in $\mathcal{E}$") then we can take as λ the multiplication by $\frac{1}{2}$, that is, a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with weak equivalences and duality always satisfies $(*)$. If $\mathcal{E}$ is the hyperbolic category $H\mathcal{A}$ of a complicial exact category with weak equivalences $\mathcal{A}$, then we can take $\lambda = (1, 0)$ the natural transformation $(1_X, 0) : (X, Y) \to (X, Y)$. So hyperbolic categories always satisfy $(*)$. If $\mathcal{A}$ is an exact category with duality for which there is a natural transformation $\lambda : id \to id$ satisfying (1) above, then the category $\mathcal{E} = \mathrm{Ch}^b(\mathcal{A})$ satisfies $(*)$. For instance, we can take as $\mathcal{A}$ the category of finitely generated projective modules over the hermitian ring $R[x]$, where R is a commutative ring with trivial involution, and $\bar{x} = 1 - x$. Here λ is multiplication with x.

2.3. **Lemma.** *Let $(\mathcal{A}, w, \eta, \sharp)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Then the functor $(w\mathcal{A})_h \to (i\mathcal{T}\mathcal{A})_h : (X, \varphi) \mapsto (X, \varphi)$ induces an isomorphism of abelian monoids of connected components*

$$\pi_0 \, (w\mathcal{A})_h \to \pi_0 \, (i\mathcal{T}\mathcal{A})_h.$$

Proof. We will show that the functor (11) below is full and essentially surjective

$$(11) \qquad\qquad w^{-1}(w\mathcal{A})_h \to (i\mathcal{T}\mathcal{A})_h.$$

This clearly implies the claim.

The functor (11) is essentially surjective on objects by the following argument. Let (A, α) be a symmetric space in $w^{-1}\mathcal{A}$. We can write α as a fraction $\alpha = a \circ s^{-1}$ with $a : B \to A^{\sharp}$ and $s : B \to A \in w$ weak equivalences in $\mathcal{A}$. Let $\varphi = \lambda s^{\sharp} a + \lambda^{\sharp} a^{\sharp} \eta s : B \to B^{\sharp}$. Then (B, φ) is a symmetric weak equivalence in $\mathcal{A}$, and $s : B \to A$ defines an isometry in $\mathcal{T}\mathcal{A}$ between (B, φ) and (A, α).

To finish, we show that the functor (11) is full. Let (A, α) and (B, β) be objects of $(w\mathcal{A})_h$, and let $[fs^{-1}]$ be an isometry in $\mathcal{T}\mathcal{A}$ from (A, α) to (B, β). In particular, f and s are weak equivalences in $\mathcal{A}$. Using a calculus of fractions in $\mathcal{T}\mathcal{A}$, we can assume the fraction fs^{-1} with $f : E \to B$ and $s : E \to A$, is such that $\alpha_0 = s^{\sharp}\alpha s$ and $\alpha_1 = f^{\sharp}\beta f$ are homotopic. Therefore, the difference $\alpha_0 - \alpha_1$ factors as $E \xrightarrow{\varepsilon} I \xrightarrow{\delta} E^{\sharp}$ where I is w-acyclic. Then the pair $(E \oplus I, \varphi)$ with

$$\varphi = \begin{pmatrix} \alpha_1 & \lambda\delta \\ \lambda^{\sharp}\delta^{\sharp}\eta & 0 \end{pmatrix} : E \oplus I \to E^{\sharp} \oplus I^{\sharp}$$

defines an object of $(w\mathcal{A})_h$. The functor (11) sends the map

$$(A, \alpha) \overset{s}{\leftarrow} (E, \alpha_0) \overset{\binom{1}{\varepsilon}}{\rightarrow} (E \oplus I, \varphi) \overset{\binom{1}{0}}{\leftarrow} (E, \alpha_1) \overset{f}{\rightarrow} (B, \beta)$$

in $w^{-1}(w\mathcal{A})_h$ to fs^{-1} in $(i\mathcal{T}\mathcal{A})_h$.   □

2.4. Lemma. *Let $(F, \varphi) : (\mathcal{A}, w) \to (\mathcal{B}, w)$ be a map of exact categories with weak equivalences and duality (φ being a symmetric weak equivalence). Suppose that there are a functor $G : (\mathcal{B}, w) \to (\mathcal{A}, w)$ of exact categories with weak equivalences (no compatibility with dualities required) and zig-zags of natural weak equivalences of functors between FG and $id_\mathcal{B}$ and between GF and $id_\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are complicial and satisfy $(*)$, then (F, φ) induces an equivalence of (-1)-connected Grothendieck-Witt theory spectra*

$$GW_{\geq 0}(\mathcal{A}, w) \overset{\sim}{\to} GW_{\geq 0}(\mathcal{B}, w).$$

Proof. To simplify notation, we will write throughout this proof $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ for the exact category with weak equivalences and duality $\mathrm{ExFun}(\mathcal{A}, w; \mathcal{B}, w)$. This is a complicial exact category with weak equivalences and duality satisfifying $(*)$ by the following definitions. One uses the functoriality of $\otimes$, $[\ ,\ ]$ and λ in $\mathcal{A}$ to define $(A \otimes F)$, $[F, A]$ and $\lambda_F : F \to F$ as $(A \otimes F)(X) = A \otimes F(X)$, $[F, A](X) = [F(X), A]$ and $\lambda_F(X) = \lambda_{F(X)} : F(X) \to F(X)$ for $F \in \mathrm{Fun}(\mathcal{E}, \mathcal{A})$. In particular, $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$ and $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ are complicial exact categories with weak equivalences and duality satisfying $(*)$.

The hypothesis of the lemma imply that $F : \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{B}) : H \mapsto F \circ H$ induces an equivalence $\mathcal{T} \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \to \mathcal{T} \mathrm{Fun}(\mathcal{B}, \mathcal{B})$ of derived categories with inverse G. By lemma 2.3, (F, φ) induces an isomorphism of abelian monoids

$$\pi_0 \, (w\,\mathrm{Fun}(\mathcal{B}, \mathcal{A}))_h \overset{\cong}{\to} \pi_0 \, (w\,\mathrm{Fun}(\mathcal{B}, \mathcal{B}))_h.$$

So there is an object $(H, \psi) \in (w\,\mathrm{Fun}(\mathcal{B}, \mathcal{A}))_h$ which under this isomorphism goes to $id_\mathcal{B} \in (w\,\mathrm{Fun}(\mathcal{B}, \mathcal{B}))_h$. This means that there is a zigzag of natural weak equivalences between $(F, \varphi) \circ (H, \psi)$ and $id_\mathcal{B}$ compatible with forms. By [Sch10, §2 Lemma 2], the functors $(F, \varphi) \circ (H, \psi)$ and $id_\mathcal{B}$ induce homotopic maps on (-1)-connected hermitian K-theory spectra. In particular, (F, φ) is surjective, and (H, ψ) is injective as maps on higher Grothendieck-Witt groups GW_i, $i \geq 0$.

By construction of H and the existence of G, there are zig-zags of weak equivalences of functors between HF and $id_\mathcal{A}$ and between FH and $id_\mathcal{B}$. By the argument above (with (H, ψ) in place of (F, φ)), we see that (H, ψ) is also surjective and hence bijective on higher Grothendieck-Witt groups. From the previous paragraph it follows that (F, φ) induces isomorphisms $GW_i(\mathcal{A}, w) \to GW_i(\mathcal{B}, w)$, $i \geq 0$.

□

2.5. The mapping cone. Let R be a commutative ring with unit. Recall that the cone $C \in \mathrm{Ch}^b(R)$ is a commutative dg-R algebra with multiplication $\mu : C \otimes C \to C$ defined in 1.6. As in 1.2, the composition $p\mu : C \otimes C \to T$ defines, by adjunction, a form $\gamma = \varphi_{p\mu} : C \to C^{\sharp T} = [C, T]$ which is symmetric, that is, $\gamma^{\sharp T} \eta^T = \gamma$ since $\mu \circ c = \mu$ (see remark 1.2). We obtain in this way a symmetric isomorphism of

exact sequences

$$
\begin{array}{ccc}
\Gamma : & \mathbb{1} \xrightarrow{\ i\ } C \xrightarrow{\ p\ } T \\
\downarrow{\scriptstyle\gamma} & \downarrow{\scriptstyle d} \quad \downarrow{\scriptstyle\gamma} \quad \downarrow{\scriptstyle id} \\
[\Gamma, T] : & [T,T] \xrightarrow{[p,1]} [C,T] \xrightarrow{[i,1]} [\mathbb{1}, T]
\end{array}
$$

from the fundamental exact sequence of 1.6 to its T-dual. In other words, the pair (Γ, γ) defines a symmetric space in the category $(S_2 \operatorname{Ch}^b(R), \sharp_T, \eta^T)$ of exact sequences of chain complexes with shifted duality $\sharp_T$.

For an exact category with weak equivalences and duality $(\mathcal{A}, w, \sharp, \eta)$, we define a functor

$$(\Delta, \delta) : (S_2 Mor\mathcal{A}, \sharp) \to (\mathcal{A}, \sharp)$$

of exact categories with weak equivalences and duality as follows. Any map $f_* : A_* \to B_*$ of exact sequences can be factored into a sequence of maps $\bar{\Delta}(f)$ of exact sequences as follows

$$
\begin{array}{ccc}
& 0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow 0 & A_* \\
& \downarrow{\scriptstyle f_0} \quad \square \quad \downarrow \qquad \| & \\
\bar{\Delta}(f) & 0 \longrightarrow B_0 \longrightarrow \Delta(f_*) \longrightarrow A_2 \longrightarrow 0 & \downarrow{\scriptstyle f_*} \\
& \| \qquad \downarrow \quad \square \quad \downarrow{\scriptstyle f_2} & \\
& 0 \longrightarrow B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow 0 & B_*,
\end{array}
$$

where the upper left and lower right squares are bicartesian. This factorization is unique up to unique isomorphism, and therefore defines a functor $\bar{\Delta}$ from $S_2 Mor\mathcal{A}$ to such diagrams. In particular, we have a functor $\Delta : S_2 Mor\mathcal{A} \to \mathcal{A}$ which is $\bar{\Delta}$ composed with evaluation at the middle spot. The duality compatibility isomorphism $\bar{\Delta}(f^\sharp) \to \bar{\Delta}(f)^\sharp$ is the unique map of diagrams inducing the identity of $f_*^\sharp$ on the composition. Its restriction to the middle spot of the diagram defines the duality compatibility isomorphism $\delta : \Delta(f^\sharp) \to \Delta(f)^\sharp$.

We define the cone functor $\mathrm{Cone} : (Mor\mathcal{A}, w, \sharp_n, \eta^n) \to (\mathcal{A}, w, \sharp_{n+1}, \eta^{n+1})$ as the composition

$$(12) \qquad \mathrm{Cone} : (Mor\mathcal{A}, \sharp_n) \xrightarrow{(\Gamma,\gamma)\otimes id} (S_2 Mor\mathcal{A}, \sharp_{n+1}) \xrightarrow{(\Delta,\delta)} (\mathcal{A}, \sharp_{n+1}).$$

For any exact category with duality and weak equivalences $\mathcal{A}$ we have a functor $(\mathcal{A}, \sharp_n) \to (Mor\mathcal{A}, \sharp_n) : A \mapsto id_A$ with duality compatibility isomorphism $id : id_{A^{\sharp_n}} \xrightarrow{=} id_A^{\sharp_n}$.

2.6. Theorem. *Let $(\mathcal{A}, w)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Then the commutative square*

$$
(13) \qquad
\begin{array}{ccc}
(\mathcal{A}, w, \sharp_n, \eta^n) & \xrightarrow{A \mapsto id_A} & (Mor\mathcal{A}, w, \sharp_n, \eta^n) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Cone}} \\
(\mathcal{A}^w, w, \sharp_{n+1}, \eta^{n+1}) & \hookrightarrow & (\mathcal{A}, w, \sharp_{n+1}, \eta^{n+1})
\end{array}
$$

induces a homotopy cartesian square up to π_0 of $GW_{\geq 0}$ spectra and the lower left corner's hermitian K-theory is contractible. In particular, there is a homotopy fibration up to π_0

$$GW_{\geq 0}(\mathcal{A}, w, \sharp_n) \longrightarrow GW_{\geq 0}(Mor\mathcal{A}, w, \sharp_n) \longrightarrow GW_{\geq 0}(\mathcal{A}, w, \sharp_{n+1}).$$

Proof. As usual, the lower left corner's hermitian K-theory is contractible by [Sch10, §2 Lemma 2] in view of the natural weak equivalence $0 \to X$ for $X \in \mathcal{A}^w$.

Let v be the set of maps f in $\text{Mor}\,\mathcal{A}$ for which $\text{Cone}(f)$ is a weak equivalence in $\mathcal{A}$. Then

$$((\text{Mor}\,\mathcal{A})^v, w, \sharp_n) \to (\text{Mor}\,\mathcal{A}, w, \sharp_n) \to (\text{Mor}\,\mathcal{A}, v, \sharp_n)$$

induces a homotopy fibration up to π_0 of $GW_{\geq 0}$ spectra by [Sch10, §4 Theorem 6]. The functor $(\mathcal{A}, w, \sharp_n) \to ((\text{Mor}\,\mathcal{A})^v, w, \sharp_n) = (\text{Mor}_w\,\mathcal{A}, w, \sharp_n) : A \mapsto id_A$ induces an equivalence of $GW_{\geq 0}$ spectra by [Sch10, §2 Lemma 2].

The induced functor $(F, \varphi) = \text{Cone} : (\text{Mor}\,\mathcal{A}, v, \sharp_n) \to (\mathcal{A}, w, \sharp_{n+1})$ induces an equivalence of $GW_{\geq 0}$ spectra by lemma 2.4, where $G : (\mathcal{A}, w) \to (\text{Mor}\,\mathcal{A}, v)$ sends X to the object $0 \to X$ of $\text{Mor}\,\mathcal{A}$. We have $FG = id_{\mathcal{A}}$ and a zigzag of natural weak equivalences $id \to H \leftarrow GF$ where H sends an object $f : X \to Y$ of $\text{Mor}\,\mathcal{A}$ to the object $\text{Cone}(id_X) \to \text{Cone}(f)$, and the natural weak equivalences are given by the natural maps from $f : X \to Y$ to $\text{Cone}(id_X) \to \text{Cone}(f)$ and from $0 \to \text{Cone}(f)$ to $\text{Cone}(id_X) \to \text{Cone}(f)$.   □

2.7. Theorem (Periodicity Fibration). *Let $\mathcal{E}$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Then forgetful and hyperbolic functors $(\mathcal{E}, w, \sharp_n) \xrightarrow{F} (H\mathcal{E}, w) \xrightarrow{H} (\mathcal{E}, w, \sharp_{n+1})$ induce a homotopy fibration up to π_0*

$$GW_{\geq 0}(\mathcal{E}, w, \sharp_n) \xrightarrow{F} K(\mathcal{E}, w) \xrightarrow{H} GW_{\geq 0}(\mathcal{E}, w, \sharp_{n+1}).$$

which is functorial for complicial exact categories with weak equivalences and duality.

Proof. This follows from theorem 2.6, additivity [Sch10, §3 Theorem 5] and the observations that the composition $(\mathcal{E}, \sharp_n) \to (\text{Mor}\,\mathcal{E}, \sharp_n) \to H\mathcal{E}$ is the forgetful functor and the composition

$$H\mathcal{E} \quad \xrightarrow{T^{-1}\otimes} \quad H\mathcal{E} \quad \longrightarrow \quad (\text{Mor}\,\mathcal{E}, \sharp_n) \quad \xrightarrow{\text{cone}} \quad (\mathcal{E}, \sharp_{n+1})$$

is naturally isometric to the hyperbolic functor. Since $T^{-1}\otimes$ is homotopic to -1 in K-theory, the pair $(F, -H)$, and hence, the pair (F, H), induce homotopy fibrations of spectra.   □

2.8. Karoubi's fundamental theorem.

2.9. Lemma (Karoubi induction). *Let $G : \mathcal{A} \to \mathcal{B}$ be a complicial functor between complicial exact categories with weak equivalences and duality satisfying $(*)$. Suppose G induces an equivalence in K-theory and isomorphisms $GW_0(\mathcal{A}, w, \sharp_n) \to GW_0(\mathcal{B}, w, \sharp_n)$, $n \in \mathbb{Z}/4$. Then G induces an equivalence for $n \in \mathbb{Z}/4$*

$$GW_{\geq 0}(\mathcal{A}, w, \sharp_n) \xrightarrow{\simeq} GW_{\geq 0}(\mathcal{B}, w, \sharp_n).$$

Proof. By the Periodicity Fibration Theorem 2.7, G induces a map of exact sequences for $n \in \mathbb{Z}$ and $i \geq 0$

$$\begin{array}{ccccccccc}
GW_{i+1}(\mathcal{A}, \sharp_n) & \longrightarrow & K_{i+1}(\mathcal{A}) & \longrightarrow & GW_{i+1}(\mathcal{A}, \sharp_{n+1}) & \longrightarrow & GW_i(\mathcal{A}, \sharp_n) & \longrightarrow & K_i(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
GW_{i+1}(\mathcal{B}, \sharp_n) & \longrightarrow & K_{i+1}(\mathcal{B}) & \longrightarrow & GW_{i+1}(\mathcal{B}, \sharp_{n+1}) & \longrightarrow & GW_i(\mathcal{B}, \sharp_n) & \longrightarrow & K_i(\mathcal{A}).
\end{array}$$

By a version of the 5-lemma, an isomorphism $GW_i(G, \sharp_n)$ implies a surjection $GW_{i+1}(G, \sharp_{n+1})$. Moreover, an isomorphism $GW_i(G, \sharp_n)$ and a surjection $GW_{i+1}(G, \sharp_n)$ imply an isomorphism $GW_{i+1}(G, \sharp_{n+1})$. $\qquad\square$

2.10. **Theorem** (Invariance under derived equivalences). *Let $G : \mathcal{A} \to \mathcal{B}$ be a complicial functor between complicial exact categories with weak equivalences and duality satisfying $(*)$. Suppose that G induces an equivalence $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B}$ of associated triangulated categories. Then G induces an equivalence for $n \in \mathbb{Z}/4$*

$$GW_{\geq 0}(\mathcal{A}, w, \sharp_n) \xrightarrow{\simeq} GW_{\geq 0}(\mathcal{B}, w, \sharp_n).$$

Proof. For a complicial exact category with weak equivalences and duality $(\mathcal{E}, w)$, the group $K_0^h(\mathcal{E}, w)$ is the co-equalizer of a diagram

$$K_0 \ \ w^{-1}(wS_2\mathcal{E})_h \quad \rightrightarrows \quad K_0 \ \ w^{-1}(w\mathcal{E})_h,$$

where $K_0(\mathcal{C})$ means "Grothendieck group of the abelian monoid of isomorphism classes of objects in the symmetric monoidal category $\mathcal{C}$"; see (8) in the proof of [Sch10, §2 Proposition 3] The two maps are induced by the two ways of associating symmetric weak equivalences in $\mathcal{E}$ to a symmetric weak equivalence in $S_2\mathcal{E}$ as in the definition of GW_o given in [Sch10, §2 Definition 1].

