The Chow-Witt group of a scheme with residual complex

by

James Plowman

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Declarations

The material contained in this thesis is original and my own work except where otherwise indicated, cited or commonly known. The material in this thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy, and has not been submitted to any other university or for any other degree.

Abstract

We give an explicit description of the boundary maps appearing in the Witt complexes of [1, 2]; providing an affirmative answer to the question posed by Balmer & Walter [1] as to whether or not their Witt complex agrees with that appearing in unpublished work [3] by Pardon. Our description of these boundary maps is made in terms of a generalisation of the classical second residue homomorphisms for Witt groups found in [4] - unlike the constructions of [1, 2] which are based on Balmer's triangular Witt groups [5], our definition of generalised second residue homormophisms does not require any assumption on the charcteristic nor any derived machinery.

Our constructions are performed using only the data of a Noetherian scheme X equipped with a *residual complex* $\mathcal{R}$ - see [6]. In this situation we define, for each immediate specialisation $x \rightsquigarrow y$ in X an abelian group homomorphism

$$\partial_2^{\mathcal{R}} = \mathrm{res}_W^{x,y}(\mathcal{R}) : W\big(x, x^\natural(\mathcal{R})\big) \longrightarrow W\big(y, y^\natural(\mathcal{R})\big)$$

where we have written for example $x^\natural(\mathcal{R})$ for the one-dimensional $\kappa(x)$-vector space $\mathrm{Hom}_{\mathcal{O}_{X,x}}\big(\kappa(x), \mathcal{R}_x^{\mu(x)}\big)$ - here $\mu(x)$ denotes the codimension of x according to the residual complex $\mathcal{R}$. This definition does not require any normalisation process as in similar generalisations such as Schmid's work [7]. It is these maps which we assemble to form a sequence of abelian group homorphisms

$$\ldots \to \bigoplus_{\mu(x)=p-1} W\big(x, x^\natural(\mathcal{R})\big) \longrightarrow \bigoplus_{\mu(x)=p} W\big(x, x^\natural(\mathcal{R})\big) \longrightarrow \bigoplus_{\mu(x)=p+1} W\big(x, x^\natural(\mathcal{R})\big) \to \ldots$$

that we verify agrees with the Witt-complex of [2] under the further assumption that $1/2 \in \Gamma(X, \mathcal{O}_X)$. The basic idea behind our definition of the residue maps $\partial_2^{\mathcal{R}}$ is also used to define further group homomorphisms

$$\mathrm{res}_{GW}^{x,y}(\mathcal{R}) : GW\big(x, x^\natural(\mathcal{R})\big) \longrightarrow W\big(y, y^\natural(\mathcal{R})\big)$$

$$\mathrm{res}_V^{x,y}(\mathcal{R}) : V\big(x, x^\natural(\mathcal{R})\big) \longrightarrow GW\big(y, y^\natural(\mathcal{R})\big)$$

where V denotes the V-theory of [8]; the V-theory of a field F provides a description of $K_1^{MW}(F)$ in terms of symmetric spaces rather than the defining presentation of [9]. With these residue maps in hand we are able to provide a definition of the Chow-Witt group $\widetilde{\mathrm{CH}}(X, \mathcal{R})$ of a Noetherian scheme X equipped with a residual complex $\mathcal{R}$ and with $1/2 \in \Gamma(X, \mathcal{O}_X)$.

Chapter 0

Introduction

Given an integral scheme V (let's say of finite type over a field k) the notion of the *order of vanishing* of a rational function on V along a subvariety Z of codimension 1 is a fundamental construction in algebraic geometry. For example, after writing F for the field of rational functions on V, in the case when the local ring $R = \mathcal{O}_{V,Z}$ of functions of V along Z is a discrete valuation ring, the order of vanishing

$$\mathrm{ord}_Z = v : F^* \longrightarrow \mathbb{Z}$$

is simply given by the valuation v on R. If X now denotes a possibly non-integral scheme, then we denote by $Z^p(X)$ the free abelian group generated by the symbols $[Z]$; one for each codimension p integral subvariety $Z \hookrightarrow X$. In this group, the elements of the form

$$\mathrm{div}(f) := \sum \mathrm{ord}_Z(f)[Z] \in Z^p(X)$$

where f is some rational function on a codimension $p-1$ subvariety of X, are called *rationally equivalent to zero* - and the quotient of $Z^p(X)$ by these cycles is the Chow group $\mathrm{CH}^p(X)$ of codimension p cycles on X; this is the main object of study in [10] and is the group on which most intersection theoretic operations act.

This thesis presents some new descriptions of analogous maps in the hermitian setting - briefly - returning to our integral scheme V this means that instead of rational functions on V we work with finite dimensional inner product spaces over the field of rational functions on V and the residue of such a space along a codimension 1 subvariety Z will be an inner product space over the rational function field of Z.

We study inner product spaces as elements of certain *Grothendieck-Witt* groups; for example to any field F we can associate the Witt group $W(F)$ whose elements can be represented by isometry classes $[M, \psi]$ of inner product spaces

$$\psi : M \xrightarrow{\simeq} \mathrm{Hom}_F(M, F)$$

and in which we set $[M, \psi] = 0$ whenever M admits a submodule $N = N^{\perp}$. For these Witt groups and our discrete valuation ring R the residue map of interest is well-understood; it is given by the *second residue homomorphism* of [4] - which after chosing a uniformiser π and writing κ for the residue field of the valuation is the map

$$\partial_2^{\pi} : W(F) \longrightarrow W(\kappa) \text{ defined by } \langle \alpha \rangle \mapsto \begin{cases} \left\langle \overline{\frac{\alpha}{\pi^{v(\alpha)}}} \right\rangle & \text{if } v(\alpha) \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

where the symbol $\langle \alpha \rangle \in W(F)$ denotes the inner product space $F \times F \to F$ given by $(x, y) \mapsto \alpha xy$ and similar remarks describe the element $\left\langle \overline{\alpha / \pi^{v(\alpha)}} \right\rangle$ of $W(\kappa)$. As suggested by the notation, these residue maps do depend on the choice of uniformizer; a problem which is resolved in [7] by having the inner products of elements in $W(F)$ and $W(\kappa)$ take values in cleverly chosen 1-dimensional vector spaces. In *loc. cit.* the above residue homomorphism is further extended to one-dimensional local domains of essentially finite type over some field of characteristic different from 2 by passing to normalisations as in [11]. In this way Schmid [7] obtains, after assuming further that $\mathrm{char}(k) \neq 2$, for our integral scheme V and *any* codimension one subvariety $Z \hookrightarrow V$ a well-defined residue map

$$\partial_2 : W\big(F, \Omega^{top}_{F/k}\big) \longrightarrow W\big(\kappa(Z), \Omega^{top}_{\kappa(Z)/k}\big)$$

where $\kappa(Z)$ denotes the field of rational functions on Z and $\Omega^{top}_{F/k}$ the highest non-zero exterior power of the module $\Omega_{F/k}$ of cotangent vectors of F over k.

However these residue maps for Witt groups are defined one desires, especially in order to establish a hermitian analogue $\widetilde{\mathrm{CH}}^p(X)$ of the Chow group, that for our scheme X they assemble to form a Gersten-type complex

$$\ldots \to \bigoplus_{x \in X^{p-1}} W\big(\kappa(x)\big) \longrightarrow \bigoplus_{x \in X^{p}} W\big(\kappa(x)\big) \longrightarrow \bigoplus_{x \in X^{p+1}} W\big(\kappa(x)\big) \to \ldots$$

where each X^i denotes the set of codimension i points of X. Quillen [12] constructs a Gersten complex for K-theory via devissage and localisations sequences - and these ideas are refined to give a Witt complex as above in [1] for regular schemes and [2] for singular schemes. The advantage of this approach is that one obtains immediately a Witt complex for X and the second residue homomorphisms would then be defined to be whatever appears in this complex - the disadvantages are that the constructions of [1, 2] require the assumption that $1/2 \in \Gamma(X, \mathcal{O}_X)$ and some derived machinery to be carried out. While Schmid's residue maps [7] do not require any derived techniques to be defined there then remains some difficulty in establishing that his residue maps assemble to form a Gersten complex as above.

The *generalised second residue homomorphisms* we define do not require any assumption on the characteristic nor derived techniques to be defined - though we will make sure that they agree with those of [1, 2]. Here is a taste of what our definition ends up being; take (A, m, κ) to be a one dimensional local domain - which is hence a Cohen Macaulay ring - and suppose it has a canonical module ω_A. This canonical module then has a minimal injective resolution

$$0 \to \omega_A \longrightarrow F \xrightarrow{d} E_A(\kappa) \to 0 \to ...$$

where F denotes the field of fractions of A and $E_A(\kappa)$ is some injective hull of the residue field κ. Let's take an element $[V, \psi] \in W(F)$ - so that $\langle \cdot, \cdot \rangle_\psi : V \times V \to F$ is a non-degenerate symmetric bilinear form. Take a finitely generated A-submodule $L \hookrightarrow V$ such that $F \otimes_A L = V$ - this will be called an *A-lattice* inside V. It's possible to choose L such that the *dual-lattice*

$$L^\flat := \{v \in V | \langle v, L \rangle_\psi \subseteq \omega_A\}$$

contains L. After writing $f.l.Mod_A$ for the category of finite length A-modules, the functor

$$* = \mathrm{Hom}_A\big(-, E_A(\kappa)\big) : f.l.Mod_A^{op} \longrightarrow f.l.Mod_A$$

together with the evaluation natural transformation $ev : \mathrm{id} \to **$ gives $f.l.Mod_A$ the structure of an *exact category with duality* - on which one may again define Witt groups. The quotient $L^\flat/L$ is a finite length A-module, which together with the bilinear map

$$L^\flat/L \times L^\flat/L \longrightarrow E_A(\kappa) \text{ given by } (\overline{x}, \overline{y}) \mapsto d\big(\langle x, y, \rangle_\psi\big)$$

represents an element in the Witt group $W\big(f.l.Mod_A, *\big)$. This element does not depend on the choice of lattice L - and further since we can find an isomorphism $W(\kappa) \cong W\big(f.l.Mod_A, *\big)$ we have hence given a suitable residue homomorphism $W(F) \to W(\kappa)$.

Our main result is that this map we construct agrees with the residue maps defined in [1, 2] - providing a positive answer to the question posed by Balmer & Walter [1] as to whether or not the boundary maps of their Witt complex agree with those produced by Pardon in the unpublished work [3]. We don't go into any detail of the construction of *loc. cit.* but mention that our residue map does indeed have the same basic idea of Pardon's construction - with differences including that we don't need the assumptions that our schemes be Cohen-Macaulay (we only need our schemes to admit residual complexes) and don't rely upon Witt groups defined in some new setting (i.e. the categories $Q\big(\mathcal{S}_i^p(X); \mathcal{C}\big)$ of *loc. cit.*) not already covered by the literature; the knowledge of Witt groups our construction requires is all very standard and long established. Our main contribution to the literature is hence a new and simpler (in for example that it doesn't require any knowledge of derived categories) description of the Witt complexes of [1, 2]. It is also not immediate that residue homomorphisms for Witt groups, like those of [7] or those we construct, can be extended to maps

$$K_1^{MW}(F) \longrightarrow K_0^{MW}(\kappa(Z))$$

$$K_0^{MW}(F) \longrightarrow K_{-1}^{MW}(\kappa(Z))$$

between the few degrees of Milnor-Witt K-theory which are required to define the Chow-Witt group. Using the same ideas underlying our explicit description of the residue maps for Witt groups coming from [1, 2] we do define residue maps as above - in particular our definition of these low degree residue maps for Milnor-Witt K-theory continues to be made in terms of symmetric spaces and not, as is perhaps more common, any presentations of Milnor-Witt K-theory - such as the defining relations of [9]. We are hence able to provide, assuming that $1/2 \in \Gamma(X, \mathcal{O}_X)$, a definition of the Chow-Witt group $\widetilde{\mathrm{CH}}(X)$ which is valid for any schemes X admitting a residual complex - we are not aware of such a definition available in this generality. Unlike the work of [1, 2], since our residue maps did not need the assumption that 2 be invertible to be defined, it is possible that our defintion of the Chow-Witt group could also make sense in the characteristic 2 case.

The structure of this work is as follows.

The first background chapter is to some degree an extended version of this introduction. Its aim is to put everything into a historical setting, together with some mathematical motivation behind the objectives we've outlined so far. Hopefully, this chapter should provide an early graduate student who somehow stumbles across this text, should they need it, some insulation against feeling at a loss as to what is being discussed. We have not in this first chapter made a huge effort give the kind of level of detail such a graduate student might require to really understand every idea they come across here - but we provide references that do so. Of course not everything in this background is purely motivational - there are a few structures we really want to work with later but nothing non-standard.

The second chapter introduces the more technical machinery we want to use - in short this consists of the settings in which we will work with Grothendieck-Witt groups and of course some elementary properties these groups enjoy. All of the dualities we make use of are extracted from, or given directly by, *residual complexes* - which the first section tries to present an understanding of in terms of their appearance in Grothendieck duality [6]. The only (we think) new result here is **Proposition 2.3.5**; though we do also in this chapter introduce a few very non-standard notations.

Having understood how to extract dualities from the terms of residual complexes, the third chapter describes how the second residue homorphism can be extracted from the boundary maps of residual complexes. Roughly the first half of this chapter is devoted to defining our residue map - without any regularity or characteristic assumption - and giving a few basic properties. We of course check that our residue map agrees with those of [4] given above, before in the second half of the chapter checking that we have agreement, when 2 is invertible, with the residue maps of [1, 2].

In the final chapter we write down how to repeat the basic ideas in our construction of a generalised second residue homomorphism for Witt groups to obtain a map which really would be an analogue of the order of vanishing of a rational function. We hence obtain a reasonable definition of the Chow-Witt group $\widetilde{\mathrm{CH}}(X)$ of singular schemes X, oringinally defined with some smoothness assumption in [13]. Given how all of our work rests upon the structure of residual complexes which are intimately involved in the constructive approach of [6] to the exceptional inverse image functor it is then natural to in our situation write down the covariant functorality of our Chow-Witt groups along proper morphisms.

One could argue that it is possible to roughly divide approaches to building Gersten complexes for K-theory, Witt groups, and Milnor-Witt K-theory into two styles - firstly those modeled on Rost's construction of cycle complexes for Milnor K-theory [11] in which residue maps are defined before the complex is assembled such as [7, 9], and secondly those following Quillen's more global construction [12], for example [1, 2] in which the complex is found without any explicit construction of the residue maps involved. We really lie in this second camp and do not compare our maps with those of [7, 9] - aside from making a few remarks which make it seem likely that we have agreement with [7]. In particular nothing we do is defined on the level of Milnor-Witt K-theory - there is ongoing work [14] which aims to generalise Rost's construction [11] of a cycle complex for Milnor K-theory to a cycle complex for Milnor-Witt K-theory for singular schemes. It appears that the work of [14] should provide definition of the Chow-Witt group of singular schemes even in characteristic 2. There is hence some overlap between the objectives of this thesis and *loc. cit.* - though in terms of how we work towards this overlap our approaches are distinct.

Notations and Conventions:

1. The term *scheme* will always refer to a universally caternary and finite dimensional Noetherian scheme. We denote by X_r and X^r respectively the dimension and codimension r points of X. For points $x, y \in X$ we use the notation $y \rightsquigarrow x$ to

denote that x is an immediate specialisation of y - meaning that x is a codimension 1 point in the closure of y in X.

2. For an irreducible scheme X we denote by ζ_X its generic point. The term *variety* will mean an integral scheme, and a *subvariety* of a scheme X is a closed embedding $Z \hookrightarrow X$ of a variety Z into X.

3. For any scheme X we denote by $QCoh_X, Coh_X$, and $Vect_X$ the categories of quasi-coherent, coherent, and finite rank locally free $\mathcal{O}_X$-modules.

4. For a point x of a scheme X, we denote by m_x the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x)$ the residue field $\mathcal{O}_{X,x}/m_x$. We denote by X_x the local scheme $\mathrm{Spec}(\mathcal{O}_{X,x})$ and by $\pi_{x,X} : \mathrm{Spec}(\kappa(x)) \to X_x$ the embedding of the closed point. So if Z is a variety then $\kappa(\zeta_Z)$ is the field of rational functions on Z, and if $Z \hookrightarrow X$ is a subvariety then we will also use $\mathcal{O}_{X,Z}$ to denote the local ring $\mathcal{O}_{X,\zeta_Z}$ of functions along Z.

5. For a ring A we denote by $f.g.Mod_A$ and $f.l.Mod_A$ the categories of finitely generated and finite length A-modules respectively. If F is a field then $Vect_F$ is the category of finite-dimensional F-vector spaces.

Chapter 1

Historical Setting & Mathematical Background

While the reader may feel that any permutation of the adjectives in the above title would not greatly change its meaning, we nonetheless make clear the intended implication of our choice on the coming subject matter - this chapter aims to in a precise manner set out some of the mathematical technology required by later chapters, while simultaneously and imprecisely sketching the evolution of these ideas through (quite recent) history. Part of this latter goal is to also provide motivation for the overall objectives of this thesis. As a result, many of the mathematical statements in this chapter are, from the perspective of this work at least, purely motivational, and are hence here not rigorously set out - but we provide references where more proper levels of detail can be found.

1.1 Chow groups and K-theory

The book [10] contains an overview of the development of intersection theory, from which we have here extracted a story most relevant to our particular topic. For schemes smooth over a field, the Chow groups are the underlying additive structure of the intersection rings in algebraic geometry. Indeed, for each subvariety $V \hookrightarrow X$ of such a smooth scheme X there corresponds an element $[V] \in \mathrm{CH}(X)$ in such a way that for another subvariety $W \hookrightarrow X$ the product $[V] \times [W] \in \mathrm{CH}(X)$ is in some way built from components of $V \cap W$ - a little more descriptively; if V is regularly embedded in X and meets W properly, then one has

$$[V] \times [W] = \sum_{j=1}^{N} i\left(C_j, V \cdot W \hookrightarrow X\right)[C_j] \in \mathrm{CH}(X) \qquad (*)$$

where the C_j are the irreducible components of the intersection $V \cap W$, and the integers $i(C_j, V \cdot W \hookrightarrow X)$ are called *intersection multiplicities* - they depend on how V,W and C_j are embedded in X. Recognisation of the importance of such intersection multiplicities probably goes back to *Bezout's Theorem* circa 1779, though in the interest of historical accuracy we should note (see [15, Chapter 3.1]) that Bezout's proof was neither the first, nor a correct proof of this classical result. Indifferent to a proper and timely attribution of mathematicians to theorems, the formula $(*)$ became the guiding principle in the construction of intersection rings.

Definition 1.1.1. Let X be any scheme satisfying our standard assumptions. We write $Z^p(X)$ for the free abelian group generated by the codimension p subvarieties of X. If $V \hookrightarrow X$ is such a subvariety, then we write $[V] \in Z^p(X)$ for the corresponding generator. An element $\alpha \in Z^p(X)$ may hence be written

$$\alpha = \sum_{V \in X^p} n_V [V]$$

where all but finitely many of the $n_V \in \mathbb{Z}$ are zero. We call $Z^p(X)$ the group of **codimension p cycles on X**, and similarly define $Z_p(X)$ to be the group of **p-dimensional cycles on X**.

The issue with an approach to intersection theory via the formula $(*)$ is finding an appropriate definition of the intersection multiplicities. But if X is a d-dimensional variety over the complex numbers, then any two subvarieties $V \hookrightarrow X$ and $W \hookrightarrow X$ of codimensions p and q determine topological homology classes $[V] \in H_{2d-p}(X, \mathbb{Z})$ and $[W] \in H_{2d-q}(X, \mathbb{Z})$ respectively. Viewed via the *Poincaré Duality isomorphism* as elements in the cohomology ring $H^*(X, \mathbb{Z})$, an intersection product

$$[V] \times [W] \in H^{p+q}(X, \mathbb{Z})$$

can be defined. In the case when V and W meet transversally, one really then has $[V] \times [W] = [V \cap W] \in H^{p+q}(X, \mathbb{Z})$, and the product may be computed for general cycles by slightly deforming one of V and W until they do meet transversally. Stimulated by this analogy, Wei-Liang Chow felt that the obstruction to extending the formula $(*)$ to arbitrary cycles was the lack of an equivalence relation on $Z^*(X)$ which could move any pair of cycles in X to a pair for which the required intersection multiplicities could be defined. Chow demonstrated that such equivalence relations do exist in [16], where he focussed on the relation of *rational equivalence* - which according to *loc. cit.* had hitherto not been given much attention.

Definition 1.1.2. Let X be a variety, and $V \hookrightarrow X$ be a codimension 1 subvariety. We define the **order of vanishing along** V to be the group homomorphism

$$\mathrm{ord}_V : \kappa(\zeta_X)^* \longrightarrow \mathbb{Z}$$

which for an element $f \in \mathcal{O}_{X,V}$ - the local ring of functions along V - is given by

$$\mathrm{ord}_V(f) = \mathrm{length}_{\mathcal{O}_{X,V}} \left(\mathcal{O}_{X,V} / f \cdot \mathcal{O}_{X,V} \right)$$

One may then define the group homomorphism div : $\kappa(\zeta_X)^* \longrightarrow Z^1(X)$ to be given by

$$\operatorname{div}(f) = \sum_{V \in X^1} \operatorname{ord}_V(f)[V]$$

Definition 1.1.3. Let X be a scheme. Then we define the subgroup $\operatorname{Rat}^p(X) \leq Z^p(X)$ of cycles **rationally equivalent to zero** to be that generated by the elements $\operatorname{div}(f)$ as f ranges over all the rational functions of codimension $p-1$ subvarieties of X. The **Chow group of codimension p cycles on X** is defined to be the quotient

$$\operatorname{CH}^p(X) = Z^p(X) / \operatorname{Rat}^p(X)$$

The groups $\operatorname{Rat}_p(X)$ and $\operatorname{CH}_p(X)$ are defined in the same way but using dimension instead of codimension to obtain the grading.

The result of Chow's paper [16] was, for smooth quasi-projective X, a well-defined ring structure on $\operatorname{CH}^*(X)$ constructed via $(*)$ where possible. A weakness in this approach is that it is not at all clear for general cycles $V, W \hookrightarrow X$ that the intersection product $[V] \times [W] \in \operatorname{CH}^*(X)$ could be written as a cycle supported on the actual intersection $V \cap W$; each cycle has potentially been moved to different subvarieties which while meeting properly do so perhaps outside of the original intersection.

Fulton & MacPherson's approach [10] radically differs from earlier approaches such as [16] in that the ring structure on $\operatorname{CH}^*(X)$ is constructed first, without requiring any quasi-projectivity hypothesis, before intersection multiplicities are defined.

Proposition 1.1.4. *Let $f : X \to Y$ be a proper morphism of schemes of finite type over some ground field. Then we write*

$$f_* : Z_p(X) \longrightarrow \operatorname{CH}_p(Y)$$

for the additive group homomorphism defined by

$$f_*([V]) = \begin{cases} [\kappa(\zeta_V) : \kappa(f(\zeta_V))]\,[f(V)] & \text{if } \dim(V) = \dim(f(V)) \\ 0 & \text{otherwise} \end{cases}$$

Then $f_*(\operatorname{Rat}_p(X)) = \{0\}$, allowing us to define (and denote by the same symbol) a pushforward map

$$f_* : \operatorname{CH}_p(X) \longrightarrow \operatorname{CH}_p(Y)$$

Definition 1.1.5. If $Z \hookrightarrow X$ is a closed subscheme of X, then we write $[Z]$ for the cycle

$$[Z] = \sum_{i=1}^{N} \operatorname{length}(\mathcal{O}_{Z,C_i})\,[C_i] \in Z^*(X)$$

where $C_1, C_2, ..., C_N$ are the irreducible components of Z, and each of the above lengths are taken over the Artinian local rings $\mathcal{O}_{Z,C_i}$ themselves. Abusing notation, we also write $[Z]$ for the image of this cycle in $\mathrm{CH}^*(X)$.

Proposition 1.1.6. *Suppose that $f : X \to Y$ is a flat morphism of schemes of finite type over some ground field, with constant relative dimension. Then we write*

$$f^* : Z^p(Y) \longrightarrow \mathrm{CH}^p(X)$$

for the additive group homomorphism defined by setting

$$f^*([Z]) = [f^{-1}(Z)]$$

Then $f^*(\mathrm{Rat}^p(Y)) = \{0\}$, allowing us to define (and denote by the same symbol) a pullback map

$$f^* : \mathrm{CH}^p(Y) \longrightarrow \mathrm{CH}^p(X)$$

The modern approach to intersection theory found in [10] is developed from a construction of Chern class actions of vector bundles on the Chow group of their base. Of all the higher Chern classes, we explicitly define here only the top Chern class of a vector bundle, as it is geometrically simplest to understand and has the greatest relevance to our story.

Proposition 1.1.7. *Let $p : E \to X$ be a vector bundle of rank r on a scheme X of finite type over a field. Then the flat pullback*

$$p^* : \mathrm{CH}^n(X) \longrightarrow \mathrm{CH}^{n-r}(E)$$

is an isomorphism for all n.

Proof. See **Theorem 3.3** of [10]. $\square$

This isomorphism allows one to perform an intersection theoretic operation important to the development of [10]. The inverse of the pullback isomorphism above can be thought of as intersecting cycles in E with the image of X under the zero section embedding $s : X \to E$. Note that it is irrelevant how a cycle in $\alpha \in \mathrm{CH}^n(E)$ meets this zero section - the intersection $(p^*)^{-1}(\alpha)$ is defined and is supported in X. The top Chern class $c_r(E)$ is a particular example of such an intersection - see **Example 3.3.2** of *loc. cit.* for the below definition.

Definition 1.1.8. Let $p : E \to X$ be a rank r vector bundle on a scheme X, and let $s : X \hookrightarrow E$ be the zero-section embedding. Then we define the **top (or $r^{\mathbf{th}}$) Chern class action** of E to be the group homomorphism

$$c_r(E) \cap (-) : \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^{*+r}(X)$$

given by $\alpha \mapsto (p^*)^{-1}s_*(\alpha) \in \mathrm{CH}^{n+r}(X)$ for each $\alpha \in \mathrm{CH}^n(X)$. We also write simply $c_r(E)$ for the **top Chern class** $c_r(E) \cap [X] \in \mathrm{CH}^r(X)$.

In the third chapter of [10], all the higher Chern classes $c_i(E)$ of a vector bundle E are defined, and they can be assembled into the *Chern character* of E, denoted $\mathrm{ch}(E)$. This Chern character takes values in the Chow group $\mathrm{CH}^*(X)_{\mathbb{Q}} := \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with rational coefficients - for a precise definition, see **Example 3.2.3** of [10]. This Chern character is additive on short exact sequences of vector bundles. Precisely, if

$$0 \to F' \longrightarrow F \longrightarrow F'' \to 0$$

is a short exact sequence of finite rank locally free sheaves on X, then we have

$$\mathrm{ch}(F) = \mathrm{ch}(F') + \mathrm{ch}(F'') \in \mathrm{CH}^*(X)_{\mathbb{Q}}$$

Hence, the Chern character defines a group homomorphism from the Grothendieck group K_0 of locally free sheaves on X to the Chow group $\mathrm{CH}^*(X)_{\mathbb{Q}}$ with rational coefficients.

Definition 1.1.9. We use the term **exact category** to refer to an additive category $\mathcal{E}$ equipped with a collection of *admissible exact sequences* which are pairs of morphisms

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \qquad (**)$$

in $\mathcal{E}$. We require of these sequences that there exists some abelian category $\mathcal{A}$ and a fully faithful embedding $\iota : \mathcal{E} \to \mathcal{A}$ of $\mathcal{E}$ as an extension closed subcategory of an abelian category $\mathcal{A}$ such that $(**)$ is an exact sequence in $\mathcal{E}$ if and only if

$$0 \to \iota(M') \xrightarrow{\iota(\alpha)} \iota(M) \xrightarrow{\iota(\beta)} \iota(M'') \to 0$$

is a short exact sequence in $\mathcal{A}$. In the exact sequence $(**)$ we call α an admissible monomorphism and β an admissible epimorphism. An admissible monomorphism is depicted by the arrow "$\rightarrowtail$" while admissible epimorphisms are depicted "$\twoheadrightarrow$".

Example 1.1.10. For any scheme X, each of the subcategories $QCoh_X, Coh_X$, and $Vect_X$ are exact subcategories of the abelian category of all $\mathcal{O}_X$-modules. The category $\mathrm{Ch}^b_{Coh}(X)$ of bounded chain complexes of quasicoherent $\mathcal{O}_X$-modules having coherent cohomology is an exact subcategory of the category of all chain complexes of $\mathcal{O}_X$-modules.

Definition 1.1.11. Let $\mathcal{E}$ be an exact category. Then the **Grothendieck group** of $\mathcal{E}$, denoted $K_0(\mathcal{E})$, is defined to be the free abelian group generated by the isomorphism classes $[E]$ of objects $E \in \mathcal{E}$, modulo the relations

$$[E] = [E'] + [E'']$$

for each short exact sequence $0 \to E' \longrightarrow E \longrightarrow E'' \to 0$ in $\mathcal{E}$.

From **Example 15.2.16(b)** of [10], one learns that for smooth X the Chern character map induces an isomorphism $\mathrm{ch} : K_0(\mathit{Vect}_X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{CH}^*(X)_{\mathbb{Q}}$. It can be argued [17] that algebraic K-theory began in part in 1957, with Grothendieck's reformulation of the Riemann-Roch theorem in terms of the correction factor required to make this Chern character commute with the pushforwards for K_0 and Chow groups. As charted in [17], higher K-groups were gradually defined over the next 20 years - mostly only for affine schemes. According to *loc. cit.*, the study of algebraic cycles via these higher K-groups was initiated by Spencer Bloch who established [18, Thm 5.15] the first instance, namely when $p = \dim(X) = 2$, of the general formula

$$H^p(X, K_p) \cong \mathrm{CH}^p(X)$$

which is valid for any regular scheme X of finite type over a field, and in which K_p is the sheaf of abelian groups on X associated to the presheaf $U \mapsto K_p(\Gamma(U, \mathcal{O}_X))$. The first proof of *Bloch's formula* in this generality came from Quillen's work [12, §7 Thm 5.19] which defined the higher algebraic K-groups as functors on exact categories. We'll give here an overview in broad strokes of Quillen's proof - a more detailed description will in later chapters be given of the analogous constructions of [1, 2] in which the higher K-groups are replaced by Balmer's *shifted Witt groups* [5].

Proposition 1.1.12. *Let $\mathcal{B}$ be a Serre subcategory (meaning closed under extensions, subobjects and quotients) of an abelian category $\mathcal{A}$. Then there is a long exact sequence*

$$\ldots \longrightarrow K_{i+1}\big(\mathcal{A}/\mathcal{B}\big) \xrightarrow{\partial_{i+1}} K_i(\mathcal{B}) \longrightarrow K_i(\mathcal{A}) \longrightarrow K_i\big(\mathcal{A}/\mathcal{B}\big) \xrightarrow{\partial_i} K_{i-1}(\mathcal{B}) \longrightarrow \ldots$$

Proof. This is [12, §5 Thm 5] - two thirds of the maps in the above sequence come from the canonical maps $\mathcal{B} \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ while, as Quillen remarks, the proof gives little information on the nature of the boundary maps ∂_i. □

Consider now, if X is our finite-type regular k-scheme, the Serre subcategories $\mathit{Coh}_X^{\geq i} \hookrightarrow \mathit{Coh}_X$ consisting of those coherent sheaves whose support is a closed subset of codimension at least i in X. From the above proposition we have boundary maps

$$\ldots \longrightarrow K_{p-i}\left(\frac{\mathit{Coh}_X^{\geq i}}{\mathit{Coh}_X^{\geq i+1}}\right) \xrightarrow{\partial_{p-i}} K_{p-i-1}\big(\mathit{Coh}_X^{\geq i+1}\big) \longrightarrow K_{p-i-1}\big(\mathit{Coh}_X^{\geq i}\big) \longrightarrow \ldots$$

from which a complex may be built on the terms $K_{p-i}\left(\frac{\mathit{Coh}_X^{\geq i}}{\mathit{Coh}_X^{\geq i+1}}\right)$ - below it appears as the diagonal maps.

$$\begin{tikzcd}
K_{p-(i-1)}\left(\frac{Coh_X^{\geq i-1}}{Coh_X^{\geq i}}\right) \arrow[r, "\partial_{p-(i-1)}"] \arrow[dr] & K_{p-i}\big(Coh_X^{\geq i}\big) \arrow[d] & \\
& K_{p-i}\left(\frac{Coh_X^{\geq i}}{Coh_X^{\geq i+1}}\right) \arrow[d, "\partial_{p-i}"'] \arrow[dr] & \\
& K_{p-(i+1)}\big(Coh_X^{\geq i+1}\big) \arrow[r] & K_{p-(i+1)}\left(\frac{Coh_X^{\geq i+1}}{Coh_X^{\geq i+2}}\right)
\end{tikzcd}$$

By devissage [12, §5 Thm 4] Quillen identifies these terms with the groups below

$$K_q\left(\frac{Coh_X^{\geq i}}{Coh_X^{\geq i+1}}\right) \simeq \bigoplus_{x \in X^i} K_q(x)$$

where $K_q(x)$ denotes the q^{th} K-group of the exact category of finite dimensional $\kappa(x)$-vector spaces - a notational convention we will stick to in various settings throughout this thesis. Given our regularity assumption, Quillen establishes that taking these sequences over each open set of X induces a flasque resolution

$$0 \to K_p \longrightarrow \iota_{\zeta_X}\big(K_p(\kappa(\zeta_X))\big) \longrightarrow \ldots \longrightarrow \bigoplus_{x \in X^{p-1}} \iota_x\big(K_1(\kappa(x))\big) \longrightarrow \bigoplus_{x \in X^p} \iota_x\big(K_0(\kappa(x))\big) \to 0$$

of the sheaf K_p; our "ι_x" notation explained below will continue to be used in later chapters.

Definition 1.1.13. Let X be a scheme and $x \in X$ a point. Then for any abelian group (respectively $\mathcal{O}_{X,x}$-module) M, we denote by $\iota_x(M)$ the sheaf of abelian groups on X (respectively the $\mathcal{O}_X$-module) $(i_x)_*(M)$, where i_x is the canonical map $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ and in the case that M is an abelian group we have denoted the constant sheaf on $\mathrm{Spec}(\mathcal{O}_{X,x})$ associated to M again by the symbol M.

Noting that the map $K_0(F) \to \mathbb{Z}$ which sends the isomorphism class of an F-vector space to the common rank of any of its representatives is an isomorphism, we see that the final non-zero term of the above complex naturally identifies itself with $Z^p(X)$; the group of codimension p cycles on X. For the previous term, one learns from [17] that constructions for the K_1 of rings had been realised by Bass and Schanuel roughly 10 years earlier in 1962. Their definition for a ring A was to take the abelianisation

$$K_1(A) = \mathrm{GL}(A)/\left[\mathrm{GL}(A), \mathrm{GL}(A)\right]$$

of the group of matrices $\mathrm{GL}(A) = \varinjlim \mathrm{GL}_n(A)$. For any field F, the determinant map induces an isomorphism $K_1(F) \cong F^*$, so that Quillen's proof of Bloch's formula may then conclude by identifying the last non-zero map in the above flasque resolution with the coproduct of the divisor maps in **Definition 1.1.2**. We would refer to the resolution itself (by which we mean the above complex with the first term K_p replaced by zero) as the p^{th} *Gersten complex for Quillen's K-theory* because it was Gersten who originally conjectured (see *Working Hypothesis 7.3* of [19]) that it would give the desired resolution. The Gersten complex itself may be constructed without any regularity assumption on X, and one still obtains essentially the definition of Chow groups by taking (co)homology in the appropriate place. This sets the precedent which leads to most definitions of Chow-Witt groups - one first constructs a Gersten-type complex, in effect for *Milnor-Witt K-theory* [9], and then takes (co)homology in the appropriate degree. Most of the effort in this thesis is devoted to giving an explicit description of Balmer's analogue [5] of the boundary maps ∂_i of **Proposition 1.1.12** in the setting of shifted Witt groups as they appear in [1, 2].

Among the assemblage of definitions of the higher K-groups of rings prior to Quillen's 1973 paper [12] is the below presentation for a field F of $K_2(F)$ which is given in [20].

$$K_2(F) = \frac{\mathbb{Z}[F^* \otimes_{\mathbb{Z}} F^*]}{\langle a \otimes b \mid a, b \in F^* \text{ and } a + b = 1 \rangle}$$

In [21] Milnor takes the above relation, which is commonly referred to as the *Steinberg relation*, to be the basis for defining an approximation to the K-theory of fields - which, despite the ad hoc nature of its definition in higher degrees, has become an important algebra in its own right.

Definition 1.1.14. The *Milnor K-theory* of a field F is defined to be the graded ring obtained as the quotient

$$K_*^M(F) = \mathrm{Tens}_{\mathbb{Z}}(F^*)/\left\langle \mathrm{St} \right\rangle$$

of the tensor algebra by the two sided ideal generated by the Steinberg relations

$$\mathrm{St} = \{a \otimes b \mid a, b \in F^* \text{ and } a + b = 1\}$$

We denote the image of $a_1 \otimes a_2 \otimes ... \otimes a_n$ in $K_n^M(F)$ by $\{a_1, a_2, ..., a_n\}$.

Clearly by design the Milnor K-theory of fields agrees with actual K-theory in degrees 0,1 and 2. Further, Bloch's formula is known to hold between Milnor K-theory and the Chow group in certain situations; see [22] for the case of zero dimensional cycles and [23] for cycles of any dimension on regular schemes over an infinite field. The construction of the Gersten complex for Milnor K-theory made in [22] is somewhat of a template for later constructions of other Gersten-type complexes. The ingredients for this template are the following *residue* and *norm* maps - which are defined in [21] and [24] respectively.

Definition 1.1.15. Let F be a field and $v : F^* \to \mathbb{Z}$ a discrete valuation on F with ring of integers $\mathcal{O}_v$ and residue field $\kappa(v)$. Then we define the **residue homomorphism**

for Milnor K-theory (for each integer n) to be the unique group homomorphism

$$\partial_v^{n+1} : K_{n+1}^M(F) \to K_n^M(\kappa(v))$$

which takes $\{f, u_1, ..., u_n\} \mapsto v(f)\{\overline{u_1}, ..., \overline{u_n}\}$ for any $f \in F^*$ and units $u_1, ..., u_n \in \mathcal{O}_v^*$.

Definition 1.1.16. For each integer n, there exists a unique family of natural homomorphisms, which we'll call the **norm maps for Milnor K-theory**

$$N_{L/F}^n : K_n^M(L) \to K_n^M(F)$$

indexed by finite field extensions L/F such that $N_{F/F} = \mathrm{id}_F$, and for any field F the sum

$$\sum_v N_{\kappa(v)/F}^n \partial_v^{n+1} = 0$$

taken over all discrete valuations v of $F(t)$ which are trivial on F^*, vanishes as above.

Remark 1. *For any finite extension L/F the norm $N_{L/F}^0$ is multiplication by the degree $[L : F]$, while $N_{L/F}^1$ is given by the usual norm function - a generator $\{l\} \in K_1^M(L)$ is sent to the generator of $K_1^M(F)$ corresponding to the determinant of the F-linear map $L \to L$ given by multiplication with l.*

The boundary maps

$$\bigoplus_{x \in X^r} \iota_x\big(K_n^M(\kappa(x))\big) \longrightarrow \bigoplus_{x \in X^{r-1}} \iota_x\big(K_{n-1}^M(\kappa(x))\big)$$

of the Gersten complex for Milnor K-theory are constructed in [22] via a normalisation process which is repeated in at least [7, 11].

Definition 1.1.17. If x and y are points of a scheme X with y an immediate specialisation of x, then for each integer n we denote by

$$d_{n,y}^{M,x} : K_n^M\big(\kappa(x)\big) \longrightarrow K_{n-1}^M\big(\kappa(y)\big)$$

the group homomorphism defined by taking a normalisation $\eta : Z \to \overline{\{x\}}$ and setting $d_{n,y}^{M,x}$ to be the following sum

$$d_{n,y}^{M,x} = \sum_{\eta(\tilde{y})=y} N_{\kappa(\tilde{y})/\kappa(y)}^{n-1} \circ \partial_{\tilde{y}}^n$$

where each $\partial_{\tilde{y}}^n$ denotes the residue homomorphism for Milnor K-theory attached to the discrete valuation on $\kappa(x)$ given by the order of vanishing along $\tilde{y}$. We commit the abuse of notation which is to denote the corresponding morphism between sheaves

$$d_{n,y}^{M,x} : \iota_x\big(K_n^M(\kappa(x))\big) \longrightarrow \iota_y\big(K_{n-1}^M(\kappa(y))\big)$$

in exactly the same way.

Definition 1.1.18. For a scheme X we define the **Gersten complex for Milnor K-theory** (in degree n) to be the cochain complex $C^*\left(X, K_n^M\right)$ consisting of the sheaves of abelian groups

$$C^*\left(X, K_n^M\right)^r = \bigoplus_{x \in X^r} \iota_x\left(K_{n-r}^M(\kappa(x))\right)$$

with boundary maps

$$d^r_{C^*(X,K_n^M)} = \bigoplus_{x \in X^r} \sum_{x \rightsquigarrow y} d_{n,y}^{M,x}$$

That the above really does define a complex was first written down in [22], and later Rost [11] identifies the properties *cycle modules* (of which Milnor K-theory is the principal example) need to have in order for a complex to be constructed via such a normalisation process. In Chapter 3, we construct a similar residue map after replacing Milnor K-theory by *Witt groups* - while in our situation we do have a notion of "norm maps" we won't need them to construct the residue map; there is no normalisation process required. Despite the fact that we explicitly construct this residue map for Witt groups before trying to put them together into a complex, our approach to constructing the resulting Gersten-Witt complex is really closer to Quillen's construction of the Gersten complex for K-theory via localilsation and devisage - indeed after defining our residue maps (for which we do not require any assumption on the characteristic) we check that they agree with those obtained for Witt groups in [1, 2] via an argument analogous to Quillen's. All of the constructions of [1, 2, 5] involve *triangular Witt groups* and hence require some assumption that 2 be invertible for anything to even be defined - hence we ultimately also have to make the assumption that $1/2 \in \Gamma(X, \mathcal{O}_X)$. There is ongoing work [14] which aims to construct similar residue maps by generalising, even for singular schemes, Rost's axoims for Milnor K-theory to Milnor-Witt K-theory. It appears that this work will produce a definition of the Chow-Witt group for singular schemes without requiring any assumption on the characteristic. Our definitions probably agree, but we only make a few scattered, non-commital remarks comparing our work with the approaches [7, 9, 14] built on Rost's cycle-complex type arguments.

1.2 A quadratic refinement - the oriented Chow groups of Barge and Morel

The top Chern class

$$c_r(E) \cap [X] \in \mathrm{CH}^r(X)$$

of **Definition 1.1.8** will vanish if E has a nowhere vanishing global section, and conversely Murthy [25] has shown that for smooth affine X over an *algebraically closed* field the vanishing of this top Chern class is also sufficient for E to have such a section. This result does not hold over general fields however - a counterexample is given by the tangent bundle to the real 2-sphere.

Barge and Morel [13] proposed a modification to the Chow groups which they conjectured would make them suitable target for Euler classes. Their *Chow group of oriented cycles* (also called the *Chow-Witt groups*) of X, which in codimension d is written $\widetilde{\mathrm{CH}}^d(X)$, can be roughly described as codimension d cycles on X with quadratic forms (instead of integers) for coefficients, satisfying some compatibility condition and modulo a quadratic analogue of rational equivalence. The definition of *loc. cit.* is to construct a Gersten complex for what would later be identified as *Milnor-Witt K-theory* [9] and then take cohomology. The complex is obtained as a fibre product between the Gersten complex for Milnor K-theory of **Definition 1.1.18** and a subcomplex of the *Witt complex* of X.

Definition 1.2.1. If L is a one-dimensional F-vector space, then we call a pair (V, ψ) where V is a finite dimensional F-vector space and $\psi : V \times V \to L$ is a non-degenerate symmetric bilinear map a **symmetric space** over (F, L). The collection of all such symmetric spaces is denoted $Sym(F, L)$. Two symmetric spaces (V_1, ψ_1) and (V_2, ψ_2) are called **isometric** when there exists an isomorphism $f : V_1 \to V_2$ such that

$$\psi_2 \circ (f \times f) = \psi_1$$

We define the sum

$$[V, \psi] + [V'\psi'] = [V \oplus V', \psi \oplus \psi']$$

of two isometry classes of symmetric spaces to be the above orthogonal sum.

Definition 1.2.2. Let L be a one-dimensional F-vector space, and $(V, \psi) \in Sym(F, L)$. Then for any subspace $N \leq V$ we define its **orthogonal** to be

$$N^{\perp} = \{v \in V |\ \psi(v, n) = 0 \text{ for any } n \in N\}$$

we call the space (V, ψ) **metabolic** when it has a subspace $N \leq V$ with $N = N^{\perp}$. We define the **Witt group of F with values in L**, or sometimes the **Witt group twisted by L**, written $W(F, L)$, to be the quotient of the Grothendieck group of the abelian monoid of isometry classes $[V, \psi]$ of symmetric spaces (V, ψ) under the orthogonal sum of the previous definition by the subgroup generated by the metabolic spaces.

We write simply $W(F)$ for the *ring* $W(F, F)$ in which multiplication is given by tensor product - for each one-dimensional F-vector space L we have further that $W(F, L)$ is under the tensor product a free $W(F)$-module of rank 1. The spaces L which arise for us will sometimes be referred to as the **twisting space** of the Witt group; this is non-standard terminology we've just made up on the spot. Our main reference for the Witt groups of fields is [4], the fourth chapter of which contains the following structural presentation and resulting residue map.

Proposition 1.2.3. *The group $W(F)$ is additively generated by the spaces $\langle \alpha \rangle$ - which denotes for $\alpha \in F^*$ the 1-dimensional space $(x, y) \mapsto \alpha xy$ - modulo the relations*

1. *For any $\alpha, \beta \in F^*$ we have $\langle \alpha\beta^2 \rangle = \langle \alpha \rangle$.*
2. *For any $\alpha \in F^*$ we have $\langle \alpha \rangle + \langle -\alpha \rangle = 0$.*
3. *For any $\alpha, \beta \in F^*$ with $\alpha + \beta \neq 0$ we have $\langle \alpha \rangle + \langle \beta \rangle = \langle \alpha + \beta \rangle + \langle \alpha\beta(\alpha + \beta) \rangle$.*

Definition 1.2.4. Let $v : F^* \to \mathbb{Z}$ be a discrete valuation on a field F with valuation ring $\mathcal{O}_v$, maximal ideal m_v, and residue field $\kappa(v)$. Then fixing a uniformizer π we have a **second residue homomorphism** $\partial_2^\pi : W(F) \to W(\kappa(v))$, which is defined to be the group homomorphism sending for $i \in \mathbb{Z}$ and unit u of the valuation

$$\langle \pi^i u \rangle \mapsto \begin{cases} \langle \overline{u} \rangle & \text{if } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Note that if π and $u\pi$ are two uniformizers of v then

$$\langle \overline{u} \rangle \partial_2^\pi = \partial_2^{u\pi}$$

so these maps really do depend on the choice of uniformizer - unlike the corresponding residue maps for Milnor K-theory of **Definition 1.1.15**. For this reason Witt groups do not form a cycle module as in [11] and the Witt complex of a scheme is not immediately obtained from the arguments therein. The residue maps can be made canonical by twisting the Witt groups involved by certain one dimensional vector spaces after which the boundary maps of the Witt complex may be defined via the normalisation process of *loc. cit.* - this approach is carried out in [7].

Definition 1.2.5. Let $A \to B$ be a ring map. Then we write $\Omega_{B/A}$ for the module of cotangent vectors of B over A. In the case when $\Omega_{B/A}$ is a free B-module of finite rank, we write $\Omega^{top}_{B/A}$ for the highest non-zero exterior power of $\Omega_{B/A}$.

The approach of [7] to constructing a canonical second residue map extending that of **Definition 1.2.4** is to fix some perfect ground field k and then to, for each finitely generated field extension $k \hookrightarrow F$, work with the Witt group $W\big(F, \Omega^{top}_{F/k}\big)$. A quick description of the resulting residue map relies on the below *lemma 2.2.7* of *loc. cit.* which we supply a proof of for no particular reason.

Lemma 1.2.6. *Suppose that F is a finitely generated field extension of a ground field k, with transcendence degree $n+1$. Let $v : F^* \to \mathbb{Z}$ be a discrete valuation, vanishing on k, with valuation ring $\mathcal{O}_v$, maximal ideal m_v, and residue field $\kappa(v)$. If $s_1, s_2, ..., s_n \in \mathcal{O}_v^*$ lift to elements which form a $\kappa(v)$-basis $d\overline{s_1}, d\overline{s_2}, ..., d\overline{s_n}$ of $\Omega_{\kappa(v)/k}$, and π is any uniformizer of v, then $d\pi, ds_1, ..., ds_n$ form an $\mathcal{O}_v$-basis of $\Omega_{\mathcal{O}_v/k}$.*

Proof. Since it is a localisation of a finitely generated k-algebra, $\Omega_{\mathcal{O}_v/k}$ is a finitely generated $\mathcal{O}_v$-module. The conormal exact sequence

$$\kappa(v)\otimes_{\mathcal{O}_v} m_v \xrightarrow{1\otimes d_{\mathcal{O}_v/k}} \kappa(v)\otimes\Omega_{\mathcal{O}_v/k} \xrightarrow{\overline{f}\otimes da\mapsto\overline{f}d_{\kappa(v)/k}\overline{a}} \Omega_{\kappa(v)/k}\to 0$$

tells us that the $\mathcal{O}_v$-module

$$\frac{\Omega_{\mathcal{O}_v/k}}{m_v\cdot\Omega_{\mathcal{O}_v/k}+d_{\mathcal{O}_v/k}(m_v)}$$

is generated by $\overline{ds_1},...,\overline{ds_n}$. Hence any element of

$$\frac{\Omega_{\mathcal{O}_v/k}}{m_v\cdot\Omega_{\mathcal{O}_v/k}}$$

may be written as an $\mathcal{O}_v$-linear combination of $\overline{ds_1},...,\overline{ds_n}$ plus some element of $d(m_v)$. An arbitrary element of $d(m_v)$ may be written $d(u\pi^m)=mu\pi^{m-1}ud\pi+\pi^m du$, for some unit $u\in\mathcal{O}_v^*$ and positive integer m. Modulo $m_v\cdot\Omega_{\mathcal{O}_v/k}$, this is simply $mu\pi^{m-1}u\cdot\overline{d\pi}$. Hence the $\mathcal{O}_v$-module $\kappa(v)\otimes\Omega_{\mathcal{O}_v/k}$ is generated by the images of $d\pi,ds_1,...,ds_n$. Hence by Nakayama's lemma, $\Omega_{\mathcal{O}_v/k}$ is generated by $d\pi,ds_1,...,ds_n$. □

Immediately after the aforementioned lemma in [7], we obtain the following description of the generalisation of the second residue map constructed in *loc.cit.*

Definition 1.2.7. Let F be a finitely generated, transcendence degree $n+1$ field extension of some ground field k, and $v:F^*\to\mathbb{Z}$ a discrete valuation trivial on k as above. Then there is a second residue map

$$\partial_2: W\left(F,\Omega^{top}_{F/k}\right)\longrightarrow W\left(\kappa(v),\Omega^{top}_{\kappa(v)/k}\right)$$

which is the unique group homomorphism with

$$\langle\alpha\rangle\langle d\pi\wedge ds_1\wedge...\wedge ds_n\rangle\mapsto\partial_2^{\pi}\left(\langle\alpha\rangle\right)\langle d\overline{s_1}\wedge d\overline{s_2}\wedge...\wedge d\overline{s_n}\rangle$$

for any uniformizer π of v and $s_1,...,s_n\in\mathcal{O}_v^*$ which lift to a basis $d\overline{s_1},d\overline{s_2},...,d\overline{s_n}$ of $\Omega_{\kappa(v)/k}$.

Note that above we've extended the notation of **Proposition 1.2.3** - to each $l\in\Omega^{top}_{F/k}$ we denote by $\langle l\rangle\in W\left(F,\Omega^{top}_{F/k}\right)$ the space $F\times F\to\Omega^{top}_{F/k}$ given by $(x,y)\mapsto xyl$. After constructing similarly twisted norm maps for Witt groups analogous to those for Milnor K-theory of **Definition 1.1.16**, a Witt complex

$$...\to\bigoplus_{x\in X^r}W\left(\kappa(x),\Omega^{top}_{\kappa(x)/k}\right)\longrightarrow\bigoplus_{x\in X^{r+1}}W\left(\kappa(x),\Omega^{top}_{\kappa(x)/k}\right)\to...$$

is constructed in [7] for schemes X of finite type over some perfect ground field k whose characteristic is different from 2 using the same normalisation process of **Definition**

1.1.17. To distinguish this complex from our later generalisation we adopt the following non-standard terminology.

Definition 1.2.8. For a scheme X of finite type over a perfect ground field k with $\mathrm{char}(k) \neq 2$, we write $C(X, W, \Omega)$ for the **Witt complex of X with Ω twisting** to be the cochain complex

$$C(X, W, \Omega)^r = \bigoplus_{x \in X^r} W\left(\kappa(x), \Omega^{top}_{\kappa(x)/k}\right)$$

with the boundary maps of [7, Ch.3].

Perhaps the main construction of this thesis is that of a similar Witt complex, whose boundary maps are defined without passing to any normalisations - they are rather inherited from the *residual complexes* from which we also obtain our local twisting spaces to use in place of the $\Omega^{top}_{\kappa(x)/k}$ above. The construction of the Witt complex of [7] is modelled around Rost's construction of the Gersten complex for Milnor K-theory of **Definition 1.1.18**. Our construction is closest to those of [1, 2] which is why we don't expend much energy describing the norm maps of [7].

For any field F, the Witt ring $W(F)$ has a unique ideal $I(F)$ for which the quotient $W(F)/I(F)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [4, Ch.III §3]. This ideal consists precisely of the isometry classes of symmetric spaces with even rank; note that while the rank of a symmetric space in the Witt group is not well defined, since any metabolic space has even rank [4, Ch.III lem.1.2], the parity of its rank is well defined.

Definition 1.2.9. For any field F we define the **rank homomorphism**

$$\mathrm{rk} : W(F, L) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

to be the ring homomorphism which takes each symmetric space $[V, \psi]$ to the dimension of V modulo 2. We denote the kernel of this homomorphism by $I(F, L)$ and call it the **fundamental ideal** of the Witt ring $W(F)$. When $L = F$, we denote by $I^n(F)$ the nth power of this ideal. In general we write $I^n(F, L)$ for $I^n(F) \cdot W(F, L)$.

Proposition 1.2.10. *For each non-negative integer n, there is a unique surjective group homomorphism, which we'll call the* **Pfister form map***,*

$$s_n : K^M_n(F) \to I^n(F)/I^{n+1}(F)$$

defined by sending each symbol $\{a_1, a_2, ..., a_n\}$ to the associated Pfister form

$$\{a_1, a_2, ..., a_n\} \mapsto \langle\langle a_1, a_2, ..., a_n\rangle\rangle := \prod_{i=1}^{n} (\langle 1\rangle - \langle a_i\rangle) \pmod{I^{n+1}(F)}$$

Proof. This is [21, Thm 4.1]. □

For our first encounter with the Chow-Witt groups we adopt the notation of [13]. There, the quotient group $I^n(F)/I^{n+1}(F)$ is denoted by $j^n(F)$ and it is observed that for any one-dimensional F-vector space L there is a canonical surjective map $I^n(F,L) \to j^n(F)$. The limit of the diagram

$$\begin{array}{ccc} & & I^n(F,L) \\ & & \downarrow \quad (*) \\ K_n^M(F) & \longrightarrow & j^n(F) \end{array}$$

of abelian groups is written $J^n(F,L)$. The cases when $n = 0, 1$, or -1 will become of particular importance to us. Clearly for any negative n we can identify $J^n(F,L)$ with the Witt group $W(F,L)$. For $n = 0$, the lowest degree in which the Milnor K-theory of the above diagram is non-zero, we have the following description.

Definition 1.2.11. For a field F and one-dimensional F-vector space L, we define the **Grothendieck-Witt group of F with values in** L, written $GW(F,L)$, to be the Grothendieck group of the abelian monoid of isometry classes $[V,\psi]$ of symmetric spaces $(V,\psi) \in Sym(F,L)$ under orthogonal sum. To be precise, the elements of $GW(F,L)$ are formal differences $[V,\psi] - [V',\psi']$ of such isometry classes, and we identify

$$[V,\psi] - [V',\psi'] = [U,\phi] - [U',\phi']$$

when there is a third space $(P,\theta) \in Sym(F,L)$ such that

$$[V,\psi] + [U',\phi'] + [P,\theta] = [U,\phi] + [V',\psi'] + [P,\theta]$$

in the abelian monoid structure.

This definition gives the wrong impression of what the Grothendieck-Witt group should be in more general situations - it is on account of $Vect_F$ being a split exact category that we do not need to enforce any further relations between metabolic spaces. Note then that the Witt group $W(F,L)$ is then the quotient of $GW(F,L)$ by the subgroup generated by metabolic spaces. Hence we have a canonical epimorphism $GW(F,L) \twoheadrightarrow W(F,L)$.

Proposition 1.2.12. *The group $GW(F)$ is additively generated by the spaces $\langle\alpha\rangle$ for $\alpha \in F^*$, modulo the relations*

1. *For any $\alpha, \beta \in F^*$ we have $\langle\alpha\beta^2\rangle = \langle\alpha\rangle$.*
2. *For any $\alpha \in F^*$ we have $\langle\alpha\rangle + \langle-\alpha\rangle = \langle 1\rangle + \langle-1\rangle$.*
3. *For any $\alpha, \beta \in F^*$ with $\alpha + \beta \neq 0$ we have $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha+\beta\rangle + \langle\alpha\beta(\alpha+\beta)\rangle$.*

Proof. This follows immediately from the slightly more economical presentation of [26, Ch.1, Thm 4.7]. □

Because metabolic spaces are not declared to be zero in the Grothendieck-Witt group, the natural way to take the rank of elements of $GW(F,L)$ is well defined - as the reader can quickly verify by glancing at the above presentation.

Definition 1.2.13. For any field F we define the **rank homomorphism**

$$\mathrm{rk} : GW(F,L) \longrightarrow \mathbb{Z}$$

to be the group homomorphism induced by taking each symmetric space $[V,\psi]$ to the dimension of V as an F-vector space.

Note the abuse of notation that the symbol "rk" also denotes the rank map of **Definition 1.2.9**. Hopefully this will not lead to any confusion, even in the below cartesian square

$$\begin{tikzcd} GW(F,L) \arrow[r, two heads] \arrow[d, "\mathrm{rk}"'] & W(F,L) \arrow[d, "\mathrm{rk}"] \\ K_0^M(F) \arrow[r, "s_0"'] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

which allows us to identify $J^0(F,L)$ with $GW(F,L)$ - as we do in the soon to come diagram which concludes this section. A glaring difference between the Witt complex of **Definition 1.2.8** and the Gersten complex for Milnor K-theory is that the Witt groups do not come with any grading. The following result claimed in [13] allows us to extract graded subcomplexes from the Witt complex.

Proposition 1.2.14. *For any scheme X of finite type over a perfect field k with* $\mathrm{char}(k) \neq 2$*, the degree r boundary map*

$$d^r_{C(X,W,\Omega)} : \bigoplus_{x \in X^r} W\left(\kappa(x), \Omega^{top}_{\kappa(x)/k}\right) \longrightarrow \bigoplus_{x \in X^{r+1}} W\left(\kappa(x), \Omega^{top}_{\kappa(x)/k}\right)$$

of the Witt complex with Ω twisting restricts for any integer m to a morphism

$$\bigoplus_{x \in X^r} I^m\left(\kappa(x), \Omega^{top}_{\kappa(x)/k}\right) \longrightarrow \bigoplus_{x \in X^{r+1}} I^{m-1}\left(\kappa(x), \Omega^{top}_{\kappa(x)/k}\right)$$

Definition 1.2.15. Let X be a scheme as in the above proposition. Then for each integer $n \in \mathbb{Z}$ we denote by $C^*(X, I^n, \Omega)$ the **filtered Witt complex with Ω twisting** which is defined to be the subcomplex of $C^*(X, W, \Omega)$ which in degree r is given by

$$C^*(X, I^n, \Omega)^r = \bigoplus_{x \in X^r} I^{n-r}\left(x, \Omega^{top}_{\kappa(x)/k}\right)$$

Further we denote by $C^*(X, j^n, \Omega)$ the quotient complex $C^*(X, I^n, \Omega)/C^*(X, I^{n+1}, \Omega)$.

We below give the original definition of the Chow-Witt group from [13]. First, we remark that the canonical quotient and Pfister form maps of the diagram $(*)$ assemble to form morphisms of complexes

$$\begin{tikzcd}
 & C^*(X, I^n, \Omega) \arrow[d] \\
C^*\big(X, K_n^M\big) \arrow[r] & C^*(X, j^n, \Omega)
\end{tikzcd}$$

We denote by $C^*(X, J^n, \Omega)$ the complex obtained as the limit of the diagram above - in particular for any field F and one-dimensional F-vector space L we pick up the notation $J^n(F, L) = K_n^M(F) \times_{j^n(F)} I^n(F, L)$. Then the first definition of the Chow-Witt group given in [13] is as below.

Definition 1.2.16. Let X be a smooth scheme of finite type over some perfect ground field of characteristic different from 2. The **Chow-Witt group of codimension n cycles** on X is then defined to be the cohomology group $H^n(C^*(X, J^n, \Omega)$, and is denoted $\widetilde{\mathrm{CH}}^n(X)$.

Note that the morphism of complexes $C^*(X, J^n, \Omega) \to C^*\big(X, K_n^M\big)$ induces a natural map $\widetilde{\mathrm{CH}}^n(X) \to \mathrm{CH}^n(X)$. Below we've given a picture of the complex $C^*(X, J^n, \Omega)$ around degree n in which, for the sake of simplicity, we have omitted the local twists $\Omega_{\kappa(x)/k}$. The string of triangles and horizontal bracket are the artist's depiction of cohomology being taken.

$$\widetilde{\mathrm{CH}}^n(X)$$

$$\begin{tikzcd}
 & {} \arrow[r] & \bigoplus_{x\in X^{n-1}} I(x) \arrow[rr] \arrow[dd, dashed] & & \bigoplus_{x\in X^n} W(x) \arrow[rr] \arrow[dd, dashed] & & \bigoplus_{x\in X^{n+1}} W(x) \arrow[r] \arrow[dd, dashed] & {} \\
{} \arrow[r] & \bigoplus_{x\in X^{n-1}} J^1(x) \arrow[rr] \arrow[ur] \arrow[dd] & & \bigoplus_{x\in X^n} GW(x) \arrow[rr] \arrow[ur] \arrow[dd] & & \bigoplus_{x\in X^{n+1}} W(x) \arrow[r] \arrow[ur] \arrow[dd] & {} & \\
 & {} \arrow[r, dashed] & \bigoplus_{x\in X^{n-1}} j^1(x) \arrow[rr, dashed] & & \bigoplus_{x\in X^n} \mathbb{Z}/2\mathbb{Z} \arrow[rr, dashed] & & 0 \arrow[r, dashed] & {} \\
{} \arrow[r] & \bigoplus_{x\in X^{n-1}} K_1^M(x) \arrow[rr] \arrow[ur, dashed] & & \bigoplus_{x\in X^n} K_0^M(x) \arrow[rr] \arrow[ur, dashed] & & 0 \arrow[r] \arrow[ur, dashed] & {} &
\end{tikzcd}$$

We conclude our background by remarking upon a few of the further developments and related constructions available which are important enough that they should be mentioned even though we shan't pursue these ideas further. A basic point to pick up on is that the groups $J^*(F)$ are given explicit presentations in [27] in terms of the *Milnor-Witt K-theory* of F - which is further developed in [9].

Definition 1.2.17. For any field F, we define the **Milnor-Witt K-theory** of F to be the $\mathbb{Z}$-graded ring $K_*^{MW}(F)$ generated by, for each $a \in F^*$ a symbol $[a]$ in degree 1, and one symbol η in degree -1, subject to the relations

1. For each $a \in F^*$ not equal to 1, we have $[a][1-a] = 0$.
2. For each pair $a, b \in F^*$ we have $[ab] = [a] + [b] + \eta[a][b]$.
3. The generator η lies in the center of $K_*^{MW}(F)$.
4. After writing $h = \eta[-1] + 2$, we set $\eta h = 0$.

Proposition 1.2.18. *If char$(F) \neq 2$ then the graded ring map*

$$K_*^{MW}(F) \longrightarrow J^*(F)$$

defined by sending, for each $u \in F^*$

$$[u] \mapsto \big(\{u\}, -\langle\langle u \rangle\rangle\big) \in K_1^M(F) \times_{j^1(F)} I(F)$$

and η to $\langle 1 \rangle \in W(F) = J^{-1}(F)$ is an isomorphism.

Proof. This is [27, Thm 5.3]. □

Morel constructs [9, Ch.5] for smooth X a *Rost-Schmid* complex by combining Rost's construction of a complex for Milnor K-theory [11] with Schmid's twisting arguments of [7]. The Chow-Witt groups can be defined using this complex instead [9, Thm 5.47] - which further reobtains a version of Bloch's formula

$$\widetilde{\mathrm{CH}}^n(X) \cong H^n\Big(X, \underline{K}_n^{MW}\Big) \qquad (*)$$

for smooth X, where $\underline{K}_n^{MW}$ Morel's *unramified Milnor-Witt K-theory sheaf.*

The motivation of the original paper [13] was to define for smooth affine n-dimensional varieties X (of finite type over a field) and oriented rank n vector bundle E on X an *Euler class* $e(E) \in \widetilde{\mathrm{CH}}^n(X)$ which should vanish precisely when E has a nowhere vanishing global section. This conjecture is verified in *loc. cit.* when X is 2-dimensional and has since been established in many other cases. If $n \geq 4$ then Morel constructs an Euler class

$$e(E) \in H^n_{Nis}\Big(X, \underline{K}_n^{MW}\Big)$$

and proves [9, Thm 8.14] that it is a suitable obstruction to E having a trivial factor. Note that there is no assumption that 2 be invertible here - Morel remarks that when 2 is invertible then via the version of Bloch's formula $(*)$ above, his Euler class agrees with that of [13]. If 2 is invertible then Fasel and V.Srinivas establish the case $n = 3$ in [28] - which finally gives an affirmative answer to the conjecture of [13]. Finally, Schlichting [29, Thm 6.18] extends Morel's result to the case when $n \geq 2$ and X is only supposed to be the spectrum of a Noetherian ring with infinite residue fields - there's no regularity assumption and X need not be defined over a field.

In the case when X is not even regular, the groups $H^n\big(X, \mathcal{K}_n^{MW}\big)$ in which the Euler classes of [29] live are not the Chow-Witt groups of X - but the reader will hopefully anyway feel well-motivated to study these groups and also be disappointed to learn that in this thesis we do not at all investigate any further geometric applications they may have. We only aim to give the Chow-Witt groups a new, and ideally simpler, definition - most of which focusses on building appropriate analogs to the residue maps for Milnor-Witt K-theory found in [9], as well as those of **Definitions 1.1.15** and **1.2.4**. All of these maps are defined in terms of some presentation; part of the reason we move away from Milnor-Witt K-theory is that we prefer to work directly with symmetric spaces - this is perhaps why we are more naturally able to compare our residue maps with those of Balmer [5].