It is easy to see that a functor $G : \mathcal{A} \to \mathcal{B}$ such that $G : \mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B}$ is an equivalence, also induces an equivalence $\mathcal{T}S_2\mathcal{A} \to \mathcal{T}S_2\mathcal{B}$ of triangulated categories associated with the categories of exact sequences in $\mathcal{A}$ and $\mathcal{B}$. It follows from lemma 2.3, that G induces isomorphisms on both terms in the diagram above, and thus isomorphisms $GW_0(\mathcal{A}, w, \sharp_n) \to GW_0(\mathcal{B}, w, \sharp_n)$, $n \in \mathbb{Z}$. Since, by a theorem of Thomason [TT90, ??], G induces an equivalence in K-theory, we are done by Lemma 2.9. $\qquad\square$

2.11. **Theorem.** *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying $(*)$. Assume that the associated sequence of triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B} \to \mathcal{T}\mathcal{C}$ is exact. Then the composition $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ factors through $\mathcal{C}^w$ and the commutative square of (-1)-connected Grothendieck-Witt theory spectra*

$$\begin{array}{ccc}
GW_{\geq 0}(\mathcal{A}, w, \sharp_n) & \longrightarrow & GW_{\geq 0}(\mathcal{B}, w, \sharp_n) \\
\downarrow & & \downarrow \\
0 \simeq GW_{\geq 0}(\mathcal{C}^w, w, \sharp_n) & \longrightarrow & GW_{\geq 0}(\mathcal{C}, w, \sharp_n)
\end{array}$$

is homotopy cartesian up to π_0, and the lower left corner is contractible.

Proof. This follows from the "change of weak equivalence" theorem [Sch10, §4 Theorem 6], and invariance under derived equivalences (theorem 2.10).

$\square$

3. THE GROTHENDIECK-WITT SPECTRUM

Let $(\mathcal{E}, w)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. We will construct a spectrum $GW(\mathcal{E}, w)$ and a map

$$GW_{\geq 0}(\mathcal{E}, w) \to GW(\mathcal{E}, w)$$

of spectra functorial in $\mathcal{E}$ such that $\pi_i GW_{\geq 0}(\mathcal{E}, w) \to \pi_i GW(\mathcal{E}, w)$ is an isomorphism for $i \geq 0$, and such that the fibration up to π_0 of theorem 2.11 extends to an honest fibration of Grothendieck-Witt theory spectra.

3.1. **Definition.** Define the Witt groups functors $W_0(n)$ as $W_0(n)(\mathcal{A}, w, \sharp, \eta) = W_0(\mathcal{A}, w, \sharp_n, \eta^n)$ and remark that $W_0(n)(\mathrm{Mor}\,\mathcal{A}, w, \sharp, \eta) = 0...$

3.2. **Construction.** We will construct a sequence of functors from complicial exact categories with weak equivalences and duality satisfying $(*)$ to spectra

(14) $$GW_{\geq 0} \to GW_{\geq -1} \to GW_{\geq -2} \to GW_{\geq -3} \to \cdots$$

such that for $n \leq 0$

(1) $GW_{\geq n}$ is $n - 1$ connected,
(2) for $n < 0$, the map $GW_{\geq n+1} \to GW_{\geq n}$ is an isomorphism in degrees $\geq n+1$, and $\pi_n GW_{\geq n} = W_0(n)$,
(3) theorem 2.11 holds with "up to π_n" and $GW_{\geq n}$ in place of "up to π_0" and $GW_{\geq 0}$,
(4) The map $GW_{\geq n+1} \mathrm{Mor} \to GW_{\geq n} \mathrm{Mor}$ is an equivalence for $n < 0$,
(5) [Sch10, §2 Lemma 2] holds for complicial functors F and G, and $GW_{\geq n}$ in place of $GW_{\geq 0}$.

Note that (4) implies that $GW_{\geq n}(\mathcal{A}) \simeq 0$ for $\mathcal{A} = 0$ the category consisting of only a zero object, since in this case $\mathcal{A} \cong \mathrm{Mor}\,\mathcal{A}$, and $GW_{\geq 0}(\mathcal{A}) = 0$.

We have seen that the functor $GW_{\geq 0}$ satisfies (1) - (5); (1) holds by contruction of $GW_{\geq 0}$, (2) is vacuous, (3) is theorem 2.11, (4) is vacuous, and (5) is [Sch10, §2 Lemma 2].

Assume we have constructed the functors $GW_{\geq -l}$ for $l = 0, ..., n \in \mathbb{N}$ satisfying (1) - (5). Let $(\mathcal{A}, w)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Consider diagram 2.6 (13), and define $F_{\geq -n}$ and $G_{\geq -n-1}$ as the homotopy fibres (in the category of spectra) of $GW_{\geq -n}$ of the left and right vertical maps in that diagram. More precisely, $F_{\geq -n}(\mathcal{A}, w, \sharp, \eta)$ is the homotopy fibre of

$$GW_{\geq -n}(\mathcal{A}, w, \sharp, \eta) \to GW_{\geq -n}(\mathcal{A}^w, w, \sharp_1, \eta^1),$$

and $G_{\geq -n-1}(\mathcal{A}, w, \sharp, \eta)$ is the homotopy fibre of

$$GW_{\geq -n}(\mathrm{Mor}\,\mathcal{A}, w, \sharp, \eta) \to GW_{\geq -n}(\mathcal{A}, w, \sharp_1, \eta^1).$$

The natural map $F_{\geq -n} \to GW_{\geq -n}$ is an equivalence since $GW_{\geq -n}(\mathcal{A}^w, w, \sharp_1, \eta^1)$ is contractible, by (5) and the remark thereafter. The natural map $F_{\geq n} \to G_{\geq n-1}$, induced by the commutativity of diagram 2.6 (13), is an equivalence up to π_{-n-1},

by (3). Moreover, $G_{\geq -n-1}$ is $-n-2$-connected, by (1), and $\pi_{-n-1}G_{\geq -n-1}$ is the cokernel of

$$\pi_{-n}GW_{\geq -n}(\mathrm{Mor}\,\mathcal{A},\sharp) \to \pi_{-n}GW_{\geq -n}(\mathcal{A},\sharp_1)$$

which is $W_0(\mathcal{A},\sharp_1) = W_0(1)(\mathcal{A},\sharp)$ for $n = 0$, and $W_0(n)(\mathcal{A},\sharp_1) = W_0(n+1)(\mathcal{A},\sharp)$ for $n > 0$ since $W_0(n)(\mathrm{Mor},\mathcal{A},\sharp) = 0$. In particular, the map $F_{\geq n}\,\mathrm{Mor} \to G_{\geq n-1}\,\mathrm{Mor}$ is an equivalence. Moreover, theorem 2.11 holds with "up to π_{n-1}" and $G_{\geq n-1}$ in place of "up to π_0" and $GW_{\geq 0}$, by (3) and (4), and the fact that the diagram in theorem 2.11 is homotopy cartesian when $GW_{\geq 0}$ is replaced with $GW_{\geq 0}\,\mathrm{Mor}$ which is (-1)-connected K-theory, by additivity. Finally, (5) holds for $G_{\geq -n-1}$ by functoriality of its construction.

Now we define $GW_{\geq -n-1}$ as the homotopy colimit

$$GW_{\geq -n-1} = \mathrm{hocolim}(GW_{\geq -n} \xleftarrow{\sim} F_{\geq -n} \longrightarrow G_{\geq -n-1}).$$

It comes with a natural map $GW_{\geq -n} \to GW_{\geq -n-1}$. Since the induced map $G_{\geq -n-1} \to GW_{\geq -n-1}$ is an equivalence (as $F_{\geq -n} \to GW_{\geq -n}$ is), the properties (1) - (5) for $GW_{\geq -n-1}$ readily follow from the corresponding properties of $G_{\geq -n-1}$.

3.3. Definition. Let $(\mathcal{E}, w, \sharp, \eta)$ be a complicial exact category with weak equivalences and duality satisfying $(\ast)$. Its *non-connective Grothendieck-Witt theory spectrum* $GW(\mathcal{E}, w, \sharp, \eta)$ is defined to be the colimit

$$GW(\mathcal{E}, w, \sharp, \eta) = \mathrm{colim}_{n \in \mathbb{N}} GW_{\geq -n}(\mathcal{E}, w, \sharp, \eta)$$

of the sequence (14). It is functorial for complicial functors of such categories.

The following, proposition 3.4 and theorem 3.5, follow immediately from the construction and discussion in 3.2.

3.4. Proposition. *Let $\mathcal{E}$ be a complicial exact category with weak equivalences and duality satisfying $(\ast)$. Then the natural map*

$$GW_{\geq 0}(\mathcal{E}, w) \to GW(\mathcal{E}, w)$$

is an isomorphism in degrees $i \geq 0$, and in negative degrees we have natural isomorphisms

$$GW_{-n}(\mathcal{E}, w, \sharp) = W_0(\mathcal{E}, w, \sharp_n), \qquad n > 0.$$

$\square$

3.5. Theorem (Abstract Localization). *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying $(\ast)$. Assume that the associated sequence of triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B} \to \mathcal{T}\mathcal{C}$ is exact. Then the composition $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ factors through $\mathcal{C}^w$, the commutative square of Grothendieck-Witt theory spectra*

$$
\begin{array}{ccc}
GW(\mathcal{A}, w, \sharp) & \longrightarrow & GW(\mathcal{B}, w, \sharp) \\
\downarrow & & \downarrow \\
0 \simeq GW(\mathcal{C}^w, w, \sharp) & \longrightarrow & GW(\mathcal{C}, w, \sharp)
\end{array}
$$

is homotopy cartesian, and the lower left corner is contractible. $\square$

3.6. **Karoubi periodicity** (or the fundamental theorem of hermitian K-theory). Let $\mathcal{E}$ be a complicial exact category with weak equivalences and duality. It is easy to see (and implicit in the proof of theorem 2.6) that the sequence

$$(\mathcal{E}, w, \sharp_n) \overset{X \mapsto 1_X}{\longrightarrow} (\mathrm{Mor}\,\mathcal{E}, w, \sharp_n) \overset{\mathrm{cone}}{\longrightarrow} (\mathcal{E}, w, \sharp_{n+1})$$

induces an exact sequence of associated triangulated categories. Therefore, theorem 3.5 implies (in the same way as theorem 2.6 implies Theorem 2.7) a homotopy fibration of Grothendieck-Witt theory spectra

$$(15) \qquad GW(\mathcal{E}, w, \sharp_n, \eta^n) \overset{F}{\longrightarrow} K(\mathcal{E}, w) \overset{H}{\longrightarrow} GW(\mathcal{E}, w, \sharp_{n+1}, \eta^{n+1})$$

for $(\mathcal{E}, w, \sharp, \eta)$ a complicial exact category with weak equivalences and duality satisfying $(*)$.

For $\varepsilon \in \{\pm 1\}$, write $_\varepsilon GW(\mathcal{A}, n)$ for $GW(\mathcal{A}, w, \sharp_n, \varepsilon\eta^n)$, and $_\varepsilon GW(\mathcal{A})$ for $_\varepsilon GW(\mathcal{A}, 0)$. Following Karoubi, one defines new theories $_\varepsilon U(\mathcal{A})$ and $_\varepsilon V(\mathcal{A})$ as the homotopy fibres of hyperbolic and forgetful functors

$$K(\mathcal{A}) \overset{H}{\to} {}_\varepsilon GW(\mathcal{A}) \quad \text{and} \quad {}_\varepsilon GW(\mathcal{A}) \overset{F}{\to} K(\mathcal{A}).$$

From the homotopy fibration (15), we see that

$$_\varepsilon U(\mathcal{A}) \simeq {}_\varepsilon GW(\mathcal{A}, -1) \quad \text{and} \quad {}_\varepsilon V(\mathcal{A}) \simeq \Omega\, {}_\varepsilon GW(\mathcal{A}, 1).$$

It follows from lemma 1.21 with $L = T$ (and $\mathrm{sgn}_T = -1$) that $_\varepsilon GW(\mathcal{A}, -1) \cong {}_{-\varepsilon}GW(\mathcal{A}, 1)$ so that we obtain a homotopy equivalence

$$_{-\varepsilon}V(\mathcal{A}) \simeq \Omega\, {}_\varepsilon U(\mathcal{A}).$$

For $\mathcal{A}$ the category of finitely generated projective R-modules (and the stable version of higher Grothendieck-Witt theory as defined in section 8 instead of GW), this is Karoubi's fundamental theorem [Kar80].

4. Negative Grothendieck-Witt groups are triangular Witt- and $\mathbb{L}$-groups

4.1. **The Balmer-Walter definition.** Let $\mathcal{A}$ be a complicial exact category with weak equivalences and duality. Recall from 2.5 (12) the cone functor whose associated form gives, for $n = -1$, a symmetric weak equivalence $\mathrm{Cone}(u) \to \mathrm{Cone}(u^{\sharp_{-1}})^{\sharp}$. If $u = u^{\sharp_{-1}}\eta^{-1}$, we obtain a symmetric weak equivalence $\varphi : \mathrm{Cone}(u) \to (\mathrm{Cone}(u))^{\sharp}$. If $\mathcal{A}$ is $\mathbb{Z}[\frac{1}{2}]$-complicial, it is shown in [Bal00] that the isometry class of $(\mathrm{Cone}(u), \varphi)$ in $(\mathcal{T}\mathcal{A}, \sharp)$ only depends on the image of u in $\mathcal{T}\mathcal{A}$, and that $(\mathrm{Cone}(u), \varphi)$ can be constructed entirely within the framework of a triangulated category with duality.

In [Wal03b], the *Grothendieck-Witt group* $GW(\mathcal{T})$ of a $\mathbb{Z}[\frac{1}{2}]$-linear triangulated category with dualtiy $\mathcal{T}$ is defined to be the abelian group freely generated by symmetric spaces $[X, \varphi]$ in $\mathcal{T}$ modulo the relations $[X, \varphi] + [Y, \psi] = [(X, \varphi) \perp (Y, \psi)]$ and $[\mathrm{Cone}(u), \varphi] = [H(X)]$ for any symmetric map $u : X \to X^{\sharp_{-1}}$ in $\mathcal{T}$. If $\mathcal{T}$ is the triangulated category $\mathcal{T}\mathcal{A}$ associated with a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with duality $\mathcal{A}$, we can assume u to be symmetric in $(\mathcal{A}, \sharp_{-1})$ as any symmetric map in $\mathcal{T}\mathcal{A}$ can be lifted (up to isometry in $\mathcal{T}\mathcal{A}$) to a symmetric map in $\mathcal{A}$.

4.2. **Lemma.** *Let $\mathcal{A}$ be a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with weak equivalences and duality. Then the map of abelian groups is well-defined and an isomorphism*

$$GW_0(\mathcal{A}, w) \to GW(\mathcal{T}\mathcal{A}) : [X, \varphi] \mapsto [X, \varphi]$$

Proof. Let $(\mathcal{E}, w)$ be a complicial exact category with strong duality. A symmetric space (X, φ) in $\mathcal{E}$ defines a symmetric space in $\mathcal{T}\mathcal{E}$. If $L \subset X$ is a Lagrangian, then $[X, \varphi] = [H(L)]$ in $GW(\mathcal{T}\mathcal{E})$ ([Bal99, 2.11]). So we obtain a well-defined map $GW_0(\mathcal{E}) \to GW(\mathcal{T}\mathcal{E}) : [X, \varphi] \mapsto [X, \varphi]$ which is obviously trivial on $GW_0(\mathcal{E}^w)$.

By the Simplicial Resolution Lemma [Sch10, §3 Lemma 5] the group $GW_0(\mathcal{A}, w)$ is the cokernel of $GW_0(\mathcal{A}^{\mathrm{str}, w}) \to GW_0(\mathcal{A}^{\mathrm{str}})$. Therefore, we obtain a well-defined map $GW_0(\mathcal{A}, w) \to GW(\mathcal{T}\mathcal{A}^{\mathrm{str}}) = GW(\mathcal{T}\mathcal{A}) : [X, \varphi] \mapsto [X, \varphi]$. By lemma 2.3, this map is surjective.