Chapter 2

Grothendieck-Witt groups & Residual Complexes

We set out here some results, terminologies and non-standard notations concerning how we extract from *residual complexes* local twisting spaces which replace the Ω-twisting of **Definition 1.2.8**. Spaces with symmetric structure with respect to the dualities induced by these residual complexes are stuided in the *Grothendieck-Witt groups* - which we define in two situations; firstly in the setting of exact categories with weak equivalences [30] (though we don't require the higher Grothendieck-Witt groups found therein) and secondly in the derived/triangular setting of [5]. The former is all we technically require to define second residue maps but we appeal to the latter to show that when 2 is invertible these second residue maps assemble to form a chain complex.

2.1 Residual Complexes in Grothendieck Duality

The main mathematical structure we exploit in this work is that of *residual complexes*, which as the reader will see neatly package together sufficient data to construct Witt complexes as in **Definition 1.2.8**. They arise as minimal injective resolutions of *dualising complexes* and play a central role in the original constructive approach to the exceptional inverse image pseudofunctor found in [6]. Despite the more polished realisations of this functor available at the time of writing, from the greatly detailed [31] to the non-constructive abstract approach in [32], our main reference for the subject of Grothendieck duality remains [6] as it deals in a very hands on way with residual complexes. Since the only thing we will in the later chapters require of our schemes is that they have residual complexes, we begin this section with a sketch of how they arise in Grothendieck duality.

2.1.1 Symmetric monoidal structure on $\mathrm{Ch}^b(X)$

For any scheme X we denote by $\mathrm{Ch}^b(X)$ the category of bounded cochain complexes of quasi-coherent $\mathcal{O}_X$-modules. If $F \in QCoh_X$ then we denote by $F[0]$ the chain complex which is F in degree zero and 0 elsewhere. We adopt the usual notation $Z^i(F) := \ker(d^i_F)$ and $B^i(F) = \mathrm{im}(d^F_{i+1})$ for the cycles and boundaries of a chain complex. The tensor product and internal hom give $\mathrm{Ch}^b(X)$ the structure of a closed symmetric monoidal category - the most delicate point being the choice of signs in the boundary maps in each of these chain complexes. Different choices for these signs are possible; the main motivation behind what we set out here is that our internal hom and translation agree with those of Balmer [1, 5] and Gille [2]. Such a collection of signs is detailed in [33] which we adopt.

Definition 2.1.1. For a scheme X, we define the **internal-hom** functor

$$[-,-]_X : \mathrm{Ch}^b(X)^{op} \times \mathrm{Ch}^b(X) \longrightarrow \mathrm{Ch}^b(X)$$

by setting for each pair $A, B \in \mathrm{Ch}^b(X)$

$$[A,B]^n = \prod_{s\in\mathbb{Z}} [A^s, B^{n+s}]_{\mathcal{O}_X}$$

where each $[A^s, B^{n+s}]_{\mathcal{O}_X}$ denotes the sheaf-hom $\mathcal{H}om_{\mathcal{O}_X}(A^s, B^{n+s})$, having the boundary maps

$$d^n_{[A,B]_X}\left(f_{(s)}\right)_{s\in\mathbb{Z}} = \left(f_{(s+1)}d^s_A - (-1)^n d^{n+s}_B f_{(s)}\right)_{s\in\mathbb{Z}}$$

for each collection $f_{(s)} : A^s \to B^{n+s}$ of morphisms of $\mathcal{O}_X$-modules. Given sufficient space, we will also write $\mathcal{H}om_X(A,B)$ for the internal hom $[A,B]_X$.

We define the **tensor product** to be the functor

$$(-) \otimes_X (-) : \mathrm{Ch}^b(X) \times \mathrm{Ch}^b(X) \longrightarrow \mathrm{Ch}^b(X)$$

by setting

$$\left(A \otimes_X B\right)^n = \bigoplus_{i+j=n} A^i \otimes_{\mathcal{O}_X} B^j$$

where each $A^i \otimes_{\mathcal{O}_X} B^j$ denotes the ordinary tensor product of $\mathcal{O}_X$-modules, with boundary map defined by

$$d^n_{A\otimes_X B}(a \otimes b) = d^i_A(a) \otimes b + (-1)^i a \otimes d^j_B(b)$$

whenever $a \in A^i$ and $b \in B^j$ with $i + j = n$. The functorial nature of $[-,-]_X$ and $(-) \otimes_X (-)$ is defined in the natural way without any intervention of signs.

It is worth highlighting notation we intend to stick to - if $f \in [A,B]^n$ then we write

$f_{(s)} : A^s \to B^{n+s}$ for the s^{th} component of f. For transparency we remark that the definition of the boundary maps of the tensor product has been made in terms of the presheaf tensor product which of course uniquely defines the actual boundary map on the level of sheaves. When checking the commutativity of some diagrams involving these tensor products it suffices to consider the maps on the level of these presheaves. With these boundary maps, the associativity of tensor product transformation is defined to be the natural isomorphism

$$ass_{(A,B,C)} : (A \otimes_X B) \otimes_X C \longrightarrow A \otimes_X (B \otimes_X C)$$

given in the natural way without any intervention of signs. The switch map however is the natural isomorphism

$$\mathrm{sw}_{A,B} : A \otimes_X B \to B \otimes_X A \qquad \text{defined by} \qquad \mathrm{sw}^{i+j}_{A,B}(a^i \otimes b^j) = (-1)^{ij}\ b^j \otimes a^i$$

for each pair $a^i \in A^i$, $b^j \in B^j$. The identity object for the tensor product is the structure sheaf of X concentrated in degree zero, for which we write $\mathbb{1}_X = \mathcal{O}_X[0]$. The natural isomorphisms $\mathbb{1}_X \otimes_X A \to A$ and $A \otimes_X \mathbb{1}_X \to A$ are defined without any signs. Unfortunately, our boundary maps of the internal-hom has forced our evaluation and coevaluation maps to involve some rather awkward signs; for any $A, B \in \mathrm{Ch}^b(X)$, we write

$$\mathrm{ev}_{A,B} : [A, B]_X \otimes_X A \longrightarrow B$$

for the **evaluation map** defined by

$$f^s \otimes a^t \mapsto (-1)^{s(s+1)/2} f^s_{(t)}(a^t)$$

for each pair $f^s \in [A, B]^s$ and $a^t \in A^t$. Finally we write

$$\nabla_{A,B} : A \longrightarrow [B, A \otimes_X B]_X$$

for the **coevaluation map** defined by

$$\nabla^n_{A,B}(a)_{(t)}(b) = (-1)^{n(n+1)/2} a \otimes b$$

for each $a \in A^n$, $t \in \mathbb{Z}$ and $b \in B^t$. With the maps defined so far we have given $\mathrm{Ch}^b(X)$ the structure of a closed symmetric monoidal category in the sense of [34]. In particular we obtain the tensor-hom andjunction map

$$\mathrm{adj}_{A,B,C} : \mathrm{Hom}_{\mathrm{Ch}^b(X)}(A \otimes_X B, C) \longrightarrow \mathrm{Hom}_{\mathrm{Ch}^b(X)}(A, [B, C])$$

defined by setting for $\beta : A \otimes_X B \to C$ the chain complex map $\mathrm{adj}_{A,B,C}(\beta) : A \to [B, C]_X$ to be the composition

$$A \xrightarrow{\nabla_{A,B}} [B, A \otimes_X B] \xrightarrow{[\mathrm{id}, \beta]} [B, C]_X$$

Explicitly then, the adjunction has the following signs.

Definition 2.1.2. For any $A, B, C \in \mathrm{Ch}^b(X)$ and $\beta : A \otimes_X B \to C$ we write $\hat{\beta} : A \to [B, C]_X$ for the adjoint map which is given by the formula

$$\hat{\beta}^n(a)_{(s)}(-) = (-1)^{\frac{n(n+1)}{2}} \beta^{n+s}(a \otimes (-))$$

Similarly, for $\psi : A \to [B, C]_X$ the map $\mathrm{adj}^{-1}_{A,B,C}(\psi)$ is defined to be the composite

$$A \otimes_X B \overset{\psi \otimes \mathrm{id}}{\longrightarrow} [B, C]_X \otimes_X B \overset{\mathrm{ev}_{B,C}}{\longrightarrow} C$$

which we again write out explicitly.

Definition 2.1.3. For any chain complex map $\psi : A \to [B, C]_X$, we write $\beta_\psi : A \otimes_X B \to C$ for the adjoint map, which is given in degree $i + j$ by

$$\beta_\psi(a \otimes b) = (-1)^{\frac{i(i+1)}{2}} \psi^i(a)_{(j)}(b)$$

for each pair $a \in A^i$ and $b \in B^j$.

We also obtain the signs for later double dual identifications by studying the composite

$$\begin{tikzcd}
A \arrow[rr, "\nabla_{A,[A,B]}"] \arrow[d, "\eta_{A,B}"'] & & \big[[A,B],\ A \otimes [A,B]\big] \arrow[d, "{\big[\mathrm{id}, \mathrm{sw}_{A,[A,B]}\big]}"] \\
{[[A,B],B]} & & \big[[A,B],\ [A,B] \otimes A\big] \arrow[ll, "{[\mathrm{id}, \mathrm{ev}_{A,B}]}"]
\end{tikzcd}$$

from which we initially obtain the following signs

$$\eta^n_{A,B}(a)_{(s)}(f) = (-1)^{\frac{1}{2}(n(n+1)+s(s+1))+ns} f_{(n)}(a)$$

for each $a \in A^n$ and $f \in [A, B]^s$. A quick computation reveals that these signs agree with those given in [2] which we write out below.

Definition 2.1.4. For any pair $A, B \in \mathrm{Ch}^b(X)$ we write

$$\eta_{A,B} : A \longrightarrow \big[[A, B], B\big]$$

for the transformation given by

$$\eta^n_{A,B}(a)_{(s)}(f) = (-1)^{\frac{(n+s)(n+s+1)}{2}} f_{(n)}(a)$$

for each $a \in A^n$ and $f \in [A, B]^s$.

Definition 2.1.5. For any $F \in \mathrm{Ch}^b(X)$ we write $F[1]$ for the **shifted complex** with boundary maps $d^n_{F[1]} = -d^{n+1}_F$. If $f : A \to B$ is a morphism of chain complexes then we define its **cone** to be the chain complex $\mathrm{cone}(f) \in \mathrm{Ch}^b(X)$ given in degree n by $A^{n+1} \oplus B^n$ with boundary maps

$$\begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix} : A^{n+1} \oplus B^n \longrightarrow A^{n+2} \oplus B^{n+1}$$

2.1.2 Dualising and Residual complexes

We write $D(X)$ for the derived category of $\mathrm{Ch}(X)$ - the additional decorations on this notation we will use can be described as follows; superscripts refer to some boundedness condition on the complexes involved while subscripts enforce conditions on their cohomology groups. For example $D^+(X)$ is the full subcategory of $D(x)$ consisting of those complexes $F \in D(X)$ for which there exists some $N \in \mathbb{Z}$ such that $F^n = 0$ for all $n < N$. We denote by $D^b_{Coh}(X)$ the full subcategory of those $F \in D(X)$ which are bounded chain complexes whose cohomology groups are coherent. For a complex $\omega \in D^+_{Coh}(X)$ of finite injective dimension the derived functor

$$\mathbf{R}\,\mathrm{Hom}_X(-, \omega) : D(X)^{op} \longrightarrow D(X)$$

of $\mathcal{H}om_X(-, \omega)$ interchanges $D^+_{Coh}(X)$ and $D^-_{Coh}(X)$, and takes $D^b_{Coh}(X)$ to itself [6, Ch.II Prop.3.3]. We denote by $\#_\omega$ the restriction

$$\#_\omega = \mathbf{R}\,\mathrm{Hom}_X(-, \omega) : D_{Coh}(X)^{op} \longrightarrow D_{Coh}(X)$$

After taking some quasi-isomorphism $\omega \to I$ of ω into a bounded complex I of injective $\mathcal{O}_X$-modules we may, adopting the signs of **Definition 2.1.4**, define a natural transformation $\eta_\omega : \mathrm{id}_{D_{Coh}(X)} \to (\#_\omega)^2$ by associating to each $F \in D_{Coh}(X)$ the map of chain complexes

$$\eta_\omega(F) : F \longrightarrow \mathbf{R}\,\mathrm{Hom}_X(\mathbf{R}\,\mathrm{Hom}_X(F, \omega), \omega)$$

which is given in degree n by

$$(\eta_\omega(F))^n : F^n \longrightarrow \prod_{s,t \in \mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(F^s, I^{s+t}), I^{t+n})$$

to be $(-1)^{\frac{(n+t)(n+t+1)}{2}}$ times the evaluation morphism if $s = n$ and zero otherwise.

Definition 2.1.6. A complex $\omega \in D^+_{Coh}(X)$ of finite injective dimension is called a **dualising complex** for X if the transformation η_ω described above is a natural isomorphism. We say $\omega \in D^+_{Coh}(Mod_A)$ is a dualising complex for a ring A if it is a dualising complex for $\mathrm{Spec}(A)$.

To establish that a complex $\omega \in D^+_{Coh}(X)$ is dualising it suffices to check that $\eta_\omega(\mathcal{O}_X[0])$ is an isomorphism in $D(X)$ [6, Ch.V Prop. 2.1].

Example 2.1.7.

1. If F is a field, then $F[0]$ is a dualising complex for F.
2. If X is regular, then $\mathcal{O}_X[0]$ is a dualising complex on X, see [6, Ch.V, 2.2].
3. If $f : X \to Y$ is finite type and ω is a dualising complex on Y, then there is a dualising complex $f^!(\omega)$ on X.

The *Ideal Theorem* of the introduction to [6] aspires to define a functor $f^! : D^+_{Coh}(Y) \to D^+_{Coh}(X)$ (sometimes called the **exceptional inverse image**) for any finite type map $f : X \to Y$ between general schemes in a manner behaving well with respect to compositions of morphisms. If f is further assumed to be proper, then $f^!$ should be right adjoint to the pushforward $\mathbf{R}f_* : D^+(X) \to D^+(Y)$. In this case the counit of the adjunction is called the **trace map** and written

$$\mathrm{tr}_f : (\mathbf{R}f_*)f^! \longrightarrow \mathrm{id}_{D^+_{Coh}(Y)}$$

This *Ideal Theorem* is established in [6] for schemes which admit dualising complexes. The definition of $f^!$ in this case is to first construct for each dualising complex ω over Y a dualising complex $f^!(\omega)$ on X, as alluded to in the third item of the above example, and then $f^!$ is given for more general complexes as a twist of the pullback $\mathbf{L}f^* : D^-_{Coh}(Y) \to D^-_{Coh}(X)$ by the dualities $\#_\omega$ and $\#_{f^!(\omega)}$ - explicitly the definition is

$$f^! = \mathbf{R}\operatorname{Hom}_X\left((\mathbf{L}f^*)\mathbf{R}\operatorname{Hom}_Y(-,\omega), f^!(\omega)\right)$$

Given suitable conditions on f the exceptional inverse image often has, and for more general morphisms is built from, easier descriptions (see point (a) of the *Ideal Theorem*). For example we have the following definition [6, Ch.III §2].

Definition 2.1.8. If $f : X \to Y$ is a smooth morphism of relative dimension n, then we define $f^! : D(Y) \to D(X)$ to be

$$f^!(-) = f^*(-) \otimes_X \left(\Omega^{top}_{X/Y}\right)[n]$$

where $\Omega_{X/Y}$ is the module of cotangent vectors of X over Y, and $\Omega^{top}_{X/Y}$ is its highest non-zero exterior power.

Example 2.1.9.

1. For an open embedding $i : U \hookrightarrow X$ we hence have that $i^!$ is the usual restriction i^* to the open subset U.

2. When working in the category Sm_k of smooth schemes over some field k, it becomes natural to use canonical line bundles as dualising complexes since for a smooth n-dimensional variety $f : X \to \mathrm{Spec}(k)$ we have $f^!(k[0]) = \Omega^{top}_{X/k}[n]$. This should essentially give the connection between our Witt complex in the next chapter and that of **Definition 1.2.8**.

The property of being a dualising complex is preserved (and may be checked) by localisation to the points of a scheme.

Lemma 2.1.10. *For a scheme X, we have that a complex $C \in D^+_{Coh}(X)$ of finite injective dimension is a dualising complex for X if and only if for each point $x \in X$ the localisation $C_x \in D^+_{Coh}(\mathrm{Spec}(\mathcal{O}_{X,x}))$ is a dualising complex for the local ring $\mathcal{O}_{X,x}$.*

Proof. This is Corollary 2.3 of Chapter V in [6], which uses results from Chapters I and II to first establish that each C_x is also of finite injective dimension, so that the result then follows by observing that $\eta_C(\mathcal{O}_X[0])$ is an isomorphism if and only if it is locally. $\square$

For closed embeddings, the exceptional inverse image is defined as follows [6, Ch.III §6].

Definition 2.1.11. Let $j : Z \hookrightarrow X$ be a closed embedding. Then we define $j^! : D^+_{Coh}(X) \to D^+_{Coh}(Z)$ to be

$$j^!(-) = \mathbf{R}\,\mathrm{Hom}_X(j_*\mathcal{O}_Z[0], -)$$

In the above definition we are viewing $\mathcal{H}om_X(j_*\mathcal{O}_Z[0], -)$ as a functor $Ch^+_{Coh}(X) \to Ch^+_{Coh}(Z)$ and then deriving it to obtain $j^!$. In this case the pushforward $j_* : Mod_{\mathcal{O}_Z} \to Mod_{\mathcal{O}_X}$ is exact and we have that the composition $j_*j^!$ is given by the usual $\mathbf{R}\,\mathrm{Hom}_X(j_*\mathcal{O}_Z[0], -)$ viewed as the right derived functor of $\mathcal{H}om_X(j_*\mathcal{O}_Z[0], -) : Ch^+_{Coh}(X) \to Ch^+_{Coh}(X)$. Continuing from our signs for the evaluation map of the previous subsection, we hence describe the trace map as below.

Definition 2.1.12. Let $j : Z \hookrightarrow X$ be a closed embedding. Then the **trace map**

$$\mathrm{tr}_j : j_*j^! \longrightarrow \mathrm{id}_{D^+_{Coh}(X)}$$

is defined to be the transformation which on a bounded below complex I of injectives is given in degree d to be $(-1)^{d(d+1)/2}$ times the "evaluation at one" morphism

$$\mathrm{tr}_j(I)^d : \mathcal{H}om_X(j_*\mathcal{O}_Z[0], I)^d = \mathcal{H}om_{\mathcal{O}_X}(j_*\mathcal{O}_Z, I^d) \xrightarrow{\psi \mapsto (-1)^{d(d+1)/2}\psi(1)} I^d$$

Precisely, over an open set U we send the sheaf morphism $\psi : j_*\mathcal{O}_Z|_U \to I^d|_U$ to $(-1)^{d(d+1)/2}\psi(U)(1)$.

So for a point $z \in X$ whose closure is the subvariety $j : Z \hookrightarrow X$, a combination of $j^!$ and localisation to z allows us to extract from a global dualising complex ω on X a dualising complex over the point $\mathrm{Spec}(\kappa(z))$.

Theorem 2.1.13. *Let X be a connected scheme and $\omega \in D^+_{Coh}(X)$ be a dualising complex for X and $C \in D^+_{Coh}(X)$ be any other complex. Then C is dualising if and only if there exists an invertible sheaf $\mathcal{L}$ and integer $n \in \mathbb{Z}$ such that $C \cong \omega \otimes_X \mathcal{L}[n]$ in $D(X)$. Further the integer n is uniquely determined by the isomorphism classes of ω and C - precisely we have $\mathcal{L}[n] \cong \mathbf{R}\,\mathrm{Hom}_X(\omega, C)$.*

Proof. This is Theorem 3.1 in [6, Ch.V]. □

Since $\kappa(z)[0] \in D^+_{Coh}(\mathrm{Spec}(\kappa(z)))$ is also a dualising complex over $\kappa(z)$, if we denote by $\pi_{z,X} : \mathrm{Spec}(\kappa(z)) \to \mathrm{Spec}(\mathcal{O}_{X,z})$ the embedding of the closed point, as is standard in our notation, then by this uniqueness theorem there must exist some integer d such that

$$\pi^!_{z,X}(\omega_z) = \mathbf{R}\,\mathrm{Hom}_{\mathcal{O}_{X,z}}(\kappa(z), \omega_z) \cong \kappa(z)[d]$$

in $D(\mathrm{Spec}(\kappa(z)))$, or in other words $\mathrm{Ext}^i_{\mathcal{O}_{X,z}}(\kappa(z), \omega_z)$ is a one-dimensional $\kappa(z)$-vector space if $i = -d$ and zero otherwise. In fact, this property characterises dualising complexes over local rings [6, Ch.V Prop.3.4].

Proposition 2.1.14. *Let (A, m, κ) be a local ring and $C \in D^+_{Coh}(\mathrm{Spec}(A))$. Then C is a dualising complex for A if and only if there is an integer d such that*

$$\mathrm{Ext}^i_A(\kappa, C) \cong \begin{cases} \kappa & \textit{if } i = d \\ 0 & \textit{otherwise} \end{cases}$$

Definition 2.1.15. If ω is a dualising complex on X then we define a map $\mu_\omega : X \to \mathbb{Z}$ on the points of X by sending each $x \in X$ to the unique integer d such that $\mathrm{Ext}^d_{\mathcal{O}_{X,x}}(\kappa(x), \omega_x)$ is non-zero. For a subvariety $Z \hookrightarrow X$ we will also write $\mu_\omega(Z)$ for $\mu_\omega(\zeta_Z)$.

Up to notation, for a subvariety $Z \hookrightarrow X$ the coefficients of the cycle $[Z]$ in our Chow-Witt group $\widetilde{CH}^p(X, \omega)$ are bilinear forms defined on finite dimensional $\kappa(\zeta_Z)$-vector spaces taking values in the one-dimensional $\kappa(\zeta_Z)$-vector space $\mathrm{Ext}^{\mu_\omega(Z)}_{\mathcal{O}_{X,Z}}(\kappa(\zeta_Z), \omega_{\zeta_Z})$. The following proposition [6, Ch.V Prop.7.1] justifies calling μ_ω the **codimension function** of the dualising complex.

Proposition 2.1.16. *Let x, y be points of a scheme X equipped with dualising complex ω, and suppose that x is an immediate specialisation of y. Then*

$$\mu_\omega(x) = \mu_\omega(y) + 1$$

Since dualising complexes have finite injective dimension the codimension function of any dualising complex takes only finitely many values, so from the above proposition we learn that a scheme which admits a dualising complex must be finite dimensional and caternary. In [6, Ch.V §7] the notion of a pointwise dualising complex is defined for schemes with possibly infinite dimension - though again a codimension function defined in the same way forces schemes with pointwise dualising complexes to be caternary. Note that on each connected component of a scheme the codimension function of a dualising complex differs from the usual codimension function by some integer.

Definition 2.1.17. A dualising complex ω on a scheme X is called **normalised** if $\mu_\omega(x) = \mathrm{codim}(x, X)$ for every point $x \in X$.

This terminology differs from that of [6] in which a dualising complex over a local ring is called normalised when its codimension function is zero at the closed point. The codimension function of a dualising complex ω affords some structural analysis on injective resolutions of ω. In the proposition below, we give the result of taking such a resolution, and trimming it down to a minimal form.

Definition 2.1.18. For a scheme X and point $x \in X$ we define a functor $\Gamma_x : Mod_{\mathcal{O}_X} \to Mod_{\mathcal{O}_{X,x}}$ by sending $F \in Mod_{\mathcal{O}_X}$ to the $\mathcal{O}_{X,x}$-submodule of F_x consisting of those germs whose support is contained in the closed point $\{x\}$ of $\mathrm{Spec}(\mathcal{O}_{X,x})$. This functor is left exact, and we write $H^i_x(-) : Mod_{\mathcal{O}_X} \to Mod_{\mathcal{O}_{X,x}}$ for the cohomology groups of its derived functor $\mathbf{R}\Gamma_x : D^+(X) \to D^+(Mod_{\mathcal{O}_{X,x}})$.

We are using below the notation of **Definition 1.1.13**.

Proposition 2.1.19. *Let ω be a dualising complex on X with codimension function μ. Then there is a bounded chain complex $E(\omega)$ of quasicohernt injective $\mathcal{O}_X$-modules*

$$E(\omega) = \dots \to \bigoplus_{\mu(x)=p} \iota_x(H^p_x(\omega)) \longrightarrow \bigoplus_{\mu(x)=p+1} \iota_x(H^{p+1}_x(\omega)) \to \dots$$

which is isomorphic to ω in $D(X)$.

Proof. This is a combination of several results from Chapters IV and V of [6]. Since it is ultimately the boundary maps of the complex $E(\omega)$ and local dualities arising from $H^p_x(\omega)$ which define for example our *generalised second residue homomorphism* for Witt groups, we'll go into a little more detail.

Let $Z^p = \{x \in X | \mu(x) \geq p\}$, so that if e is the minimum value of μ we have a finite filtration

$$\emptyset \subseteq Z^{e+d} \subseteq ... \subseteq Z^{e+1} \subseteq Z^e = X$$

of the points of X, where $d = \dim(X)$. For an $\mathcal{O}_X$-module F, we write [6, Ch.III Var.3] $\underline{\Gamma}_{Z^p}(F)$ for the $\mathcal{O}_X$-module which over an open set U consists of those sections of $F(U)$ whose support is contained in some finite union of closures of points in Z^p; i.e. the sections whose support is of codimension at least p according to μ. Then we have inclusions $\underline{\Gamma}_{Z^{p+1}}(F) \hookrightarrow \underline{\Gamma}_{Z^p}(F)$ and so can write

$$\underline{\Gamma}_{Z^p/Z^{p+1}}(F) := \underline{\Gamma}_{Z^p}(F)/\underline{\Gamma}_{Z^{p+1}}(F) \in Mod_{\mathcal{O}_X}$$

As in [6, Ch.III], the cohomology of the derived functor of $\underline{\Gamma}_{Z^p/Z^{p+1}}$ is denoted $\underline{H}^i_{Z^p/Z^{p+1}}$. Now if $\omega \to I$ is an isomorphism in $D(X)$ with I a bounded chain complex of injectives, then we have a filtration

$$I = \underline{\Gamma}_{Z^e}(I) \supseteq \underline{\Gamma}_{Z^{e+1}}(I) \supseteq ... \supseteq \underline{\Gamma}_{Z^{e+d}}(I) \supseteq 0$$

of the chain complex I. To such a filtration, one may associate a spectral sequence [35, 5.4], which appears as *Motif G* in [6, Ch.III]. The zeroth page has

$$E_0^{p,q} = \frac{\underline{\Gamma}_{Z^p}(I^{p+q})}{\underline{\Gamma}_{Z^{p+1}}(I^{p+q})} = \underline{\Gamma}_{Z^p/Z^{p+1}}(I^{p+q}) \xrightarrow{d_0^{p,q}} \underline{\Gamma}_{Z^p/Z^{p+1}}(I^{p+q+1})$$

with the boundary maps $d_0^{p,q}$ being d_I^{p+q} restricted to the subquotient $\underline{\Gamma}_{Z^p/Z^{p+1}}(I^{p+q})$. So on the first page we have

$$E_1^{p,q} = \underline{H}^{p+q}_{Z^p/Z^{p+1}}(\omega) \xrightarrow{d_1^{p,q}} \underline{H}^{p+q+1}_{Z^{p+1}/Z^{p+2}}(\omega)$$

where the boundary map $d_1^{p,q}$ is again d_I^{p+q} restricted to a subquotient of I^{p+q}. The zeroth row of this page is called the **Cousin complex** of ω with respect to the filtration $Z^\bullet$ [6, Ch.III §3]. We write $E(\omega)$ for this complex

$$E(\omega) = ... \to \underline{H}^p_{Z^p/Z^{p+1}}(\omega) \xrightarrow{d_1^{p,0}} \underline{H}^{p+1}_{Z^{p+1}/Z^{p+2}}(\omega) \to ...$$

Since the terms of $E(\omega)$ may be realised as subquotients of terms of an injective resolution I of ω we have that $E(\omega)$ is a bounded complex. Proposition 7.3 of [6, Ch.V] says in part that $E(\omega)$ is a complex of injectives, isomorphic in $D(X)$ to $E(\omega)$. Finally *Motif F* of [6, Ch.III] gives canonical functorial isomorphisms

$$\underline{H}^p_{Z^p/Z^{p+1}}(\omega) \xrightarrow{\simeq} \bigoplus_{\mu(x)=p} \iota_x(H^p_x(\omega))$$

□

The resolution $E(\omega)$ is also called a **minimal dualising complex** in the language of [2]. Because of this minimality, the components $H^i_x(\omega)$ of its terms manage to retain some duality properties of ω - we will examine these at the end of this chapter.

Definition 2.1.20. For a ring A and A-module M an A-**injective hull** of M is an injective A-module I for which there exists an embedding $i : M \hookrightarrow I$ which is **essential**

- meaning that for any non-zero A-submodule $N \leq I$ we have $i(M) \cap N \neq 0$. We denote A-injective hulls of M by $E_A(M)$.

While any A-module admits injective hulls, they are only unique up to non-canonical isomorphism - hence the notation $E_A(M)$ is descriptive rather than definitive.

Lemma 2.1.21. *Let A be a Noetherian ring. Then*

1. *For any $p \in \operatorname{Spec}(A)$ the annhilator of any non-zero element $e \in E_A(A/p)$ is a p-primary ideal in A.*

2. *For any $p \in \operatorname{Spec}(A)$ any element $f \in A \backslash p$ acts on $E_A(A/p)$ as an automorphism; in other words $E_A(A/p)$ is already an A_p-module.*

3. *If $\iota : M \to E_A(M)$ is an injective hull of M and $S \subseteq A$ a multiplicatively closed subset in A then $S^{-1}\iota : S^{-1}M \to S^{-1}E_A(M)$ is an $S^{-1}A$-injective hull of $S^{-1}M$.*

Proof. For the first two statements see [36, Lem.3.2], the third is [37, Lem.3.2.5]. $\square$

It is remarked [6, Ch.IV §3] that for complexes F which are isomorphic in $D(X)$ to their cousin complex $E(F)$, one does not in general have a unique or functorial choice for these isomorphisms $F \cong E(F)$. If F is a dualising complex however, then $E(F)$ is a *residual complex* - for these complexes a functorial choice is possible.

Definition 2.1.22. A **codimension function** on a scheme X is a set map $\mu : X \to \mathbb{Z}$ for which we have

$$\mu(y) = \mu(x) + 1$$

whenever y an immediate specialisation of x. A **residual complex** $\mathcal{R}$ on X is a bounded complex of quasi-coherent injective $\mathcal{O}_X$-modules having coherent cohomology, together with a codimension function μ such that whenever $\mu(x) = p$ we have that $\mathcal{R}^p_x$ is an $\mathcal{O}_{X,x}$-injective hull of $\kappa(x)$. We further require that for each $p \in \mathbb{Z}$ the map

$$\mathcal{R}^p \longrightarrow \bigoplus_{\mu(x)=p} \mathcal{R}(x)$$

induced by localisation, where each $\mathcal{R}(x)$ denotes the quasi-coherent $\mathcal{O}_X$-module $\iota_x(\mathcal{R}^p_x)$, is an isomorphism. We write $\operatorname{Res}(X)$ for the subcategory of residual complexes in $\operatorname{Ch}(X)$.

Definition 2.1.23. Let $(X, \mathcal{R})$ be a scheme with residual complex, and $x \in X$ be a point with $\mu(x) = p$. Then we write $\rho(x, X) : \mathcal{R}(x) \hookrightarrow \mathcal{R}^p$ for the embedding of $\mathcal{O}_X$-modules which is the inclusion $\mathcal{R}(x) \hookrightarrow \bigoplus_{\mu(y)=p} \mathcal{R}(y)$ followed by the inverse to the localisation isomorphism $\mathcal{R}^p \to \bigoplus_{\mu(y)=p} \mathcal{R}(y)$.

Example 2.1.24. (Taken from [38]) Let D be a Dedekind domain with field of fractions F. Then the complex

$$\cdots \longrightarrow 0 \longrightarrow F \twoheadrightarrow \bigoplus_{m \in \max(D)} F/D_m \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees 0 and 1 is a normalised residual complex for D.

It is clear that any translate of a residual complex is again a residual complex but with a different codimension function. As in **Definition 2.1.17**, when the codimension function μ of a residual complex $\mathcal{R}$ agrees with the actual codimension function - that is we have for each point $x \in X$ that $\mu(x)$ is the Krull dimension of $\mathcal{O}_{X,x}$ - we call $\mathcal{R}$ a **normalised** residual complex.

Definition 2.1.25. Let $\mu : X \to \mathbb{Z}$ be a codimension function. Then we write X^p_μ for the set of points $x \in X$ with $\mu(x) = p$. We describe a subvariety $Z \hookrightarrow X$ as being of μ-codimension p when its generic point lies in X^p_μ.

In our finite dimensional setting the relationship between residual and dualising complexes set out in [39, Lem.3.2.1] is as follows.

Proposition 2.1.26. *Let $\mu : X \to \mathbb{Z}$ be a codimension function. Denote by $Q : \mathrm{Ch}(X) \to D(X)$ the localisation functor. Then*

1. *If $\mathcal{R}$ is a residual complex on X then $Q(\mathcal{R})$ is a dualising complex on X.*
2. *If ω is a dualising complex on X then $E(\omega)$ is a residual complex on X.*

In the situation where X admits a dualising complex, we obtain an equivalence of categories

$$\begin{tikzcd} \mathrm{Dual}_\mu(X) \arrow[r, bend left, "E"] & \mathrm{Res}_\mu(X) \arrow[l, bend left, "Q"] \end{tikzcd}$$

where $\mathrm{Dual}_\mu(X)$ denotes the subcategory of dualising complexes in $D(X)$ whose associated codimension function is μ, while $\mathrm{Res}_\mu(X)$ denotes the subcategory of residual complexes in $\mathrm{Ch}(X)$ again having codimension function μ.

Since the maps in $\mathrm{Res}(X)$ are ordinary morphisms of chain complexes, global residual complexes may be obtained by glueing together a collection of residual complexes given locally on an open cover in the usual way. Hence by the above equivalence the same is true for dualising complexes in the derived category. Knowing for a finite type map $f : X \to Y$ that $f^!$ carries $\mathrm{Dual}(Y)$ into $\mathrm{Dual}(X)$ one obtains, in the notation of [6], a functor $f^\triangle : \mathrm{Res}(Y) \to \mathrm{Res}(X)$ via $f^\triangle = Ef^!Q$. If the reader's viewpoint is that $f^!$ is obtained via the constructive approach of *loc. cit.* then this is a bit backwards - $f^!$ is defined for dualising complexes by building $f^\triangle$ first; though again, even locally on X this requires in general some application of an injective resolution functor E. The point we draw the reader's attention to now is that in the situation of **Definition 2.1.11** -

that is when f is a finite map - one has that for $\mathcal{R} \in \mathrm{Res}(Y)$ that $f^!(\mathcal{R})$ defined as in **Proposition 2.1.28** naturally appears as a residual complex without any *need* to take the injective resolution $E(f^!(\mathcal{R}))$.

Proposition 2.1.27. *Let $f : X \to Y$ be a finite morphism and $\mathcal{R}$ a residual complex on Y with codimension function μ_Y. Then we define $f^\triangle(\mathcal{R})$ by setting for each affine open $U = \mathrm{Spec}(B) \subseteq Y$ with $f^{-1}(U) = \mathrm{Spec}(A)$ the restriction of $f^\triangle(\mathcal{R})$ to $f^{-1}(U)$ to be the chain complex of A-modules*

$$f^\triangle(\mathcal{R})_A = \mathcal{H}om_B(A[0], \mathcal{R}_B) = [A[0], \mathcal{R}_B]_U$$

with the obvious A-module structure. Then $f^\triangle(\mathcal{R}) \in \mathrm{Res}(X)$ and the codimension functions satisfy $\mu_X(x) = \mu_Y(f(x))$.

We then have $f_* f^\triangle(\mathcal{R}) \simeq [f_*\mathcal{O}_X, \mathcal{R}]_Y$ so that we obtain the trace map with signs as in **Definition 2.1.12** which we reiterate below - this is also a building block of the construction for the general trace map for residual complexes [6, Ch. VI, §4].

Definition 2.1.28. For a finite morphism $f : X \to Y$ and residual complex $\mathcal{R} \in \mathrm{Res}(Y)$, we write

$$\mathrm{tr}_f(\mathcal{R}) : f_* f^\triangle(\mathcal{R}) : f_* f^\triangle(\mathcal{R}) \longrightarrow \mathcal{R}$$

for the morphism of chain complexes of $\mathcal{O}_Y$-modules given over an open set U in degree d by

$$\mathrm{tr}_f(\mathcal{R})^d : \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{R}^d) \xrightarrow{\psi \mapsto (-1)^{\frac{d(d+1)}{2}} \psi_U(1)} \mathcal{R}^d$$

Proposition 2.1.29. *Let $i : U \to X$ be the localisation to a point, inclusion of an open subset, or localisation map $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ for some multiplicatively closed subset S of A. Then for any residual complex $\mathcal{R}$ on X with codimension function μ_X we have that $i^*(\mathcal{R})$ is a residual complex on U with codimension function $\mu(u) = \mu_X(i(u))$.*

Proof. These results follow immediately from **Lemma 2.1.21**. □

Suppose that $\mathcal{R}$ is a (for simplicity normalised) residual complex on X. Then for each point $x \in X^p$ we have a residual complex

$$0 \to \mathcal{R}_x^0 \longrightarrow \mathcal{R}_x^1 \longrightarrow \dots \longrightarrow \mathcal{R}_x^p \to 0$$

over the local ring $\mathcal{O}_{X,x}$. Each term decomposes

$$\mathcal{R}_x^i \cong \bigoplus_{p \in \mathrm{Spec}(\mathcal{O}_{X,x})^i} E_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/p)$$

so on account of **Proposition 2.1.27** we have that

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\kappa(x), \mathcal{R}_x^i\right) \cong \begin{cases} 0 & \text{if } i \neq p \\ \kappa(x) & \text{if } i = p \end{cases}$$

Recalling our standard notation $\pi_{x,X} : \mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ for the embedding of the closed point, then we read from the above that $\pi^{\triangle}_{X,x}(\mathcal{R}_x)$ is a complex concentrated in degree p - and further that the degree p term is isomorphic to $\kappa(x)$.

Definition 2.1.30. If $\mathcal{R}$ is a residual complex on a scheme X and $x \in X^p_\mu$, then we write $x^\natural(\mathcal{R})$ for the one-dimensional $\kappa(x)$-vector space

$$x^\natural(\mathcal{R}) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \mathcal{R}^p_x)$$

The embedding

$$x^\natural(\mathcal{R}) \hookrightarrow \mathcal{R}^p_x$$

given by $(-1)^{p(p+1)/2}$ times the evaluation at one morphism is a particular example of the trace map of Grothendieck duality - c.f. **Definition 2.1.28**.

Lemma 2.1.31. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of closed embeddings and $\mathcal{R} \in \mathrm{Res}(Z)$. Then we have a canonical isomorphism*

$$c_{f,g} : f^\triangle g^\triangle(\mathcal{R}) \xrightarrow{\simeq} (fg)^\triangle(\mathcal{R})$$

fitting into a commutative diagram

$$\begin{tikzcd}
g_* f_* f^\triangle g^\triangle(\mathcal{R}) \arrow[r, "g_* \mathrm{tr}_f(g^\triangle(\mathcal{R}))"] \arrow[d, "(gf)_* c_{f,g}"'] & g_* g^\triangle(\mathcal{R}) \arrow[d, "\mathrm{tr}_g(\mathcal{R})"] \\
(gf)_*(gf)^\triangle(\mathcal{R}) \arrow[r, "\mathrm{tr}_{gf}(\mathcal{R})"'] & \mathcal{R}
\end{tikzcd}$$

Proof. That there is an isomorphism between $f^\triangle g^\triangle(\mathcal{R})$ and $(gf)^\triangle(\mathcal{R})$ is clear - we only mention that there are some signs forced upon it by those of the trace maps. Supposing that X, Y and Z are the spectra of rings A, B and C respectively, we set

$$c_{f,g} : \mathrm{Hom}_B\big(A, \mathrm{Hom}_C(B, \mathcal{R})\big) \longrightarrow \mathrm{Hom}_C(A, \mathcal{R})$$

to be the chain map given in degree p by

$$c^p_{f,g}(\psi)(a) = (-1)^{p(p+1)/2}\psi(a)(a_B)$$

for each $a \in A$ and $\psi \in \mathrm{Hom}_B\big(A, \mathrm{Hom}_C(B, \mathcal{R})\big)$. $\square$

2.2 Grothendieck-Witt groups

We define in this section Grothendieck-Witt groups in two settings - firstly for exact categories with weak equivalences and secondly we define the Witt groups of triangulated categories. Both situations can be thought of as generalisations of that of simply exact categories with duality; but the latter requires some assumption that 2 be invertible. In the triangular setting we only define Witt groups, though the Grothendieck-Witt group can still be constructed in this setting - see [40].

2.2.1 Of exact categories with weak equivalences

Our main references for the Grothendieck-Witt groups of this section are [30, 41]. The latter we refer to for some properties of the Grothendieck-Witt groups of exact categories, while we rely on the former to make the appropriate definition of these groups given the structure below.

Definition 2.2.1. An **exact category with weak equivalences** $(\mathcal{E}, \omega)$ consists of an exact category $\mathcal{E}$ together with a collection ω of maps in $\mathcal{E}$ such that the following properties hold.

1. The collection ω contains all identity morphisms and is closed under composition and isomorphism.
2. If the composition $A \xrightarrow{s} B \xrightarrow{r} A$ is the identity and $s \in \omega$ then $r \in \omega$ also.
3. If

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, tail] & B \arrow[d, tail] \\ A' \arrow[r, "q'"'] & B' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

   is a pushout square with $q \in \omega$, then $q' \in \omega$ also. The dual statement - that weak equivalences are closed under pullback along admissible epimorphisms is also true.
4. If f, g are composable morphisms in $\mathcal{E}$ and any two of $\{f, g, fg\}$ are in ω then so is the third.

We use the arrow $\xrightarrow{\sim}$ to denote weak equivalences in diagrams.

Example 2.2.2. Any exact category $\mathcal{E}$ may be viewed as an exact category with weak equivalences by taking the weak equivalences to be the isomorphisms in $\mathcal{E}$. If X is any scheme then the category $\mathrm{Ch}^b_{Coh}(X)$ is an exact category by virtue of its embedding in $\mathrm{Ch}(X)$, in which the quasi-isomorphisms (chain maps inducing isomorphisms on cohomology) form a collection of weak equivalences.

Definition 2.2.3. If $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ are two exact categories with weak equivalences then an additive functor $F : \mathcal{E} \to \mathcal{E}'$ is called **exact** when it takes exact sequences to exact sequences and weak equivalences to weak equivalences.

Definition 2.2.4. A **duality** on an exact category with weak equivalences $(\mathcal{E}, \omega)$ is a pair $(*, \eta)$ where $* : \mathcal{E} \to \mathcal{E}$ is an exact contravariant functor and $\eta : \mathrm{id} \to **$ a natural weak equivalence. The quadruple $(\mathcal{E}, \omega, *, \eta)$ is called an **exact category with weak equivalences and duality**. When ω is the isomorphisms in $\mathcal{E}$ we omit it from the quadruple and call $(\mathcal{E}, *, \eta)$ an **exact category with duality**.

Unless stated otherwise, from now on any quadruple with notation in similar spirit to $(\mathcal{E}, \omega, *, \eta)$ is an exact category with weak equivalences and duality as above.

Definition 2.2.5. A **symmetric space** in $(\mathcal{E}, \omega, *, \eta)$ is a pair (P, ϕ) where P is an object of $\mathcal{E}$ and $\phi : P \xrightarrow{\sim} P^*$ a weak equivalence satisfying $\phi^* \eta_P = \phi$. We say two symmetric spaces (P_1, ϕ_1) and (P_2, ϕ_2) are **isometric** and write $(P_1, \phi_1) \simeq (P_2, \phi_2)$ when there is a weak equivalence $f : P_1 \xrightarrow{\sim} P_2$ for which $f^* \phi_2 f = \phi_1$. The map f is called an **isometry** between (P_1, ϕ_1) and (P_2, ϕ_2).

Definition 2.2.6. We define the **Grothendieck-Witt group** of $(\mathcal{E}, \omega, *, \eta)$, written $GW(\mathcal{E}, \omega, *, \eta)$, to be the free abelian group on the isometry classes $[P, \phi]$ of symmetric spaces in $(\mathcal{E}, \omega, *, \eta)$ modulo the relations

1. For any pair of symmetric spaces (P_i, ϕ_i), we have $[P_1, \phi_1] + [P_2, \phi_2] = [(P_1, \phi_1) \perp (P_2, \phi_2)]$.

2. For each admissible exact sequence $P_{-1} \overset{\alpha}{\rightarrowtail} P_0 \overset{\beta}{\twoheadrightarrow} P_1$, weak equivalences ϕ_{-1}, ϕ_0 and ϕ_1 fitting into a commutative diagram

$$\begin{tikzcd} P_{-1} \arrow[r, tail, "\alpha"] \arrow[d, "\wr"', "\phi_{-1}"] & P_0 \arrow[r, two heads, "\beta"] \arrow[d, "\wr"', "\phi_0"] & P_1 \arrow[d, "\wr"', "\phi_1"] \\ P_1^* \arrow[r, tail, "\beta^*"'] & P_0^* \arrow[r, two heads, "\alpha^*"'] & P_{-1}^* \end{tikzcd}$$

   with (P_0, ϕ_0) a symmetric space, $\phi_{-1}^* \eta_{P_1} = \phi_1$ and $\phi_1^* \eta_{P_{-1}} = \phi_{-1}$, we have

$$[P_0, \phi_0] = \left[P_{-1} \oplus P_1, \begin{pmatrix} 0 & \phi_1 \\ \phi_{-1} & 0 \end{pmatrix} \right]$$

The definition of the **Witt group** $W(\mathcal{E}, \omega, *, \eta)$ is the same, except that in relation (2.) one sets

$$[P_0, \phi_0] = \left[P_{-1} \oplus P_1, \begin{pmatrix} 0 & \phi_1 \\ \phi_{-1} & 0 \end{pmatrix} \right] = 0$$

Note that there is a quotient map $GW(\mathcal{E}, \omega, *, \eta) \twoheadrightarrow W(\mathcal{E}, \omega, *, \eta)$. It is established in [41] that the below is enough structure with which to establish the functorial nature for all higher Grothendieck-Witt groups.

Definition 2.2.7. An **exact form functor**

$$(F, \psi) : (\mathcal{A}, \omega, *, \alpha) \longrightarrow (\mathcal{B}, \omega, *, \beta)$$

consists of an exact functor $F : (\mathcal{A}, \omega) \to (\mathcal{B}, \omega)$ together with a **duality compatibility morphism** $\psi : F* \to *F$ such that for any $A \in \mathcal{A}$ the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\beta_{F(A)}"] \arrow[d, "F(\alpha_A)"'] & F(A)^{**} \arrow[d, "\psi_A^*"] \\ F(A^{**}) \arrow[r, "\psi_{A^*}"'] & F(A^*)^* \end{tikzcd}$$

commutes. The exact form functor is further called **non-singular** when ψ is a natural weak equivalence.

Proposition 2.2.8. *Any non-singular exact form functor*

$$(F, \psi) : (\mathcal{A}, \omega, *, \alpha) \longrightarrow (\mathcal{B}, \omega, *, \beta)$$

between exact categories with weak equivalences and duality induces group homomorphisms

$$(F, \psi)_* : GW(\mathcal{A}, \omega, *, \alpha) \longrightarrow GW(\mathcal{B}, \omega, *, \beta)$$

$$(F, \psi)_* : W(\mathcal{A}, \omega, *, \alpha) \longrightarrow W(\mathcal{B}, \omega, *, \beta)$$

defined by setting

$$(F, \psi)_* ([P, \phi]) = [P, \psi_P F(\phi)]$$

Most of our Grothendieck-Witt groups will be those of exact categories with duality - i.e. the collection of weak equivalences will be simply the isomorphisms. In particular, we adopt the following shorthand.

Notation 2.2.9. Let A be a ring and $\mathcal{M} \hookrightarrow Mod_A$ some subcategory of A-modules exact by virtue of its embedding into Mod_A. If $E \in Mod_A$ is such that the functor $\mathrm{Hom}_A(-, E) : \mathcal{M} \to \mathcal{M}$ is well defined and the evaluation functor $ev : \mathrm{id} \to \mathrm{Hom}_A(\mathrm{Hom}_A(-, E), E)$ a natural isomorphism, then we write simply $(\mathcal{M}, E)$ for the resulting exact category with duality.

Further, in this situation, symmetric spaces $\phi : P \xrightarrow{\cong} \mathrm{Hom}_A(P, E)$ may be identified with non-degenerate symmetric bilinear maps $P \times P \to E$. For a symmetric space (P, ψ) we write $\langle \cdot, \cdot \rangle_\psi$ for the bilinear form associated.

When working in an exact category with duality, we make the following repetition of **Definition 1.2.2**.

Definition 2.2.10. Let (P, ϕ) be a symmetric space in $(\mathcal{E}, *, \eta)$, and $\alpha : L \rightarrowtail P$ be an admissible monomorphism. Then we define the **orthogonal** to L in P to be the admissible monomorphism $L^{\perp} = \ker(\alpha^* \phi) \rightarrowtail P$. Precisely, let $L \xrightarrow{\alpha} P \xrightarrow{\beta} P/L$ be an exact sequence in $\mathcal{E}$. Then $\phi^{-1}\beta^* : (P/L)^* \to P$ is a kernel of $\alpha^*\phi$.

Definition 2.2.11. Let (P, ϕ) be a symmetric space in $(\mathcal{E}, *, \eta)$. Then an admissible monomorphism $\alpha : L \rightarrowtail P$ is called a **sublagrangian** of (P, ϕ) when $\alpha^*\phi\alpha = 0$ and the induced map $L \to L^{\perp}$ is an admissible monomorphism. We call α a **Lagrangian** of (P, ϕ) when this induced monomorphism $L \rightarrowtail L^{\perp}$ is an isomorphism. We say that (P, ϕ) is **metabolic** when it has a Lagrangian.

In an exact category with weak equivalences and duality $(\mathcal{E}, \omega, *, \eta)$ one can recover symmetric spaces from objects of $\mathcal{E}$. The symmetric spaces obtained in this way are called hyperbolic.

Definition 2.2.12. Let P be an object of $\mathcal{E}$. Then the **hyperbolic** space associated to P, denoted $\mathcal{H}(P)$, is the symmetric space with underlying object $P \oplus P^*$ and form given by

$$P \oplus P^* \xrightarrow{\begin{pmatrix} 0 & id \\ \eta_P & 0 \end{pmatrix}} P^* \oplus P^{**}$$

Note that in an exact category with duality any hyperbolic space $\mathcal{H}(P)$ is metabolic with a Lagrangian given by the first component $P \rightarrowtail P \oplus P^*$. Further, a space (P, ϕ) is metabolic if and only if the sequence

$$0 \to L \xrightarrow{\alpha} P \xrightarrow{\alpha^*\phi} L^* \to 0$$

is exact in $\mathcal{E}$. These exact sequences play the role of the arbitrary exact sequences of **Definition 1.1.11** in the definition of the Grothendieck-Witt group of an exact category with duality. Indeed, the defining relation of $GW(\mathcal{E}, *, \eta)$ in **Definition 2.2.6** reads that for any metabolic space (P, ϕ) as above we have

$$[P, \phi] = \mathcal{H}(L) = \left[L \oplus L^*, \begin{pmatrix} 0 & id \\ \eta_P & 0 \end{pmatrix} \right]$$

which on underlying spaces reads as the defining relation of $K_0(\mathcal{E})$. To be clear we have the following description of the Witt and Grothendieck-Witt groups of an exact category with duality

Lemma 2.2.13. *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Then its Grothendieck-Witt group can be identified with the Grothendieck group of the abelian monoid of isometry classes $[P, \psi]$ of symmetric spaces $(P, \psi) \in Sym(\mathcal{E}, *, \eta)$ modulo the subgroup of* *elements $[M,\phi] - [\mathcal{H}(L)]$ for each metabolic space (M,ϕ) with Lagrangian $L \rightarrowtail M$. Similarly, its Witt group is the quotient of $GW(\mathcal{E},*,\eta)$ by the subgroup generated by the hyperbolic spaces.*

The above redefinition is essentially that of [42, Ch.I §4], which establishes that if $\mathcal{E}$ is the category of finitely generated modules over some ring, then the subgroup cut out by the defining relations $[M,\phi] - [\mathcal{H}(L)]$ is zero. We hence obtain in this case agreement with **Definition 1.2.2** and **Definition 1.2.11**. The following classical result is part of [41, Lem.2.8], for example.

Proposition 2.2.14. *(Sublagrangian reduction) Let $(\mathcal{E},*,\eta)$ be an exact category with duality, $(P,\psi) \in Sym(\mathcal{E},*,\eta)$ with $L \rightarrowtail P$ a sublagrangian. Let $q : L^\perp \twoheadrightarrow L^\perp/L$ denote the quotient map. Then there is a unique form*

$$\overline{\psi} : L^\perp/L \xrightarrow{\cong} \left(L^\perp/L\right)^*$$

with $\psi\big|_{L^\perp} = q^*\overline{\psi}q$. Further, in $GW(\mathcal{E},*,\eta)$ we have the equation

$$[P,\psi] = [L^\perp/L, \overline{\psi}\,] + [\mathcal{H}(L)]$$

We will make such frequent use of the following non-singular exact form functor that it is worth highlighting and giving special notation to now. Let $f : A \to B$ be a ring map, and let $\mathcal{M}_A \subseteq Mod_A$ and $\mathcal{M}_B \subseteq Mod_B$ be a pair of full exact subcategories such that the forgetful functor induced by f restricts to a forgetful functor $F : \mathcal{M}_B \to \mathcal{M}_A$. Suppose further that objects $I \in Mod_A$ and $J \in Mod_B$ give exact categories with duality $(\mathcal{M}_A, I)$ and $(\mathcal{M}_B, J)$ and that we have an A-module map $s : J \to I$ which is such that the induced postcomposition with s map $s_* : \mathrm{Hom}_B(M,J) \to \mathrm{Hom}_A(M,I)$ is an isomorphism of A-modules for any $M \in \mathcal{M}_B$. Then the pair (F, s_*) is a non-singular exact form functor

$$(F, s_*) : (\mathcal{M}_B, J) \longrightarrow (\mathcal{M}_A, I)$$

so that we hence obtain the following **transfer maps** - sometimes referred to as **Scharlau transfers** in the literature.

Definition 2.2.15. In the situation just described, we denote by

$$s_* : GW(\mathcal{M}_B, J) \to GW(\mathcal{M}_A, I)$$
$$s_* : W(\mathcal{M}_B, J) \to W(\mathcal{M}_A, I)$$

the group homomorphisms induced by the non-singular exact form functor (F, s_*).

Note that our notation doesn't distinguish between the transfer for Grothendieck-Witt groups and that for Witt groups - we hope it will be clear what is meant from context. A further abuse of notation is that the symbol s_* not only denotes both the group homomorphisms, but also the duality compatibility map $\mathrm{Hom}_B(M,J) \to \mathrm{Hom}_A(M,I)$. So we have for a symmetric space (P,ψ) in $(\mathcal{M}_B, J)$

$$s_*([P,\psi]) = [P, s_*\psi]$$

for both the Witt and Grothendieck-Witt groups.

2.2.2 Overview of derived Witt groups

The first published construction of Witt complexes for some class of schemes was made by Balmer & Walter in [1] for regular schemes, and their argument was later repeated to cover the singular case [2]. The strategy follows that of Quillen's construction of a Gersten sequence for K-theory; localisation sequences are developed for short exact sequences of suitable categories and then repeatedly applied to a coniveau filtration of an appropriate category. The categories in question are the bounded derived categories of vector bundles in the regular case and of coherent sheaves in the singular case. A generalisation of Witt groups to such triangular framework, including the localisation sequence, is constructed in [5]. To define this localisation sequence Balmer requires that the triangulated categories in question have uniquely 2-divisible hom-groups. In the context of a derived category of a scheme X, the resulting restriction is that $1/2 \in \Gamma(X, \mathcal{O}_X)$. We begin our overview by recalling the definition of Balmer's shifted Witt groups of triangulated categories following [5] - from which we also obtain the following quick recollection of the basic properties of triangulated categories.

Definition 2.2.16. A **pre-triangulated category** (K,T) consists of an additive category K together with an automorphism $T : K \to K$ called the **translation functor** and a collection

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A) \qquad (*)$$

of triples of morphisms, called **exact triangles**, satisfying the following properties.

- **(TR1)** For any $A \in K$ the sequence

  $$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow T(A)$$

  is an exact triangle. For any morphism $u : A \to B$ in K there exists some $C \in K$ and morphisms v, w such that

  $$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

  is an exact triangle.

- **(TR2)** A sequence as in $(*)$ is an exact triangle if and only if the sequence

$$B \xrightarrow{v} C \xrightarrow{w} T(A) \xrightarrow{-T(u)} T(B)$$

is an exact triangle.

- **(TR3)** For any diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed] & T(A) \arrow[d, "T(f)"] \\ D \arrow[r, "u'"'] & E \arrow[r, "v'"'] & F \arrow[r, "w'"'] & T(D) \end{tikzcd}$$

in which both rows are exact triangles and the left square commutes, there exists a morphism $h : C \to F$ filling in the dotted line making the whole diagram commute.

We stress that the morphism h in the above definition is not uniquely determined. The property of triangulated categories most important for us already holds on the level of pre-triangulated categories.

Lemma 2.2.17. *Let (K, T) be a triangulated category and*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"'] & T(A) \arrow[d, "T(f)"] \\ D \arrow[r, "u'"'] & E \arrow[r, "v'"'] & F \arrow[r, "w'"'] & T(D) \end{tikzcd}$$

be a commutative diagram with both rows exact triangles. Then if any two of f, g and h are isomorphisms, so is the third.

An important implication is that the object C appearing in any extension of the morphism $u : A \to B$ to an exact triangle as in **(TR1)** is unique up to non-canonical isomorphism. Such an object C is usually referred to as the **cone** of u, and the only property that a **triangulated category** requires over those of a pre-triangulated category is the *octahedral axiom* which captures how cones should behave over compositions of morphisms. Balmer further requires triangulated categories to adhere to an enriched version of this axiom in order to construct a triangular analogue of sublagrangians which are used in establishing that the localisation sequence really is exact. Since our overview won't go into these details we don't repeat these axioms here - if necessary the reader can find them as **(TR4)** and **(TR4$^+$)** of [5, §0]. Whenever we talk about a **triangulated category** we assume it satisfies both the octahedral axiom and its enhancement; certainly every example of a triangulated category we want to use does.

Definition 2.2.18. Let (K_1, T_1) and (K_2, T_2) be two triangulated categories. Then a **triangulated functor** $(F, c) : (K_1, T_1) \to (K_2, T_2)$ consists of an additive functor $F : K_1 \to K_2$ together witha natural isomorphism $c : FT_1 \to T_2F$ such that for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

in K_1, the sequence

$$F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \xrightarrow{c_A F(w)} T(F(A))$$

is an exact triangle in K_2.

Definition 2.2.19. For a triangulated category (K, T) and $\delta = \pm 1$, an additive functor $\# : K^{op} \to K$ is said to be **δ-exact** if $T\#T = \#$ and for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

the sequence

$$C^{\#} \xrightarrow{v^{\#}} B^{\#} \xrightarrow{u^{\#}} A^{\#} \xrightarrow{\delta T(w^{\#})} T(C^{\#})$$

is again an exact triangle. If $\omega : \mathrm{id}_K \to \#\#$ is a natural isomorphism such that $\omega_{T(M)} = T(\omega_M)$ and $(\omega_M)^{\#} \circ \omega_{M^{\#}} = \mathrm{id}_{M^{\#}}$ for any $M \in K$ then we call the quadruple $(K, \#, \delta, \omega)$ a **triangulated category with δ-duality**.

If the δ being used is in some way clear from context we omit it from the terminology and refer to simply a triangulated category with duality. The above is the structure on which the 0^{th} derived Witt groups is defined; a sequence of Witt groups behaving in some ways like a cohomology theory is obtained by taking the 0^{th} derived Witt group of the same triangulated category but with a shifted duality attached.

Definition 2.2.20. Let $(K, \#, \delta, \omega)$ be a triangulated category with duality. Then we define

$$\#_n = T^n\#, \quad \delta_n = (-1)^n\delta, \quad \omega_n = (-1)^{\frac{n(n+1)}{2}}\delta^n\omega$$

for which the quadruple $(K, \#_n, \delta_n, \omega_n)$ is a triangulated category with δ_n-duality. The triple $(\#_n, \delta_n, \omega_n)$ is called the **n^{th}-shifted duality** of $(\#, \delta, \omega)$.

Definition 2.2.21. Let $(K, \#, \delta, \omega)$ be a triangulated category with duality. Then a morphism $u : A \to A^{\#}$ is called **symmetric** when $u = u^{\#}\omega$. A **symmetric space** (P, ψ) consists of an object $P \in K$ together with a symmetric isomorphism $\psi : P \to P^{\#}$ which is as usual called the **symmetric form** on P.

An isometry of symmetric spaces is defined in the usual way; it is simply an isomorphism in K commuting with the symmetric forms. Orthogonal sums are again defined as usual.

Definition 2.2.22. For any additive category C we write $1/2 \in C$ to mean that for any $A, B \in C$ the group $\mathrm{Hom}_C(A, B)$ is *uniquely 2 divisible*; precisely this means that for any $f \in \mathrm{Hom}_C(A, B)$ there exists a unique $g \in \mathrm{Hom}_C(A, B)$ such that $f = g + g$.

The below is [5, Thm 1.6] and plays a central role not only in the definition of triangular Witt groups themselves but also in the definition of the connecting homomorphism in the localisation sequence attached to short exact sequences of triangulated categories.

Proposition 2.2.23. *Let $(K, \#, \delta, \omega)$ be a triangulated category with duality and suppose that $1/2 \in K$. Then for any morphism $u : A \to A^{\#_{n-1}}$ symmetric with respect to the $(n-1)^{st}$-shifted duality there exists an isomorphism ψ fitting into a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\delta_{n-1}\omega_{n-1}"'] & A^{\#_{n-1}} \arrow[r, "v"] \arrow[d, "\mathrm{id}"'] & C \arrow[r, "w"] \arrow[d, "\psi"'] & T(A) \arrow[d, "\delta_{n-1}T(\omega_{n-1})"] \\
A^{\#_{n-1}\#_{n-1}} \arrow[r, "\delta_{n-1}u^{\#_{n-1}}"'] & A^{\#_{n-1}} \arrow[r, "-w^{\#_n}"'] & C^{\#_n} \arrow[r, "v^{\#_n}"'] & T(A^{\#_{n-1}\#_{n-1}})
\end{tikzcd}$$

in which both rows are exact triangles, and further (C, ψ) is a symmetric space with respect to the n^{th}-shifted duality. The space (C, ψ) is uniquely determined by u up to isometry.

Definition 2.2.24. Let $(K, \#, \delta, \omega)$ be a triangulated category with duality and $u : A \to A^{\#_{n-1}}$ a morphism symmetric with respect to the $(n-1)^{st}$-shifted duality. Then we define the **cone** of u to be the symmetric space

$$\mathrm{cone}(u) = (C, \psi)$$

appearing in the above proposition.

These cones play the role of metabolic spaces in the definition of triangular Witt groups. Precisely, we call two spaces (P, ψ_1) and (P_2, ψ_2) symmetric with respect to the n^{th}-shifted duality *Witt-equivalent* when there exists morphisms u_1 and u_2 symmetric with respect to the $(n-1)^{st}$-shifted duality and an isometry

$$(P_1, \psi_1) \perp \mathrm{cone}(u_1) \simeq (P_2, \psi_2) \perp \mathrm{cone}(u_2)$$

Definition 2.2.25. Let $(K, \#, \delta, \omega)$ be a triangulated category with duality. Then we define its n^{th}**-shifted Witt group**, written $W^n(K, \#, \delta, \omega)$ to be the group of Witt equivalence classes $[P, \psi]$ of spaces symmetric with respect to the n^{th}-shifted duality under orthogonal sum.

Below we give framework, analogous to **Definition 2.2.7** and taken from [43], on which functorial properties of these triangular Witt groups can be defined.

Definition 2.2.26. For $i = 1, 2$ let $(K_i, \#_i, \delta_i, \omega_i)$ be a triangulated category with duality. Then a **duality preserving functor**

$$(F, \tau) : (K_1, \#_1, \delta_1, \omega_1) \longrightarrow (K_2, \#_2, \delta_2, \omega_2)$$

consists of a covariant triangulated functor $F : K_1 \to K_2$ together with a natural isomorphism

$$\tau : F\#_1 \xrightarrow{\simeq} \#_2 F$$

such that for any $A \in K_1$ we have the following.

1. The commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\omega_1(A))"] \arrow[d, "\omega_2(F(A))"'] & F(A^{\#_1\#_1}) \arrow[d, "\tau(A^{\#_1})"] \\ F(A)^{\#_2\#_2} \arrow[r, "\tau(A)^{\#_2}"'] & F(A^{\#_1})^{\#_2} \end{tikzcd}$$

2. The identity $T_2^{-1}\tau(A) = \delta_1\delta_2\omega(T_1(A))$ in which T_1 and T_2 denote the translation functors of K_1 and K_2 respectively.

That these functors induce homomorphisms on all the shifted Witt groups associated to these triangulated categories with duality is established by [43, Thm 2.7] - which gives us the result below. Note that as in *loc. cit.* our notation for the induced map on Witt groups suppresses the shift and duality compatibility transformation.

Proposition 2.2.27. *Let*

$$(F, \tau) : (K_1, \#_1, \delta_1, \omega_1) \longrightarrow (K_2, \#_2, \delta_2, \omega_2)$$

be a duality preserving functor between triangulated categories with duality. Then we obtain group homomorphisms

$$F_* : W^i(K_1, \#_1, \delta_1, \omega_1) \longrightarrow W^i(K_2, \#_2, \delta_2, \omega_2)$$

by setting $F_*([A, \psi]) = [F(A), \tau(A) \circ F(\psi)]$. If F is further an equivalence of categories then each F_* is an isomorphism.

To understand these triangular Witt groups as a generalisation of the ordinary Witt groups of exact categories, we ideally would like an isomorphism between the Witt groups of an exact category with duality $(\mathcal{E}, *, \eta)$ and one of the Witt groups of a related triangulated category. This is established in [44] under the assumption that $\mathcal{E}$

be *semi-saturated*; meaning that any morphism in $\mathcal{E}$ which admits a right inverse is an admissible epimorphism. It is remarked upon in the outset of [1] that the result of [44] in fact holds for any exact category - which while being a greater generality than we actually need, this does make for a simpler statement of the isomorphism between usual and derived Witt groups.