To see that this map is also injective, we construct its inverse. Consider a symmetric map $u : X \to X^{\sharp-1}$ in $(\mathcal{A}, \sharp_{-1})$. The definition of the cone functor in 2.5 yields a symmetric weak equivalence of exact sequences

$$
\begin{array}{ccccc}
[X, T^\sharp] & \rightarrowtail & \mathrm{Cone}(u) & \twoheadrightarrow & TX \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle T\eta} \\
[X, T^\sharp] & \rightarrowtail & (\mathrm{Cone}(u))^\sharp & \twoheadrightarrow & T(X^{\sharp\sharp})
\end{array}
$$

Using the isomorphism of lemma 2.3 we obtain a well-defined surjective map of abelian groups $GW(\mathcal{T}\mathcal{A}, \sharp) \to GW_0(\mathcal{A}, w, \sharp)$ such that the composition $GW_0(\mathcal{A}, w) \to GW(\mathcal{T}\mathcal{A}) \to GW_0(\mathcal{A}, w)$ is the identity map. $\qquad\square$

4.3. Lemma. *Let $\mathcal{A}$ be a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with weak equivalences and duality. Then for all $n < 0$, there are natural isomorphisms*

$$
GW_n(\mathcal{A}, w) \to W^{-n}(\mathcal{T}\mathcal{A}).
$$

Proof. This follows from Lemma 4.2 and the second part of Proposition 3.4. $\qquad\square$

4.4. Remark (Extension of Walter's exact sequence). Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between $\mathbb{Z}[\frac{1}{2}]$-complicial exact categories with duality such that the associated sequence of triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B} \to \mathcal{T}\mathcal{C}$ is exact. In [Wal03b], Walter extends Balmer's localization exact sequence of Witt-groups [Bal00] to an exact sequence

$$GW(\mathcal{T}\mathcal{A}, \sharp_n) \to GW(\mathcal{T}\mathcal{B}, \sharp_n) \to GW(\mathcal{T}\mathcal{C}, \sharp_n) \to W(\mathcal{T}\mathcal{A}, \sharp_{n+1}) \to W(\mathcal{T}\mathcal{B}, \sharp_{n+1}) \to \cdots$$

which continues to the right with triangular Witt groups. By theorem 3.5 and lemma 4.2, this sequence now extends to the left with higher Grothendieck-Witt groups. In fact, it is the negative degree part of the long exact sequence of homotopy groups associated with the homotopy fibre square of theorem 3.5.

4.5. The functor $F^{(\infty)}$. Let F be a functor from complicial exact categories with weak equivalences and duality. Besides the Grothendieck-Witt theory functor GW introduced in section 3, we have in mind the functors $K_{h\mathbb{Z}/2}$ and $K^{h\mathbb{Z}/2}$ – the homotopy orbit and homotopy fixed point spectra of K-theory under a $\mathbb{Z}/2$-action explained below. Let $F^{(1)}$ be the homotopy cofibre of $F\,\mathrm{Mor}(-1) \to F$ induced by the cone map 12. This comes with a map $F \to F^{(1)}$. Now set $F^{(n+1)} = (F^{(n)})^{(1)}$ and $F^{(0)} = F$. Then we have a sequence of functors

$$F = F^{(0)} \to F^{(1)} \to F^{(2)} \to F^{(3)} \to \cdots \to F^{(\infty)} = \operatorname{colim}_n F^{(n)}$$

whose (homotopy) colimit is defined to be $F^{(\infty)}$. We will calculate $F^{(\infty)}$ for F the functors GW, $K_{h\mathbb{Z}/2}$ and $K^{h\mathbb{Z}/2}$.

4.6. **Reminder on $\mathbb{L}$-theory.**

4.7. **Lemma.** *On the category of complicial exact categories with weak equivalences and duality satisfying* $(*)$*, there is a natural equivalence*

$$GW^{(\infty)} \simeq \mathbb{L}$$

where $\mathbb{L}$ *denotes Ranicki's $\mathbb{L}$-theory spectrum.*

5. Relation to $\mathbb{Z}/2$-equivariant homotopy theory

5.1. **The $\mathbb{Z}/2$-action on K-theory.** Let $(\mathcal{E}, *, w)$ be an exact category with weak equivalences and duality. Without loss of generality we may assume the duality to be strict and to satisfy $(X \oplus Y)^* = Y^* \oplus X^*$ and $0 \oplus X = X = X \oplus 0$ for all objects X of $\mathcal{E}$. This is because any exact category with weak equivalences and duality can (functorially) be replaced by one of this sort without changing K-theory and hermitian K-theory. Recall that the hyperbolic category $\mathcal{HE} = \mathcal{E} \times \mathcal{E}^o$ is equipped with the duality $(X, Y) \mapsto (Y, X)$ (ignoring the duality $*$ on $\mathcal{E}$). The functor

$$\sigma : \mathcal{HE} \to \mathcal{HE} : (X, Y) \mapsto (Y^*, X^*)$$

is duality preserving (as the duality on $\mathcal{E}$ is strict) and induces a $\mathbb{Z}/2$ action on the exact category with weak equivalences and duality $\mathcal{HE}$. By functoriality, we obtain an induced $\mathbb{Z}/2$-action on the Grothendieck-Witt spectrum $GW(\mathcal{HE}, w)$. Since this spectrum is naturally equivalent to the K-theory spectrum $K(\mathcal{E}, w)$, by [Sch10, §2 Proposition 1], we will write in what follows $K(\mathcal{E}, w)$ for $GW(\mathcal{HE}, w)$, and obtain a $\mathbb{Z}/2$-action on the K-theory spectrum $K(\mathcal{E}, w) = GW(\mathcal{HE}, w)$. Of course, this action depends on the duality on $\mathcal{E}$.

5.2. **The hypernorm in K-theory.** Let G be a finite group. For any spectrum X with G-action its Tate spectrum $\hat{\mathbb{H}}(\mathbb{Z}/2, X)$ is the homotopy cofibre of the hypernorm map $\tilde{N} : X_{hG} \to X^{hG}$ from the homotopy orbit spectrum X_{hG} to the homotopy fix point spectrum X^{hG}; see Appendix D. In Lemma 5.3 we give another description of this map for the $\mathbb{Z}/2$-spectrum $K(\mathcal{E}, w)$ associated with an exact category with weak equivalences and duality $\mathcal{E}$ (where the action was defined in §5.1).

Let $(\mathcal{E}, w, *)$ be an exact category with weak equivalences and strict duality. We consider $(\mathcal{E}, w, *)$ equipped with the trivial $\mathbb{Z}/2$-action. Recall from §5.1 that the hyperbolic category $(\mathcal{HE}, w, *)$ is equipped with a $\mathbb{Z}/2$-action (in the category of exact categories with weak equivalences and strict duality). We will construct a sequence

$$(16) \qquad\qquad \mathcal{HE} \xrightarrow{\ H\ } \bar{\mathcal{E}} \xleftarrow{\ \simeq\ } \mathcal{E} \xrightarrow{\ F\ } \mathcal{HE}$$

of $\mathbb{Z}/2$ equivariant maps between exact categories with weak equivalences and strict duality in which the backward arrow is an equivalence of categories (forgetting the $\mathbb{Z}/2$ action).

Let $\bar{\mathcal{E}}$ be the category whose objects are isomorphisms $a : A \xrightarrow{\simeq} B$ in $\mathcal{E}$ and whose maps are commutative squares in $\mathcal{E}$. We define a strict duality on $\bar{\mathcal{E}}$ on objects by $(a : A \to B)^* = ((a^*)^{-1} : A^* \to B^*)$ and on morphisms by $(f, g)^* = (f^*, g^*)$. The equivalence of categories $\mathcal{E} \to \bar{\mathcal{E}} : X \mapsto (1 : X \to X), f \mapsto (f, f)$ is duality preserving. This allows us to equip $\bar{\mathcal{E}}$ with the structure of an exact category with

weak equivalences and strict duality in such a way that the functor $\mathcal{E} \to \bar{\mathcal{E}}$ becomes an equivalence of such categories. The functor

$$\bar{\mathcal{E}} \to \bar{\mathcal{E}} : (f : X \to Y) \mapsto (f^{-1} : Y \to X)$$

is duality preserving and induces a $\mathbb{Z}/2$-action on $(\bar{\mathcal{E}}, w, *, id)$. Note that the duality preserving equivalence $(\mathcal{E}, w, *) \to (\bar{\mathcal{E}}, w, *)$ is $\mathbb{Z}/2$-equivariant. The functors H and F in diagram (16) are defined as

$$H : \mathcal{H}\mathcal{E} \to \bar{\mathcal{E}} : (X, Y) \mapsto (X \oplus Y^* \xrightarrow{\tau} Y^* \oplus X) \qquad \text{and}$$

$$F : \mathcal{E} \to \mathcal{H}\mathcal{E} : X \mapsto (X, X^*)$$

where τ denotes the map switching the two factors. Both functors are duality preserving, exact and $\mathbb{Z}/2$-equivariant.

Applying Grothendieck-Witt spectra to diagram (16) yields a diagram

$$(17) \qquad K(\mathcal{E}, w) \xrightarrow{H} GW(\bar{\mathcal{E}}, w) \xleftarrow{\simeq} GW(\mathcal{E}, w) \xrightarrow{F} K(\mathcal{E}, w)$$

of $G = \mathbb{Z}/2$ -equivariant spectra. Taking homotopy (co-) limits, we obtain a sequence of spectra

$$(18) \qquad K_{hG} \xrightarrow{H} \overline{GW}_{hG} \xleftarrow{\sim} GW_{hG} \longrightarrow GW \longrightarrow GW^{hG} \xrightarrow{F} K^{hG}$$

where we have suppressed the entries $(\mathcal{E}, w)$ and where we wrote $\overline{GW}$ for $GW(\bar{\mathcal{E}}, w)$. The non-labelled maps in the diagram are the natural maps $GW_{hG} \to GW_G = GW = GW^G \to GW^{hG}$.

5.3. **Lemma.** *The map $K(\mathcal{E}, w)_{hG} \to K(\mathcal{E}, w)^{hG}$ in (18) is naturally equivalent to the hypernorm map D.8 (34) for the $G = \mathbb{Z}/2$-spectrum $K(\mathcal{E}, w)$ defined in §5.1.*

Proof. Let $G_1 = G_2 = G = \mathbb{Z}/2$ and consider the sequence (17) as a sequence of $G_1 \times G_2$-spectra where $G_1 \times G_2$ acts on the left two spectra via the projection $G_1 \times G_2 \to G_1$ on the first factor. The action on $GW(\mathcal{E}, w)$ is trivial and the action on the last spectrum is via the projection $G_1 \times G_2 \to G_2$ onto the second factor. The map in (18) is the application of the functor $X \mapsto (X^{hG_2})_{hG_1}$ to the $G_1 \times G_2$-equivariant string (17) preceeded by $(X_2^G)_{hG} \to (X^{hG_2})_{hG_1}$ and followed by $(X^{hG_2})_{hG_1} \to (X^{hG_2})_{G_1}$. We have to show that the sequence (17) is $G_1 \times G_2$-equivariantly weakly equivalent to the sequence D.8 (33) where $X = GW(\mathcal{H}\mathcal{E}, w)$.

By naturality of the G_1-equivariant map $\mathcal{E} \to \bar{\mathcal{E}}$ applied to the G_2-equivariant map $F : \mathcal{E} \to \mathcal{H}\mathcal{E}$, we have a $G_1 \times G_2$-equivariant commutative diagram of exact categories with weak equivalences and duality

$$(19) \qquad \begin{array}{ccc} \mathcal{E} & \xrightarrow{\sim} & \bar{\mathcal{E}} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \bar{F}} \\ \mathcal{H}\mathcal{E} & \xrightarrow{\sim} & \overline{\mathcal{H}\mathcal{E}}. \end{array}$$

Define a functor $J : \mathcal{H}\mathcal{E} \times \mathcal{H}\mathcal{E} \to \overline{\mathcal{H}\mathcal{E}}$ by sending the object $((X, Y), (U, V))$ to object $(\tau, \tau) : (X \oplus U, Y \oplus V) \longrightarrow (U \oplus X, V \oplus Y)$ and on maps in the obvious way. The functor is exact, duality preserving and $G_1 \times G_2$-equivariant where G_1 acts on $\mathcal{H}\mathcal{E} \times \mathcal{H}\mathcal{E}$ by interchanging the two fators $(x, y) \mapsto (y, x)$ and G_2 acts on $\mathcal{H}\mathcal{E} \times \mathcal{H}\mathcal{E}$

via $(x, y) \mapsto (\sigma y, \sigma x)$. We have the following $G_1 \times G_2$-equivariant commutative diagram of exact categories with weak equivalences and duality

$$
\begin{array}{ccccc}
\mathcal{HE} & \xrightarrow{1,\sigma} & \mathcal{HE} \times \mathcal{HE} & \xleftarrow{\sim} & \mathcal{HE} \vee \mathcal{HE} \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle 1\vee 1} \\
\bar{\mathcal{E}} & \xrightarrow{\bar{F}} & \overline{\mathcal{HE}} & \xleftarrow{\sim} & \mathcal{HE}
\end{array}
$$

where, strictly speaking, the upper right corner only makes sense after application of the functor GW using the convention $GW(\mathcal{A} \vee \mathcal{B}) = GW(\mathcal{A}) \vee GW(\mathcal{B})$. Going down then right is the $G_1 \times G_2$-equivariant map (16) in view of the commutative square (19). Going right then down is the map that defines the hypernorm map. $\qquad \square$

5.4. **Lemma.** *Let $(\mathcal{A}, w)$ be an exact category with weak equivalences. Then all maps in the sequence (18) are stable equivalences when applied to the hyperbolic category $(\mathcal{E}, w, *) = \mathcal{H}(\mathcal{A}, w)$ with weak equivalences and duality:*

$$
K(\mathcal{HA}, w)_{h\mathbb{Z}/2} \xrightarrow{\sim} GW(\mathcal{HA}, w) \xrightarrow{\sim} K(\mathcal{HA}, w)^{h\mathbb{Z}/2}.
$$

Proof. write something $\qquad \square$

For the next theorem recall that $\mathbb{L}(\mathcal{E}, w, *)$ denotes the $\mathbb{L}$-theory spectrum whose homotopy groups are naturally isomorphic to Balmer's higher Witt groups (see **??**) and $\hat{\mathbb{H}}(\mathbb{Z}/2, K(\mathcal{E}, w, *))$ denotes the Tate spectrum (D.10) of the K-theory spectrum $K(\mathcal{E}, w, *) = GW(\mathcal{HE}, w)$ with $\mathbb{Z}/2$-action $\sigma : \mathcal{HE} \to \mathcal{HE} : (X, Y) \mapsto (Y^*, X^*)$.

5.5. **Theorem.** *On the category of complicial exact categories with weak equivalences and duality satisfying $(*)$ the sequence of functors (18) can be completed to a commutative diagram of spectra*

$$
\begin{array}{ccccc}
K_{h\mathbb{Z}/2} & \longrightarrow & GW & \longrightarrow & K^{h\mathbb{Z}/2} \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \mathbb{L} & \longrightarrow & \hat{\mathbb{H}}(\mathbb{Z}/2, K)
\end{array}
$$

in which all squares are homotopy cartesian.

Proof. For a functor F from complicial exact categories with duality write $F(n)$ for the n-th twisted functor $F(n)(\mathcal{E}, w, \sharp) = F(\mathcal{E}, w, \sharp_n)$. All functors in (18) satisfy localization and so do their twisted versions. Therefore, they send the sequence $(\mathcal{E}, w, \sharp_n) \to (\mathrm{Mor}\,\mathcal{E}, w, \sharp_n) \to (\mathcal{E}, w, \sharp_{n+1})$ to a homotopy fibration of spectra for every complicial $(\mathcal{E}, w, \sharp)$. Also, all functors in (18) satisfy additivity (and so do their twisted versions). Therefore, they induce stable equivalences when applied to the functor $\mathcal{HE} \to \mathrm{Mor}\,\mathcal{E} : (X, Y) \mapsto (X \xrightarrow{0} Y^*)$. It follows that for every $n \in \mathbb{Z}$ we

have a commutative diagram of functors

$$
\begin{array}{ccccc}
K_{h\mathbb{Z}/2}(n) \circ \mathcal{H} & \longrightarrow & GW(n) \circ \mathcal{H} & \longrightarrow & K^{h\mathbb{Z}/2}(n) \circ \mathcal{H} \\
\downarrow & & \downarrow & & \downarrow \\
K_{h\mathbb{Z}/2}(n+1) & \longrightarrow & GW(n+1) & \longrightarrow & K^{h\mathbb{Z}/2}(n+1) \\
\downarrow & & \downarrow & & \downarrow \\
S^1 \wedge K_{h\mathbb{Z}/2}(n) & \longrightarrow & S^1 \wedge GW(n) & \longrightarrow & S^1 \wedge K^{h\mathbb{Z}/2}(n)
\end{array}
$$

in which the columns are homotopy fibrations. By lemma 5.4 all maps in the top row are stable equivalences. It follows that the lower two squares are homotopy cartesian. Therefore, the composition

$$F \to F^{\infty} = \operatorname{colim}(F \to S^1 \wedge F(-1) \to S^2 \wedge F(-2) \to \cdots)$$

induces a diagram

$$
\begin{array}{ccccc}
K_{h\mathbb{Z}/2} & \longrightarrow & GW & \longrightarrow & K^{h\mathbb{Z}/2} \\
\downarrow & & \downarrow & & \downarrow \\
(K_{h\mathbb{Z}/2})^{\infty} & \longrightarrow & GW^{\infty} & \longrightarrow & (K^{h\mathbb{Z}/2})^{\infty}
\end{array}
$$

of homotopy cartesian squares. Since $K(\mathcal{E}, w)$ is (-1)-connected (that is, has trivial negative homotopy groups), so is $K_{h\mathbb{Z}/2}$ and all its twisted versions. Therefore $S^n \wedge K_{h\mathbb{Z}/2}(-n)$ is $n-1$-connected and the colimit $(K_{h\mathbb{Z}/2})^{\infty}$ is contractible. In view of Lemma 5.3 it follows that $(K_{h\mathbb{Z}/2})^{\infty}$ is the Tate spectrum $\hat{\mathbb{H}}(\mathbb{Z}/2, K)$. Finally, the spectrum GW^{∞} was identified with the $\mathbb{L}$-theory spectrum in **??**.   □

5.6. **Lemma.** *Let F be a functor from complicial exact categories with weak equivalences and duality satisfying (*) to spectra. Assume that F satisfies localization and additivity. Then:*

(1) *The functor F^{∞} also satisfies localization and additivity.*
(2) *The functor F^{∞} is trivial on hyperbolic categories, that is, for any exact category with weak equivalences $(\mathcal{A}, w)$ we have $F^{\infty}(\mathcal{H}\mathcal{A}, w) \simeq 0$.*
(3) *The natural map $F^{\infty}(n) \to S^4 \wedge F^{\infty}(n-4) \cong S^4 \wedge F^{\infty}(n)$ is an equivalence.*

Proof. The first item is clear. The second item holds because for $\mathcal{E} = \mathcal{H}\mathcal{A}$ the sequence $(\mathcal{E}, \sharp_n) \to (\operatorname{Mor}\mathcal{E}, \sharp_n) \to (\mathcal{E}, \sharp_{n+1})$ splits as it is equivalent to $\mathcal{H}\mathcal{A} \to \mathcal{H}\mathcal{A} \times \mathcal{H}\mathcal{A} \to \mathcal{H}\mathcal{A}$. Therefore, the functor F sends this sequence to a split homotopy fibration and the associated boundary map $F(n) \to S^1 \wedge F(n-1)$ is zero on hyperbolic categories. In the limit we obtain $F^{\infty}(\mathcal{H}\mathcal{A}) \simeq 0$. For the third item, in the homotopy fibration $F^{\infty}\mathcal{H} \to F^{\infty}(n) \to S^1 \wedge F^{\infty}(n-1)$ the second map is an equivalence because the left hand side is contractible. Finally, recall that there is a natural equivalence $(\mathcal{E}, w, \sharp_n) \cong (\mathcal{E}, w, \sharp_{n-4})$ which induces the equivalence $F^{\infty}(n) \cong F^{\infty}(n-4)$.   □

5.7. **Remark.** In particular, for every complicial exact category $(\mathcal{E}, w, \sharp)$ with weak equivalences and duality satisfying (*) we have a natural isomorphism

$$\pi_n F^{\infty}(\mathcal{E}, w, \sharp) \cong \pi_{n+4} F^{\infty}(\mathcal{E}, w, \sharp).$$

5.8. **Remark.** It follows from Lemma 5.6 that the spectra $GW^\infty(\mathcal{E}, w, \sharp) = \mathbb{L}(\mathcal{E}, w, \sharp)$ and $(K^{h\mathbb{Z}/2})^\infty(\mathcal{E}, w, \sharp) = \hat{\mathbb{H}}(\mathbb{Z}/2, K)(\mathcal{E}, w, \sharp)$ are 4-periodic and that in the diagram of Theorem 5.5 the map $\mathbb{L} \to \hat{\mathbb{H}}(\mathbb{Z}/2, K)$ is 4-pediodic, too.