Following [44], we observe that for an exact category with duality $(\mathcal{E}, *, \eta)$ we obtain an exact category with duality $\big(\mathrm{Ch}^b(\mathcal{E}), \#, \omega\big)$ when for each $A \in \mathrm{Ch}^b(\mathcal{E})$ we set

$$(A^{\#})^n = (A^{-n})^* \quad \text{with boundary maps} \quad d^n_{A^{\#}} = \big(d_A^{-n-1}\big)^*$$

and the double dual identification to be $\omega^n_A = \eta_{A^n}$ in degree n. Being exact functors these localise immediately to the derived category $D^b(\mathcal{E})$ which is the usual localisation of $\mathrm{Ch}^b(\mathcal{E})$ by the collection of quasi-isomorphisms. The category $D^b(\mathcal{E})$ obtains the structure of a triangulated category with duality in terms of the "ordinary" cones of chain complex morphisms as those described in **Definition 2.1.5**, see [31, Ex.1.4.4] for example. Briefly, we define a translation functor $T : \mathrm{Ch}^b(\mathcal{E}) \to \mathrm{Ch}^b(\mathcal{E})$ by setting $T(A) = A[1]$ to be the chain complex with $d^n_{T(A)} = -d^{n+1}_A$ for boundary maps - this functor immediately localises to $D^b(\mathcal{E})$. Then the cone of any chain complex morphism $f : A \to B$ extends canonically to a diagram

$$A \xrightarrow{f} B \hookrightarrow \mathrm{cone}(f) \twoheadrightarrow T(A)$$

and the triangulation of $D^b(\mathcal{E})$ is defined by taking the exact triangles to be any diagram in $D^b(\mathcal{E})$ isomorphic to the image of the sequence above for some morphism f of chain complexes. We hence obtain from $(\mathcal{E}, *\eta)$ a triangulated category $(D^b(\mathcal{E}), \#, 1, \omega)$ with 1-exact duality; its Witt group is related to the ordinary Witt group $W(\mathcal{E}, *, \eta)$ by the below proposition - which is the main result of [44].

Proposition 2.2.28. *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality and suppose that $1/2 \in \mathcal{E}$. Then the functor*

$$c_0 : \mathcal{E} \longrightarrow D^b(\mathcal{E}), \quad E \mapsto E[0]$$

which concentrates $\mathcal{E}$ in degree zero induces an isomorphism

$$c_{0*} : W(\mathcal{E}, *, \eta) \longrightarrow W^0(D^b(\mathcal{E}), \#, 1, \omega)$$

on Witt groups. Precisely, an isometry class of $\psi : P \to P^*$ is sent to the Witt equivalence class of $c_0(\psi) : P[0] \to P[0]^{\#} = P^*[0]$, where $c_0(\psi)$ is simply the morphism ψ concentrated in degree zero.

Let us now set up the context in which we can consider a localisation sequence, following the notation of [1]. Suppose that $\mathbf{C}$ is a triangulated category and $\mathbf{D}$ a strictly full triangulated subcategory. Then we call $\mathbf{D}$ *saturated* in $\mathbf{C}$ when it is closed under direct summands. We write $\mathbf{C}/\mathbf{D}$ for the triangulated category obtained as the localisation of $\mathbf{C}$ by the class $S(\mathbf{D})$ of those morphisms in $\mathbf{C}$ that have their cone in $\mathbf{D}$. Our reference for these localisations is [45] - briefly $\mathbf{C}/\mathbf{D}$ may be constructed via a calculus of fractions; its objects are the same as those of $\mathbf{C}$ and the morphisms $\mathrm{Mor}_{\mathbf{C}/\mathbf{D}}(X, Y)$ are the equivalence classes, as described in Definition 2.1.11 of *loc. cit.*, of diagrams

$$X \xleftarrow{u} Z \xrightarrow{f} Y$$

with $u \in S(\mathbf{D})$. We have the localisation functor $q : \mathbf{C} \to \mathbf{C}/\mathbf{D}$ which is the identity on objects and takes $f \in \mathrm{Mor}_{\mathbf{C}}(X, Y)$ to $X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y$. We give $\mathbf{C}/\mathbf{D}$ a triangulation by setting the exact triangles to be the sequences which are isomorphic to the image under q of some exact triangle in $\mathbf{C}$. The functor $q : \mathbf{C} \to \mathbf{C}/\mathbf{D}$ is then initial among all triangulated functors carrying morphisms in $S(\mathbf{D})$ to isomorphisms.

Definition 2.2.29. By a **short exact sequence of triangulated categories** we mean a sequence

$$\mathbf{D} \hookrightarrow \mathbf{C} \xrightarrow{q} \mathbf{C}/\mathbf{D}$$

where $\mathbf{D} \hookrightarrow \mathbf{C}$ is a saturated strictly full triangulated subcategory of $\mathbf{C}$ and q is the universal localisation functor.

The following terminology is taken from [5, Def.4.4].

Definition 2.2.30. Let $(\mathbf{C}, \#, \delta, \omega)$ be a triangulated category with duality and $\mathbf{D} \subseteq \mathbf{C}$ be a triangulated subcategory. Then an **$S(\mathbf{D})$-space** in $\mathbf{C}$ is a pair (A, s) with $s : A \to A^{\#}$ a symmetric morphism lying in $S(\mathbf{D})$. Two $S(\mathbf{D})$-spaces (A_1, s_1) and (A_2, s_2) are called isometric when there exists an object $B \in \mathbf{C}$ and morphisms $u : B \to A_1$ and $v : B \to A_2$, both lying in $S(\mathbf{D})$, such that $u^{\#} s_1 u = v^{\#} s_2 v$.

Suppose that our triangulated subcategory $\mathbf{D} \subseteq \mathbf{C}$ is stable with respect to # - i.e. suppose that $\#(\mathbf{D}) \subseteq \mathbf{D}$. Then the triple $(\#, \delta, \omega)$ restricts to a duality on $\mathbf{D}$ and localises to a duality on $\mathbf{C}/\mathbf{D}$. As in [1] we denote this restriction and the localisation by the same symbols $(\#, \delta, \omega)$ and also omit this triple from our notation when it is clear what duality is being used. The following description [5, Prop.4.5] of the Witt groups of $\mathbf{C}/\mathbf{D}$ in terms of $S(\mathbf{D})$-spaces allows the connecting homomorphism in the localisation sequence to be defined in terms of the cones of $S(\mathbf{D})$-spaces [5, Thm 4.8].

Proposition 2.2.31. *Let $(\mathbf{C}, \#, \delta, \omega)$ be a triangulated category with duality and suppose that $1/2 \in \mathbf{C}$. We write $\mathbf{D} \subseteq \mathbf{C}$ for some saturated full triangulated subcategory of $\mathbf{C}$ and denote by $q : \mathbf{C} \to \mathbf{C}/\mathbf{D}$ the localisation map. Then any element of $W^0(\mathbf{C}/\mathbf{D})$ can be written $[q(A), q(s)]$ for some $S(\mathbf{D})$-space (A, s). Further, we have a well defined group homomorphism*

$$\partial^0 : W^0\left(\mathbf{C}/\mathbf{D}\right) \longrightarrow W^1(\mathbf{D})$$

given by $[q(A), q(s)] \mapsto \mathrm{cone}(s)$ *for each* $S(\mathbf{D})$*-space* (A, s).

For clarification, the form on the above $\mathrm{cone}(s)$ is taken as in **Definition 2.2.24** so that $\mathrm{cone}(s)$ really makes sense as an element of $W^1(\mathbf{C}/\mathbf{D})$. One of course equally defines

$$\partial^n : W^n\left(\mathbf{C}/\mathbf{D}\right) \longrightarrow W^{n+1}(\mathbf{D})$$

by using the n^{th}-shifted duality in place of $(\#, \delta, \omega)$.

Theorem 2.2.32. *Let* $(\mathbf{C}, \#, \delta, \omega)$ *be a triangulated category with duality and suppose that* $1/2 \in \mathbf{C}$. *Then for any full saturated triangulated subcategory* $\mathbf{D} \subseteq \mathbf{C}$ *stable under* $\#$ *we obtain a long exact sequence*

$$\ldots \to W^{n-1}\left(\frac{\mathbf{C}}{\mathbf{D}}\right) \overset{\partial^{n-1}}{\longrightarrow} W^n(\mathbf{D}) \longrightarrow W^n(\mathbf{C}) \longrightarrow W^n\left(\frac{\mathbf{C}}{\mathbf{D}}\right) \overset{\partial^{n}}{\longrightarrow} W^{n+1}(\mathbf{D}) \to \ldots$$

Proof. If $\mathbf{C}/\mathbf{D}$ is assumed to be *weakly cancelative* then this is the main result (Theorem 5.2) of [5]. This assumption is removed in [1, §2]. $\square$

2.3 Matlis Duality from Residual Complexes

Let us first remark that if X is a scheme with a residual complex $\mathcal{R}$ then since this residual complex is in the derived category a dualising complex we obtain immediately that

$$\left(\mathrm{Ch}^b_{Coh}(X), *_{\mathcal{R}}, \mathrm{qis}, \eta\right)$$

is an exact category with weak equivalences (being the quasi-isomorphisms of chain complexes) and duality given by

$$*_{\mathcal{R}} = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}) : \mathrm{Ch}^b_{Coh}(X)^{op} \longrightarrow \mathrm{Ch}^b_{Coh}(X)$$

with η defined as in **Definition 2.1.6**. One might suggest that it is due to the minimality of $\mathcal{R}$ as a chain complex, in the sense of [2], that the individual terms of $\mathcal{R}$ also induce local dualities on X.

Recall that we call a scheme X Cohen-Macaulay when the sections of $\mathcal{O}_X$ over any affine open set are a Cohen-Macaulay ring, or equivalently when all the local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay.

Definition 2.3.1. A **dualising module/sheaf** for a Cohen-Macaulay scheme X is a sheaf $\omega_X \in Coh_X$ such that for each $x \in X$ the stalk $\omega_{X,x}$ is a canonical module for the Cohen-Macaulay local ring $\mathcal{O}_{X,x}$.

We avoid refering to a dualising sheaf as a “canonical” sheaf because it is only unique up to tensoring with line bundles, see [37, Ch.3.3] or **Theorem 2.1.13**. However we will use the term canonical module when working with a Cohen-Macaulay local ring.

Proposition 2.3.2. *Let A be a Cohen Macaulay ring of finite dimension. If A has a canonical module ω_A, then any minimal injective resolution of ω_A is a residual complex for* $\operatorname{Spec}(A)$. *Conversely, any residual complex on A is a minimal injective resolution of a canonical module.*

Proof. This is a restatement of [46, Thm 6.2]. □

Proposition 2.3.3. *Let $\mathcal{R}$ be a residual complex on a scheme X and $x \in X$ a point with codimension p according to $\mathcal{R}$. Then the functor*

$$\operatorname{Hom}_{\mathcal{O}_{X,x}}(-\mathcal{R}_x^p) : Mod_{\mathcal{O}_{X,x}}^{op} \longrightarrow Mod_{\mathcal{O}_{X,x}}$$

is length preserving, and forms an exact category with duality

$$\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x^p\right)$$

Further, the trace map

$$\operatorname{tr}_{\pi_{x,X}}(\mathcal{R}_x)^p : x^\natural(\mathcal{R}) \longrightarrow \mathcal{R}_x^p$$

induces an isomorphism

$$\left(\operatorname{tr}_{\pi_{x,X}}\right)_* : GW\left(x, x^\natural(\mathcal{R})\right) \longrightarrow GW\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x^p\right)$$

and again for Witt groups.

Proof. Since by definition $\mathcal{R}_x^p$ is an $\mathcal{O}_{X,x}$-injective hull of $\kappa(x)$, this is part of *Matlis duality* - see [37, Prop.3.2.12]. It is further clear that the trace map is a suitable transfer map as in **Definition 2.2.15**, and the fact that one obtains the above *devisage* isomorphisms is contained in [47] for example. □

Corollary 2.3.3.1. *Let $(X, \mathcal{R})$ be a scheme with a residual complex and $j : Z \hookrightarrow X$ a closed embedding. Then for any point $z \in Z$ with $\mu(z) = p$ say, we have a commutative square of isomorphisms*

$$\begin{tikzcd}
GW\left(f.l.Mod_{\mathcal{O}_{X,z}}, \mathcal{R}_z^p\right) & GW\left(f.l.Mod_{\mathcal{O}_{Z,z}}, j^\triangle(\mathcal{R})_z^p\right) \arrow[l, "(\operatorname{tr}_j(\mathcal{R})_z^p)_*"'] \\
GW\left(z, z^\natural(\mathcal{R})\right) \arrow[u, "\left(\operatorname{tr}_{\pi_{z,X}}(\mathcal{R})_z^p\right)_*"] \arrow[r, leftrightarrow] & GW\left(z, z^\natural(j^\triangle(\mathcal{R}))\right) \arrow[u, "\left(\operatorname{tr}_{\pi_{z,Z}}(j^\triangle(\mathcal{R}))_z^p\right)_*"']
\end{tikzcd}$$

and the same again for Witt groups.

Proof. Given that the vertical maps are isomorphisms, the remaining statement follows by the compatibility **Lemma 2.1.31**. $\square$

Slightly more generally, the above devisage result also holds for semi-local rings.

Proposition 2.3.4. *Let $X = \mathrm{Spec}(A)$ with A a semi-local ring that has maximal ideals $p_1, p_2, ..., p_n$, all of the same height, and let $\mathcal{R}$ be a residual complex on X with codimension function μ, and let's fix $d = \mu(p_i)$. We write $\pi_i : \mathrm{Spec}(\kappa(p_i)) \to X$ for the closed embedding, similar to our standard notation $\pi_{p_i,X} : \mathrm{Spec}(\kappa(p_i)) \to \mathrm{Spec}(A_{p_i})$.*

Then $\big(f.l.Mod_A, \mathcal{R}^d\big)$ and each of the $\big(f.l.Mod_{A_{p_i}}, \mathcal{R}^d_{p_i}\big)$ are exact categories with duality, and we denote by

$$loc_i : \big(f.l.Mod_A, \mathcal{R}^d\big) \longrightarrow \big(f.l.Mod_{A_{p_i}}, \mathcal{R}^d_{p_i}\big)$$

the non-singular exact form functor induced by localisation to the point p_i, and by

$$(s_i)_* : \big(f.l.Mod_{A_{p_i}}, \mathcal{R}^d_{p_i}\big) \to \big(f.l.Mod_A, \mathcal{R}^d\big)$$

the nonsingular exact form functor attached to the section $s_i = \rho(p_i) : \mathcal{R}^d_{p_i} \to \mathcal{R}^d$ induced by the isomorphism $\mathcal{R}^d \xrightarrow{\simeq} \bigoplus \mathcal{R}^d_{p_i}$. Then we have a commutative triangle of isomorphisms

$$\begin{tikzcd}
 & GW\big(f.l.Mod_A, \mathcal{R}^d\big) \arrow[d, bend right, "\sum(loc_i)_*"'] \\
\bigoplus_{i=1}^n GW\big(p_i, \pi_i^{\triangle}(\mathcal{R})^d\big) \arrow[ur, "\bigoplus\big(\mathrm{tr}_{\pi_i}(\mathcal{R})^d\big)_*"] \arrow[r, "\bigoplus\Big(\mathrm{tr}_{\pi_{p_i,X}}(\mathcal{R})^d\Big)_*"'] & \bigoplus_{i=1}^n GW\Big(f.l.Mod_{A_{p_i}}, \mathcal{R}^d_{p_i}\Big) \arrow[u, bend right, "\bigoplus(s_i)_*"']
\end{tikzcd}$$

in which $\sum(loc_i)_*$ and $\bigoplus(s_i)_*$ are inverse to eachother. The same is true with Witt groups in place of Grothendieck-Witt groups.

The next result extends the spirit of the information in **Proposition 3.2.5** to the non-derived, characteristic 2 inclusive case. We have included it here, despite the fact that we won't make use of it in this thesis, because it seems not to have been clearly written out and potentially useful - especially given how useful **Proposition 3.2.5** will be to us.

Proposition 2.3.5. *Let (A, m, κ) be a local ring with residual complex $\mathcal{R}$ for which we suppose $\mu(m) = 0$. Then any symmetric space $(M, \psi) \in Sym\big(\mathrm{Ch}^b_{f.l.}(Mod_A), \mathcal{R}\big)$ induces canonical isomorphisms*

$$\theta^n : H^n(M) \longrightarrow \mathrm{Hom}_A(H^{-n}(M), \mathcal{R}^0)$$

satisfying for each n the symmetry condition

$$\theta^n = (\theta^{-n})^{*_0}\, ev$$

where $*_0 = \mathrm{Hom}_A(-, \mathcal{R}^0)$ is the Matlis duality functor and ev the evaluation map.

Proof. Take some $\alpha \in Z^n(M)$ so that $\psi(\alpha) \in Z^n(\mathcal{H}om_A(M, \mathcal{R}))$ is a morphism $M \longrightarrow \mathcal{R}[n]$ of chain complexes; we'll denote by $g^i : M^i \to \mathcal{R}^{n+i}$ the components of this morphism. We begin by constructing, for each $i \geq 0$, a map

$$s^{-n-i} : M^{-n-i} \longrightarrow \mathcal{R}[n]^{-n-i-1}$$

such that

$$g^{-n-i} = s^{-n-i+1} d_M^{-n-i} + d_{\mathcal{R}[n]}^{-n-i-1} s^{-n-i} \qquad (*)$$

whenever $i \geq 1$. The maps s^{-n-i} themselves will not be uniquely determined, however the restrictions of $d_{\mathcal{R}[n]}^{-n-i-1} s^{-n-i}$ to the kernels $Z^{-n-i}(M)$ will be for all $i \geq 0$.

We may fix an integer $N \geq 1$ such that $M^{-n-i} = 0$ whenever $i > N$, and we set $s^{-n-i} = 0$ for all $i \geq N$; we have then equation $(*)$ trivially for $i > N$. Further, if $i \geq N$ then the modules $Z^{-n-i}(M) = H^{-n-i}(M)$ are finite length, so because $N \geq 1$ the only maps $Z^{-n-i} \to \mathcal{R}^{-i-1}$ are zero - hence that the restrictions of the maps s^{-n-i} themselves to these kernels are unique.

Suppose that the maps s^{-n-i} have been constructed for $i \geq k \geq 1$ and they satisfy $(*)$ for $i \geq k+1$. So we have in particular

$$g^{-n-k-1} = s^{-n-k} d_M^{-n-k-1} + d_{\mathcal{R}[n]}^{-n-k-2} s^{-n-k-1}$$

into which we may substitute $d_{\mathcal{R}[n]}^{-n-k-1} g^{-n-k-1} = g^{-n-k} d_M^{-n-k-1}$ to obtain

$$g^{-n-k} d_M^{-n-k-1} = d_{\mathcal{R}[n]}^{-n-k-1} s^{-n-k} d_M^{-n-k-1}$$

Hence the map

$$g^{-n-k} - d_{\mathcal{R}[n]}^{-n-k-1} s^{-n-k} : M^{-n-k} \longrightarrow \mathcal{R}^{-k}$$

vanishes on $B^{-n-k}(M)$ and hence also on $Z^{-n-k}(M)$. Thus $g^{-n-k} - d_{\mathcal{R}[n]}^{-n-k-1} s^{-n-k}$ lifts to a map $B^{-n-k+1}(M) \to \mathcal{R}^{-k}$ which, because $\mathcal{R}^{-k}$ is injective, extends uniquely to a map $Z^{-n-k+1}(M) \to \mathcal{R}^{-k}$ but non-uniquely to a map

$$s^{-n-k+1} : M^{-n-k+1} \longrightarrow \mathcal{R}^{-k}$$

Different choices of this extension will differ by $f d_M^{-n-k+1}$ for some map $f : B^{-n-k+2}(M) \to$

$\mathcal{R}^{-k}$. We complete our construction by checking that when $k \geq 2$ the map

$$d_{\mathcal{R}[n]}^{-n-k+1} s^{-n-k+2} : M^{-n-k+2} \longrightarrow \mathcal{R}^{-k+2}$$

restricted to $Z^{-n-k+2}(M)$ does not depend on the choice of s^{-n-k+1} above. It suffices to show that the map $d_{\mathcal{R}[n]}^{-n-k+1} s^{-n-k+2}$ restricted to the boundary $B^{-n-k+2}(M)$ is uniquely determined. To that end, take a map $f : B^{-n-k+2}(M) \to \mathcal{R}^{-k}$ so that

$$s^{-n-k+1} + f d_M^{-n-k+1} : M^{-n-k+1} \longrightarrow \mathcal{R}^{-k}$$

represents a different choice of s^{-n-k+1}. Our construction of a map s^{-n-k+2} from this new choice starts by lifting the map

$$g^{-n-k+1} - d_{\mathcal{R}[n]}^{-n-k} \left(s^{-n-k+1} + f d_M^{-n-k+1} \right) : M^{-n-k+1} \longrightarrow \mathcal{R}^{-k+1}$$

to $B^{-n-k+2}(M)$. This lift can be written

$$s^{-n-k+2}|_{B^{-n-k+2}(M)} + d_{\mathcal{R}[n]}^{-n-k} f$$

where $s^{-n-k+2} : M^{-n-k+2} \longrightarrow \mathcal{R}^{-k+1}$ denotes any extension of

$$g^{-n-k+1} - d_{\mathcal{R}[n]}^{-n-k} s^{-n-k+1} : B^{-n-k+2}(M) \longrightarrow \mathcal{R}^{-k+1}$$

Now it is immediate that

$$d_{\mathcal{R}[n]}^{-n-k+1} \left(s^{-n-k+2}|_{B^{-n-k+2}(M)} + d_{\mathcal{R}[n]}^{-n-k} f \right) = d_{\mathcal{R}[n]}^{-n-k+1} s^{-n-k+2}|_{B^{-n-k+2}(M)}$$

The upshot of this construction is that we obtain a well defined map

$$\left(g^{-n} - d_{\mathcal{R}[n]}^{-n-1} s^{-n} \right) |_{Z^{-n}(M)} : Z^{-n}(M) \longrightarrow \mathcal{R}^0$$

which vanishes on $B^{-n}(M)$. We set $\theta^n(\alpha)$ to be the lift of this map to $H^{-n}(M)$. Of course we must mention why this assignment carries $B^n(M)$ to zero - if $\alpha \in B^n(M)$ then $\psi(\alpha) \in B^n(\mathcal{H}om_A(M, \mathcal{R}))$ is a chain map $M \to \mathcal{R}[n]$ homotopic to zero. In this case our maps s^{-n-i} can be taken to form a chain homotopy; in particular including a map $s^{-n+1} : M^{-n+1} \to \mathcal{R}^0$. We have then

$$g^{-n} - d_{\mathcal{R}[n]}^{-n-1} s^{-n} = s^{-n+1} d_M^{-n}$$

so that $(g^{-n} - d_{\mathcal{R}[n]}^{-n-1} s^{-n})|_{Z^{-n}(M)} = 0$. Conversely, we observe that if $\theta^n(\alpha) = 0$, then

$$g^{-n} - d_{\mathcal{R}[n]}^{-n-1} s^{-n} : M^{-n} \longrightarrow \mathcal{R}^0$$

vanishes on $Z^{-n}(M)$ hence lifting to a map $B^{-n}(M) \to \mathcal{R}^0$ which may be extended to a final term $s^{-n+1} : M^{-n+1} \to \mathcal{R}^0$ of a chain homotopy for $\psi(\alpha)$. Hence $\psi(\alpha) \in$

$B^n(\mathcal{H}om_A(M, \mathcal{R}))$ so that $\alpha \in B^n(M)$ because ψ is a quasi-isomorphism. So our map θ^n is injective.

Conversely again, if $\overline{g} : H^{-n}(M) \to \mathcal{R}^0$ is any map then we obtain a map $g : Z^{-n}(M) \to \mathcal{R}^0$ which can be extended to a map $g^{-n} : M^{-n} \to \mathcal{R}^0$. Setting $g^t : M^t \to \mathcal{R}^{n+t}$ to be the zero map if $t \neq n$, we obtain a chain map $M \to \mathcal{R}[n]$, which must equal $\psi^n(\alpha)$ for some $\alpha \in Z^n(M)$. When computing $\theta^n(\alpha)$, we may take $s^{-n-i} = 0$ for all $i \geq 0$ - so we do have $\theta^n(\alpha) = \overline{g}$. Observe a consequence of this surjectivity argument; for any element $\overline{\alpha} \in H^n(M)$, one may choose a representative $\alpha \in Z^n(M)$ such that the chain map $\psi^n(\alpha)$ is zero in all degrees but $-n$.

Finally, we extract our symmetry condition for θ^n from the symmetry of ψ - which reads for any $x \in M^l$ and $y \in M^t$ as below.

$$[\psi^l(x)]_{(t)}(y) = (-1)^{(l+t)(l+t+1)/2}[\psi^t(y)]_{(l)}(x) \qquad (***)$$

Now take $\overline{\alpha} \in H^n(M)$ and $\overline{\beta} \in H^{-n}(M)$ - we aim to show that $\theta^n(\overline{\alpha}) = (-1)^n\theta^{-n}(\overline{\beta})$. To this end, take representatives $\alpha \in Z^n(M)$ and $\beta \in Z^{-n}(M)$ such that $\psi(\alpha)$ and $\psi(\beta)$ are chain maps zero outside of the degrees $-n$ and n respectively. Then $\theta^n(\alpha)$ is induced by the map $\psi^{-n}(\alpha)$ while $\theta^{-n}(\beta)$ is induced by the map $\psi^n(\beta)$, so that our desired symmetry result for the θ^n can now be read off from $(***)$ by setting $x = \alpha, y = \beta$ where then of course $l = n$ while $t = -n$. $\square$

We conclude this chapter with a diagram which may help describe how the residue homomorphism of the next section fits into a residual complex. The setup for the following commutative diagram is that X is a scheme with residual complex $\mathcal{R}$, we denote by $j : Z \hookrightarrow X$ a subvariety of X with generic point z and $x \in Z^1$. We have then (as is standard in our notation) the maps $\pi_{x,X} : \mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ and $\pi_{x,Z} : \mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\mathcal{O}_{Z,x})$ which denote the embeddings of the closed points, and we have written $j_x : \mathrm{Spec}(\mathcal{O}_{Z,x}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ for the localisation of j at x. Finally we write p for the codimension of z according to $\mathcal{R}$.

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathcal{R}^{p-1} \arrow[rr] \arrow[dd, dashed, two heads] & & \mathcal{R}^{p} \arrow[rr] \arrow[dd, dashed, two heads] & & \mathcal{R}^{p+1} \arrow[rr] \arrow[dd, dashed, two heads] & & \mathcal{R}^{p+2} \arrow[r] \arrow[dd, dashed, two heads] & \cdots \\
& \cdots \arrow[r] & 0 \arrow[r] \arrow[ul] \arrow[dd, two heads] & & j^{\triangle}(\mathcal{R})^{p} \arrow[rr] \arrow[ul, "\mathrm{tr}_j(\mathcal{R})^p"'] \arrow[dd, two heads] & & j^{\triangle}(\mathcal{R})^{p+1} \arrow[rr] \arrow[ul, "\mathrm{tr}_j(\mathcal{R})^{p+1}"'] \arrow[dd, two heads] & & j^{\triangle}(\mathcal{R})^{p+2} \arrow[r] \arrow[ul, "\mathrm{tr}_j(\mathcal{R})^{p+2}"'] \arrow[dd, two heads] & \cdots \\
\cdots \arrow[r, dashed] & \mathcal{R}^{p-1}_x \arrow[rr, dashed] & & \mathcal{R}^{p}_x \arrow[rr, dashed] & & \mathcal{R}^{p+1}_x \arrow[rr, dashed] & & 0 \arrow[r, dashed] & \cdots \\
& \cdots \arrow[r] & 0 \arrow[r] \arrow[ul, dashed] & & j^{\triangle}(\mathcal{R})^{p}_x \arrow[rr, "d^p_{j^{\triangle}(\mathcal{R})_x}"] \arrow[ul, dashed, "\mathrm{tr}_{j_x}(\mathcal{R}_x)^p"] \arrow[dr, leftrightarrow, "\cong"] & & j^{\triangle}(\mathcal{R})^{p+1}_x \arrow[rr] \arrow[ul, dashed, "\mathrm{tr}_{j_x}(\mathcal{R}_x)^{p+1}"] & & 0 \arrow[r] & \cdots \\
& & & & & z^{\natural}(\mathcal{R}) & & x^{\natural}(\mathcal{R}) \arrow[ul, "\mathrm{tr}_{\pi_{x,\mathrm{Z}}}(j^{\triangle}(\mathcal{R})_x)^{p+1}"'] \arrow[uull, dashed, bend right, "\mathrm{tr}_{\pi_{x,\mathrm{X}}}(\mathcal{R}_x)^{p+1}"'] & &
\end{tikzcd}
$$

We have extracted from $\mathcal{R}$ a residual complex $j^{\triangle}(\mathcal{R})_x$ over the 1-dimensional local ring $\mathcal{O}_{Z,x}$. If $\langle\cdot,\cdot\rangle : V \times V \to j^{\triangle}(\mathcal{R})^p_x$ is some symmetric space over $\kappa(z)$ then, up to devisage, the form on the image of $\big(V, \langle\cdot,\cdot\rangle\big)$ under our residue map

$$\partial_2 : W\left(z, z^{\natural}(\mathcal{R})\right) \longrightarrow W\left(x, x^{\natural}(\mathcal{R}))\right)$$

will be given by $d^p_{j^{\triangle}(\mathcal{R})_x}\big(\langle\cdot,\cdot\rangle\big)$. In total then, our useage of the structure $(X, \mathcal{R})$ of a residual complex on a scheme can be summarised as follows; from the *terms* $\mathcal{R}^*$ of $\mathcal{R}$ we extract the local twisting spaces $(-)^{\natural}(\mathcal{R})$ which replace the Ω-twisting of **Definition 1.2.8** and then from the *boundary maps* $d^*_{\mathcal{R}}$ extract the second residue maps.

Chapter 3

The Witt Complex

Let X be a scheme with a residual complex $\mathcal{R}$ that has a codimension function μ. We construct in this section a sequence of abelian group homomorphisms

$$... \to \bigoplus_{\mu(x)=p} W\left(x, x^{\natural}(\mathcal{R})\right) \longrightarrow \bigoplus_{\mu(x)=p+1} W\left(x, x^{\natural}(\mathcal{R})\right) \to ... \tag{3.1}$$

which we'll denote $W(X, \mathcal{R})$. The maps are defined as a coproduct over *second residue homomorphisms*

$$\partial_2 : W\left(z, z^{\natural}(\mathcal{R})\right) \longrightarrow W\left(x, x^{\natural}(\mathcal{R})\right)$$

generalising those of **Definition 1.2.4**, taken over the immediate specialisations $z \rightsquigarrow x$ with $\mu(z) = p$. In the case when $1/2 \in \Gamma(X, \mathcal{O}_X)$, we are able to show that our residue maps agree with those of [2] - hence in this case $W(X, \mathcal{R})$ really forms a complex. The difference between the residue maps we define in this chapter and those of [5] which are used to construct the Witt complexes of [1, 2] is that they don't *need* the assumption $1/2 \in \Gamma(X, \mathcal{O}_X)$ or any machinery of derived categories to be defined.

3.1 A generalised second residue homomorphism

Let's suppose for a moment that F is the fraction field of a 1-dimensional Noetherian local domain (A, m, κ) which has a residual complex $\mathcal{R}$ that is concentrated in degrees p and $p+1$. We'll write $X = \mathrm{Spec}(A)$ and then ζ for the generic point of X. In this situation, our *second residue homomorphism* $\partial_2^{\mathcal{R}}$ is defined so that it sits in a commutative diagram as below.

$$\begin{array}{ccc} W\big(F,\mathcal{R}^p\big) & \xrightarrow{d_W(\mathcal{R})} & W\big(f.l.Mod_A,\mathcal{R}^{p+1}\big) \\ {\scriptstyle \big(\operatorname{tr}_{\pi_{\zeta,X}}(\mathcal{R}_\zeta)^p\big)_*}\Big\uparrow & & \Big\uparrow{\scriptstyle \big(\operatorname{tr}_{\pi_{m,X}}(\mathcal{R})^{p+1}\big)_*} \\ W\big(F,\zeta^\natural(\mathcal{R})\big) & \xrightarrow[\partial_2^{\mathcal{R}}]{} & W\big(\kappa,m^\natural(\mathcal{R})\big) \end{array}$$

Both vertical maps are isomorphisms by **Proposition 2.3.4**; the right hand isomorphism being non-trivial while the one on the left is essentially just an annoyance of the signs

$$\operatorname{tr}_{\pi_{\zeta,X}}(\mathcal{R}_\zeta)^p : \zeta^\natural(\mathcal{R}) = \operatorname{Hom}_F\left(F,\mathcal{R}^p_\zeta\right) \xrightarrow{(-1)^{p(p+1)/2}\mathrm{ev}@1_F} \mathcal{R}^p_\zeta = \mathcal{R}^p$$

intervening from **Proposition 2.1.28**. So our main construction is really of the map $d_W(\mathcal{R}) : W\big(F,\mathcal{R}^p\big) \to W\big(f.l.Mod_A,\mathcal{R}^{p+1}\big)$ - the most difficult thing involved being choosing the right generality on which it should be defined.

There are two ways one might try create a more general setting from what we have so far - firstly one can drop the assumption that X be connected and replace the Witt group $W\big(F,\mathcal{R}^p\big)$ with the group $W\big(f.l.Mod_{S^{-1}A},\mathcal{R}^p\big)$; here S is the collection of elements of the maximal ideal m not contained in any of the minimal primes $q_1,...,q_n$ of A. Denoting by $j_i : \operatorname{Spec}(A/q_i) \hookrightarrow X$ the irreducible components of X, the resulting map $d_W(\mathcal{R})$ is then a simple description of the sum of the maps $d_W\left(j_i^\triangle(\mathcal{R})\right)$ we would define anyway.

Secondly, one might decide that A is only supposed to be a semi-local ring - for which the group $W\big(f.l.Mod_A,\mathcal{R}^{p+1}\big)$ makes sense without any notational change. Again, in this instance the resulting map $d_W(\mathcal{R})$ describes the sum, after denoting by m_i the maximal ideals in A, of the maps $d_W(\mathcal{R}_{m_i})$ defined in both cases.

In either situation, the construction can be performed in exactly the same way and knowledge of both generalities has at various points in the production of this work appeared at least potentially useful - hence we mention both possibilities in this introduction. Ultimately, only the latter case appears to be in some sense "*needed*" for the remainder of our objectives; in the interest of brevity that is the one we work towards.

3.1.1 Finite length modules and $\mathcal{O}_X$-lattices

Arguably all the real *action* of our second residue map happens along the morphism $d_W(\mathcal{R})$ described above. One outcome of this subsection is the definition of, for a scheme X equipped with a residual complex $\mathcal{R}$ and immediate specialisation $x \rightsquigarrow x'$ with $\mu(x) = p$, a homomorphism

$$d_W^{x,x'}(\mathcal{R}) : W\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x^p\big) \longrightarrow W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}_{x'}^{p+1}\big)$$

of which the map $d_W(\mathcal{R})$ appearing in the most recent commutative square above is an example obtained by taking $X = \mathrm{Spec}(A)$. Perhaps more importantly, the construction really describes for each $x \in X_\mu^p$ the sum, taken over all the immediate specialisations $x \rightsquigarrow x'$ in X, of the maps

$$\sum_{x \rightsquigarrow x'} d_W^{x,x'}(\mathcal{R}) : W\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x^p\big) \longrightarrow \bigoplus_{\mu(y)=p+1} W\big(f.l.Mod_{\mathcal{O}_{X,y}}, \mathcal{R}_y^{p+1}\big)$$

in terms of a single $\mathcal{O}_X$-bilinear map on a coherent module. In this subsection, we will write $*_p$ for the exact functor

$$*_p = \mathcal{H}om_{\mathcal{O}_X}\left(-, \mathcal{R}^p\right) : QCoh_X^{op} \longrightarrow QCoh_X$$

Given a symmetric space $(M, \phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x^p\big)$ we obtain an isomorphism

$$(i_x)_*(\phi) : \iota_x(M) \to \iota_x\left(\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(M, \mathcal{R}_x^p\right)\right)$$

of quasi-coherent $\mathcal{O}_X$-modules, where $i_x : \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is the canonical map.

Lemma 3.1.1. *Let M be a finite length $\mathcal{O}_{X,x}$-module. Then we have canonical isomorphisms*

$$\iota_x\left(\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(M, \mathcal{R}_x^p\right)\right) \xrightarrow[h(x)]{\simeq} \mathcal{H}om_{\mathcal{O}_X}\left(\iota_x(M), \mathcal{R}(x)\right) \xrightarrow[(\rho(x))_*]{\simeq} \mathcal{H}om_{\mathcal{O}_X}\left(\iota_x(M), \mathcal{R}^p\right)$$

Proof. We describe first the map

$$h(x)_U : \iota_x\left(\mathrm{Hom}_{\mathcal{O}_{X,x}}(M, \mathcal{R}_x^p)\right)(U) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\big(\iota_x(M), \mathcal{R}(x)\big)(U)$$

of the $\mathcal{O}_X$-module map $h(x)$ over some open set U. If $x \notin U$ then both sides are zero so $h(x)_U = 0$. Otherwise we define $h(x)_U(f)$ for each $f \in \mathrm{Hom}_{\mathcal{O}_{X,x}}(M, \mathcal{R}_x^p)$ to be the sheaf morphism $\iota_x(M)|_U \to \mathcal{R}(x)|_U$ which over each open $V \subseteq U$ is f if $x \in V$ and zero otherwise.

Secondly, let us note that the map

$$\big(\rho(x)\big)_* : \mathcal{H}om_{\mathcal{O}_X}\big(\iota_x(M), \mathcal{R}(x)\big) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\big(\iota_x(M), \mathcal{R}^p\big)$$

given by postcomposition with $\rho(x) : \mathcal{R}(x) \hookrightarrow \mathcal{R}^p$ is certainly injective. Suppose that U is some open neighbourhood of x and take some $f \in \mathcal{H}om_{\mathcal{O}_X}\big(\iota_x(M), \mathcal{R}^p\big)(U)$. We have an isomorphism

$$\mathcal{R}^p|_U \xrightarrow{\simeq} \bigoplus_{y \in U^p_\mu} \mathcal{R}(y)|_U$$

of sheaves of $\mathcal{O}_U$-modules coming straight from the definition of residual complexes; so for each $y \in U^p_\mu$ not equal to x we may consider the map

$$f(y) : \iota_x(M)|_U \longrightarrow \mathcal{R}(y)|_U$$

which is f postcomposed with the projection $\mathcal{R}^p|_U \twoheadrightarrow \mathcal{R}(y)|_U$. Since $\mu(x) = \mu(y)$, we can find some open neighbourhood $V \subseteq U$ of y not containing x. Then for any open set $x \in U' \subseteq U$ and $m \in M = \iota_x(M)(U')$ we have

$$f(y)_{U'}(m)|_{V \cap U'} = f(y)_{V \cap U'}(m|_{V \cap U'}) = 0$$

so $f(y) = 0$ for any $y \in U^p_\mu$ not equal to x and hence f is in the image of $(\rho(x))_*$. $\square$

It is only up to this isomorphism that our ι_x notation does not just refer to the push-forward along the canonical map $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$.

Definition 3.1.2. If $(M, \phi) \in \mathit{Sym}\big(f.l.\mathit{Mod}_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$ is a symmetric space, then we write

$$\iota_x(\phi) : \iota_x(M) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\big(\iota_x(M), \mathcal{R}^p\big)$$

for the isomorphism of quasi-coherent $\mathcal{O}_X$-modules which is the postcomposition of $(i_x)_*(\phi)$ by the isomorphism of the above lemma.

Definition 3.1.3. If M is a finite length $\mathcal{O}_{X,x}$-module, then we define an $\mathcal{O}_X$**-lattice** inside M to be an inclusion $i : \mathcal{L} \hookrightarrow \iota_x(M)$ of a coherent submodule such that the localisation $i_x : \mathcal{L}_x \to M$ is an isomorphism.

Remark 2. *It is clear that any finite length $\mathcal{O}_{X,x}$-module admits an $\mathcal{O}_X$-lattice. Indeed, for any generating set of M as an $\mathcal{O}_{X,x}$-module the submodule of $\iota_x(M)$ globally generated as an $\mathcal{O}_X$-module by that generating set is an $\mathcal{O}_X$-lattice inside M.*

It may be helpful to also note that the property of a coherent submodule $\mathcal{L} \hookrightarrow \iota_x(M)$ being a lattice may be checked affine locally around x; we simply require for any (equivalently an) affine open neighbourhood $U = \mathrm{Spec}(A)$ of x and $m \in M$ that there exists some $a \in A \setminus x$ such that $am \in \mathcal{L}(U)$.

Definition 3.1.4. Let $(M, \phi) \in \mathit{Sym}\big(f.l.\mathit{Mod}_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$, and suppose that $i : \mathcal{L} \hookrightarrow \iota_x(M)$ is an $\mathcal{O}_X$-lattice. Then we define its **dual** (with respect to ϕ) to be the preimage of $\mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{L}, Z^p(\mathcal{R})\big)$ under the map

$$i^{*_p}\iota_x(\phi) : \iota_x(M) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{L}, \mathcal{R}^p\big)$$

We denote this submodule by $i^\flat : \mathcal{L}^\flat \hookrightarrow \iota_x(M)$.

Proposition 3.1.5. *For any symmetric space $(M, \phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$ and $\mathcal{O}_X$-lattice $i : \mathcal{L} \hookrightarrow \iota_x(M)$ we have that the dual $\mathcal{L}^\flat \hookrightarrow \iota_x(M)$ is again an $\mathcal{O}_X$-lattice inside M. Further, the map $i^{*_p}\iota_x(\phi)$ induces an isomorphism*

$$i^{*_p}\iota_x(\phi)i^\flat : \mathcal{L}^\flat \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{L}, Z^p(\mathcal{R})\big)$$

of $\mathcal{O}_X$-modules.

Proof. Everything in the statement can be checked affine locally; so let's suppose that $X = \operatorname{Spec}(A)$. As depicted in **Definition 3.1.4** the map

$$i^{*_p}\iota_x(\phi)i^\flat : \mathcal{L}^\flat \to \operatorname{Hom}_A\big(\mathcal{L}, Z^p(\mathcal{R})\big)$$

is an epimorphism because $\mathcal{R}^p$ is injective. Since the A-module $\mathcal{L}$ contains a generating set for M as an A_x-module, the non-degeneracy of ϕ further implies that it is injective.

Next suppose that M has length N as an A_x-module, and write I for the N^{th} power of the ideal in A corresponding to x. Write $j : Z = \operatorname{Spec}(A/I) \hookrightarrow X$ for the closed embedding so that by **Proposition 2.1.27** we have that $j^\triangle(\mathcal{R}) = \operatorname{Hom}_A(A/I, \mathcal{R})$ is a residual complex over Z. Since $\mathcal{L}$ is a module over A/I we have a canonical isomorphism

$$\big(\operatorname{tr}_j(\mathcal{R})^p\big)_* : \operatorname{Hom}_{A/I}\big(\mathcal{L}, j^\triangle(\mathcal{R})^p\big) \xrightarrow{\simeq} \operatorname{Hom}_A\big(\mathcal{L}, \mathcal{R}^p\big)$$

induced by postcomposition with $\operatorname{tr}_j(\mathcal{R})^p$. Since $\operatorname{tr}_j(\mathcal{R})$ is a monomorphism of chain complexes, the submodule of interest $\operatorname{Hom}_A\big(\mathcal{L}, Z^p(\mathcal{R})\big)$ corresponds under this isomorphism to $\operatorname{Hom}_A\big(\mathcal{L}, Z^p(j^\triangle(\mathcal{R})\big)$. Further, the generic point of Z has codimension p according to $j^\triangle(\mathcal{R})$, so we have that $Z^p(j^\triangle(\mathcal{R})) = H^p(j^\triangle(\mathcal{R}))$ is a coherent A/I-module; hence also a coherent A-module. We may now conclude that $\operatorname{Hom}_A\big(\mathcal{L}, Z^p(\mathcal{R})\big)$ is a coherent A-module, so too must $\mathcal{L}^\flat$ be given that they are isomorphic. $\square$

On account of this proposition, we are justified in referring to $\mathcal{L}^\flat$ as the **dual lattice** of $\mathcal{L}$ with respect to ϕ. For each affine open neighbourhood $U = \operatorname{Spec}(A)$ of x we can describe $\mathcal{L}^\flat|_U$ as being the A-submodule of M consisting of those $m \in M$ such that

$$\langle m, l\rangle_\phi \in Z^p\big(\mathcal{R}^p|_U\big)$$

for any $l \in \mathcal{L}(U)$.

Definition 3.1.6. Let $(M, \phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$ be a symmetric space and $\mathcal{L}$ an $\mathcal{O}_X$-lattice inside M. Then when $\mathcal{L} \leq \mathcal{L}^\flat$ we call $\mathcal{L}$ **self-dual** with respect to ϕ. If $\mathcal{L} = \mathcal{L}^{\flat\flat}$ then we call $\mathcal{L}$ **non-degenerate** with respect to ϕ.

Lemma 3.1.7. *Let (M, ϕ) be a symmetric space in $\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x\big)$ and $y \in X$ be a specialisation of x. Then for any lattice $i : \mathcal{L} \hookrightarrow \iota_x(M)$ we have that the localisation*

$i_y^\flat : \mathcal{L}_y^\flat \hookrightarrow M$ is precisely the dual lattice of $i_y : \mathcal{L}_y \hookrightarrow M$ with respect to ϕ on the local scheme $X_y = \mathrm{Spec}(\mathcal{O}_{X,y})$. In particular the properties of being non-degenerate or self-dual are both stable under localisation to y.

The non-degenerate self-dual $\mathcal{O}_X$-lattices are the ones of greatest utility for us. It is hence essential to establish that they can always be found and the remaining items below are used to establish that our construction of $d_W^{x,x'}(\mathcal{R})(M,\phi)$ is independent of the choice of lattice involved.

Proposition 3.1.8. *For any symmetric space $(M,\phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}_x^p\big)$ we have the following relations for any pair of $\mathcal{O}_X$-lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ inside M.*

1. *If $\mathcal{L}_1 \leq \mathcal{L}_2$ then the reverse containment $\mathcal{L}_2^\flat \leq \mathcal{L}_1^\flat$ holds on dual lattices.*
2. *The containment $\mathcal{L}_1 \leq \mathcal{L}_1^{\flat\flat}$ always holds.*
3. *The intersection $\mathcal{L}_1 \cap \mathcal{L}_2$ is again a lattice, which is non-degenerate and self dual if both $\mathcal{L}_1$ and $\mathcal{L}_2$ are.*

Finally, there exists an $\mathcal{O}_X$-lattice non-degenerate and self-dual with respect to ϕ.

Proof. The first two items are clear. For the third we suppose that L_1 and L_2 are both non-degenerate and self-dual. Then taking the dual of the outer terms in the inclusions

$$\mathcal{L}_1 \cap \mathcal{L}_2 \leq \mathcal{L}_1 \leq \mathcal{L}_1^\flat$$

we obtain $(\mathcal{L}_1 \cap \mathcal{L}_2)^\flat \geq \mathcal{L}_1^{\flat\flat} = \mathcal{L}_1 \geq (\mathcal{L}_1 \cap \mathcal{L}_2)$ which reveals that $\mathcal{L}_1 \cap \mathcal{L}_2$ is self-dual. Secondly, taking the double dual of the inclusion $\mathcal{L}_1 \cap \mathcal{L}_2 \leq \mathcal{L}_1$ and using that $\mathcal{L}_1$ is non-degenerate we obtain that $(\mathcal{L}_1 \cap \mathcal{L}_2)^{\flat\flat} \leq \mathcal{L}_1$. Similarly one obtains $(\mathcal{L}_1 \cap \mathcal{L}_2)^{\flat\flat} \leq \mathcal{L}_2$ - revealing that $\mathcal{L}_1 \cap \mathcal{L}_2$ is also non-degenerate.

For the final point we prove that given *any* lattice $\mathcal{L}$ the intersection $\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat}$ is self-dual and non-degenerate. Since M certainly admits a lattice, this gives us the required existence. First, we take the dual of the inclusion $\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat} \leq \mathcal{L}^\flat$ to learn that

$$\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat} \leq \mathcal{L}^{\flat\flat} \leq (\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat})^\flat$$

and so that $\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat}$ is self-dual. Further, since $\mathcal{L} \leq \mathcal{L}^{\flat\flat}$ we have from the above that $\mathcal{L} \leq (\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat})^\flat$, the dual of which is

$$(\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat})^{\flat\flat} \leq \mathcal{L}^\flat$$

We secondly take the dual of the inclusion $\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat} \leq \mathcal{L}^{\flat\flat}$ to obtain

$$\mathcal{L}^\flat \leq \mathcal{L}^{\flat\flat\flat} \leq (\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat})^\flat$$

Taking the dual of the outer most inclusion above we obtain that also

$$(\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat})^{\flat\flat} \leq \mathcal{L}^{\flat\flat}$$

so that $(\mathcal{L}^\flat \cap \mathcal{L}^{\flat\flat})$ is non-degenerate. $\square$

If $\mathcal{L}$ is an $\mathcal{O}_X$-lattice inside M self-dual with respect to some form $\phi : M \to \mathrm{Hom}_A\left(M, \mathcal{R}^p_x\right)$, then the map

$$(i^\flat)^{*_p}\iota_x(\phi)i^\flat : \mathcal{L}^\flat \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{L}^\flat, \mathcal{R}^p\right)$$

where $i^\flat : \mathcal{L}^\flat \hookrightarrow \iota_x(M)$ is the embedding of the dual lattice, has an image consisting of those morphisms $\mathcal{L}^\flat \to \mathcal{R}^p$ which carry $\mathcal{L}$ into $Z^p(\mathcal{R})$.

Definition 3.1.9. Let $(M, \phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$ be a symmetric space with respect to which we suppose that $\mathcal{L}$ is a self-dual $\mathcal{O}_X$-lattice. Then we write $\phi^{\mathcal{L}}$ for the morphism

$$\phi^{\mathcal{L}} = \overline{(i^\flat)^{*_p}\iota_x(\phi)i^\flat} : \frac{\mathcal{L}^\flat}{\mathcal{L}} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\left(\frac{\mathcal{L}^\flat}{\mathcal{L}}, \frac{\mathcal{R}^p}{Z^p(\mathcal{R})}\right)$$

lifting that described above. We similarly write

$$\langle \cdot, \cdot \rangle^{\mathcal{L}}_\phi : \frac{\mathcal{L}^\flat}{\mathcal{L}} \times \frac{\mathcal{L}^\flat}{\mathcal{L}} \longrightarrow \frac{\mathcal{R}^p}{Z^p(\mathcal{R})}$$

for the associated bilinear map of $\mathcal{O}_X$-modules. Finally, we write

$$\left(d^p_{\mathcal{R}}\right)_* \phi^{\mathcal{L}} : \mathcal{L}^\flat/\mathcal{L} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{L}^\flat/\mathcal{L},\ \mathcal{R}^{p+1}\right)$$

for the map $\phi^{\mathcal{L}}$ followed by the map $\left(d^p_{\mathcal{R}}\right)_* = \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{L}^\flat/\mathcal{L}, d^p_{\mathcal{R}}\right)$ and similarly we obtain a bilinear map

$$d^p_{\mathcal{R}}\big(\langle \cdot, \cdot \rangle^{\mathcal{L}}_\phi\big) : \mathcal{L}^\flat/\mathcal{L} \times \mathcal{L}^\flat/\mathcal{L} \longrightarrow \mathcal{R}^{p+1}$$

Note that the coherent module $\mathcal{L}^\flat/\mathcal{L}$ is supported in μ-codimension $\geq p+1$. Indeed, the only point of μ-codimension less than $p+1$ at which the stalk of $\mathcal{L}^\flat$ didn't vanish was x - so since both $\mathcal{L}$ and $\mathcal{L}^\flat$ localise at x to give M we have that $\mathcal{L}^\flat/\mathcal{L}$ vanishes at x. Secondly, we note that if $\mathcal{L}$ is further taken to be non-degenerate the map $\phi^{\mathcal{L}}$, and hence also $\left(d^p_{\mathcal{R}}\right)_*\phi^{\mathcal{L}}$, is an injective morphism of $\mathcal{O}_X$-modules.

Definition 3.1.10. For a space $(M, \phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$ and self-dual $\mathcal{O}_X$-lattice $\mathcal{L}$ inside M, we write $d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M, \phi)\big)$ for the pair

$$\Big(\mathcal{L}^\flat/\mathcal{L},\ (-1)^{p+1} d^p_{\mathcal{R}}\big(\langle \cdot, \cdot \rangle^{\mathcal{L}}_\phi\big)\Big)$$

For any immediate specialisation $x \rightsquigarrow x'$ we write $d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M,\phi)\big)_{x'}$ for the localisation of the above; explicitly then, this is the map

$$(-1)^{p+1} d^p_{\mathcal{R}_{x'}}\big(\langle\cdot,\cdot\rangle^{\mathcal{L}_{x'}}_{\phi}\big) : \frac{\mathcal{L}^\flat_{x'}}{\mathcal{L}_{x'}} \times \frac{\mathcal{L}^\flat_{x'}}{\mathcal{L}_{x'}} \xrightarrow{(\overline{\alpha},\overline{\beta}) \mapsto (-1)^{p+1} d^p_{\mathcal{R}_{x'}}\left(\rho(x,X_{x'})\left(\langle\alpha,\beta\rangle_\phi\right)\right)} \mathcal{R}^{p+1}_{x'}$$

In the above explicit description, the use of the map $\rho(x, X_{x'}) : \mathcal{R}^p_x \hookrightarrow \mathcal{R}^p_{x'}$ is perhaps a little pedantic; the fact that this map should in theory be included could've been deduced from the use of the boundary map $d^p_{\mathcal{R}_{x'}}$ immediately afterwards. In the future, we will omit such appearances of ρ-maps whenever, as above, the gained ease of notation outweighs the additional potential for confusion. In this description, we were also driving at highlighting that we have

$$d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M,\phi)\big)_{x'} = d^p_{\mathcal{R}_{x'}}\big(\mathcal{L}_{x'} \hookrightarrow (M,\phi)\big)$$

and we further note that if the lattice $\mathcal{L}$ in the above definition is taken to be non-degenerate, then the map

$$\big(d^p_{\mathcal{R}_{x'}}\big)_* \phi^{\mathcal{L}_{x'}} : \frac{\mathcal{L}^\flat_{x'}}{\mathcal{L}_{x'}} \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x'}}\left(\frac{\mathcal{L}^\flat_{x'}}{\mathcal{L}_{x'}}, \mathcal{R}^{p+1}_{x'}\right)$$

associated to the bilinear map $d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M,\phi)\big)_{x'}$ is non-degenerate. Indeed, the non-degeneracy of ϕ ensures that this map is injective and it is then further seen to be surjective because the functor $\mathrm{Hom}_{\mathcal{O}_{X,x'}}\big(-, \mathcal{R}^{p+1}_{x'}\big)$ preserves lengths.

Proposition 3.1.11. *For any $(M,\phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$ the isometry class of the symmetric space $d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M,\phi)\big)_{x'}$ in the Witt group $W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\big)$ does not depend on the self-dual non-degenerate $\mathcal{O}_X$-lattice $\mathcal{L} \hookrightarrow \iota_x(M)$.*

Proof. We may assume that $X = \mathrm{Spec}(\mathcal{O}_{X,x'})$, and first suppose we have an inclusion $P \leq L$ of non-degenerate self-dual $\mathcal{O}_{X,x'}$-lattices in M. Then we have $P \leq L \leq L^\flat \leq P^\flat$ and since L is self-dual, L/P is a sublagrangian of $d^p_{\mathcal{R}}\big(P \hookrightarrow (M,\phi)\big)$ with orthogonal $L^\flat/P$. The sublagrangian reduction **Proposition 2.2.14** then yields

$$d^p_{\mathcal{R}}\big(P \hookrightarrow (M,\phi)\big) = d^p_{\mathcal{R}}\big(L \hookrightarrow (M,\phi)\big) \in W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}\big)$$

For two non-degenerate self-dual lattices L_1 and L_2, we may apply this result to the pair of inclusions $L_1 \cap L_2 \leq L_1, L_2$ to obtain the statement for L_1 and L_2. ☐

For a symmetric space $(M,\phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big)$, we may hence write simply $d^p_{\mathcal{R}}(M,\phi)_{x'}$ for the class in $W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\big)$ of $d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M,\phi)\big)_{x'}$ where $\mathcal{L} \hookrightarrow \iota_x(M)$ is some $\mathcal{O}_X$-lattice non-degenerate and self-dual with respect to ϕ. It is immediately seen that the map $(M,\phi) \mapsto d^p_{\mathcal{R}}(M,\phi)_{x'}$ respects isometry and orthogonal sums.

Proposition 3.1.12. *The assignment*

$$[M,\psi]\mapsto d^p_{\mathcal{R}}(M,\psi)_{x'}\in W\big(f.l.Mod_{\mathcal{O}_{X,x'}},\mathcal{R}^{p+1}_{x'}\big)$$

of isometry classes of symmetric spaces in $\big(f.l.Mod_{\mathcal{O}_{X,x}},\mathcal{R}^p_x\big)$ *induces group homomorphisms*

$$d^{x,x'}_{W}(\mathcal{R}):W\big(f.l.Mod_{\mathcal{O}_{X,x}},\mathcal{R}^p_x\big)\longrightarrow W\big(f.l.Mod_{\mathcal{O}_{X,x'}},\mathcal{R}^{p+1}_{x'}\big)$$

$$d^{x,x'}_{GW}(\mathcal{R}):GW\big(f.l.Mod_{\mathcal{O}_{X,x}},\mathcal{R}^p_x\big)\longrightarrow W\big(f.l.Mod_{\mathcal{O}_{X,x'}},\mathcal{R}^{p+1}_{x'}\big)$$

Proof. We only need to check that metabolic spaces are carried to zero in the common Witt groups on the right hand side, and it suffices to suppose that $X=\mathrm{Spec}(R)$ where $R=\mathcal{O}_{X,x'}$. So take $(M,\psi)\in Sym\big(f.l.Mod_{R_x},\mathcal{R}^p_x\big)$ to be such a space, having Lagrangian

$$0\to F\xrightarrow{i}M\xrightarrow{i^*\psi}F^*\to 0$$

and take some R-lattice $L\hookrightarrow M$ non-degenerate and self-dual with respect to ψ. Then we obtain the short exact sequence of short exact sequences below.

$$\begin{tikzcd}
L\cap F \arrow[r, tail] \arrow[d, tail] & L \arrow[r, two heads] \arrow[d, tail] & i^*\psi(L) \arrow[d, tail] \\
L^\flat\cap F \arrow[r, tail] \arrow[d, two heads] & L^\flat \arrow[r, two heads] \arrow[d, two heads] & i^*\psi(L^\flat) \arrow[d, two heads] \\
\frac{L^\flat\cap F}{L\cap F} \arrow[r, tail] & \frac{L^\flat}{L} \arrow[r, two heads] & \frac{i^*\psi(L^\flat)}{i^*\psi(L)}
\end{tikzcd}$$

The submodule

$$\frac{L^\flat\cap F}{L\cap F}=\frac{L^\flat\cap F+L}{L}\longrightarrow L^\flat/L$$

is a sublagrangian in the space $d^p_{\mathcal{R}}\big(L\hookrightarrow(M,\psi)\big)$. We observe $P=L^\flat\cap F+L\leq M$ is again a lattice in M, for which we have the inclusions $L\leq P\leq P^\flat\leq L^\flat$. The submodule $P^\flat/L\leq L^\flat/L$ is the orthogonal space to the sublagrangian P/L, hence by sublagrangian reduction we simultaneously learn that P must be non-degenerate and reobtain the well-definedness result of the previous proposition for the lattices L and P, namely that

$$d^p_{\mathcal{R}}\big(L\hookrightarrow(M,\psi)\big)=d^p_{\mathcal{R}}\big(P\hookrightarrow(M,\psi)\big)$$

We now consider two possibilities; if $P=L$ then

$$L^\flat=P^\flat=\big\{x\in L^\flat|\langle x,L^\flat\cap F\rangle_\psi\subseteq Z^p(\mathcal{R})\big\}\quad(*)$$

and from the lowest short exact sequence in our short exact sequence of short exact sequences, we obtain in this case an isometry

$$d^p_{\mathcal{R}}(M,\psi) = \left[\frac{i^*\psi(L^\flat)}{i^*\psi(L)}, \left(d^p_{\mathcal{R}}\right)_*\right]$$

where $\left(d^p_{\mathcal{R}}\right)_*$ denotes postcomposition with $d^p_{\mathcal{R}}$. But from the condition $(*)$ we read that $\left(d^p_{\mathcal{R}}\right)_*(i^*\psi(L^\flat)) = 0$, so we're done in this case.

Otherwise, we repeat the argument with L replaced by P. On account of the inclusions $L \lneq P \leq P^\flat \lneq L^\flat$ we have that the length of the R-module $P^\flat/P$ is strictly less than that of $L^\flat/L$. So with repetition of this argument we are bound to eventually win. □

Definition 3.1.13. Let $(X,\mathcal{R})$ be a scheme with a residual complex. Then for any immediate specialisation $x \rightsquigarrow x'$ in X with $\mu(x) = p$ say, we write

$$\mathrm{res}^{x,x'}_{GW}(\mathcal{R}) : GW\big(x, x^\natural(\mathcal{R})\big) \longrightarrow W\big(x', x'^\natural(\mathcal{R})\big)$$

for the **residue homomorphism** which is defined to be the map forming a commutative diagram

$$\begin{tikzcd}
GW\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big) \arrow[r, "d^{x,x'}_{GW}(\mathcal{R})"] & W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\big) \\
W\big(x, x^\natural(\mathcal{R})\big) \arrow[u, "\left(\mathrm{tr}_{\pi_{x,X}}(\mathcal{R}_x)^p\right)_*"] \arrow[r, "\mathrm{res}^{x,x'}_{GW}(\mathcal{R})"'] & W\big(x', x'^\natural(\mathcal{R})\big) \arrow[u, "\left(\mathrm{tr}_{\pi_{x',X}}(\mathcal{R}_{x'})^{p+1}\right)_*"']
\end{tikzcd}$$

We define the map

$$\mathrm{res}^{x,x'}_{W}(\mathcal{R}) : W\big(x, x^\natural(\mathcal{R})\big) \longrightarrow W\big(x', x'^\natural(\mathcal{R})\big)$$

in the same way.

Proposition 3.1.14. *Let $j : Z \hookrightarrow X$ be a closed embedding containing the immediate specialisation $x \rightsquigarrow x'$, and let $(M,\phi) \in Sym\big(f.l.Mod_{\mathcal{O}_{Z,x}}, j^\triangle(\mathcal{R})^p_x\big)$. Then for any $\mathcal{O}_Z$-lattice $\mathcal{L}$ inside M we have that $j_*\mathcal{L}$ is an $\mathcal{O}_X$-lattice inside M with*

$$\big(j_*\mathcal{L}\big)^\flat\big((\mathrm{tr}_j(\mathcal{R})^p_x)_*\phi\big) = j_*\big(\mathcal{L}^\flat(\phi)\big)$$

As a result, we obtain the following commutative square.

$$
\begin{tikzcd}
GW\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big) \arrow[rr, "d^{x,x'}_{GW}(\mathcal{R})"] & & W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\big) \\
GW\big(f.l.Mod_{\mathcal{O}_{Z,x}}, j^{\triangle}(\mathcal{R})^p_x\big) \arrow[u, "\big(\mathrm{tr}_j(\mathcal{R})^p_x\big)_*"] \arrow[rr, "d^{x,x'}_{GW}\big(j^{\triangle}(\mathcal{R})\big)"'] & & W\big(f.l.Mod_{\mathcal{O}_{Z,x'}}, j^{\triangle}(\mathcal{R})^{p+1}_{x'}\big) \arrow[u, "\big(\mathrm{tr}_j(\mathcal{R})^{p+1}_{x'}\big)_*"']
\end{tikzcd}
$$

Proof. It is perhaps inconvenient that our ι_x notation does not distinguish between the schemes Z and X. Nonetheless, we make further abuses of notation by writing the identity $\iota_x(M) = j_*\iota_x(M)$ where on the left hand side it might have been more correct to write $\iota_x\big((j_x)_*(M)\big)$ in which $j_x : \mathrm{Spec}(\mathcal{O}_{Z,x}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is the fibre of j. It is then clear that $j_*\mathcal{L} \hookrightarrow j_*\iota_x(M) = \iota_x(M)$ is an $\mathcal{O}_X$-lattice inside M. After noting that in the below commutative square

$$
\begin{tikzcd}
j_*j^{\triangle}(\mathcal{R})(x) \arrow[r, "j_*\rho(x,Z)"] \arrow[d] & j_*j^{\triangle}(\mathcal{R})^p \arrow[d, "\mathrm{tr}_j(\mathcal{R})^p"] \\
\mathcal{R}(x) \arrow[r, "\rho(x,X)"'] & \mathcal{R}^p
\end{tikzcd}
$$

the map $j_*j^{\triangle}(\mathcal{R})(x) \to \mathcal{R}(x)$ is over each open neighbourhood of x given by the localisation $\mathrm{tr}_j(\mathcal{R})^p_x$, we find that we have a commutative diagram

$$
\begin{tikzcd}
j_*\iota_x(M) \arrow[r, "j_*\iota_x(\phi)"] \arrow[dd, equal] & j_*\big[\iota_x(M), j^{\triangle}(\mathcal{R})^p\big]_{\mathcal{O}_Z} \arrow[r, two heads] \arrow[d, equal] & j_*\big[\mathcal{L}, j^{\triangle}(\mathcal{R})^p\big]_{\mathcal{O}_Z} \arrow[dd, "\big(\mathrm{tr}_j(\mathcal{R})^p\big)_*"] & j_*\big[\mathcal{L}, Z^p(j^{\triangle}(\mathcal{R}))\big]_{\mathcal{O}_Z} \arrow[l, hook'] \arrow[dd, "\big(\mathrm{tr}_j(\mathcal{R})^p\big)_*"] \\
 & \big[j_*\iota_x(M), j_*j^{\triangle}(\mathcal{R})^p\big]_{\mathcal{O}_X} \arrow[d, "\big(\mathrm{tr}_j(\mathcal{R})^p\big)_*"] & & \\
\iota_x(M) \arrow[r, "\iota_x\big(\big(\mathrm{tr}_j(\mathcal{R}^p_x)\big)_*\phi\big)"'] & \big[\iota_x(M), \mathcal{R}^p\big]_{\mathcal{O}_X} \arrow[r, two heads] & \big[j_*\mathcal{L}, \mathcal{R}^p\big]_{\mathcal{O}_X} & \big[j_*\mathcal{L}, Z^p(\mathcal{R})\big]_{\mathcal{O}_X} \arrow[l, hook']
\end{tikzcd}
$$

in which every vertical map is an isomorphism. In detail, the commutative square we gave above is used to build the square on the left, while the fact the every vertical map is an isomorphism follows from two observations - firstly that because j is a closed embedding the pushforward j_* commutes with the sheaf-homs, and secondly that because the $j_*j^{\triangle}(\mathcal{R})$ is affine locally isomorphic to the subcomplex of elements annihilated by the ideal sheaf of Z we can factor any morphism $\iota_x(M) \to \mathcal{R}^p$ or $j_*\mathcal{L} \to \mathcal{R}^p$ through the trace map $\mathrm{tr}_j(\mathcal{R})^p$. Now since $\big(j_*(\mathcal{L})\big)^\flat\big((\mathrm{tr}_j(\mathcal{R})^p_x)_*\phi\big)$ is the preimage of the lower rightmost term $\big[j_*\mathcal{L}, Z^p(\mathcal{R})\big]_{\mathcal{O}_X}$ under the epimorphism $\iota_x(M) \twoheadrightarrow$

$\big[j_*\mathcal{L},\mathcal{R}^p\big]_{\mathcal{O}_X}$ while $j_*\big(\mathcal{L}^\flat(\phi)\big)$ is the preimage under the epimorphism along the top of the diagram of the upper rightmost term $j_*\big[\mathcal{L},Z^p(j^\triangle(\mathcal{R}))\big]_{\mathcal{O}_Z}$ we do indeed have

$$\big(j_*\mathcal{L}\big)^\flat\big((\mathrm{tr}_j(\mathcal{R})^p_x)_*\phi\big)=j_*\big(\mathcal{L}^\flat(\phi)\big)$$

To check the commutativity of the remaining square, let's continue with our symmetric space $(M,\phi)\in Sym\big(f.l.Mod_{\mathcal{O}_{Z,x}},j^\triangle(\mathcal{R})^p_x\big)$ and take an $\mathcal{O}_Z$-lattice $\mathcal{L}$ non-degenerate and self-dual with respect to ϕ. So we have that $\big(\mathrm{tr}_j(\mathcal{R})^{p+1}_{x'}\big)_*\big(d^{x,x'}_{GW}(j^\triangle(\mathcal{R}))(M,\phi)\big)$ is the form on $\mathcal{L}^\flat_{x'}/\mathcal{L}_{x'}$ given by

$$\langle\overline{\alpha},\overline{\beta}\rangle=(-1)^{p+1}\,\mathrm{tr}_j(\mathcal{R}_{x'})^{p+1}\Big(d^p_{j^\triangle(\mathcal{R})_{x'}}\Big(\rho\big(x,Z_{x'}\big)\big(\langle\alpha,\beta\rangle_\phi\big)\Big)\Big)$$

To compare this space with $d^{x,x'}_{GW}(\mathcal{R})\big(M,\big(\mathrm{tr}_j(\mathcal{R})^p_x\big)_*\phi\big)$ we of course use $j_*\mathcal{L}$ as our non-degenerate self-dual $\mathcal{O}_X$ lattice. We find then that this second space is the form on $j_*\mathcal{L}_{x'}/j_*\mathcal{L}_{x'}$ - which as an $\mathcal{O}_{X,x'}$ is just the $\mathcal{O}_{Z,x'}$-module $\mathcal{L}^\flat_{x'}/\mathcal{L}_{x'}$ viewed as an $\mathcal{O}_{X,x'}$-module - given by

$$\langle\overline{\alpha},\overline{\beta}\rangle'=(-1)^{p+1}d^p_{\mathcal{R}_{x'}}\Big(\rho\big(x,X_{x'}\big)\big(\mathrm{tr}_j(\mathcal{R})^p_x(\langle\alpha,\beta\rangle_\phi)\big)\Big)$$

After confirming that the map $\rho\big(x,Z_{x'}\big):\mathrm{Hom}_{\mathcal{O}_{X,x}}\big(\mathcal{O}_{Z,x},\mathcal{R}^p_x\big)\to\mathrm{Hom}_{\mathcal{O}_{x,x'}}\big(\mathcal{O}_{Z,x'},\mathcal{R}^p_{x'}\big)$ is given by precomposition with the localisation map $\mathcal{O}_{Z,x'}\to\mathcal{O}_{Z,x}$ and postcomposition with $\rho\big(x,X_{x'}\big)$ one finds that $\mathrm{tr}_j(\mathcal{R})^p_{x'}\circ\rho\big(x,Z_{x'}\big)=\rho\big(x,X_{x'}\big)\circ\mathrm{tr}_j(\mathcal{R})^p_x$. This observation, together with the fact that $\mathrm{tr}_j(\mathcal{R}_{x'})$ is a morphism of chain complexes reveals that the two forms just given are the same. $\square$

The reader will note that we have not yet checked that the condition our lattices be non-degenerate is not superfluous. In defining an analogue of rational equivalence for the Chow-Witt group, we will require a residue map whose domain is *V-theory*; the elements of which are isometry classes of spaces equipped with not one but two symmetric forms. It does not appear to be clear that one can always find a lattice non-degenerate and self-dual with respect to both forms at once; the following result, as well as being a slight simplification to our description of the residue maps we have so far, is hence also of structural necessity for our later definition of Chow-Witt groups.