6. Cofinality revisited

6.1. **Cofinality in K-theory.** Let $\mathcal{A} \to \mathcal{B}$ be a map of complicial exact categories with weak equivalences such that the induced map on associated triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B}$ is cofinal, that is, it is fully faithful and every object of $\mathcal{T}\mathcal{B}$ is a direct factor of an object of $\mathcal{T}\mathcal{A}$. Let $K_0(\mathcal{B}, \mathcal{A})$ be the monoid of isomorphism classes in $\mathcal{T}\mathcal{B}$ under direct sum operation, modulo the submonoid of isomorphisms classes of objects in $\mathcal{T}\mathcal{A}$. There is a canonical map $K_0(\mathcal{B}, w) \to K_0(\mathcal{B}, \mathcal{A}) : [B] \mapsto [B]$ which induces an isomorphism of the cokernel of $K_0(\mathcal{A}, w) \to K_0(\mathcal{B}, w)$ with $K_0(\mathcal{B}, \mathcal{A})$. By a theorem of Thomason [TT90, 1.10.1], there is a homotopy fibration

$$(20) \qquad\qquad K(\mathcal{A}) \to K(\mathcal{B}) \to K_0(\mathcal{B}, \mathcal{A})$$

where $K_0(\mathcal{B}, \mathcal{A})$ denotes the Eilenberg-MacLane spectrum whose only homotopy group is in degree 0 where it is $K_0(\mathcal{B}, \mathcal{A})$.

6.2. **Theorem** (Cofinality). *Let $\mathcal{A} \to \mathcal{B}$ be a map of complicial exact categories with weak equivalences and duality satisfying* (∗). *If $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B}$ is cofinal, then there are homotopy fibrations of spectra*

$$
\begin{array}{ccccc}
GW(\mathcal{A}, w) & \to & GW(\mathcal{B}, w) & \to & \mathbb{H}^\bullet(\mathbb{Z}/2, K_0) \\
\mathbb{L}(\mathcal{A}, w) & \to & \mathbb{L}(\mathcal{B}, w) & \to & \hat{\mathbb{H}}(\mathbb{Z}/2, K_0)
\end{array}
$$

where K_0 denotes the $\mathbb{Z}/2$-module $K_0(\mathcal{T}\mathcal{B}, \mathcal{T}\mathcal{A})$ with action induced by the duality functor on $\mathcal{B}$.

Proof. Write $GW^?$ and $\mathbb{L}^?$ for the homotopy cofibres of $GW(\mathcal{A}, w) \to GW(\mathcal{B}, w)$ and $\mathbb{L}(\mathcal{A}, w) \to \mathbb{L}(\mathcal{B}, w)$. By cofinality in algebraic K-theory (20) and theorem 5.5, we have a homotopy cartesian square

$$
\begin{array}{ccc}
GW^? & \longrightarrow & \mathbb{H}^\bullet(\mathbb{Z}/2, K_0) \\
\downarrow & & \downarrow \\
\mathbb{L}^? & \longrightarrow & \hat{\mathbb{H}}(\mathbb{Z}/2, K_0).
\end{array}
$$

By Cofinality for higher Grothendieck-Witt groups [Sch10, Theorem 7] and invariance under derived equivalences (Theorems 2.10 and 3.5), the upper horizontal map is an isomorphism on homotopy groups in positive degrees since in this range both groups are zero. It follows that the lower horizontal map is an isomorphism on homotopy groups in degrees ≥ 2. Since the lower horizontal map is periodic, by Remark 5.8, it induces an isomorphism in all degrees. It is therefore a stable equivalence. It follows that the upper horizontal map is also a stable equivalence. $\qquad\square$

6.3. **Remark.** The $\mathbb{L}$-theory fibration of theorem 6.2 is of course not new. It is called "Rothenberg sequence" and is proved, for instance, in [**?**].

7. Product structures and the connecting map in the Periodicity Fibration

7.1. Theorem. *Let $n = 2k$ be an even integer, and let $\mathcal{E}$ be an exact category with duality (no assumption on the characteristic!). Then the following sequence is a homotopy fibration of spaces*

$$GW^n(\mathcal{E}) \xrightarrow{F} K(\mathcal{E}) \xrightarrow{H} GW^{n+1}(\mathcal{E}).$$

Proof. Additivity

7.2. Remark. In general, the sequences in Theorem 7.1 are not homotopy fibrations for n odd, by appendix C.

7.3. Theorem. *The boundary map $GW_i^n \to GW_{i-1}^{n-1}$ in the periodicity fibration of Theorem 2.7 is given by the cup-product with the element $\eta \in GW_{-1}^{-1}(\mathbb{Z}')$ corresponding to $1 \in W(\mathbb{Z}')$ under the isomorphism $GW_{-1}^{-1}(\mathbb{Z}') \cong W(\mathbb{Z}')$.*

8. The Karoubi-Grothendieck-Witt spectrum

Define the spectrum $\mathbb{G}W(\mathcal{E}, w, \sharp)$ via suspensions. Then we have

8.1. Proposition. *The canonical map of spectra $GW(\mathcal{E}, w) \to \mathbb{G}W(\mathcal{E}, w)$ is an equivalence if $K_i(\mathcal{E}, w) = 0$ for all $i < 0$.*

8.2. Theorem (Localization). *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying 2.1 $(*)$. Assume that the associated sequence of triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B} \to \mathcal{T}\mathcal{C}$ is exact up to factors. Then the sequence of spectra*

$$\mathbb{G}W(\mathcal{A}) \to \mathbb{G}W(\mathcal{B}) \to \mathbb{G}W(\mathcal{C})$$

is a homotopy fibration.

9. Vector bundle Grothendieck-Witt groups

Here we list immediate consequences of our main theorems. The implications are well-known from K-theory and Witt-theory. All our schemes here are assusmed to be over $\mathbb{Z}[\frac{1}{2}]$. Note that for smooth schemes, we have $GW^n(X, L) = \mathbb{G}W^n(X, L)$ since smooth schemes have trivial negative K-groups.

9.1. Theorem (Nisnevich descent). *Given an elementary Nisnevich square of $\mathbb{Z}[\frac{1}{2}]$-schemes*

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} ,$$

that is, the square is cartesian, the horizontal maps are open inclusions, the vertical maps are etale, and the map $Y - V \to X - U$ on reduced induced subschemes is an isomorphism. Assume that X has an ample family of line-bundles. Then, for any

line-bundle L on X and any integer $n \in \mathbb{Z}$, the square of Karoubi-Grothendieck-Witt spectra

$$\begin{array}{ccc}
\mathbb{G}W^n(X,L) & \longrightarrow & \mathbb{G}W^n(U,L) \\
\downarrow & & \downarrow \\
\mathbb{G}W^n(Y,L) & \longrightarrow & \mathbb{G}W^n(V,L)
\end{array}$$

is homotopy cartesian.

Proof. This follows from Theorem 8.2 in view of Thomason's results in [TT90] □

9.2. **Remark.** In case the vertical maps in the square of schemes are also open immersions, the theorem was proved in [Sch10] more generally for any scheme (admitting an ample family of line bundles).

projective bundle theorem, homotopy invariance, rigidity, obstruction to excision in triangular Witt theory

10. Coherent Grothendieck-Witt groups

Dfn, localization, Poincare duality, homotopy invariance, Brown-Gersten spectral sequence, Gersten-conjecture and oriented Chow groups

Appendix A. Homology of the infinite orthogonal group

A.1. **Group completions.** Let X be a homotopy commutative H-space. A group completion of X is an H-space map $f : X \to Y$ into a homotopy commutative H-spaces Y such that $\pi_0 Y$ is a group and such that the map

$$(\pi_0 X)^{-1} H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$$

induced by f is an isomorphism. If $(\mathcal{S}, \oplus, 0)$ is a symmetric monoidal category such that all translations $\oplus B : \mathcal{S} \to \mathcal{S} : A \mapsto A \oplus B$ are faithful, Quillen constructs a symmetric monoidal category $\mathcal{S}^{-1}\mathcal{S}$ and a monoidal map $\mathcal{S} \to \mathcal{S}^{-1}\mathcal{S}$ which induces a group completion of associated classifying spaces [Gra76, Theorem, p. 221]. If $(\mathcal{S}, \oplus, 0)$ is a symmetric strict monoidal category (up to equivalence of categories, this can always be achieved [May74]), a group completion is also the map $B\mathcal{S} \to \Omega \bar{B}\mathcal{S}$ where $\bar{B}\mathcal{S}$ is the Bar construction (classifying space) of the topological monoid $B\mathcal{S}$ [**?**]. The two group completions are equivalent via the zigzag $\Omega \bar{B}\mathcal{S} \xrightarrow{\sim} \Omega \bar{B}\mathcal{S}^{-1}\mathcal{S} \xleftarrow{\sim} B\mathcal{S}^{-1}\mathcal{S}$ of homotopy equivalences.

Let $(\mathcal{A}, *, \eta)$ be an additive category with duality. Inclusion of degree zero simplices yields a map $(i\mathcal{A})_h \to (iS_\bullet^e \mathcal{A})_h$ such that its composition with $(iS_\bullet^e \mathcal{A})_h \to iS_\bullet \mathcal{A}$ is the zero map. Therefore, we obtain an induced map into the homotopy fibre of the last map

$$(21) \qquad\qquad\qquad (i\mathcal{A})_h \longrightarrow GW(\mathcal{A}).$$

In [Sch04b], we have shown that this map is a group completion if 2 is invertible in $\mathcal{A}$. Our proof in [Sch04b] uses Karoubi's fundamental theorem [Kar80]. The purpose of this section is to give a direct proof of that fact avoiding the fundamental theorem, so that our proof of theorem 2.7 (see also §3.6) gives not only a generalization but also a new proof of Karoubi's theorem and its topological counterpart, classical Bott periodicity; cf. §B.

A.2. **Theorem.** *Let $(\mathcal{A}, *, \eta)$ be a split exact category with duality such that $\frac{1}{2} \in \mathcal{A}$. Then the natural map (21) is a group completion. In particular, there is a natural homotopy equivalence*

$$(i\mathcal{A}_h)^{-1}(i\mathcal{A}_h) \simeq GW(\mathcal{A}).$$

The proof of theorem A.2 occupies most of this section. Before going into the details we derive the statement that makes the link between the homology of the infinite orthogonal group and higher Grothendieck-Witt groups. For that, let R be a ring with involution $r \mapsto \bar{r}$ and $\varepsilon = \pm 1$. We denote by

$$_\varepsilon\mathcal{P}(R) = (\mathcal{P}(R), *, \varepsilon \operatorname{can}).$$

the additive category with duality, where $\mathcal{P}(R)$ is the category of finitely generated projective right R-modules, the dual P^* of a right R-module P is the left R-module $P^* = \operatorname{Hom}_R(P, R)$ considered as a right module via the involution, and $\operatorname{can}_P : P \to P^{**}$ is the double dual identification with $\operatorname{can}_P(x)(f) = \overline{f(x)}$. The associated category $(i_\varepsilon\mathcal{P}(R))_h$ is the usual category of non-degenerated ε-symmetric bilinear forms on finitely generated projective R-modules with isometries as morphisms. We denote by $_\varepsilon GW(R)$, the Grothendieck-Witt space

$$_\varepsilon GW(R) = GW(\mathcal{P}(R), *, \varepsilon \operatorname{can}).$$

By theorem A.2, this space is a group completion of $(i_\varepsilon\mathcal{P}(R))_h$. Let

$$_\varepsilon O(R) = \operatorname{colim}_n Aut(_\varepsilon H(R^n))$$

denote the infinite ε-orthogonal group of R, where $Aut(_\varepsilon H(P))$ denotes the group of isometries of the ε-hyperbolic space $_\varepsilon H(P) = (P \oplus P^*, \left(\begin{smallmatrix} 0 & 1 \\ \varepsilon \operatorname{can} & 0 \end{smallmatrix}\right))$ and the inclusion $Aut(_\varepsilon H(R^n)) \to Aut(_\varepsilon H(R^{n+1}))$ is induced by the functor $\perp {_\varepsilon H(R)}$.

A.3. **Corollary.** *For any ring with involution R such that $\frac{1}{2} \in R$, there is a natural homotopy equivalence*

$$B_\varepsilon O(R)^+ \xrightarrow{\sim} {_\varepsilon GW(R)_0}$$

where $_\varepsilon GW(R)_0$ denotes the connected component of 0 of the space $_\varepsilon GW(R)$ and $B_\varepsilon O(R)^+$ is Quillen's plus construction applied to $B_\varepsilon O(R)$.

Proof. By theorem A.2, the space $_\varepsilon GW(R)$ is a group completion of $(i_\varepsilon\mathcal{P}(R))_h$. The proof now is the same as in [Gra76, Theorem, p. 224]. The main point is that if $\frac{1}{2} \in R$ then every non-degenerated ε-symmetric bilinear form is a direct factor of the hyperbolic form $_\varepsilon H(R^n)$ for some n, see also [Wei81, Proposition 3]. $\square$

The proof of theorem A.2 is based on the following proposition. We introduce some notation. For an exact category with duality $(\mathcal{E}, *, \eta)$, the category $E(\mathcal{E})$ of exact sequences in $\mathcal{E}$ is an exact category with duality where the dual $(E_\bullet)^*$ of an exact sequence $E_\bullet = (E_{-1} \rightarrowtail E_0 \twoheadrightarrow E_1)$ in $\mathcal{E}$ is the exact sequence $E_1^* \rightarrowtail E_0^* \twoheadrightarrow E_{-1}^*$ and the double dual identification $\eta_{E_\bullet}$ is $(\eta_{E_{-1}}, \eta_{E_0}, \eta_{E_1})$.

A.4. **Proposition.** *Let $\mathcal{A}$ be a split exact category with duality such that $\frac{1}{2} \in \mathcal{A}$. Then the map $F : (iE\mathcal{A})_h \to i\mathcal{A} : (E_\bullet, \varphi) \mapsto E_{-1}$ of symmetric monoidal categories induces a homotopy equivalence after group completion.*

Proof. The functor F has a section, the hyperbolic functor

$$\mathcal{H} : i\mathcal{A} \to (iE\mathcal{A})_h : \quad A \quad \mapsto \quad \mathcal{H}(A) = \left(A \overset{\binom{1}{0}}{\rightarrowtail} A \oplus A^* \overset{(\,0\ 1\,)}{\twoheadrightarrow} A, \left(\eta, \left(\begin{smallmatrix} 0 & 1 \\ \eta & 0 \end{smallmatrix} \right), 1 \right) \right)$$

$$g \quad \mapsto \quad \mathcal{H}(g) = \left(g, \left(\begin{smallmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{smallmatrix} \right), (g^*)^{-1} \right)$$

which is symmetric monoidal and thus induces an action of $i\mathcal{A}$ on $(iE\mathcal{A})_h$ in the sense of [Gra76]. In view of the lemma A.6 3 below, the hyperbolic functor induces on π_0 an isomorphism of abelian monoids

$$\pi_0(\mathcal{H}) : S = \pi_0 i\mathcal{A} \overset{\cong}{\longrightarrow} \pi_0(iE\mathcal{A})_h.$$

To abbreviate, we write S for this monoid. The actions of the symmetric monoidal category $i\mathcal{A}$ on $(iE\mathcal{A})_h$ and on $i\mathcal{A}$ induce actions of the abelian monoid S on the homology groups of $(iE\mathcal{A})_h$ and of $i\mathcal{A}$ compatible with F. We will show that the localized map

(22) $F : S^{-1} H_*((iE\mathcal{A})_h, k) \to S^{-1} H_*(i\mathcal{A}, k)$

is an isomorphism when the coefficient group is $k = \mathbb{F}_p, \mathbb{Q}$. This implies that (22) is an isomorphism for $k = \mathbb{Z}$, that is, the functor $F : (iE\mathcal{A})_h \to i\mathcal{A}$ induces a homology (hence homotopy) equivalence after group completion.

In order to prove that (22) is an isomorphism, we will analize the spectral sequence associated with the functor $F : (iE\mathcal{A})_h \to i\mathcal{A}$. We first review some elementary properties of that spectral sequence in the more general context of an arbitrary functor $F : \mathcal{B} \to \mathcal{C}$ of small categories. For that, we fix an abelian group k (which we will suppress in most formulas), and write $H_p(\mathcal{C}) = H_p(\mathcal{C}, k)$ for the homology of the category $\mathcal{C}$ with coefficients in k. More generally, we write $H_p(\mathcal{C}, A)$ for the homology of $\mathcal{C}$ with coefficients in a functor $A : \mathcal{C} \to \langle \text{ab gps} \rangle$ from $\mathcal{C}$ to abelian groups. This is the homology of the chain complex associated with the simplicial abelian group

$$p \mapsto \bigoplus_{C_0 \to \ldots \to C_p} A(C_0)$$

of chains on $N\mathcal{C}$ with coefficients in A.

A.5. **Lemma.**
  (1) *Let $F : \mathcal{B} \to \mathcal{C}$ be a functor between small categories. There is a strongly convergent first quadrant spectral sequence*

$$E^2_{p,q}(F) = H_p(\mathcal{C}, H_q(F \downarrow \_\,)) \Rightarrow H_{p+q}(\mathcal{B})$$

with differentials $d^n_{p,q} : E^n_{p,q} \to E^n_{p-n,q+n-1}$ and edge morphism

$$H_p(\mathcal{B}) \twoheadrightarrow E^\infty_{p,0} \subset E^2_{p,0} = H_p(\mathcal{C}, H_0(F \downarrow \_\,)) \to H_p(\mathcal{C})$$

the map $H_p(F)$ induced by F on homology. Here, the last map is the change-of-coefficient map sending $(F \downarrow \_\,)$ to the constant one-object-one-morphism category.
  (2) *If $F' : \mathcal{B}' \to \mathcal{C}'$ is another functor, and $b : \mathcal{B} \to \mathcal{B}'$ and $c : \mathcal{C} \to \mathcal{C}'$ are functors such that $F'b = cF$, then there is a map of spectral sequences $E(b,c) : E(F) \to E(F')$ compatible with edge-maps and which, on E^2-term and abutment, is the usual maps on homology induced by b and c.*

(3) *Given two pairs of functors $b_i : \mathcal{B} \to \mathcal{B}'$, $c_i : \mathcal{C} \to \mathcal{C}'$ such that $F'b_i = c_iF$, $i = 0, 1$. If there are natural transformations $\beta : b_0 \Rightarrow b_1$ and $\gamma : c_0 \Rightarrow c_1$ such that $F'\beta_B = \gamma_{FB}$ for all objects B of $\mathcal{B}$, then the two induced maps of spectral sequences agree from the E^2-page on:*

$$E_{p,q}^n(b_0, c_0) = E_{p,q}^n(b_1, c_1) : E_{p,q}^n(F) \to E_{p,q}^n(F'), \ \ n \geq 2$$

Proof (sketch). The spectral sequence is obtained by filtering the bicomplex $K_{p,q} = \bigoplus_{X_{p,q}} k$ associated with the bisimplicial set $X_{\bullet,\bullet}$ whose p, q simplices is the set

$$X_{p,q} = \{B_0 \to ... \to B_q, \ FB_q \to C_0 \to ... \to C_p \mid B_i \in \mathcal{B}, \ C_i \in \mathcal{C}\}.$$

The filtration is by subcomplexes $K_{\leq n,\bullet} \subset K_{\leq n+1,\bullet} \subset K$ with $(K_{\leq n,\bullet})_{p,q} = K_{p,q}$ if $p \leq n$ and $(K_{\leq n,\bullet})_{p,q} = 0$ if $p > n$. As in the proof of Quillen's theorem A [Qui73, §1], the bisimplicial set $X_{\bullet,\bullet}$ is homotopy equivalent to the bisimplicial set which is constant in the p-direction and is the nerve $N_\bullet\mathcal{B}$ of $\mathcal{B}$ in the q-direction. In particular, the abuttment of the spectral sequence computes the homology of $\mathcal{B}$. It is straight forward to identify the E^2-term as in 1. Functoriality as in 2 is clear since the pair (b, c) induces a map of the corresponding bicomplexes. The edge map for the spectral sequence of the identity functor id_C of $\mathcal{C}$ is the identity map, so that the functoriality in 2, applied to the map $(F, id_{\mathcal{C}}) : F \to id_{\mathcal{C}}$ of functors, yields the description of the edge map in 1. For the somewhat extended functoriality in 3, one constructs an explicit homotopy between the two maps

$$(b_i, c_i) : \bigoplus_{C_0 \to ... \to C_p} H_j(F \downarrow C_0) \longrightarrow \bigoplus_{C'_0 \to ... \to C'_p} H_j(F' \downarrow C'_0), \quad i = 0, 1,$$

of simplicial abelian groups which compute the E^2-terms of the spectral sequences. Again this is straight forward, and we omit the details. $\square$

We apply the lemma to the functor $F : (iE\mathcal{A})_h \to i\mathcal{A}$ and obtain the spectral sequence

$$(23) \qquad E_{p,q}^2 = H_p(i\mathcal{A}, H_q(F \downarrow \_)) \Rightarrow H_{p+q}((iE\mathcal{A})_h).$$

Every object A of $i\mathcal{A}$ induces functors $\oplus\mathcal{H}(A) : (iE\mathcal{A})_h \to (iE\mathcal{A})_h$ and $\oplus A : i\mathcal{A} \to i\mathcal{A}$ compatible with F. By lemma A.5 2, these functors induce a map of spectral sequences. Every map $A \to A'$ in $i\mathcal{A}$ induces natural transformations $\oplus\mathcal{H}(A) \Rightarrow \oplus\mathcal{H}(A')$ and $\oplus A \Rightarrow \oplus A'$ compatible with F. By lemma A.5 3, the objects A and A' induce the same map of spectral sequences from E^2 on. In other words, we obtain an induced action of the monoid S on the spectral sequence (23) from E^2 on. Since localization w.r.t. a commutative monoid is exact, we obtain a strongly convergent first quadrant spectral sequence

$$(24) \qquad S^{-1}E_{p,q}^2 = S^{-1}H_p(i\mathcal{A}, H_q(F \downarrow \_)) \Rightarrow S^{-1}H_{p+q}((iE\mathcal{A})_h).$$

In order to analize this spectral sequence, we will employ the symmetric monoidal automorphism

$$\Sigma : (iE\mathcal{A})_h \to (iE\mathcal{A})_h : (M, \varphi) \mapsto (M, \frac{1}{2}\varphi), g \mapsto g.$$

Note that $F \circ \Sigma = F$, so that Σ induces an action of the spectral sequence (23). For every object A in $i\mathcal{A}$, there is a natural transformation of functors

$$\Sigma \circ (\oplus\mathcal{H}(A)) \Rightarrow (\oplus\mathcal{H}(A)) \circ \Sigma$$

compatible with F. It is induced by the isometry $\Sigma \mathcal{H}(A) \cong \mathcal{H}(A)$ given by the map

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} : A \oplus A^* \to A \oplus A^*.$$

Therefore, the action of S on the sectral sequence (23) commutes with the action of Σ on it from the E^2-page on. In particular, Σ induces an action on the spectral sequence (24).

A.6. **Lemma.**

(1) *For every object A of $i\mathcal{A}$, there is an equivalence of categories between the comma category $(F \downarrow A)$ and the one-object category whose endomorphism set is the set*

$$\{a : A^* \to A \mid \eta\, a + a^* = 0\} \subset Hom(A^*, A)$$

with addition as composition. This is a uniquely 2-divisible abelian group.

(2) *Under the equivalence in 1, the automorphism $\Sigma : (F \downarrow A) \to (F \downarrow A)$ corresponds to $a \mapsto 2a$.*

(3) *The functor F induces an isomorphism of abelian monoids*

$$\pi_0(iE\mathcal{A})_h \xrightarrow{\cong} \pi_0 i\mathcal{A}.$$

Proof. Choose for every inflation $M_{-1} \overset{i}{\rightarrowtail} M_0$ in $\mathcal{A}$ a retraction $r : M_0 \twoheadrightarrow M_{-1}$, so that we have $ri = 1$. For the inflation $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : A \rightarrowtail A \oplus A^*$ we choose $r = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$. The choice of these retractions defines for every object

$$(25) \qquad\qquad M_{-1} \overset{i}{\rightarrowtail} M_0 \overset{p}{\twoheadrightarrow} M_1, \ (\varphi_{-1}, \varphi_0, \varphi_1), \quad M_{-1} \overset{g}{\to} A$$

of the comma category $(F \downarrow A)$ an isomorphism to the object

$$(26) \qquad\qquad A \overset{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}{\rightarrowtail} A \oplus A^* \overset{\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)}{\twoheadrightarrow} A^*, \ \left(\eta, \left(\begin{smallmatrix} 0 & 1 \\ \eta & 0 \end{smallmatrix}\right), 1\right), \quad A \overset{1}{\to} A$$

of that category given by the maps

$$(27) \qquad g : M_{-1} \to A, \quad \begin{pmatrix} gr + g\delta p \\ (g^*)^{-1}\varphi_1 p \end{pmatrix} : M_0 \to A \oplus A^*, \quad (g^*)^{-1}\varphi_1 : M_1 \to A^*,$$

where the map $r : M_0 \twoheadrightarrow M_{-1}$ is the chosen retraction of $i : M_{-1} \rightarrowtail M_0$, the map $\delta : M_1 \to M_{-1}$ is defined as $\delta = \frac{1}{2}(\varphi_1^* \eta)^{-1}\beta$, and $\beta : M_1 \to M_1^*$ is the unique map satisfying $p^* \beta p = \psi$ for the symmetric map $\psi = \varphi_0 - p^*\varphi_1^* \eta r - r^* \varphi_1 p : M_0 \to M_0^*$, the existence of β being assured since $\psi i = 0$. Note that the map just defined is an isomorphism in $E\mathcal{A}$ since it is an isomorphism on subobject and quotient object. All these isomorphisms together define an equivalence of categories between $(F \downarrow A)$ and the full subcategory of $(F \downarrow A)$ consisting of the single object (26). The endomorphism set of (26) is the set consisting of the maps

$$(28) \qquad 1 : A \to A, \quad \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : A \oplus A^* \to A \oplus A^*, \quad 1 : A^* \to A^*,$$

where $\eta\, a + a^* = 0$. Composition corresponds to matrix multiplication, hence to addition of the a's. By our assumption $\frac{1}{2} \in \mathcal{A}$, the group $Hom(A^*, A)$ is uniquely 2-divisible, so is the kernel of the map $a \mapsto \eta\, a + a^*$. This proves part 1 of the lemma.

The existence of the isomorphism from (25) to (26) above (with $g = 1$) shows that every object of $(iE\mathcal{A})_h$ is (up to isomorphism) in the image of the hyperbolic functor $\mathcal{H} : i\mathcal{A} \to (iE\mathcal{A})_h$. Since $F \circ \mathcal{H} = id$, this proves 3.

For part 2, we note that Σ sends the object (26) to the object

(29) $$ A \overset{\binom{1}{0}}{\rightarrowtail} A \oplus A^* \overset{(0\ 1)}{\twoheadrightarrow} A^*, \quad \frac{1}{2}\left(\eta, \left(\begin{smallmatrix} 0 & 1 \\ \eta & 0 \end{smallmatrix}\right), 1\right), \quad A \overset{1}{\rightarrow} A $$

for which, according to our choice of $r = (\,1\ 0\,)$ (implying $\psi = 0$ and $\delta = 0$), the isomorphism (27) is given by the maps

$$ 1 : A \rightarrow A, \quad \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/2 \end{smallmatrix}\right) : A \oplus A^* \rightarrow A \oplus A^*, \quad 1/2 : A^* \rightarrow A^*. $$

The functor Σ sends the endomorphism (28) of (26) to the endomorphism of (29) given by the same maps as in (28). Under the equivalence of categories, this corresponds to the endomorphism of (26) given by the maps

$$ 1 : A \rightarrow A, \quad \left(\begin{smallmatrix} 1 & 2a \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)^{-1} : A \oplus A^* \rightarrow A \oplus A^*, \quad 1 : A^* \rightarrow A^*. $$

This proves the lemma. $\qquad\square$

Let $k = \mathbb{Z}/2$. Then lemma A.6 1 implies that the spectral sequences (23) and (24) degenerate at the E^2-page to yield the isomorphism (22) since for a uniquely 2-divisible abelian group G, we have $H_q(G, \mathbb{Z}/2) = 0$, $q > 0$.

From now on we will assume $k = \mathbb{F}_p$, p prime $\neq 2$, or $k = \mathbb{Q}$.

A.7. **Lemma.** *Let k be one of the fields $\mathbb{F}_p$, $p \neq 2$, or $\mathbb{Q}$. Let G be an abelian group and $t : G \rightarrow G : a \mapsto 2a$. Then for all $q \geq 0$,*

$$ H_q(G, k) = \bigoplus_{q/2 \leq i \leq q,\ i \in \mathbb{N}} V_i $$

where $H_q(t)$ acts on the k-vector space V_i as multiplication by 2^i.

Proof. This follows from the explicit description of $H_*(G)$ given, for instance, in [Bro94, Theorem 6.4 (ii), Theorem 6.6]. Alternatively, it can be proved as follows. The statement is clearly true for $q = 0$ and $q = 1$ (for the latter use the natural isomorphism $G \cong \pi_i BG \cong H_1(BG, \mathbb{Z})$), hence it is true for $G = \mathbb{Z}$ (as $B\mathbb{Z} = S^1$). Using the Serre spectral sequence associated with the fibration $B\mathbb{Z} \rightarrow B\mathbb{Z} \rightarrow B(\mathbb{Z}/l)$ we see that the statement is true for $G = \mathbb{Z}/l$. Using the Künneth formula, if the statement is true for G_1 and G_2 then it is true for $G_1 \times G_2$. Therefore, the lemma holds for finitely generated abelian groups. Passing to limits, we conclude that it is true for all abelian groups. $\qquad\square$

Lemma A.7 applies to the endomorphism Σ of the k-module $H_q(F \downarrow A)$, in view of lemma A.6 2. We obtain a direct sum decomposition of the spectral sequences (23) and (24) into the eigen-space spectral sequences of Σ. Note that the eigen-space spectral sequence of (24) corresponding to the eigen-value 1 has $E^2_{p,q}$-term $S^{-1}H_p(i\mathcal{A}, H_0(F \downarrow \_\,)) = H_p(i\mathcal{A}, k)$ if $q = 0$ and $E^2_{p,q} = 0$ for $q \neq 0$. In particular, it degenerates completely at the E^2-page.

The next lemma shows that Σ acts as the identity on the abuttment of (24), so that the eigen-space spectral sequence of (24) corresponding to the eigen-value 1 has abuttment the abuttment of (24). Degeneration of this spectral sequence at the E^2-page shows that its edge map (A.5 1) is an isomorphism, that is, (22) is also an isomorphism for $k = \mathbb{F}_p$, p prime $\neq 2$ and $\mathbb{Q}$.

A.8. **Lemma.** *Let k be one of the fields $\mathbb{F}_p$, p prime $\neq 2$, or $\mathbb{Q}$.*

(1) *Let $(\mathcal{C}, \oplus, 0)$ be a unital symmetric monoidal category with $\pi_0 \mathcal{C} = 0$. Then the map $id \oplus id : \mathcal{C} \to \mathcal{C} : A \mapsto A \oplus A$ induces an isomorphism on homology*

$$H_*(\mathcal{C}, k) \xrightarrow{\sim} H_*(\mathcal{C}, k).$$

(2) *The functor $\Sigma : (iE\mathcal{A})_h \to (iE\mathcal{A})_h$ induces the identity map on the abuttment $S^{-1}H_*(iE\mathcal{A})_h$ of the spectral sequence (24).*

Proof. For part 1, consider $X = Y = B\mathcal{C}$ as pointed spaces with base-point $0 \in \mathcal{C}$, and write $f : X \to Y$ for the map on classifying spaces induced by $id \oplus id : \mathcal{C} \to \mathcal{C}$. Note that f preserves base points and that $\pi_*(f)$ is multiplication by 2, so that $\pi_*(f) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism. Let G be the cokernel of the map $\pi_1(f) : \pi_1 X \to \pi_1 Y$ and write $\tilde{Y} \to Y$ for the covering of Y corresponding to the quotient map $\pi_1 Y \to G$, that is, $\tilde{Y}$ is the pullback of EG along $Y \to B\pi_1 Y \to BG$. By construction, the map $f : X \to Y$ factors through $\tilde{Y} \to Y$, and the induced map $X \to \tilde{Y}$ is surjective on π_1. Let F be the homotopy fibre of $X \to \tilde{Y}$. By construction, F is a connected nilpotent space (in fact an infinite loop space as all maps $X \to \tilde{Y} \to Y \to BG$ are infinite loop space maps) with $\pi_* F \otimes \mathbb{Z}[\frac{1}{2}] = 0$. It follows that $H_q(F, \mathbb{Z}[\frac{1}{2}]) = 0$, hence $H_q(F, k) = 0$, $q > 0$. By the Serre spectral sequence, $X \to \tilde{Y}$ induces an isomorphism in homology with coefficients in k. Finally, the principal G-fibration $\tilde{Y} \to Y$ induces an isomorphism in homology with coefficients in k, since every element in G has order invertible in k.