Proposition 3.1.15. *Let $(V,\psi)\in Sym\big(f.l.Mod_{\mathcal{O}_{X,x}},\mathcal{R}^p_x\big)$ be a symmetric space with V a $\kappa(x)$-vector space. Then for any $\mathcal{O}_X$-lattice $\mathcal{L}\hookrightarrow\iota_x(V)$ and immediate specialisation $x\rightsquigarrow x'$ we have that the $\mathcal{O}_{X,x'}$-lattice $\mathcal{L}_{x'}$ is non-degenerate with respect to ψ. Hence if $\mathcal{L}$ is only supposed to be self-dual, then*

$$d^{x,x'}_{GW}(\mathcal{R})(V,\psi)=d^p_{\mathcal{R}}\big(\mathcal{L}\hookrightarrow(V,\psi)\big)_{x'}$$

Proof. We can suppose that $X=\mathrm{Spec}(\mathcal{O}_{X,x'})$ and for ease of notation write $\mathcal{O}_{X,x'}=R$ and $\mathcal{L}_{x'}=L$ for the R-lattice coming from L. Then we have the closed embedding of the subvariety $j:\mathrm{Spec}(R/x)\hookrightarrow X$ with generic point x - by the previous proposition we can further replace R by R/x and hence X by $\mathrm{Spec}(R/x)$.

Now R is a Cohen Macaulay local ring, hence by **Proposition 2.3.2** we know that $\mathcal{R}$ is an injective resolution of a canonical module. So let's write ω_R for the module $Z^p(\mathcal{R})$. Then as in **Proposition 3.1.5** we have isomorphisms $L^\flat \cong \mathrm{Hom}_R(L, \omega_R)$ and $L^{\flat\flat} \cong \mathrm{Hom}_R(\mathrm{Hom}_R(L, \omega_R), \omega_R)$ induced by ψ. Using this isomorphism, we see that the natural inclusion $L \hookrightarrow L^{\flat\flat}$ factors

$$\begin{tikzcd} L \arrow[rr, hook] \arrow[dr, "ev"', "\sim"] & & L^{\flat\flat} \\ & \mathrm{Hom}_R(\mathrm{Hom}_R(L, \omega_R), \omega_R) \arrow[ur, "\cong"'] & \end{tikzcd}$$

Here, we have noted that since L is a maximal Cohen-Macaulay R-module, the evaluation map is an isomorphism by [37, Thm 3.3.10]. $\square$

Notation 3.1.16. When $X = \mathrm{Spec}(\mathcal{R})$ with (R, m, κ) a DVR having field of fractions F and ζ denoting the zero ideal, we also write

$$\partial_2^{\mathcal{R}} : W\big(F, \zeta^\natural(\mathcal{R})\big) \longrightarrow W\big(\kappa, m^\natural(\mathcal{R})\big)$$

for the map $\mathrm{res}_W^{\zeta,m}(\mathcal{R})$.

We've included this piece of special notation to respect the fact that this is the setting in which both this work and [4] have a second residue homomorphism defined for Witt groups. Indeed, the notation is supposed to be reminiscent of the ∂_2^π notation of *loc. cit.* - the map ∂_2^π depends on a choice of uniformizer π while our map $\partial_2^{\mathcal{R}}$ depends on the choice of residual complex $\mathcal{R}$. Then *c.f.* **Example 2.1.24** we have the following comparison result.

Lemma 3.1.17. *Let (R, m, κ) be a DVR with field of fractions F and write π : $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(R)$ for the embedding of the closed point. Let $\mathcal{R}$ be the residual complex*

$$... \to 0 \to F \longrightarrow F/R \to 0 \to ...$$

concentrated in degrees -1 and 0, and take $\lambda \in m$ to be some uniformizer. Then, denoting by ζ the zero ideal of R, we have a commutative square

$$\begin{tikzcd} W\left(\zeta, \zeta^\natural(\mathcal{R})\right) \arrow[r, "\partial_2^{\mathcal{R}}"] & W\left(\kappa, m^\natural(\mathcal{R})\right) \\ W(F) \arrow[u, "||"] \arrow[r, "\partial_2^\lambda"'] & W(\kappa) \arrow[u, "\lambda_*"'] \end{tikzcd}$$

where ∂_2^λ is the residue map of [4] and λ_* is the isomorphism attached to the transfer map $\lambda_* : \kappa \to m^\natural(\mathcal{R})$ defined by

$$\mathrm{tr}_\pi(\mathcal{R})^0\big(\lambda_*(\overline{r})\big) = \frac{\overline{r}}{\lambda} \in F/R$$

for each $\overline{r} \in \kappa$.

Proof. The equals sign on the lefthand side of the square is a slight abuse of notation which we have made because the trace map

$$\mathrm{tr}_{\pi_{\zeta,\mathrm{Spec}(R)}}(\mathcal{R}_\zeta)^{-1} : \zeta^\natural(\mathcal{R}_\zeta) = \mathrm{Hom}_F(F, F) \longrightarrow F = \mathcal{R}^{-1}$$

has positive sign - that is it is simply the evaluation at one morphism. Let $u \in R^*$ and consider the generator $\langle u\lambda^n\rangle$ of $W(F)$. A self-dual lattice is here given by

$$L = R \cdot \lambda^{\lceil -\frac{n}{2} \rceil} \hookrightarrow F$$

We may then compute that

$$L^\flat = R \cdot \lambda^{\lceil -n - \lceil -\frac{n}{2} \rceil \rceil} \hookrightarrow F$$

So if n is even, we have $L^\flat = L$ and hence $\partial_2^{\mathcal{R}}(\langle u\lambda^n\rangle) = 0$. If n is odd, then let's set $m = -(n+1)/2$ so that $L = R \cdot \lambda^{m+1}$ and $L^\flat = R \cdot \lambda^m$, revealing that $L^\flat/L \cong \kappa$; precisely we take the isomorphism $\kappa \to L^\flat/L$ given by sending $1 \in \kappa$ to the image of λ^m in $L^\flat/L$. Since in the space $\langle u\lambda^n\rangle$ we have

$$\langle \lambda^m, \lambda^m\rangle = u\lambda^n\lambda^{2m} = u/\lambda$$

we learn that $\partial_2^{\mathcal{R}}(\langle u\lambda^n\rangle) = \lambda_*\big(\langle\overline{u}\rangle\big)$ when n is even. □

Remark 3. *Huang [38] provides an explicit construction of residual complexes which may be the simplest way to compare our twisting by residual complexes with the arguments of Schmid [7] which we in our background chapter christened Ω-twisting.*

Let $X = \mathrm{Spec}(A)$ with (A, m, κ) a one-dimensional local domain with field of fractions F, and $\mathcal{R}$ be a residual complex on A concentrated in degrees -1 and 0. Again, we denote by $\pi : \mathrm{Spec}(\kappa) \to X$ the embedding of the closed point and write ζ for the zero ideal of A. Then as in [4, Ch.1, 5.4] one may check that the tensor product gives both $W\left(F, \zeta^\natural(\mathcal{R})\right)$ and $W\left(\kappa, m^\natural(\mathcal{R})\right)$ the structure of $W(A)$-modules. For clarity we elucidate the first of these structure, the second being defined in the same way. If $B : M \times M \to A$ is some non-degenerate symmetric bilinear form on a finitely generated free A-module M, and $[V, \psi] \in W\left(F, \zeta^\natural(\mathcal{R})\right)$, then we set $[M, B][V, \psi] \in W\left(F, \zeta^\natural(\mathcal{R})\right)$ to be the inner product space over F with underlying vector space $M \otimes_A V$ and form defined by

$$\langle m_1 \otimes v_1, m_2 \otimes v_2 \rangle = B(m_1, m_2)\langle v_1, v_2\rangle_\psi \in \zeta^\natural(\mathcal{R})$$

Lemma 3.1.18. *If (A, m, κ) is a one-dimensional local domain as described in the paragraph above, then the second residue homomorphism*

$$\partial_2^{\mathcal{R}} : W\left(F, \zeta^\natural(\mathcal{R})\right) \longrightarrow W\left(\kappa, m^\natural(\mathcal{R})\right)$$

is $W(A)$-linear.

Proof. It is clear that $W\left(f.l.Mod_A, \mathcal{R}^0\right)$ inherits a $W(A)$-module structure via the trace map from $W\big(\kappa, m^\natural(\mathcal{R})\big)$, and one may check that this module structure is again given by the tensor product over A. We hence have that the isomorphism

$$\left(\mathrm{tr}_\pi(\mathcal{R})^0\right)_* : W\left(\kappa, m^\natural(\mathcal{R})\right) \longrightarrow W\left(f.l.Mod_A, \mathcal{R}^0\right)$$

is $W(A)$-linear so it suffices to check that $d_W^{\zeta,m}(\mathcal{R})$ is similarly linear. For this it is convenient to fix an isomorphism $\mathcal{R}^{-1} \cong F$, and it suffices to consider a rank one space $\langle \alpha \rangle \in W(F, \mathcal{R}^{-1})$. Since A is Cohen-Macaulay, the kernel $Z^{-1}(\mathcal{R})$ is a canonical module for A - let's denote it by ω. Further, without loss of generality, we may assume that $\alpha \cdot \omega \trianglelefteq A$ - thus we may assume that $\omega \hookrightarrow F$ is a self-dual A-lattice inside the space $\langle \alpha \rangle$.

Now let $B : M \times M \to A$ be an inner product on a finite rank free A-module M. We'll write

$$B\langle \cdot, \cdot \rangle_\psi : M \otimes_A F \times M \otimes_A F \longrightarrow \mathcal{R}^{-1}$$

for the inner product of $[M, B] \cdot [V, \psi]$. One may quickly check that $M \otimes \omega \hookrightarrow M \otimes F$ is a self-dual A-lattice inside this inner product space. We hence have that

$$d_{\mathcal{R}}^{-1}\big([M, B] \cdot [V, \psi]\big) = \left[\frac{(M \otimes \omega)^\flat}{M \otimes \omega}, d_{\mathcal{R}}^{-1}\Big(\big(B\langle \cdot, \cdot \rangle_\psi\big)^{M \otimes \omega}\Big)\right] \in W\left(f.l.Mod_A, \mathcal{R}^0\right)$$

On the other hand, we may write

$$[M, B] \cdot d_{\mathcal{R}}^{-1}\big([V, \psi]\big) = \left[M \otimes \frac{\omega^\flat}{\omega}, B d_{\mathcal{R}}^{-1}\big(\langle \cdot, \cdot \rangle_\psi^\omega\big)\right] \in W\left(f.l.Mod_A, \mathcal{R}^0\right)$$

Note that the composition

$$M \otimes \tfrac{\omega^\flat}{\omega} \xrightarrow{\ \cong\ } \tfrac{M \otimes \omega^\flat}{M \otimes \omega} \longrightarrow \tfrac{(M \otimes \omega)^\flat}{M \otimes \omega}$$

respects these symmetric forms, where the first map comes from the exactness of $M \otimes (-)$, while the second comes from the natural inclusion of A-lattices $M \otimes \omega^\flat \leq (M \otimes \omega)^\flat$ in $[M, B] \cdot [V, \psi]$. It hence suffices to show that this composition is an isomorphism, for which it in turn suffices to show that the inclusion $M \otimes \omega^\flat \leq (M \otimes \omega)^\flat$ is the identity. We establish this by observing that the inclusion fits into the below commutative diagram

$$\begin{tikzcd}
M \otimes \omega^\flat \arrow[rr] \arrow[dd, "\mathbf{Prop.\ 3.1.5}\ \wr\|"'] & & (M \otimes \omega)^\flat \arrow[d, "\mathbf{Prop.\ 3.1.5}"', "\|\wr"] \\
& & \mathrm{Hom}_A(M \otimes \omega, \omega) \arrow[d, "\|\wr\ \mathrm{adj}"] \\
M \otimes \mathrm{Hom}_A(\omega, \omega) \arrow[d, "\wr\|"', "{[37,\ \mathrm{Thm}\ 3.3.4]}"] & & \mathrm{Hom}_A(M, \mathrm{Hom}_A(\omega, \omega)) \arrow[d, "\|\wr\ {[37,\ \mathrm{Thm}\ 3.3.4]}"] \\
M \arrow[rr, "\overline{B}"'] & & \mathrm{Hom}_A(M, A)
\end{tikzcd}$$

where the maps are isomorphisms for the reasons referenced. □

While the map of $\mathcal{O}_X$-modules

$$(-1)^{p+1} \left(d^p_{\mathcal{R}}\right)_* \phi^{\mathcal{L}} : \mathcal{L}^\flat/\mathcal{L} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{L}^\flat/\mathcal{L}, \mathcal{R}^{p+1}\right) \qquad (*)$$

represents a compact description of the sum

$$\sum_{x \rightsquigarrow x'} d^{x,x'}_W(\mathcal{R})(M, \phi)$$

it is not immediately clear how to identify a useful category with duality in which $(*)$ becomes a genuine symmetric space. A tempting candidate would be the exact category $QCoh^{\geq p+1}_X$ of quasi-coherent sheaves supported in codimension $p+1$ with duality given by $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}^{p+1})$ and weak equivalences being those morphisms of $\mathcal{O}_X$-modules whose stalks are isomorphisms at any point of μ-codimension $p+1$. The problem here is that localisation to points of codimension $p+1$ is not a non-singular form functor. Such localisations are form functors

$$\left(\mathrm{Ch}^{b, \geq p+1}_{Coh}(X), qis, \mathcal{R}_X[p+1], \eta_{\mathcal{R}_X[p+1]}\right) \longrightarrow \left(\mathrm{Ch}^b_{f.l.}(Mod_{\mathcal{O}_{X,x'}}), qis, \mathcal{R}^{p+1}_{X,x'}, ev\right)$$

but concentrating $(-1)^{p+1} d^p_{\mathcal{R}}\left(\phi^{\mathcal{L}}\right)$ in degree zero does not form a symmetric space in the left hand side above - essentially because the module $\mathcal{L}^\flat/\mathcal{L}$ could admit maps into some $\mathcal{R}^i_X$ with $i > p+1$. Our comparison with Balmer's residue maps will identify a suitable symmetric space in the derived analogue of the left hand side. First, we give the following special case in which the object $d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M,\phi)\big)$ can be naturally identified as a symmetric space.

Proposition 3.1.19. *Let A be a semi-local domain with residual complex $\mathcal{R}$ concentrated in degrees p and $p+1$ and field of fractions F. Then after writing ζ for the zero ideal and $m_1, ..., m_n$ for the maximal ideals of A we have morphisms on Witt and Grothendieck-Witt groups*

$$d_{GW}(\mathcal{R}) : GW\big(F, \mathcal{R}^p\big) \longrightarrow W\big(f.l.Mod_A, \mathcal{R}^{p+1}\big)$$

defined by $[V,\psi] \mapsto d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (V,\psi)\big) \in W\big(f.l.Mod_A, \mathcal{R}^{p+1}\big)$. These maps fit into commutative diagrams

$$\begin{tikzcd} GW\big(F,\mathcal{R}^p\big) \arrow[rr, "d_{GW}(\mathcal{R})"] \arrow[d, equal] & & W\big(f.l.Mod_A, \mathcal{R}^{p+1}\big) \\ GW\big(F,\mathcal{R}^p\big) \arrow[rr, "\sum d^{\zeta,m_i}_{GW}(\mathcal{R})"'] & & \bigoplus_{i=1}^n W\big(f.l.Mod_{A_{m_i}}, \mathcal{R}^{p+1}_{m_i}\big) \arrow[u, "\bigoplus(\rho(m_i))_*"'] \end{tikzcd}$$

As usual, we also write $d_W(\mathcal{R})$ for the lift of the above map to the Witt group $W\big(F,\mathcal{R}^p\big)$.

Proof. The well-definedness of $d_{GW}(\mathcal{R})$ follows exactly as in **Propositions 3.1.11** and **3.1.12**. By the devisage **Proposition 2.3.4** the map $\bigoplus\big(\rho(m_i)\big)_*$ is an isomorphism whose inverse is given by the sum of localisation maps. The commutativity of the square hence follows immediately from the definition of $d^{\zeta,m_i}_{GW}(\mathcal{R})$. $\square$

3.2 Comparison with Balmer's residue maps

Let's fix for this section $(X,\mathcal{R})$ to be a scheme with a residual complex $\mathcal{R}$ that has codimension function μ which for simplicity we assume to be normalised. Then after writing $\# = [-,\mathcal{R}]_X$ we have that the quadruple $\big(D^b_{Coh}(X), \#, 1, \eta\big)$ is a triangulated category with duality as in **Definition 2.2.19**, with the double dual identification η being as in **Definition 2.1.6**. We write as usual $(\#_i, \delta_i, \eta_i)$ for the i^{th} shift of this duality as in **Definition 2.2.20**. To save space we'll denote this quadruple simply as $\big(D^b_{Coh}(X), \mathcal{R}\big)$.

We begin with an overview of the results of [2] to understand how Balmer's residue map of **Proposition 2.2.31** can be used to construct a Witt complex on X, and then argue that our residue map agrees with - or in other words is an explicit/alternative description of - the boundary maps appearing in this complex. We hence further assume throughout this section that $1/2 \in \Gamma(X, \mathcal{O}_X)$.

Definition 3.2.1. For each $n \in \mathbb{Z}$, we denote by $D^{\geq n}(X)$ the full subcategory of $D^b_{Coh}(X)$ consisting of those complexes supported in codimension at least n according to μ, precisely the objects are

$$D^{\geq n}(X) := \{M \in D^b_{Coh}(X) |\ M_x \text{ is acyclic whenever } \mu(x) < n\}$$

Each $D^{\geq n}(X)$ is then a saturated strictly full triangulated subcategory of $D^b_{Coh}(X)$ preserved by the duality $(\#, 1, \eta)$. We hence have in the sense of **Definition 2.2.29** short exact sequences

$$D^{\geq n+1}(X) \longrightarrow D^{\geq n}(X) \longrightarrow D^{\geq n}(X)/D^{\geq n+1}(X)$$

of triangulated categories with duality - and we continue our shorthand notation $(-, \mathcal{R})$ for these categories. We write

$$\partial^n : W^n \left(\frac{D^{\geq n}(X)}{D^{\geq n+1}(X)}, \mathcal{R} \right) \longrightarrow W^{n+1} \left(D^{\geq n+1}(X), \mathcal{R} \right)$$

for the connecting homomorphism of **Proposition 2.2.31** appearing in the localisation sequence associated to the above short exact sequence. If we replaced X by $\mathrm{Spec}(\mathcal{O}_{X,x})$ and $\mathcal{R}$ by its localisation at x, then as a particular case of the fact that $\#$ preserves the subcategories $D^{\geq n}(X)$ we are justified in making the following definition.

Definition 3.2.2. For each point $x \in X$, we write $\left(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}_x \right)$ for the triangulated category with duality on the full subcategory $D^b_{f.l.}(\mathcal{O}_{X,x})$ of $D(Mod_{\mathcal{O}_{X,x}})$ which consists of those complexes whose cohomology groups are all finite length, together with the 1-exact duality $[-, \mathcal{R}_x]_{X_x}$ and usual double dual identification.

Proposition 3.2.3. *For each $n \in \mathbb{Z}$ we have an isomorphism*

$$\mathrm{loc}^n : W^n \left(\frac{D^{\geq n}(X)}{D^{\geq n+1}(X)}, \mathcal{R} \right) \longrightarrow \bigoplus_{\mu(x)=n} W^n \left(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}_x \right)$$

induced by localisation.

Proof. The fact that we have this isomorphism is covered in [2, §5] - we here just describe how the map loc^n is defined. Certainly, we have for each $x \in X^n_\mu$ that localisation to x is a 1-exact functor

$$\mathrm{loc}_x : D^{\geq n}(X) \longrightarrow D^b_{f.l.}(\mathcal{O}_{X,x})$$

between triangulated categories with dualities on each; the pair $(\mathrm{loc}_x, \mathrm{id})$ is a duality preserving functor in the sense of **Definition 2.2.26**. Note further that if $M \in D^{\geq n}(X)$, then the points $x \in X^n_\mu$ with $\mathrm{loc}_x(M) = M_x \neq 0 \in D^b_{f.l.}(\mathcal{O}_{X,x})$ are among the generic points of the irreducible components of the support of the closed subset $\bigcup_{i \in \mathbb{Z}} \mathrm{Supp}(H^i(M))$ of X. Hence $\mathrm{loc}_x(M)$ vanishes for all but finitely many

$x \in X^n_\mu$.

If f is a morphism in $D^{\geq n}(X)$ with cone lying in $D^{\geq n+1}(X)$, then for each $x \in X^n_\mu$ we have that $\mathrm{cone}(f_x) = \mathrm{cone}(f)_x = 0$ so that f_x is an isomorphism in $D^b_{f.l.}(\mathcal{O}_{X,x})$. Hence each localisation map lifts to a morphism

$$\mathrm{loc}_x : D^{\geq n}(X)/D^{\geq n+1}(X) \longrightarrow D^b_{f.l.}(\mathcal{O}_{X,x})$$

still a duality preserving functor in the sense of **Definition 2.2.26**. We hence obtain the map

$$\mathrm{loc}^n := \bigoplus_{\mu(x)=n} (\mathrm{loc}_x)_* : W^n\left(\frac{D^{\geq n}(X)}{D^{\geq n+1}(X)}, \mathcal{R}\right) \longrightarrow \bigoplus_{\mu(x)=n} W^n\left(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}\right)$$

in the statement of the proposition. $\square$

Lemma 3.2.4. *For any point $x \in X$ we have that the inclusion*

$$D^b(f.l.Mod_{\mathcal{O}_{X,x}}) \hookrightarrow D^b_{f.l.}(\mathcal{O}_{X,x})$$

is an equivalence of triangulated categories.

Proof. We can factor the inclusion in question as

$$\begin{tikzcd}
D^b_{f.l.}\left(f.g.Mod_{\mathcal{O}_{X,x}}\right) \arrow[r] & D^b_{f.l.}(\mathcal{O}_{X,x}) \\
D^b(f.l.Mod_{\mathcal{O}_{X,x}}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where $D^b_{f.l.}(f.g.Mod_{\mathcal{O}_{X,x}})$ is the full subcategory of $D^b(f.g.Mod_{\mathcal{O}_{X,x}})$ consisting of those complexes whose cohomology groups are finite length. We have by [48, §1.15, Example (b)] that the inclusion $D^b(f.l.Mod_{\mathcal{O}_{X,x}}) \to D^b_{f.l.}(f.g.Mod_{\mathcal{O}_{X,x}})$ is an equivalence. By [49, Prop.3.5] we have that the inclusion $D^b(f.g.Mod_{\mathcal{O}_{X,x}}) \to D^b_{f.g.}(Mod_{\mathcal{O}_{X,x}})$ is an equivalence - hence so is its restriction to $D^b_{f.l.}\left(f.g.Mod_{\mathcal{O}_{X,x}}\right) \to D^b_{f.l.}(\mathcal{O}_{X,x})$. $\square$

Proposition 3.2.5. *Let $x \in X$ with $\mu(x) = n$. Then we have an isomorphism*

$$\Delta_x : W\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^n_x\right) \longrightarrow W^n\left(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}_x\right)$$

which sends an isometry class of a symmetric space $\psi : M \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(M, \mathcal{R}^n_x)$ to the class of

$$\psi[0] : M[0] \longrightarrow \left[M[0], \mathcal{R}_x\right][n]$$

where $[M[0], \mathcal{R}_x]$ *denotes the internal hom of chain complexes of* $\mathcal{O}_{X,x}$*-modules and we have identified* $\big[M[0], \mathcal{R}_x\big][n] = \mathrm{Hom}_{\mathcal{O}_{X,x}}\big(M, \mathcal{R}^n_x\big)$.

Proof. The map written down is just a description of the composition of isomorphisms

$$\begin{tikzcd} W\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^n_x\right) \arrow[d, "(c_0)_*"'] \arrow[drr, "\Delta_x"] & & \\ W\left(D^b(f.l.Mod_{\mathcal{O}_{X,x}}), *\right) \arrow[rr] & & W^n\left(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}_x\right) \end{tikzcd}$$

where the map

$$(c_0)_* : W\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^n_x\right) \longrightarrow W\left(D^b(f.l.Mod_{\mathcal{O}_{X,x}}), *\right)$$

is the isomorphism between a "usual" and its derived Witt group of [44] - which we gave in **Proposition 2.2.28** - while the map

$$W\left(D^b(f.l.Mod_{\mathcal{O}_{X,x}}), *\right) \longrightarrow W^n\left(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}_x\right)$$

is induced by the equivalence $D^b(f.l.Mod_{\mathcal{O}_{X,x}}) \hookrightarrow D^b_{f.l.}(\mathcal{O}_{X,x})$ of the previous lemma - with the duality compatibility transformation as specified in [2, §3]. Precisely, if $C \in D^b(f.l.Mod_{\mathcal{O}_{X,x}})$ then for each degree i we have

$$\big(C^*\big)^i = \mathrm{Hom}_{\mathcal{O}_{X,x}}(C^{-i}, \mathcal{R}^n_x) = \Big([C, \mathcal{R}_x][n]\Big)^i$$

and the duality compatibility map $C^* \to [C, \mathcal{R}_x][n]$ is given in degree i to be $(-1)^{in}$ times the identity map. Note in particular that if C were concentrated in degree 0, then this duality compatibility introduces no extra signs - hence none are present in our description of Δ_x. $\square$

We hence have for each $n \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{\mu(x)=n} W\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^n_x\right) \xrightarrow{\left(\mathrm{loc}^n\right)^{-1} \circ \bigoplus\limits_{\mu(x)=n} \Delta_x} W^n\left(\frac{D^{\geq n}(X)}{D^{\geq n+1}(X)}, \mathcal{R}\right)$$

so the boundary maps of a Witt complex can be defined in terms of the connecting homomorphisms ∂^n of Balmer's localisation sequence. Precisely, the boundary maps are those appearing in the picture below where we have omitted the $\mathcal{R}$'s in the notation for each of the Witt groups $W(?, \mathcal{R})$.

$$
\begin{tikzcd}
 & & (2) \arrow[d, dashed, no head] & \\
 & W^{n-1}\left(\frac{D^{\geq n-1}(X)}{D^{\geq n}(X)}\right) \arrow[r, "\partial^{n-1}"] \arrow[dr, "d^{n-1}_{W(X,\mathcal{R})}"'] & W^{n}\left(D^{\geq n}(X)\right) \arrow[d] \arrow[r, dashed, no head] & (1) \\
 & & W^{n}\left(\frac{D^{\geq n}(X)}{D^{\geq n+1}(X)}\right) \arrow[d, "\partial^{n}"'] \arrow[dr, "d^{n}_{W(X,\mathcal{R})}"] & \\
(3) \arrow[rr, dashed, no head] & & W^{n+1}\left(D^{\geq n+1}(X)\right) \arrow[r] & W^{n+1}\left(\frac{D^{\geq n+1}(X)}{D^{\geq n+2}(X)}\right)
\end{tikzcd}
$$

In this picture, along each of the dashed lines (1),(2) and (3) we have part of the localisation sequences of **Definition 2.2.32** attached to the short exact sequences

$$
\begin{aligned}
D^{\geq n}(X) \to & D^{\geq n-1}(X) \to D^{\geq n-1}(X)/D^{\geq n}(X) \\
D^{\geq n+1}(X) \to & D^{\geq n}(X) \to D^{\geq n}(X)/D^{\geq n+1}(X) \\
D^{\geq n+2}(X) \to & D^{\geq n+1}(X) \to D^{\geq n+1}(X)/D^{\geq n+2}(X)
\end{aligned}
$$

respectively. Note that the fact that $d^{n}_{W(X,\mathcal{R})}d^{n-1}_{W(X,\mathcal{R})} = 0$ follows immediately from the composition along the dashed line (2) being zero.

Definition 3.2.6. If $x \rightsquigarrow y$ is some immediate specialisation in X with $\mu(x) = n$ say, then we write

$$d^{x,y}_{\Delta} : W\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^{n}_{x}\right) \longrightarrow W\left(f.l.Mod_{\mathcal{O}_{X,y}}, \mathcal{R}^{n+1}_{y}\right)$$

for the map

$$\operatorname{loc}^{n+1} \partial^{n} (\operatorname{loc}^{n})^{-1} \Delta_{x} : W\left(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^{n}_{x}\right) \longrightarrow \bigoplus_{\mu(x')=n+1} W\left(D^{b}_{f.l.}(\mathcal{O}_{X,x'}), \mathcal{R}_{x'}\right)$$

followed by projection onto the component of the right hand side indexed by y and then by the inverse to the isomorphism

$$\Delta_{y} : W\left(f.l.Mod_{\mathcal{O}_{X,y}}, \mathcal{R}^{n+1}_{y}\right) \longrightarrow W^{n+1}\left(D^{b}_{f.l.}(\mathcal{O}_{X,y}), \mathcal{R}_{y}\right)$$

For the remainder of this section we suppose that X is affine, let's say $X = \operatorname{Spec}(A)$, and take a point $p \in X^{n-1}_{\mu}$. Consider now a symmetric space $(V, \psi) \in Sym\big(f.l.Mod_{A_p}, \mathcal{R}^{n-1}_{p}\big)$ - our objective now is to show that for any immediate specialisation $p \rightsquigarrow q$ we have

$$d_W^{p,q}(\mathcal{R})(M,\psi) = d_\Delta^{p,q}(M,\psi)$$

Since the map

$$\left(\operatorname{tr}_{\pi_{p,X}}(\mathcal{R}_p)^{n-1}\right)_* : W(p,p^\natural(\mathcal{R})) \longrightarrow W\left(f.l.Mod_{A_p}, \mathcal{R}_p^{n-1}\right)$$

is an isomorphism, we may assume that V is a one-dimensional $\kappa(p)$-vector space. We then take an $\mathcal{O}_X$ lattice

$$i : L \hookrightarrow \iota_p(V)$$

self-dual with respect to ψ in the sense of **Definition 3.1.4**. Let's quickly argue that we can take L to be generated by a single element.