For part 2, write $\mathcal{S} = (iE\mathcal{A})_h$. The abuttment of (24) is the homology of the symmetric monoidal category $\mathcal{C} = \mathcal{S}^{-1}\mathcal{S}$, in view of A.6 3 and [Gra76, ???], see also A.1. Note that $\pi_0 \Sigma : \pi_0 \mathcal{S} \to \pi_0 \mathcal{S}$ is the identity because $F \circ \Sigma = F$ and $\pi_0 F$ is an isomorphism, by A.6 3. In particular, Σ induces the identity on $\pi_0 \mathcal{C}$. Let $\mathcal{C}_0 \subset \mathcal{C}$ be the connected component of 0. In view of the natural isomorphism $H_*(\mathcal{C}_0) \otimes_k k\pi_0 \mathcal{C} \cong H_*\mathcal{C}$, it suffices to show that the restriction Σ_0 of $\Sigma : \mathcal{C} \to \mathcal{C}$ to $\mathcal{C}_0$ induces the identity on homology with coefficients in k. To that end, we consider the functor $T = id \oplus id : \mathcal{S} \to \mathcal{S}$ and the natural transformation $\Sigma T \cong T$ given by

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} : (M, \varphi/2) \perp (M, \varphi/2) \xrightarrow{\cong} (M, \varphi) \perp (M, \varphi).$$

The functors T and Σ and the natural transformation induce functors T and Σ on $\mathcal{C} = \mathcal{S}^{-1}\mathcal{S}$ and a natural transformation $\Sigma T \cong T$ of endofunctors of $\mathcal{C}$. Restricted to the connected component $\mathcal{C}_0$ of $\mathcal{C}$, we obtain functors $T_0, \mathcal{S}_0 : \mathcal{C}_0 \to \mathcal{C}_0$ and a natural transformation $\Sigma_0 T_0 \cong T_0$. By part 1, T_0 induces an isomorphism on homology with coefficients in k. Since $H_*(\Sigma_0, k) \circ H_*(T_0, k) = H_*(T_0, k)$, by the existence of the natural transformation $\Sigma_0 T_0 \cong T_0$, we have $H_*(\Sigma_0, k) = id$. $\square$

This finishes the proof of proposition A.4. $\square$

For the statement of the next proposition, recall that for an exact category with duality $(\mathcal{A}, *, \eta)$, the category $S_n\mathcal{A}$ is an exact category with duality, where the dual A^* of an object A of $S_n\mathcal{A}$ is $(A^*)_{i,j} = (A_{n-j,n-i})^*$ and the double dual identification η_A satisfies $(\eta_A)_{i,j} = \eta_{A_{i,j}}$.

A.9. Proposition. *Let $\mathcal{A}$ be an additive category with duality. Assume $\frac{1}{2} \in \mathcal{A}$. Then the symmetric monoidal functors*

$$(iS_{2n}\mathcal{A})_h \to (i\mathcal{A})^n : \qquad\qquad (A_\bullet, \varphi_\bullet) \mapsto (A_{0,1}, A_{1,2}, ..., A_{n-1,n})$$

$$(iS_{2n+1}\mathcal{A})_h \to (i\mathcal{A})^n \times (i\mathcal{A})_h : \quad (A_\bullet, \varphi_\bullet) \mapsto (A_{0,1}, A_{1,2}, ..., A_{n-1,n}), (A_{n,n+1}, \varphi_{n,n+1})$$

are homotopy equivalences after group completion.

Before we give the proof, we note the following corollary of proposition A.4.

A.10. Corollary (of proposition A.4). *Let $(A, *, \eta)$, $(\mathcal{B}, *, \eta)$ be additive categories with duality such that $\frac{1}{2} \in \mathcal{A}, \mathcal{B}$.*

(1) *Given a non-singular exact form functor $(F_\bullet, \varphi_\bullet) : \mathcal{A} \to E\mathcal{B}$, that is, a commutative diagram of additive functors $F_i : (\mathcal{A}, w) \to (\mathcal{B}, w)$, $i = -1, 0, 1$,*

$$
\begin{array}{ccccccc}
F_\bullet : & & F_1 & \xrightarrow{\;\;d_1\;\;} & F_0 & \xrightarrow{\;\;d_0\;\;} & F_{-1} \\[4pt]
\cong \Big\downarrow \varphi_\bullet & & \cong \Big\downarrow \varphi_1 & & \cong \Big\downarrow \varphi_0 & & \cong \Big\downarrow \varphi_{-1} \\[4pt]
F_\bullet^\sharp : & & F_{-1}^\sharp & \xrightarrow[d_0^\sharp]{\;\;} & F_0^\sharp & \xrightarrow[d_1^\sharp]{\;\;} & F_1^\sharp
\end{array}
$$

with exact rows and $(\varphi_1, \varphi_0, \varphi_{-1}) = (\varphi_{-1}^\sharp\eta, \varphi_0^\sharp\eta, \varphi_1^\sharp\eta)$ an isomorphism. Then the two non-singular exact form functors

$$(F_0, \varphi_0) \quad \text{and} \quad H \circ F_1$$

induce homotopic maps $(i\mathcal{A})_h \to (i\mathcal{B})_h$ after group completions.

(2) *Given a non-singular exact form functor $(F_\bullet, \varphi_\bullet) : \mathcal{A} \to S_3\mathcal{B}$, then the two non-singular exact form functors*

$$(F_{03}, \varphi_{03}) \quad \text{and} \quad (F_{12}, \varphi_{12}) \perp H \circ F_{01}$$

induce homotopic maps $(i\mathcal{A})_h \to (i\mathcal{B})_h$ after group completions.

Proof. Part 1 is a formal consequence of Proposition A.4. In a more general context, the implication is explained, for instance, in [Sch10, 3.10. Proof of theorem 3.3].

For 2, write $\sim$ for "homotopic after group completion". For any additive form functor $(F, \varphi) : \mathcal{A} \to \mathcal{B}$, we have a form functor $\mathcal{A} \to E\mathcal{B}$ given by

$$F \xrightarrow{\binom{1}{1}} F \oplus F \xrightarrow{(1\;-1)} F, \quad (\varphi, \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}, \varphi),$$

so that $H(F) \sim (F, \varphi) \perp (F, -\varphi)$, by 1.

Now consider the given non-singular exact form functor $(F_\bullet, \varphi_\bullet) : \mathcal{A} \to S_3\mathcal{B}$. It induces the non-singular exact form functor $\mathcal{A} \to E\mathcal{B}$ given by

$$F_{02} \xrightarrow{\binom{F_{02\leq 12}}{F_{02\leq 03}}} F_{12} \oplus F_{03} \xrightarrow{(-F_{12\leq 23}\;\; F_{03\leq 23})} F_{23}, \quad (\varphi_{02}, \begin{pmatrix} -\varphi_{12} & 0 \\ 0 & \varphi_{03} \end{pmatrix}, \varphi_{23}),$$

so that $H(F_{02}) \sim (F_{12}, -\varphi_{12}) \perp (F_{03}, \varphi_{03})$, by 1. By additivity in K-theory and the remark above, we have $H(F_{02}) \sim H(F_{01}) \perp H(F_{12}) \sim H(F_{01}) \perp (F_{12}, \varphi_{12}) \perp (F_{12}, -\varphi_{12})$. Cancelling $(F_{12}, -\varphi_{12})$, the claim follows. $\qquad\square$

Proof of proposition A.9. The cases S_0 and S_1 are trivial, and the case S_2 is proposition A.4 since $E\mathcal{A} = S_2\mathcal{A}$. Recall that the assignement $[n] \mapsto S_n\mathcal{A}$ is a simplicial category, where a monotonic map $\vartheta : [n] \to [p]$ induces the functor $\vartheta^! : S_p\mathcal{A} \to S_n\mathcal{A} : A \mapsto \vartheta^!A$ with $(\vartheta^!A)_{i,j} = A_{\vartheta(i),\vartheta(j)}$. If $\vartheta(n-i) = p - \vartheta(i)$, then $\vartheta^!$ preserves dualities.

We consider the following monotonic maps:

$$\theta : [n-2] \to [n] : i \mapsto i+1, \quad \rho : [n] \to [n-2] : i \mapsto \begin{cases} 0 & i = 0 \\ i-1 & 1 \le i \le n-1 \\ n-2 & i = n, \end{cases}$$

$$\iota : [1] \to [n] : i \mapsto i, \qquad \pi : [n] \to [1] : i \mapsto \begin{cases} 0 & i = 0 \\ 1 & 1 \le i \le n. \end{cases}$$

Note that $\theta^!$ and $\rho^!$ preserve dualities whereas, $\iota^!$ and $\pi^!$ don't. We will show that the functor

$$F = (\iota^!, \theta^!) : (iS_n\mathcal{A})_h \to i\mathcal{A} \times (iS_{n-2}\mathcal{A})_h : (A, \varphi) \mapsto \iota^!A, \theta^!(A, \varphi)$$

is a homotopy equivalence after group completion with inverse the functor

$$G = \rho^! \perp H\pi^! : i\mathcal{A} \times (iS_{n-2}\mathcal{A})_h \to (iS_n\mathcal{A})_h : A, (B, \varphi) \mapsto \rho^!(B, \varphi) \perp H(\pi^!A),$$

where $H : iS_n\mathcal{A} \to (iS_n\mathcal{A})_h$ denotes the hyperbolic functor $A \mapsto A \oplus A^*, \left(\begin{smallmatrix} 0 & 1 \\ \eta & 0 \end{smallmatrix} \right)$. One readily verifies $F \circ G = id$. For the other composition, consider the following monotonic maps

$$\sigma_{01} = \iota\pi : [n] \to [n], \qquad \sigma_{02} : [n] \to [n] : i \mapsto \begin{cases} i & 0 \le i \le n-1 \\ n-1 & i = n, \end{cases}$$

$$\sigma_{03} = id : [n] \to [n] : i \mapsto i, \quad \sigma_{13} : [n] \to [n] : i \mapsto \begin{cases} 1 & i = 0 \\ i & 1 \le i \le n, \end{cases}$$

$$\sigma_{12} = \theta\rho : [n] \to [n], \qquad \sigma_{23} : [n] \to [n] : i \mapsto \begin{cases} n-1 & 0 \le i \le n-1 \\ n & i = n, \end{cases}$$

and note that $(i, j) \mapsto \sigma_{i,j}^!$ defines a duality preserving functor $\sigma^! : S_n\mathcal{A} \to S_3S_n\mathcal{A}$. By corollary A.10 2, the functors $G \circ F = \sigma_{12}^! \perp H\sigma_{01}^!$ and $\sigma_{03}^! = id$ induce homotopic maps after taking group completions. $\square$

Proof of theorem A.2. This is the same as the proof of [Sch04b, Theorem 4.2], or its version [Sch04b, Corollary 4.6]. The only place where Karoubi's fundamental theorem was used in [Sch04b] was in [Sch04b, Lemma 4.4] which asserts that a certain monoidal functor $\beta_n : (i\mathcal{A})^n \times (i\mathcal{A})_h \to (iS_{2n+1}\mathcal{A})_h$ is a homotopy equivalence after group completion. But this functor has a retraction which is a homotopy equivalence after group completion, by proposition A.9. Thus β_n is a homotopy equivalence after group completion. $\square$

APPENDIX B. APPLICATION: CLASSICAL BOTT PERIODICITY

This section is independent of the rest of the article and can safely be skipped. Here, we show how the Periodicity Fibration 2.7 implies classical Bott periodicity.

Let A be a Banach algebra with involution. For a compact topological space K, we denote by AK the Banach algebra of continuous functions $K \to A$. It inherits an involution from A.

B.1. **Lemma.** *Let K be a contractible compact topological space. Then the map of Banach algebras $A \to AK$ induced by $K \to$ pt yields isomorphisms*

$$K_0(A) \cong K_0(AK),$$

$$GW_0^n(A) \cong GW^n(AK).$$

Proof. classical, insert reference.

Let Δ_{top}^n be the standard topological n simplex. This is a contractible compact topological space. Varying n yields a cosimplicial space $n \mapsto \Delta_{top}^n$, hence a simplicial Banach algebra $A\Delta_{top}^*$. We set

$$K_{top}(A) = |K(A\Delta_{top}^*)|$$

$$GW_{top}^n(A) = |GW^n(A\Delta_{top}^*)|.$$

Recall that for $\varepsilon = \pm 1$, we may write $_\varepsilon GW(A)$ instead of $GW^{\varepsilon-1+4k}(A)$.

B.2. **Lemma.** *There are canonical homotopy fibrations of spaces*

$$BGl^{top}A \to K_{top}(A) \to K_0(A)$$

$$B_\varepsilon O_{\infty,\infty}^{top}(A) \to {}_\varepsilon GW_{top}(A) \to {}_\varepsilon GW_0(A),$$

where G^{top} denotes the group G with its usual Euclidian topology. In particular, $K_{top}(A)$ and $_\varepsilon GW_{top}(A)$ are the usual topological K-theory and ε-hermitian K-theory spaces of A.

Proof. We only explain the hermitian K-theory case, the K-theory case being similar. By Corollary A.3, Lemma B.1 and the Bousfield-Friedlander Theorem [BF78], the homotopy fibre of $_\varepsilon GW_{top}(A) \to {}_\varepsilon GW_0(A)$ is $|B_\varepsilon O_{\infty,\infty}(A\Delta^*)^+|$. In the zigzag of maps

$$B_\varepsilon O_{\infty,\infty}^{top}(A) \leftarrow |B \operatorname{Sing}_* {}_\varepsilon O_{\infty,\infty}^{top}(A)| = |B_\varepsilon O_{\infty,\infty}(A\Delta_{top}^*)| \to |B_\varepsilon O_{\infty,\infty}(A\Delta^*)^+|,$$

the left map is the usual homotopy equivalence between a space (of the homotopy type of a CW complex) and the topological realization of its simplicial complex and the right map is a homology isomorphism of nilpotent spaces (in fact of H-spaces), hence it is also a homotopy equivalence. $\square$

B.3. **Proposition.** *Let A be a Banach algebra with involution, and let $n \in \mathbb{Z}$ be an integer. Then there are homotopy fibrations*

$$GW_{top}^n(A) \xrightarrow{F} K_{top}(A) \xrightarrow{H} GW_{top}^{n+1}(A)$$

induced by the respective forgetful and hyperbolic functors.

Proof. Let $n \in \mathbb{Z}$. By the Periodicity Theorem 2.7, we have a sequence of simplicial spaces

$$GW^n(A\Delta^*_{top}) \xrightarrow{F} K(A\Delta^*_{top}) \xrightarrow{H} GW^{n+1}(A\Delta^*_{top})$$

which is a homotopy fibration in each degree. In view of Lemma B.1, the Bousfield-Friedlander Theorem [BF78] implies that its topological realization is still a homotopy fibration.

$\square$

B.4. **Classical Bott periodicity.** As in §2.8, Proposition B.3 implies the topological version of Karoubi's fundamental theorem

(30) $$_{-\varepsilon}V_{top}(A) \simeq \Omega_\varepsilon U_{top}(A)$$

where V_{top} and U_{top} are the homotopy fibres of the topological forgetful and hyperbolic functors $F : {}_\varepsilon GW_{top}(A) \to K_{top}(A)$ and $H : K_{top}(A) \to {}_\varepsilon GW_{top}(A)$.

Consider $\mathbb{R}$ and $\mathbb{C}$ as Banach algebras with trivial involution, and consider the quaternions $\mathbb{H}$ as Banach algebra with its usual involution sending i, j, k to $-i, -j, -k$, respectively. Then we obtain

$$V(\mathbb{R}) \sim \mathbb{Z} \times BO \qquad V(\mathbb{C}) \sim U/O \qquad V(\mathbb{H}) \sim \mathbb{Z} \times BSp$$

$$U(\mathbb{R}) \sim O \qquad U(\mathbb{C}) \sim O/U \qquad U(\mathbb{H}) \sim Sp$$

$$_-V(\mathbb{R}) \sim O/U \qquad _-V(\mathbb{C}) \sim U/Sp \qquad _-V(\mathbb{H}) \sim \mathbb{Z} \times Sp/U$$

$$_-U(\mathbb{R}) \sim U/O \qquad _-U(\mathbb{C}) \sim Sp/U \qquad _-U(\mathbb{H}) \sim U/Sp$$

in view of the classical homotopy equivalences

$$O_n \xrightarrow{\sim} Gl_n(\mathbb{R}) \qquad U_n \xrightarrow{\sim} Gl_n(\mathbb{C}) \qquad Sp_n \xrightarrow{\sim} Gl_n(\mathbb{H})$$

$$O_m \times O_n \xrightarrow{\sim} O_{m,n}(\mathbb{R}) \quad O_{m+n} \xrightarrow{\sim} O_{m,n}(\mathbb{C}) \quad Sp_m \times Sp_n \xrightarrow{\sim} O_{m,n}(\mathbb{H})$$

$$U_n \xrightarrow{\sim} Sp_n(\mathbb{R}) \qquad Sp_n \xrightarrow{\sim} Sp_n(\mathbb{C}) \qquad U_n \xrightarrow{\sim} Sp_n(\mathbb{H})$$

induced by the polar decompositions of $Gl_n(\mathbb{R})$, $Gl_n(\mathbb{C})$ and $Gl_n(\mathbb{H})$. Using the topological version of Karoubi-periodicity (30), we obtain the following table of homotopy equivalences

$$O \sim \Omega(\mathbb{Z} \times BO)$$

$$_-V(\mathbb{R}) \sim \Omega_+ U(\mathbb{R}) \qquad O/U \sim \Omega O$$

$$_-V(\mathbb{C}) \sim \Omega_+ U(\mathbb{C}) \qquad U/Sp \sim \Omega O/U$$

$$_+V(\mathbb{H}) \sim \Omega_- U(\mathbb{H}) \quad \mathbb{Z} \times BSp \sim \Omega U/Sp$$

$$Sp \sim \Omega(\mathbb{Z} \times BSp)$$

$$_-V(\mathbb{H}) \sim \Omega_+ U(\mathbb{H}) \qquad Sp/U \sim \Omega Sp$$

$$_+V(\mathbb{C}) \sim \Omega_- U(\mathbb{C}) \qquad U/O \sim \Omega Sp/U$$

$$_+V(\mathbb{R}) \sim \Omega_- U(\mathbb{R}) \quad \mathbb{Z} \times BO \sim \Omega U/O$$

which yields the homotopy equivalence

$$\mathbb{Z} \times BO \sim \Omega^8(\mathbb{Z} \times BO).$$

Using the complex numbers $\mathbb{C}$ with its usual involution $i \mapsto -i$, the topological version of Karoubi periodicity yields

$$\mathbb{Z} \times BU \sim \Omega^2(\mathbb{Z} \times BU).$$

Appendix C. A counterexample to invariance under derived equivalences

We recall from [Sch10] the definition of the Grothendieck-Witt group of an exact category with weak equivalences and duality.

C.1. **Definition.** The *Grothendieck-Witt group*

$$GW_0(\mathcal{E}, w, *, \eta)$$

of an exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$ is the free abelian group generated by isomorphism classes $[X, \varphi]$ of symmetric spaces (X, φ) in $(\mathcal{E}, w, *, \eta)$, subject to the following relations

(1) $[X, \varphi] + [Y, \psi] = [X \oplus Y, \varphi \oplus \psi]$
(2) if $g : X \to Y$ is a weak equivalence, then $[Y, \psi] = [X, g^*\psi g]$, and
(3) if $(E_\bullet, \varphi_\bullet)$ is a symmetric space in the category of exact sequences in $\mathcal{E}$, that is, a map

$$
\begin{array}{ccccccc}
E_\bullet : & & E_{-1} & \overset{i}{\rightarrowtail} & E_0 & \overset{p}{\twoheadrightarrow} & E_1 \\
& & \wr\downarrow \varphi_\bullet & & \wr\downarrow \varphi_{-1} \quad \wr\downarrow \varphi_0 & & \wr\downarrow \varphi_1 \\
E_\bullet^* : & & E_1^* & \underset{p^*}{\rightarrowtail} & E_0^* & \underset{i^*}{\twoheadrightarrow} & E_{-1}^*
\end{array}
$$

of exact sequences with $(\varphi_{-1}, \varphi_0, \varphi_1) = (\varphi_1^*\eta, \varphi_0^*\eta, \varphi_{-1}^*\eta)$ a weak equivalence, then

$$[E_0, \varphi_0] = \left[E_{-1} \oplus E_1, \begin{pmatrix} 0 & \varphi_1 \\ \varphi_{-1} & 0 \end{pmatrix} \right].$$

C.2. **Definition.** The *Witt* group

$$W_0(\mathcal{E}, w, *, \eta)$$

of an exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$ is the free abelian group generated by isomorphism classes $[X, \varphi]$ of symmetric spaces (X, φ) in $(\mathcal{E}, w, *, \eta)$, subject to the relations C.1 1, 2 and

(c') if $(E_\bullet, \varphi_\bullet)$ is a symmetric space in the category of exact sequences in $(\mathcal{E}, w, *, \eta)$, then $[E_0, \varphi_0] = 0$.

C.3. **Lemma.** *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. The natural quotient map $GW_0(\mathcal{E}, w, *, \eta) \to W_0(\mathcal{E}, w, *, \eta) : [X] \mapsto [X]$ and the hyperbolic map $H : K_0(\mathcal{E}, w) \to GW_0(\mathcal{E}, w, *, \eta) : [X] \mapsto [X \oplus X^*, \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix}]$ fit into an exact sequence*

$$K_0(\mathcal{E}, w) \overset{H}{\longrightarrow} GW_0(\mathcal{E}, w, *, \eta) \longrightarrow W_0(\mathcal{E}, w, *, \eta) \longrightarrow 0.$$

Proof.

C.4. **Context for the counter example.** Let R be commutative ring, and write $(\mathrm{sPerf}(R), \mathrm{quis}, \sharp_n)$ for the exact category of bounded chain complexes of finitely generated projective R-modules with weak equivalences the quasi-isomorphisms, and equipped with the n-th shifted duality $E^{\sharp_n} = [E, R[n]]$ and canonical double dual identification. Further, let $(\mathrm{Mor}\,\mathrm{sPerf}(R), \sharp_n)$ be the exact category of morphisms in $\mathrm{sPerf}(R)$ with duality $(a : X \to Y) \mapsto (a^{\sharp_n} : Y^{\sharp_n} \to X^{\sharp_n})$. Call a map $f : a \to b$ in $\mathrm{Mor}\,\mathrm{Ch}^b\,\mathcal{E}$ a weak equivalence of its cone $C(f) : C(a) \to C(b)$ is a quasi-isomorphism in $\mathrm{sPerf}(R)$. The cone functor

$$(31) \qquad \mathrm{cone} : (\mathrm{Mor}\,\mathrm{sPerf}(R), w, \sharp_{n-1}) \to (\mathrm{sPerf}(R), \mathrm{quis}, \sharp_n)$$

can be made into a non-singular exact form functor (**??**)

C.5. **Proposition.** *Let R be a commutative ring.*

(1) *The cone functor (31) induces an equivalence of triangulated categories*

$$w^{-1}\mathrm{Mor}\,\mathrm{sPerf}(R) \overset{\cong}{\to} \mathrm{quis}^{-1}\,\mathrm{sPerf}(R)$$

(2) *The cone functor (31) satisfies the hypothesis of Waldhausen's approximation theorem.*

(3) *If 2 is not a unit in R, then the map*

$$GW_0(\mathrm{Mor}\,\mathrm{sPerf}(R), w, \sharp_{-1}) \to GW_0(\mathrm{sPerf}(R), \mathrm{quis}, \sharp_0)$$

induced by the cone functor (31) is not surjective.

Proof. For part 1, one constructs an explicit inverse, or uses (2). For 2, let $a : X \to Y$ be an object of $\mathrm{Mor}\,\mathrm{sPerf}(R)$, and $\alpha : \mathrm{cone}(a) \to Z$ a map in $\mathrm{sPerf}(R)$. Write C for the cone of $\mathbb{Z}$ in $\mathrm{sPerf}(\mathbb{Z})$ and $CX = C \otimes X$ for the cone of $X \in \mathrm{sPerf}(R)$. It is easy to check that C is a dga over $\mathbb{Z}$ (with unique multiplication). We denote by $\eta : \mathbb{Z} \to C$ the unit map. Write $\bar{a} : CX \to \mathrm{cone}(a)$ for the push-out of a along the inflation $\eta \otimes 1_X : X \to CX$. Consider the following commutative diagram in $\mathrm{sPerf}(R)$

$$
\begin{array}{ccccccc}
X & \xrightarrow{f_0} & CX & \xrightarrow{\eta \otimes 1_C \otimes 1_X} & CCX & \xrightarrow{\mu \otimes 1} & CX \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \alpha\bar{a}} & & \downarrow & & \downarrow{\scriptstyle \alpha\bar{a}} \\
Y & \xrightarrow{f_1} & Z & \xrightarrow{j} & \mathrm{cone}(\alpha\bar{a}) & \xrightarrow{\beta} & Z
\end{array}
$$

in which the right two squares are cocartesian, the map $\mu : C \otimes C \to C$ is the multiplication in C, so that the composition $CX \to CCX \to CX$ is the identity on CX, and thus, the composition $Z \to \mathrm{cone}(\alpha\bar{a}) \to Z$ is also the identity. It follows that we have the identity $\alpha = \beta \circ \mathrm{cone}(f_0, f_1)$, and β is a homotopy equivalence since j is, and $\beta \circ j = 1$.

For part 3, recall that $GW_0(\mathrm{sPerf}(R), \mathrm{quis}, \sharp_0)$ is the usual Grothendieck-Witt group $GW_0(R)$ via the map that sends a finitely generated projective R-module equipped with a non-degenerated symmetric bilinear form to the complex concentrated in degree zero where it is that R-module together with the induced form on the complex. Let $m \subset R$ be a maximal ideal containing 2 which exists since 2 is not a unit in R. Then $k = R/m$ is a field of characteristic 2. The composition $\mathrm{rk}_m : GW_0(R) \to GW_0(k) \overset{\mathrm{rk}}{\to} \mathbb{Z} : [M, \varphi] \mapsto \dim(M \otimes_R k)$ is surjective since $[R, 1]$ is sent to 1. We will show that for every symmetric space (M, φ) in

$(\operatorname{Mor} \operatorname{sPerf}(R), w, \sharp_{-1})$, the rank $\operatorname{rk}_m(\operatorname{cone} M)$ of cone M at m is even. For that, we can assume $R = k$ and it suffices to show that the composition

$$(32) \qquad W_0(\operatorname{Mor} \operatorname{sPerf}(k), w, \sharp_{-1}) \xrightarrow{\operatorname{cone}} W_0(k) \xrightarrow{\operatorname{rk}} \mathbb{Z}/2$$

is zero. Since k has characteristic 2, we will ignore all signs. As mentioned above, inclusion as complexes concentrated in degree zero induces an isomorphism $W_0(k) \to W_0(\operatorname{sPerf}(k), \operatorname{quis}, \sharp_0)$. The inverse $W_0(\operatorname{sPerf}(k), \operatorname{quis}, \sharp_0) \to W_0(k)$ is given by the zero-homology functor $[M, \varphi] \mapsto [H_0 M, H_0 \varphi]$ (exercise!). Let $(M, \varphi) = (P \xrightarrow{f} Q, \varphi^P, \varphi^Q)$ be a symmetric space in $(\operatorname{Mor} \operatorname{sPerf}(R), w, \sharp_{-1})$. We have to show that the symmetric space

$$(N, \psi) = H_0 \operatorname{cone}(M, \varphi)$$

in $(\operatorname{Vect}_k, *)$ has even rank. Symmetry of the map $(\varphi^P, \varphi^Q) : (P, Q) \to (Q^*[-1], P^*[-1])$ means that $\varphi_i^Q = (\varphi_{1-i}^P)^* \eta_{Q_i}$. The cone symmetric space of (M, φ) in degree zero is the symmetric map of k-vector spaces

$$\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^* \eta \end{smallmatrix} \right) : Q_0 \oplus P_1 \to P_1^* \oplus Q_0^*, \qquad \text{where} \quad \alpha = \varphi_0^Q.$$

The particular shape of the symmetric map shows that $Q_0 \oplus P_1$ has a basis $v_1, ..., v_n$, $n = \dim Q_0 + \dim P_1$, of isotropic vectors, that is, of vectors v_i for which the (possibly singular) associated symmetric bilinear form satisfies $\langle v_i, v_i \rangle = 0$, $i = 1, ..., n$. The field k has characteristic 2 which implies that every vector $v \in Q_0 \oplus P_1$ has to be isotropic. As a subquotient, the (non-singular) symmetric space (N, ψ) also consists of isotropic vectors, so that its Gram-matrix (with respect to any basis) has diagonal entries all zero. Since the symmetric form on (N, ψ) is non-singular, its Gram-matrix is invertible, so that the dimension of N has to be even, by the following well-known lemma. In particular, the map (32) is the zero map. $\qquad \square$

C.6. **Lemma.** *Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix with entries in a commutative ring R. If n is odd and $a_{ii} = 0$ for $i = 1, ..., n$, then $\det A \in 2R$.*

Proof. Let S_n be the permutation group on n letters, that is, the group of bijections of the set $\{1, ..., n\}$. Let $S_n^+ \subset S_n$ be the subset of fix point free permutations σ, that is, of those $\sigma \in S_n$ for which $\sigma(i) \neq i$ for all $i = 1, ..., n$. The two element group $\mathbb{Z}/2$ acts on S_n^+ via $\sigma \mapsto \sigma^{-1}$. This action is free because we assume n odd (if $\sigma = \sigma^{-1}$, then $\sigma^2 = id$, so that all cycles in σ have length one or two, but σ is a fix point free permutation, thus no cycle of length one occurs, and n has to be even). Let $\bar{S}_n^+ \subset S_n^+$ be a set of representatives for the factor set $S_n^+/(\mathbb{Z}/2)$. Then we have

$$
\begin{aligned}
\det A &= \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \; a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&= \sum_{\sigma \in S_n^+} \operatorname{sgn} \sigma \; a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&= 2 \sum_{\sigma \in \bar{S}_n^+} \operatorname{sgn} \sigma \; a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&\in 2R.
\end{aligned}
$$

$\qquad \square$

APPENDIX D. TATE SPECTRA

D.1. **Spectra.** In this article we shall work with spectra in the sense of Bousfield and Friedlander [BF78]. Recall [BF78] that a *spectrum* is a sequence X_n, $n \in \mathbb{N}$, of pointed simplicial sets together with pointed simplicial maps $e_n : S^1 \wedge X_n \to X_{n+1}$ called *bonding maps*. A map of spectra $X \to Y$ is a sequence of pointed simplicial maps $X_n \to Y_n$ commuting with the bonding maps. The category of spectra is denoted by Sp^Σ. A spectrum X is called Ω-*spectrum* if the simplicial sets X_n are Kan sets and if the adjoints $X_n \to \Omega X_{n+1}$ of the bonding maps e_n are weak equivalences.

The category of spectra is a simplicial category. Let K be a pointed simplicial set and X a spectrum. Their *smash-product* is the spectrum $X \wedge K$ with n-th space $X_n \wedge K$ and bonding maps $e_n \wedge K : S^1 \wedge X_n \wedge K \to X_{n+1} \wedge K$. The *mapping (or power) spectrum* from K to X is the spectrum $\mathrm{Sp}^\Sigma(K, X)$ with n-th space the mapping space (a simplicial set) in the category of pointed simplicial sets $\mathrm{Map}(K, X_n)$ and bonding maps

$$
\begin{array}{ccccccc}
S^1 & \wedge & \mathrm{Map}(K, X_n) & \longrightarrow & \mathrm{Map}(K, S^1 \wedge X_{n+1}) & \xrightarrow{e_n} & \mathrm{Map}(K, X_{n+1}) \\
t & \wedge & f & \mapsto & (x \mapsto t \wedge f(x)) & &
\end{array}
$$

If X is an Ω-spectrum then so is $\mathrm{Sp}^\Sigma(K, X)$.

The *homotopy groups* $\pi_n X$, $n \in \mathbb{Z}$, of a spectrum X are the groups

$$\pi_n X = \mathrm{colim}_k \, \pi_{n+k}(|X_k|)$$

where the colimit is taken over the system

$$\pi_m(|X_k|) \to \pi_{m+1}(|X_{k+1}|) : f \mapsto e_k \circ (S^1 \wedge f).$$

A map of spectra $X \to Y$ is called *stable equivalence* if it induces an isomorphism $\pi_n X \to \pi_n Y$ for all $n \in \mathbb{Z}$.

Define $CX = X \wedge I$, $Z(f)$ and cones of zigzag of maps (functorial in the zigzag of fixed length). (this avoids the lengthy dfn of a functorial Tate spectrum) The htpy cat of spectra is triangulated ...

There are several simplicial model category structures on the category of spectra with stable equivalences as the weak equivalences. In the injective model structure [**?**] cofibrations are the monomorphisms of spectra and thus, every spectrum is cofibrant in that model structure. In the projective model structure [BF78], the fibrant spectra are precisely the Ω-spectra and the fibrations $X \to Y$ between Ω-spectra are precisely the maps of spectra for which $X_n \to Y_n$ is a Kan fibration of simplicial sets for all $n \in \mathbb{N}$.

D.2. **Homotopy Orbit spectra.** Let G be a group and let EG be the usual contractible simplicial set on which G acts freely on the left. If $Ob[n]$ denotes the set of objects of the category $[n]$ then $E_n G = G^{n+1} = \mathrm{Map}_{Sets}(Ob[n], G)$ with simplicial structure induced by the cosimplicial set $n \mapsto Ob[n]$. The left action is given by $gf(x) = g(f(x))$ for $g \in G$, $f : Ob[n] \to G$ and $x \in Ob[n]$.

Let X be a spectrum with right G-action. The *homotopy orbit spectrum* X_{hG} is the spectrum

$$X_{hG} = X \wedge_G EG_+$$

where for a pointed simplicial set K with left G-action the spectrum $X \wedge_G K$ is defined by the coequalizer diagram

$$\bigvee_{g \in G} X \wedge K \underset{g \wedge 1}{\overset{1 \wedge g}{\rightrightarrows}} X \wedge K \longrightarrow X \wedge_G K$$

There is a string of maps

$$X = X \wedge S^0 \overset{\sim}{\longrightarrow} X \wedge EG_+ \longrightarrow X \wedge_G EG_+ \longrightarrow X \wedge_G S^0 = X/G$$

induced by the non-equivariant map $\mathrm{pt} \to G = E_0 G \to EG : \mathrm{pt} \mapsto 1$ and the G-equivariant $EG \to \mathrm{pt}$ where pt denotes the one point space. Thus, the homotopy orbit spectrum is equipped with natural maps

$$X \to X_{hG} \to X/G$$

whose composition is the quotient map $X \to X/G$. By the coequalizer definition of homotopy orbits, the two compositions

$$X \wedge S^0 \underset{g \wedge 1}{\overset{1 \wedge g}{\rightrightarrows}} X \wedge EG_+ \longrightarrow X \wedge_G EG_+$$

are equal for all $g \in G$. The two maps $1 \wedge g, 1 \wedge 1 : X = X \wedge S^0 \to X \wedge EG_+$ are homotopic, by contractibility of EG. It follows that the diagram

is homotopy commutative for all $g \in G$. In particular, for all $n \in \mathbb{Z}$ the map $\pi_n(X) \to \pi_n(X_{hG})$ factors through $\pi_n(X)/G \to \pi_n(X_{hG})$. In general, this map is far from being an isomorphism.

D.3. **Proposition.**

(1) *If $X \to Y$ is a G-equivariant map of spectra which, forgetting the G-action, is a stable equivalence, then $X_{hG} \to Y_{hG}$ is also a stable equivalence.*

(2) *If X is a non-equivariant spectrum then $X \wedge G_+$ is a spectrum with right G-action and the natural map $(X \wedge G_+)_{hG} \to (X \wedge G_+)/G = X$ is a stable equivalence.*

(3) *For a spectrum X with right G-action there is a homological spectral sequence*

$$E_{p,q}^2 = H_p(G, \pi_q X) \Rightarrow \pi_{p+q}(X_{hG})$$

with differentials $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$. The spectral sequence converges strongly if $\pi_i X = 0$ for $i << 0$.

Proof. In any simplicial model category $\mathcal{M}$, the functor $? \wedge EG_+$ preserves weak equivalences between point-wise cofibrant objects [Hir03, Theorem 18.5.3.(1)]. In the injective model structure on Sp^Σ every spectrum is cofibrant. This proves part (1). For (2), we note that the functor $X \wedge ?$ sends weak equivalences of simplicial sets to stable equivalences. It follows that

$$(X \wedge G_+) \wedge_G EG_+ = X \wedge EG_+ \to X \wedge S^0 = X$$

is a stable equivalence. The spectral sequence in part (3) is the spectral sequence of a filtered spectrum obtained by applying the functor $X \wedge_G (\ )_+$ to the sekeletal filtration of EG. $\square$

D.4. Remark. If X is a spectrum with $\pi_i X = 0$ for $i < 0$, then the spectral sequence shows that the natural map $\pi_0(X)/G \to \pi_0(X_{hG}) = H_0(G, \pi_0 X)$ is an isomorphism. If X is an Eilenberg-MacLane spectrum, that is, $\pi_i X = 0$ for $i \neq 0$, then the spectral sequence in D.3 (3) collapses and yields an isomorphism for all $i \in \mathbb{Z}$

$$\pi_i(X_{hG}) = H_i(G, \pi_0 X).$$

D.5. Homotopy fix point spectra. Whereas in the injective model structure every spectrum is cofibrant, not every spectrum is fibrant in the projective model structure (nor in any other known model structure on the category of spectra based on simplicial sets). Therefore, we will have to choose a functorial fibrant replacement $X \to X_f$ for the projective model structure. That is, X_f is an Ω-spectrum and $X \to X_f$ is a natural stable equivalence. In particular, if X has a right G-action then so has X_f and the map $X \to X_f$ is G-equivariant.

In the definition of homotopy fix points below we consider the simplicial set EG equipped with the free right G-action defined by $fg(x) = f(x)g$ for $g \in G$, $f : Ob[n] \to G$ and $x \in Ob[n]$. Let X be a spectrum with right G-action. The *homotopy fix point spectrum X^{hG}* is the spectrum

$$X^{hG} = \mathrm{Sp}^{\Sigma G}(EG_+, X_f)$$

where for a pointed simplicial set K and a spectrum Y both with right G-actions the spectrum $\mathrm{Sp}^{\Sigma G}(K, Y)$ is defined by the equalizer diagram

$$\mathrm{Sp}^{\Sigma G}(K, Y) \longrightarrow \mathrm{Sp}^{\Sigma}(K, Y) \underset{(1,g)}{\overset{(g,1)}{\rightrightarrows}} \prod_{g \in G} \mathrm{Sp}^{\Sigma}(K, Y).$$

The G-equivariant map $EG \to \mathrm{pt}$ and the non-equivariant map $\mathrm{pt} \to EG : \mathrm{pt} \mapsto 1$ induce a string of maps

$$X^G \to (X_f)^G = \mathrm{Sp}^{\Sigma G}(S^0, X_f) \to \mathrm{Sp}^{\Sigma G}(EG_+, X_f) \to \mathrm{Sp}^{\Sigma}(EG_+, X_f) \to \mathrm{Sp}^{\Sigma}(S^0, X_f) = X_f$$

such that the diagram

$$\begin{array}{ccc} X^G & \longrightarrow & X^{hG} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_f \end{array}$$

commutes.

According to the equalizer definition of the homotopy fix point spectrum, the two compositions

$$X^{hG} = \mathrm{Sp}^{\Sigma G}(EG_+, X_f) \longrightarrow \mathrm{Sp}^{\Sigma}(EG_+, X_f) \underset{(1,g)}{\overset{(g,1)}{\rightrightarrows}} \mathrm{Sp}^{\Sigma}(S^0, X_f) = X_f$$

are equal. By contractibility of EG, the map $(g, 1)$ in the diagram is homotopic to $(1, 1)$. Therefore, for all $g \in G$ the diagram

$$
\begin{array}{ccc}
X^{hG} & \longrightarrow & X_f \\
& \searrow & \downarrow g \\
& & X_f
\end{array}
$$

homotopy commutes. In particular, the map $\pi_n(X^{hG}) \to \pi_n(X_f) = \pi_n(X)$ factors through the fix points $\pi_n(X)^G$ and we obtain a natural map $\pi_n(X^{hG}) \to \pi_n(X)^G$ for all $n \in \mathbb{Z}$. Of course, this map is far from being an isomorphism, in general.

D.6. Proposition.

(1) If $X \to Y$ is a G-equivariant map of spectra which, forgetting the G-action, is a stable equivalence, then $X^{hG} \to Y^{hG}$ is a stable equivalence.

(2) If X is (non-equivariant) spectrum then the spectrum $\mathrm{Sp}^\Sigma(G_+, X)$ has a right G-action and the map $X = \mathrm{Sp}^\Sigma(G_+, X)^G \to \mathrm{Sp}^\Sigma(G, X)^{hG}$ is a stable equivalence.

(3) For a spectrum X with right G-action there is a spectral sequence

$$
E_2^{p,q} = H^p(G, \pi_{-q}X) \Rightarrow \pi_{-p-q}(X^{hG})
$$

which converges strongly if $\pi_i X = 0$ for $i >> 0$.

Proof. The proof is dual to the proof of D.3 using the projective model structure on the category of spectra. We omit the details. $\square$

D.7. Remark. If X is a spectrum with $\pi_i X = 0$ for $i > 0$ then the spectral sequence shows that $H^0(G, \pi_0 X) = \pi_0(X^{hG})$, and the map $\pi_0(X^{hG}) \to (\pi_0 X)^G$ is an isomorphism. If X is an Eilenberg-MacLane spectrum ($\pi_i X = 0$, $i \neq 0$) with a G-action, then the spectral sequence in D.6 (3) collapses and yields an isomorphism for all $i \in \mathbb{Z}$

$$
\pi_{-i}(X^{hG}) = H^i(G, \pi_0 X).
$$

D.8. The hypernorm map. Let G be a finite group acting on a spectrum X. The *Tate spectrum* $\hat{\mathbb{H}}(G, X)$ is the homotopy cofibre of the hyper norm map $\tilde{N} : X_{hG} \to X^{hG}$; see [ACD89], [WW89], [Gre01], [Jar97]. In case of a right G-module A, the Tate spectrum $\hat{\mathbb{H}}(G, A)$ of the Eilenberg-MacLane spectrum associated with A has n-th homotopy group naturally isomorphic to ordinary Tate cohomology $\hat{H}^{-n}(G, A)$ of G with coefficients in A. We review the relevant definitions and facts in case $G = \mathbb{Z}/2$.

Let $G = \mathbb{Z}/2 = \{1, \sigma\}$ where $\sigma \in G$ is the unique element of order 2, and let X be a spectrum with a G-action. We consider $X \times X$ as a spectrum with $G_1 \times G_2$-action where $G_1 = G_2 = G$ and $(\sigma, 1) \in G_1 \times G_2$ acts as the switch map $(x, y) \mapsto (y, x)$ and $(1, \sigma)$ acts as $(x, y) \mapsto (\sigma y, \sigma x)$. This also induces a $G_1 \times G_2$-action on $X \vee X \subset X \times X$. There is a string of $G_1 \times G_2$-equivariant spectra

$$
(33) \qquad X \xrightarrow{(1,\sigma)} X \times X \xleftarrow{\sim} X \vee X \xrightarrow{1 \vee 1} X
$$

where $G_1 \times G_2$ acts on the source X via the projection $G_1 \times G_2 \to G_1$ onto the first factor and $G_1 \times G_2$ acts on the target X of the map via the second projection $G_1 \times G_2 \to G_2$. The arrow in the wrong direction is a stable equivalence. We write $N : X \to X$ for this string of maps and call it *norm map*. The norm map is an

honest map in the homotopy category of spectra and induces $1 + \sigma : \pi_n X \to \pi_n X$ on homortopy groups.

For a spectrum Y with $G_1 \times G_2$-action, we can consider Y as a G_i-spectrum via the group homomorphisms $G_1 \to G_1 \times G_2 : x \mapsto (x, 1)$ and $G_2 \to G_1 \times G_2 : x \mapsto (1, x)$, and the two actions commute. In particular, the homotopy fix point spectrum Y^{hG_2} is a G_1-spectrum and thus has a homotopy orbit spectrum $(Y^{hG_2})_{hG_1}$. To abbreviate we write $FY = (Y^{hG_2})_{hG_1}$ for this spectrum. Note that the functor F preserves stable equivalences, and comes equipped with natural maps $(Y^{G_2})_{hG_1} \to FY \to (Y^{hG_2})_{G_1}$. Since G_2 acts trivially on the sourse of (33) and G_1 acts trivially on the target of (33), we obtain the *hyper norm map*

$$(34) \qquad \tilde{N} : X_{hG} = (X^{G_2})_{hG} \to FX \xrightarrow{FN} FX \to (X^{hG})_{G_1} = X^{hG}.$$

More precisely, the hypernorm map is the string of maps $\tilde{N}$:

$$X_{hG} = (X^{G_2})_{hG_1} \to FX \to F(X \times X) \xleftarrow{\sim} F(X \vee X) \to FX \to (X^{hG_2})_{G_1} = X^{hG}$$

where the arrow in the wrong direction is a stable equivalence.

D.9. Lemma. *The two maps maps $X \to X_{hG} \xrightarrow{\tilde{N}} X^{hG} \to X_f$ and $X \xrightarrow{N} X \to X_f$ are equal in the homotopy category of spectra.*

Proof. For clarity of exposition, write X and Y for source and target of (33) with their respective $G_1 \times G_2$ actions. So, the norn map is a $G_1 \times G_2$-equivariant string of maps $X \to Y$ with the arrow pointing in the wrong direction a stable equivalence. By functoriality, we have a commutative diagram

$$
\begin{array}{ccccccc}
X & \longrightarrow & X_f & \longleftarrow & X^{hG_2} & \longrightarrow & FX \\
\downarrow N & & \downarrow N & & \downarrow N & & \downarrow N \\
Y & \longrightarrow & Y_f & \longleftarrow & Y^{hG_2} & \longrightarrow & FY
\end{array}
$$

where the vertical maps are actually strings of maps with the arrows in the wrong direction being stable equivalences. Every spectrum in the top row receives a natural map from X^{G_2} such that all triangles commute. Similarly, every spectrum in the bottom row naturally maps to $(Y_f)_{G_1}$ such that all triangles commute. The resuting composition $X^{G_2} \to FX \to FY \to (Y_f)_{G_1}$ is the first map in the lemma, and the map $X^{G_2} \to X \to Y \to (Y_f)_{G_1}$ is the second map in the lemma. By the commutativity of the diagram, these two maps are equal. $\qquad\square$

D.10. Definition. The *Tate spectrum*

$$\hat{\mathbb{H}}(\mathbb{Z}/2, X)$$

of a spectrum X with $G = \mathbb{Z}/2$ action is the homotopy cofibre of the hypernorm map $\tilde{N} : X_{hG} \to X^{hG}$. Since $\tilde{N}$ involves a stable weak equivalence in the wrong direction, we give the following more precise and functorial version on the level of spectra. The Tate spectrum is defined to be the lower right corner in the diagram

$$
\begin{array}{ccccccc}
X_{hG} & \longrightarrow & F(X \times X) & \xleftarrow{\sim} & F(X \vee X) & \longrightarrow & X^{hG} \\
\downarrow & \ulcorner & \downarrow & & \downarrow & \ulcorner & \downarrow \\
X_{hG} \wedge I & \longrightarrow & P & \xleftarrow{\sim} & Z(f) & \longrightarrow & \hat{\mathbb{H}}(\mathbb{Z}/2, X)
\end{array}
$$

where left and right squares are push-outs, all vertical arrows are cofibrations, the left vertical map is the inclusion $X \to X \wedge I : x \mapsto x \wedge 1$ of X into its cone (0 being the base-point of $I = \Delta^1$), and where for a map $f : X \to Y$ the factorization via the mapping cylinder $X \to Z(f) \to Y$ is given by $X \to Y \times (X \wedge I) \to Y$...

The functor $\hat{\mathbb{H}}(\mathbb{Z}/2, \quad)$ preserves stable equivalences and it sends sequences of $\mathbb{Z}/2$-spectra which (forgetting the action) are homotopy fibre sequences to homotopy fibre sequences.

D.11. **Example.** If X is an Eilenberg-McLane spectrum ($\pi_i X = 0$ for $i \neq 0$) equipped with a $\mathbb{Z}/2$-action, then the long exact sequence associated with the homotopy fibration $X_{hG} \to X^{hG} \to \hat{\mathbb{H}}(G, X)$ together with the calculations of the homotopy groups of X_{hG} and X^{hG} in Remarks D.4 and D.7 yield isomorphisms

$$
\pi_i \hat{\mathbb{H}}(\mathbb{Z}/2, X) \cong \begin{cases} H_i(\mathbb{Z}/2, \pi_0 X) & i \geq 2 \\ H^{-i}(\mathbb{Z}/2, \pi_0 X) & i \leq -1 \end{cases}
$$

and an exact sequence

$$
0 \to \pi_1 \hat{\mathbb{H}}(G, X) \to \pi_0(X_{hG}) \xrightarrow{\tilde{N}} \pi_0(X^{hG}) \to \pi_0 \hat{\mathbb{H}}(G, X) \to 0.
$$

By Remarks D.4, D.7 and Lemma D.9 the middle map $\pi_0 \tilde{N}$ is the usual norm map $1 + \sigma : (\pi_0 X)/G \to (\pi_0 X)^G$. These properties characterize Tate cohomology of $G = \mathbb{Z}/2$ with coefficients in the G-module $\pi_0 X$. Therefore, we have natural isomorphisms for $i \in \mathbb{Z}$

$$
\pi_i \hat{\mathbb{H}}(\mathbb{Z}/2, X) \cong \hat{H}^{-i}(\mathbb{Z}/2, \pi_0 X)
$$

References

[ACD89] A. Adem, R. L. Cohen, and W. G. Dwyer. Generalized Tate homology, homotopy fixed points and the transfer. In *Algebraic topology (Evanston, IL, 1988)*, volume 96 of *Contemp. Math.*, pages 1–13. Amer. Math. Soc., Providence, RI, 1989.

[Bal99] Paul Balmer. Derived Witt groups of a scheme. *J. Pure Appl. Algebra*, 141(2):101–129, 1999.

[Bal00] Paul Balmer. Triangular Witt groups. I. The 12-term localization exact sequence. *K-Theory*, 19(4):311–363, 2000.

[BF78] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 80–130. Springer, Berlin, 1978.

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[Gil02] Stefan Gille. On Witt groups with support. *Math. Ann.*, 322(1):103–137, 2002.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[Gre01] J. P. C. Greenlees. Tate cohomology in axiomatic stable homotopy theory. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 149–176. Birkhäuser, Basel, 2001.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Jar97] J. F. Jardine. *Generalized étale cohomology theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[Kar80] Max Karoubi. Le théorème fondamental de la K-théorie hermitienne. *Ann. of Math. (2)*, 112(2):259–282, 1980.

[Kel94]   Bernhard Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[Kel96]   Bernhard Keller. Derived categories and their uses. In *Handbook of algebra, Vol. 1*, pages 671–701. North-Holland, Amsterdam, 1996.

[KML71] G. M. Kelly and S. Mac Lane. Coherence in closed categories. *J. Pure Appl. Algebra*, 1(1):97–140, 1971.

[Kob99] Damjan Kobal. K-theory, Hermitian K-theory and the Karoubi tower. *K-Theory*, 17(2):113–140, 1999.

[May74] J. P. May. E_∞ spaces, group completions, and permutative categories. In *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pages 61–93. London Math. Soc. Lecture Note Ser., No. 11. Cambridge Univ. Press, London, 1974.

[ML98]   Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Ran92] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[Sch04a] Marco Schlichting. Delooping the K-theory of exact categories. *Topology*, 43(5):1089–1103, 2004.

[Sch04b] Marco Schlichting. Hermitian K-theory on a theorem of Giffen. *K-Theory*, 32(3):253–267, 2004.

[Sch05] Marco Schlichting. Negative K-theory of derived categories. *Math. Z.*, 2005. accepted for publication.

[Sch08] Marco Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *http://www.math.lsu.edu/~mschlich/research.html*, 2008.

[Sch10] Marco Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.*, 179(2):349–433, 2010.

[TT90]   R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

[Wal03a] Charles Walter. Grothendieck-Witt groups of projective bundles. *preprint*, 2003. www.math.uiuc.edu/K-theory/.

[Wal03b] Charles Walter. Grothendieck-Witt groups of triangulated categories. *preprint*, 2003. www.math.uiuc.edu/K-theory/.

[Wei81] Charles A. Weibel. K-theory of Azumaya algebras. *Proc. Amer. Math. Soc.*, 81(1):1–7, 1981.

[Wil05] Bruce Williams. *Quadratic K-theory and geometric topology*, chapter in Handbook of algebraic K-theory. Springer, 2005.

[WW89] Michael Weiss and Bruce Williams. Automorphisms of manifolds and algebraic K-theory. II. *J. Pure Appl. Algebra*, 62(1):47–107, 1989.

Marco Schlichting, Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

E-mail address: m.schlichting@warwick.ac.uk