Lemma 3.2.7. *There exists a non-zero $v \in V$ such that the image of $\langle v,v\rangle_\psi$ under the inclusion $\rho(p) : \mathcal{R}_p^{n-1} \hookrightarrow \mathcal{R}^{n-1}$ lies inside $Z^{n-1}(\mathcal{R})$.*

Proof. Take any non-zero $v \in V$, then since $Z^{n-1}(\mathcal{R})_p = \mathcal{R}_p$ we have that there is some $f \in A \setminus p$ such that $f\langle v,v\rangle_\psi = z \in Z^{n-1}(\mathcal{R})$. Then $fv \neq 0$ and $\langle fv, fv\rangle_\psi = fz \in Z^{n-1}(\mathcal{R})$. □

The A-submodule L of V generated by an element v as in the above lemma is self-dual and further isomorphic to A/p. We may view this lattice as a complex concentrated in degree zero, obtaining an element $L[0] \in D^{\geq n-1}(X)$ and via ψ a map

$$\theta : L[0] \longrightarrow L[0]^{\#_{n-1}}$$

Precisely, we define $\theta^0 : L \to (L[0]^{\#_{n-1}})^0 = \operatorname{Hom}_A(L, \mathcal{R}^{n-1})$ to be $i^*\iota_p(\psi)i$ where i^* denotes the epimorphism $\operatorname{Hom}_A(i, \mathcal{R}^{n-1})$. Explicitly then, for each $l \in L$ we have that $\theta^0(l)$ is the restriction of $\psi(l) : V \to \mathcal{R}_p^{n-1}$ to L viewed as a map into $\mathcal{R}^{n-1}$. The self-dual nature of L ensures that θ really is a morphism of chain complexes.

Lemma 3.2.8. *A chain complex morphism $\theta : M \to M^{\#_i}$ is symmetric with respect to the i^{th}-shifted duality if and only if*

$$(-1)^{\frac{i(i+1)}{2}}\theta^s(\alpha)_{(t)}(\beta) = (-1)^{\frac{(i+t+s)(i+t+s+1)}{2}}\theta^t(\beta)_{(s)}(\alpha)$$

for any $\alpha \in M^i$ and $\beta \in M^j$.

Proof. The definition of i-symmetry reads that

$$\theta^s(\alpha)_{(t)}(\beta) = \Big((\theta^{\#_i})^s\eta_i^s(\alpha)\Big)_{(t)}(\beta)$$

and all we're doing is expanding the term on the right. To be clear; we're using that if $g \in (M^{\#_i})^k$ then

$$\eta_i^s(\alpha)_{(k)}(g) = (-1)^{\frac{i(i+1)}{2}} \eta^s(\alpha)_{(k+i)}(g)$$

with the term on the right involving

$$\eta^s(\alpha)_{k+i}(g) = (-1)^{\frac{(k+s+i)(k+s+i+1)}{2}} g_{(s)}(\alpha)$$

while the map $(\theta^{\#_i})^s : M^{\#_i\#_i} \to M^{\#_i}$ is precomposition with θ. $\square$

With this lemma, one sees that θ is a symmetric morphism with respect to the $(n-1)^{th}$-shifted duality.

Lemma 3.2.9. *The chain complex map $\theta : L[0] \to L[0]^{\#_{n-1}}$ is an $S(D^{\geq n}(X))$-space for the $(n-1)^{th}$-shifted duality of $(\#, 1, \eta)$ on $D^{\geq n-1}(X)$.*

Proof. It only remains to show that $\mathrm{cone}(\theta) \in D^{\geq n}(X)$ - or equivalently that θ_x is a quasi-isomorphism for any $x \in X$ with $\mu(x) < n$. If x is any such point other than p then the localisations to x of both $L[0]$ and $L[0]^{\#_{n-1}}$ are zero - so θ_x is trivially a quasi-isomorphism here. At p we have $(L[0])_p = V[0]$ and θ_p is just ψ concentrated in degree zero - which is again a quasi-isomorphism. $\square$

So after writing $Q : D^{\geq n-1}(X) \to D^{\geq n-1}(X)/D^{\geq n}(X)$ for the localisation map we have as in **Proposition 2.2.31** that the pair $(Q(L[0]), Q(\theta))$ represents an element of $W^{n-1}\big(D^{\geq n-1}(X)/D^{\geq n}(X), \mathcal{R}\big)$, for which we have

$$\mathrm{loc}^{n-1}\left([Q(L[0]), Q(\theta)]\right) = \sum_{\mu(x)=n-1} \big[L[0]_x, \theta_x\big] \in \bigoplus_{\mu(x)=n-1} W^{n-1}\big(D^b_{f.l.}(\mathcal{O}_{X,x}), \mathcal{R}_x\big)$$

The proof of the above lemma essentially just observes that

$$\mathrm{loc}^{n-1}\left([Q(L[0]), Q(\theta)]\right) = \big[V[0], \psi[0]\big] \in W^{n-1}\big(D^b_{f.l.}(\mathcal{O}_{X,p}), \mathcal{R}_p\big)$$

So $\left(\mathrm{loc}^{n-1}\right)^{-1}\Delta_p(M,\psi) = [Q(L[0]), Q(\theta)]$ telling us that to compute $d_\Delta^{p,q}(M,\psi)$ we should find the cone of θ as in **Definition 2.2.24**. Below we give an explicit description of this cone - after replacing $L[0]$ by an arbitrary complex M.

Proposition 3.2.10. *Let $\theta : M \to M^{\#_{n-1}}$ be a morphism in $\mathrm{Ch}(X)$ which is symmetric with respect to the $(n-1)^{th}$-shifted duality. Then denoting by $\mathrm{cone}(\theta)$ the standard cone of a morphism of chain complexes, we define the morphism*

$$\chi : \mathrm{cone}(\theta) \longrightarrow \mathrm{cone}(\theta)^{\#_n}$$

to be given in degree N by setting for each $m \in M^{N+1}$ and $\lambda \in \left(M^{\#_{n-1}}\right)^N$ the image of $(m,\lambda) \in \mathrm{cone}(\theta)^N$ under χ^N to be the collection of morphisms $\mathrm{cone}(\theta)^k \to \mathcal{R}^{N+n+k}$

$$\chi^N(m,\lambda)_{(k)}(\alpha, f) = -\lambda_{(k+1)}(\alpha) + (-1)^{[N,k]} f_{(N+1)}(m)$$

for each pair $\alpha \in M^{k+1}$ *and* $f \in \left(M^{\#_{n-1}}\right)^k$, *where*

$$[N,k] = \frac{1}{2}\Big((n-1)(n+2) + (N+k+n)(N+k+n+1)\Big)$$

Then χ *is a quasi-isomorphism of chain complexes, symmetric with respect to the* n^{th}-*shifted duality, fitting into the commutative diagram*

$$\begin{tikzcd}
M \arrow[r, "\theta"] \arrow[d, "\delta_{n-1}\eta_{n-1}"'] & M^{\#_{n-1}} \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & \mathrm{cone}(\theta) \arrow[r, "e"] \arrow[d, "\chi"'] & T(M) \arrow[d, "\delta_{n-1}T(\eta_{n-1})"] \\
M^{\#_{n-1}\#_{n-1}} \arrow[r, "\delta_{n-1}\theta^{\#_{n-1}}"'] & M^{\#_{n-1}} \arrow[r, "-e^{\#_n}"'] & \mathrm{cone}(\theta)^{\#_n} \arrow[r, "u^{\#_n}"'] & T(M^{\#_{n-1}\#_{n-1}})
\end{tikzcd}$$

where the top row is the "standard" exact triangle extending θ.

Proof. The hardest part to check is that χ is a morphism of chain complexes - so that's the part we'll write down here; to ease notation we put $* = \#_{n-1}$. Take $m \in M^{N+1}$ and $\lambda \in (M^*)^N$ so that (m,λ) represents an arbitrary object of $\mathrm{cone}(\theta)^N$. Then

$$d^N_{\mathrm{cone}(\theta)}(m,\lambda) = \Big(-d^{N+1}_M(m),\ \theta^{N+1}(m) + d^N_{M^*}(\lambda)\Big)$$

So for each $(\beta, g) \in \mathrm{cone}(\theta)^k$ we have

$$\begin{aligned}
\Lambda_1 := \chi^{N+1}\Big(d^N_{\mathrm{cone}(\theta)}(m,\lambda)\Big)_{(k)}(\beta,g) = -\Big(\theta^{N+1}(m) + d^N_{M^*}(\lambda)\Big)_{(k+1)}(\beta) \\
+ (-1)^{[N+1,k]} g_{(N+2)}\Big(-d^{N+1}_M(m)\Big)
\end{aligned}$$

The most complicated term on the right hand side can be written out as

$$\begin{aligned}
d^N_{M^*}(\lambda)_{(k+1)}(\beta) &= (-1)^{n-1} d^{N+n-1}_{M^\#}(\lambda)_{(k+1)}(\beta) \\
&= (-1)^{n-1}\Big(\lambda_{(k+2)} d^{k+1}_M(\beta) - (-1)^{N+n-1} d^{k+N+n}_{\mathcal{R}} \lambda_{(k+1)}(\beta)\Big) \\
d^N_{M^*}(\lambda)_{(k+1)}(\beta) &= (-1)^{n-1}\lambda_{(k+2)} d^{k+1}_M(\beta) + (-1)^{N+1} d^{k+N+n}_{\mathcal{R}} \lambda_{(k+1)}(\beta)
\end{aligned}$$

So we obtain the following expression for $\chi^{N+1}\Big(d_{\mathrm{cone}(\theta)}(m,\lambda)\Big)$

$$\begin{aligned}
\Lambda_1 = -\theta^{N+1}(m)_{(k+1)}(\beta) + (-1)^n \lambda_{(k+2)} d^{k+1}_M(\beta) + (-1)^N d^{k+N+n}_{\mathcal{R}} \lambda_{(k+1)}(\beta) \\
+ (-1)^{[N+1,k]+1} g_{(N+2)}\Big(d^{N+1}_M(m)\Big)
\end{aligned}$$

which we aim to compare with $d^N_{\text{cone}(\theta)\#n}\Big(\chi^N(m,\lambda)\Big)$. One carefully finds that

$$\begin{aligned}
d^N_{\text{cone}(\theta)\#n}\Big(\chi^N(m,\lambda)\Big) &= (-1)^n d^{n+N}_{\text{cone}(\theta)\#}\Big(\chi^N(m,\lambda)\Big) \\
&= (-1)^n\Big(\chi^N(m,\lambda)_{(t+1)}d^t_{\text{cone}(\theta)} - (-1)^{n+N}d^{t+n+N}_{\mathcal{R}}\chi^N(m,\lambda)_{(t)}\Big)_{t\in\mathbb{Z}}
\end{aligned}$$

To evaluate the k^{th} component of this collection of morphisms at $(\beta, g) \in \text{cone}(\theta)^k$, we first compute that

$$\begin{aligned}
\chi^N(m,\lambda)_{(k+1)}d^k_{\text{cone}(\theta)}(\beta,g) &= \chi^N(m,\lambda)_{(k+1)}\Big(-d^{k+1}_M(\beta),\ \theta^{k+1}(\beta) + d^k_{M^*}(g)\Big) \\
\chi^N(m,\lambda)_{(k+1)}d^k_{\text{cone}(\theta)}(\beta,g) &= \lambda_{(k+2)}\Big(d^{k+1}_M(\beta)\Big) + (-1)^{[N,k+1]}\Big(\theta^{k+1}(\beta) + d^k_{M^*}(g)\Big)_{(N+1)}(m)
\end{aligned}$$

The rightmost term above is given by

$$\begin{aligned}
d^k_{M^*}(g)_{(N+1)}(m) &= (-1)^{n-1}d^{k+n-1}_{M\#}(g)_{(N+1)}(m) \\
&= (-1)^{n-1}\Big(g_{(N+2)}d^{N+1}_M(m) - (-1)^{k+n-1}d^{N+k+n}_{\mathcal{R}}g_{(N+1)}(m)\Big) \\
d^k_{M^*}(g)_{(N+1)}(m) &= (-1)^{n-1}g_{(N+2)}d^{N+1}_M(m) + (-1)^{k+1}d^{N+k+n}_{\mathcal{R}}g_{(N+1)}(m)
\end{aligned}$$

So we have

$$\begin{aligned}
\chi^N(m,\lambda)_{(k+1)}d^k_{\text{cone}(\theta)}(\beta,g) = \lambda_{(k+2)}\Big(d^{k+1}_M(\beta)\Big) &+ (-1)^{[N,k+1]}\theta^{k+1}(\beta)_{(N+1)}(m) \\
&+ (-1)^{[N,k+1]+n-1}g_{(N+2)}d^{N+1}_M(m) \\
&\quad + (-1)^{[N,k+1]+k+1}d^{N+k+n}_{\mathcal{R}}g_{(N+1)}(m)
\end{aligned}$$

After reminding ourselves that $\chi^N(m,\lambda)_{(k)}(\beta,g) = -\lambda_{(k+1)}(\beta) + (-1)^{[N,k]}g_{(N+1)}(m)$ we obtain the expression

$$\begin{aligned}
\Lambda_2 = (-1)^n\lambda_{(k+2)}\Big(d^{k+1}_M(\beta)\Big) &+ (-1)^{n+[N,k+1]}\theta^{k+1}(\beta)_{(N+1)}(m) \\
&+ (-1)^{[N,k+1]-1}g_{(N+2)}d^{N+1}_M(m) + (-1)^{[N,k+1]+n+k+1}d^{N+k+n}_{\mathcal{R}}g_{(N+1)}(m) \\
&+ (-1)^N d^{k+n+N}_{\mathcal{R}}\lambda_{(k+1)}(\beta) + (-1)^{[N,k]+N+1}d^{k+n+N}_{\mathcal{R}}g_{(N+1)}(m)
\end{aligned}$$

The reader can check that the $d^{k+n+N}_{\mathcal{R}}g_{(N+1)}(m)$ terms above cancel out, and further that every other term not involving θ agrees with the corresponding term in Λ_1. So to see that χ is a morphism of chain complexes, it only remains to see that

$$\theta^{N+1}(m)_{(k+1)}(\beta) = (-1)^{n+1+[N,k+1]}\theta^{k+1}(\beta)_{(N+1)}(m)$$

which follows from the $(n-1)$-symmetry of θ using **Lemma 3.2.8** above - which one can also use to check that χ is symmetric with respect to the n^{th} shifted duality. $\square$

So our element $\partial^{n-1}\big(\mathrm{loc}^{n-1}\big)^{-1}\Delta_x(M,\psi) \in W^n\big(D^{\geq n}(X),\mathcal{R}\big)$ is represented by the isometry class of the quasi-isomorphism

$$\begin{array}{ccccccccccccc}
\mathrm{cone}(\theta) = & \cdots \longrightarrow 0 & \longrightarrow & L & \xrightarrow{i^*\iota_p(\psi)i} & [L,\mathcal{R}^{n-1}] & \longrightarrow & [L,\mathcal{R}^n] & \longrightarrow & [L,\mathcal{R}^{n+1}] & \cdots \\
\Big\downarrow \chi & \Big\downarrow & & \Big\downarrow \chi^{-1} & & \Big\downarrow \chi^0 & & \Big\downarrow \chi^1 & & \Big\downarrow \chi^2 & \\
\mathrm{cone}(\theta)^{\#_n} = & \cdots \longrightarrow 0 & \longrightarrow & (\mathrm{cone}(\theta)^{\#_n})^{-1} & \longrightarrow & (\mathrm{cone}(\theta)^{\#_n})^{0} & \longrightarrow & (\mathrm{cone}(\theta)^{\#_n})^{1} & \longrightarrow & (\mathrm{cone}(\theta)^{\#_n})^{2} & \cdots
\end{array}$$

The component of $\mathrm{loc}^n\,\partial^{n-1}\Delta_p(M,\psi)$ corresponding to q is then just the localisation of χ at q. Note that in the localisation of $\mathrm{cone}(\theta)$ at q, all terms in degree greater than 1 vanishes, i.e.

$$\mathrm{cone}(\theta)_q = ... \to 0 \to L_q \longrightarrow \big[L_q,\mathcal{R}_q^{n-1}\big] \longrightarrow \big[L_q,\mathcal{R}_q^{n}\big] \to 0 \to ...$$

Lemma 3.2.11. *Let $C_L \in D^b_{f.l.}(\mathcal{O}_{X,q})$ be the complex*

$$C_L = ... \to 0 \to L_q \longrightarrow L_q^\flat \to 0 \to ...$$

concentrated in degrees -1 and 0, where $i^\flat : L^\flat \hookrightarrow V$ is the dual lattice of **Definition 3.1.4** *and $d_{C_L}^{-1}$ is the natural inclusion obtained from the self-dual nature of L. Then the map*

$$\begin{array}{ccccccccccc}
C_L = & \cdots \longrightarrow 0 & \longrightarrow & L_q & \longrightarrow & L_q^\flat & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \cdots \\
\Big\downarrow \Phi & \Big\downarrow & & \Big\downarrow \mathrm{id} & & \Big\downarrow i_q^*\iota_p(\psi)_q i_q^\flat & & \Big\downarrow & & \Big\downarrow \\
\mathrm{cone}(\theta)_q = & \cdots \longrightarrow 0 & \longrightarrow & L_q & \longrightarrow & \big[L_q,\mathcal{R}_q^{n-1}\big] & \longrightarrow & \big[L_q,\mathcal{R}_q^{n}\big] & \longrightarrow & 0 \longrightarrow \cdots
\end{array}$$

is a quasi-isomorphism.

Proof. It is immediate that Φ is a morphism of chain complexes inducing isomorphisms on cohomology in degrees ≤ 0. The only tricky point now is seeing that $H^1(\mathrm{cone}(\theta)_q) = 0$ - for which we remind ourselves that L was taken to be isomorphic to A/p, so that the truncated complex

$$... \to 0 \to \big[L_q,\mathcal{R}_q^{n-1}\big] \longrightarrow \big[L_q,\mathcal{R}_q^{n}\big] \to 0 \to ...$$

is from **Proposition 2.1.27** a residual complex over the Cohen-Macaulay local ring $(A/p)_q$. Hence by **Proposition 2.3.2** it has non-zero cohomology only in degree 0. $\square$

We hence have that

$$\big[\operatorname{cone}(\theta)_q, \chi_q\big] = \big[C_L, \Phi^\star \chi_q \Phi\big] \in W^n\left(D^b_{f.l.}(\mathcal{O}_{X,q})\mathcal{R}_q\right)$$

where $\star$ denotes the n^{th}-shift of the duality $\big[-, \mathcal{R}_q\big]$ obtained by localising the duality $(\#, 1, \eta)$ at q.

Lemma 3.2.12. *The map $\Phi^\star \chi_q \Phi : C_L \longrightarrow C_L^\star$ is given by*

$$\big(\Phi^\star \chi_q \Phi\big)^{-1}(l)_{(k)} = \begin{cases} (-1)^{[-1,0]}\langle l, -\rangle_\psi|_{L_q^\flat} : C_L^0 = L_q^\flat \to \mathcal{R}_q^{n-1} & \textit{if } k=0 \\ 0 & \textit{otherwise} \end{cases}$$

$$\big(\Phi^\star \chi_q \Phi\big)^{0}(l^\flat)_{(k)} = \begin{cases} -\langle l^\flat, -\rangle_\psi|_{L_q} : C_L^{-1} = L_q \to \mathcal{R}_q^{n-1} & \textit{if } k=-1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. This is an exercise in unraveling the formula of **Proposition 3.2.10** for our map $\theta : L[0] \to \big(L[0]\big)^{\#_{n-1}}$ after localisation at q. Let's begin with degree -1, where the map $\Phi^{-1} : L_q \to \operatorname{cone}(\theta)_q^{-1} = L_q \oplus (L[0]^{\#_{n-1}})_q^{-1}$ is the inclusion into the first component. Of course $(L[0]^{\#_{n-1}})^{-1}$=0 but we include this second component in our notation so that it is easier to read off the formula for χ^{-1}; we have for any $l \in L_q = C_L^{-1}$ that

$$\Phi^{-1}(l) = (l, 0) \in L_q \oplus (L[0]^{\#_{n-1}})_q^{-1} = \operatorname{cone}(\theta)_q^{-1}$$

so $\chi_q^{-1}\Phi^{-1}(l) \in \operatorname{cone}(\theta)_q^{\#_n}$ is the collection of maps $\operatorname{cone}(\theta)_q^k \to \mathcal{R}_q^{k+n-1}$

$$\chi_q^{-1}\Phi^{-1}(l)_{(k)}(\alpha, f) = (-1)^{[-1,k]} f_{(0)}(l)$$

for each $\alpha \in L[0]^{k+1}$ and $f \in \big(L^{\#_{n-1}}\big)_q^k = \operatorname{Hom}_{A_q}\big(L_q, \mathcal{R}^{n+k-1}\big)$. Much of the information here is redundant - if $k \leq -1$ then there are no non-zero maps $\operatorname{cone}(\theta)^k \to \mathcal{R}^{k+n-1}$ while if $k > 1$ then $\mathcal{R}_q^{k+n-1} = 0$. This leaves only the components with $k = 0, 1$ and in both of these cases $L[0]^{k+1} = 0$. So $\chi_q^{-1}\Phi^{-1}(l)$ consists of two maps

$$\begin{aligned} \chi_q^{-1}\Phi^{-1}(l)_{(0)} : \operatorname{cone}(\theta)_q^0 \longrightarrow \mathcal{R}_q^{n-1}, \qquad & f \in \big(L^{\#_{n-1}}\big)_q^0 \mapsto (-1)^{[-1,0]} f_{(0)}(l) \\ \chi_q^{-1}\Phi^{-1}(l)_{(1)} : \operatorname{cone}(\theta)_q^1 \longrightarrow \mathcal{R}_q^{n}, \qquad & f \in \big(L^{\#_{n-1}}\big)_q^1 \mapsto (-1)^{[-1,1]} f_{(0)}(l) \end{aligned}$$

The collection of maps $\big(\Phi^\star\big)^{-1}\chi_q^{-1}\Phi^{-1}(l)_{(k)} : C_L^k \to \mathcal{R}_q^{k+n-1}$ is given by precompositions with Φ^k. We hence have that

$$\big(\Phi^\star \chi_q \Phi\big)^{-1}(l)_{(-1)} : C_L^{-1} = L_q \longrightarrow \mathcal{R}_q^{n-2}$$

is zero because $\big(\chi_q\Phi\big)^{-1}(l)_{(-1)} = 0$ while

$$\big(\Phi^\star\chi_q\Phi\big)^{-1}(l)_{(0)} = (-1)^{[-1,0]}\langle l, -\rangle_\psi : C_0^L = L_q^\flat \longrightarrow \mathcal{R}_q^{n-1}$$

exactly as in the statement of the lemma.

Now in degree 0 we have for each $l^\flat \in L_q^\flat = C_L^0$ that $\Phi^0(l^\flat) \in \operatorname{cone}(\theta)_q^0 = \big(L[0]\big)_q^1 \oplus \big(L[0]^{\#_{n-1}}\big)_q^0$ is given by

$$\Phi^0(l^\flat) = \Big(0,\ \langle l^\flat, -\rangle_\psi|_{L_q} : L_q \to \mathcal{R}_q^{n-1}\Big)$$

Then $\chi_q^0\Phi^0(l^\flat)$ is the collection of maps $\operatorname{cone}(\theta)_q^k \to \mathcal{R}_q^{k+n}$

$$\big(\chi_q\Phi\big)^0(l^\flat)_{(k)}(\alpha, f) = -\big(\langle l^\flat, -\rangle_\psi|_{L_q}\big)_{(k+1)}(\alpha)$$

for any $\alpha \in L[0]^{k+1}$ and $f \in \big(L^{\#_{n-1}}\big)_q^k$. The only component which survives is that with $k = -1$, for which we have $f = 0$ and

$$\big(\chi_q\Phi\big)^0(l^\flat)_{(-1)}(\alpha) = -\langle l^\flat, l\rangle_\psi \in \mathcal{R}_q^{n-1}$$

Again, we obtain $\big(\Phi^\star\chi_q\Phi\big)^0(l^\flat)$ as precomposition with Φ^{-1} which gives us the stated formula in degree zero. □

Taking stock, we now know that after writing $\overline{\chi} = \Phi^\star\chi_q\Phi$ we have

$$d_\Delta^{p,q}(V, \psi) = \Delta_q^{-1}\big([C_L, \overline{\chi}]\big) \in W^n\big(D_{f.l.}^b(\mathcal{O}_{X,q}), \mathcal{R}_q\big)$$

Meanwhile, we have from **Proposition 3.1.15** that

$$d_W^{p,q}(\mathcal{R})(V, \psi) = d_{\mathcal{R}}^p\big(\mathcal{L} \hookrightarrow (V, \psi)\big)_q$$

where we recall from **Definition 3.1.10** that the form on this space is the map

$$(-1)^n\big(d_{\mathcal{R}_q}^{n-1}\big)_*\psi_q^L : L_q^\flat/L_q \longrightarrow \operatorname{Hom}_{A_q}\big(L_q^\flat/L_q, \mathcal{R}_q^n\big)$$

which written out explicitly is given by

$$\overline{l^\flat} \mapsto (-1)^n d_{\mathcal{R}_q}^{n-1}\big(\langle l^\flat, -\rangle_\psi\big)$$

As defined in **Proposition 3.2.5** we have that $\Delta_q\big(d_W^{p,q}(\mathcal{R})(V, \psi)\big)$ is the above map concentrated in degree 0. We have an obvious quasi-isomorphism $e : C_L \to L^\flat/L[0]$ and hence that

$$\Delta_q\big(d_W^{p,q}(\mathcal{R})(V, \psi)\big) = \big[C_L, (-1)^n e^\star\big(d_{\mathcal{R}_q}^{n-1}\big)_*\psi_q^L e\big] \in W^n\big(D_{f.l.}^b(\mathcal{O}_{X,q}), \mathcal{R}_q\big)$$

Lemma 3.2.13. *The chain complex maps $(-1)^n e^\star\big(d^{n-1}_{\mathcal{R}_q}\big)_* \psi^L_q e : C_L \to C_L^\star$ and $\overline{\chi} : C_L \to C_L^\star$ are homotopic.*

Proof. The map $e^\star\big(d^{n-1}_{\mathcal{R}_q}\big)_* \psi^L_q e : C_L \to C_L^\star$ is zero in degree -1 while in degree 0 we have for each $l^\flat \in L^\flat_q = C^0_L$ that $\big(e^\star\big(d^{n-1}_{\mathcal{R}_q}\big)_* \psi^L_q e\big)^0(l^\flat)$ is the collection of morphisms $C^k_L \to \mathcal{R}^{n+k}_q$ which are zero for every non-zero value of k while the map $C^0_L : L^\flat_q \to \mathcal{R}^n_q$ is given by $d^{n-1}_{\mathcal{R}_q}\big(\langle l^\flat, -\rangle_\psi\big)$. We hence have that the difference $\overline{\chi} - (-1)^n e^\star\big(d^{n-1}_{\mathcal{R}_q}\big)_* \psi^L_q e$ is the chain complex map as written out below.

$$
\begin{tikzcd}
C_L = \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & L_q \arrow[r] \arrow[d, "l \mapsto (-1)^{[-1,0]}\langle l, -\rangle_\psi"'] & L^\flat_q \arrow[r] \arrow[d, "l^\flat \mapsto -\left(\langle l^\flat, -\rangle_\psi|_{L_q},\, (-1)^n d^{n-1}_{\mathcal{R}_q}\langle l^\flat, -\rangle_\psi|_{L^\flat}\right)"] \arrow[dl, "s^0"'] & 0 \arrow[r] \arrow[d] & {} \\
C_L{}^\star = & \cdots \arrow[r] & 0 \arrow[r] & \big[L^\flat_q, \mathcal{R}^{n-1}_q\big] \arrow[r] & \big[L_q, \mathcal{R}^{n-1}_q\big] \oplus \big[L^\flat_q, \mathcal{R}^n_q\big] \arrow[r] & \big[L_q, \mathcal{R}^n_q\big] \arrow[r] & {}
\end{tikzcd}
$$

and to give a chain homotopy we only need to specify the map $s^0 : L^\flat_q \to \big[L^\flat_q, \mathcal{R}^{n-1}_q\big]$; we take

$$s^0(l^\flat) := (-1)^{n+1}\langle l^\flat, -\rangle_\psi$$

It is then straightforward to verify that this defines the desired chain homotopy. $\square$

In total, we have proved the following comparison result.

Theorem 3.2.14. *Let $(X, \mathcal{R})$ be an affine scheme equipped with a residual complex that has codimension function μ. Then for any point $p \in X^{n-1}_\mu$ and immediate specialisation $p \rightsquigarrow q$ we have that the maps $d^{p,q}_\Delta$ of* **Definition 3.2.6** *and $d^{p,q}_W(\mathcal{R})$ of* **Proposition 3.1.12** *are identically equal.*

3.3 The Witt complex

Let X be a scheme with a residual complex $\mathcal{R}$ that has codimension function μ. Then for each immediate specialisation $z \rightsquigarrow x$ we have from **Definition 3.1.13** a group homomorphism

$$\mathrm{res}^{z,x}_W(\mathcal{R}) : W\left(z, z^\natural(\mathcal{R})\right) \longrightarrow W\left(x, x^\natural(\mathcal{R})\right)$$

From these residue maps we build our Witt complex in the usual way.

Notation 3.3.1. When the global residual complex $\mathcal{R}$ is clearly understood, we will denote the residue map $\mathrm{res}^{z,x}_W(\mathcal{R})$ simply as $\mathrm{res}^{z,x}_W$. Similarly, we write $d^{z,x}_W$ in place of the maps $d^{z,x}_W(\mathcal{R})$ of **Proposition 3.1.12**.

Definition 3.3.2. For X a scheme with residual complex $\mathcal{R}$ that has codimension function μ, we define the boundary map

$$d^p_{W(X,\mathcal{R})} : \bigoplus_{\mu(x)=p} W\left(x, x^\natural(\mathcal{R})\right) \longrightarrow \bigoplus_{\mu(x)=p+1} W\left(x, x^\natural(\mathcal{R})\right)$$

by the formula

$$d^p_{W(X,\mathcal{R})} = \bigoplus_{\mu(z)=p} \sum_{z \rightsquigarrow x} \mathrm{res}^{z,x}_W$$

Of course, we must now check that $d^p_{W(X,\mathcal{R})}$ really takes values in the coproduct.

Proposition 3.3.3. *Let $(X, \mathcal{R})$ be a scheme with a residual complex $x \in X$ be a point with μ-codimension p and $(V, \psi) \in Sym\big(x, x^\natural(\mathcal{R})\big)$. Then there are only finitely many points $x' \in X^{p+1}_\mu$ with $\mathrm{res}^{x,x'}_W \left([V, \psi]\right) \neq 0$.*

Proof. Let $\mathcal{L} \hookrightarrow \iota_x(V)$ be an $\mathcal{O}_X$-lattice self-dual with respect to $\iota_x(\psi)$ in the sense of **Definition 3.1.4**. Then we have from **Definition 3.1.13** for any $x' \in X^{p+1}_\mu$ that

$$\big(\mathrm{tr}_{\pi_{x',X}}(\mathcal{R}_{x'})^{p+1}\big)_* \mathrm{res}^{x,x'}_W([V, \psi]) = d^{x,x'}_W(\mathcal{R})\big(\mathrm{tr}_{\pi_{x,X}}(\mathcal{R}_x)^p\big)_*([V, \psi]) \in W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\big)$$

If we take $\mathcal{L}$ to be some $\mathcal{O}_X$-lattice self-dual with respect to $\iota_x(\psi)$ then the space on the righthand side is defined on the localisation of $\mathcal{L}^\flat/\mathcal{L}$ at x'. Hence the $x' \in X^{p+1}_\mu$ with $\mathrm{res}^{x,x'}_W \left([V, \psi]\right) \neq 0$ are among the irreducible components of the support of $\mathcal{L}^\flat/\mathcal{L}$ - of which there are finitely many. $\square$

Proposition 3.3.4. *Let $(X, \mathcal{R})$ be a scheme with a residual complex and suppose that $1/2 \in \Gamma(X, \mathcal{O}_X)$. Then the sequence of maps $W(X, \mathcal{R})$ (3.1) forms a complex of abelian groups.*

Proof. As we pointed out in the proof of the previous proposition, via the trace maps $\big(\mathrm{tr}_{\pi_{x,X}}(\mathcal{R}_x)^{\mu(x)}\big)_* : W\big(x, x^\natural(\mathcal{R})\big) \to W\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^{\mu(x)}_x\big)$ we have that $W(X, \mathcal{R})$ is isomorphic to the sequence of maps

$$\ldots \to \bigoplus_{\mu(x)=p} W\big(f.l.Mod_{\mathcal{O}_{X,x}}, \mathcal{R}^p_x\big) \xrightarrow{\bigoplus_x \sum_{x \rightsquigarrow x'} d^{x,x'}_W(\mathcal{R})} \bigoplus_{\mu(x')=p+1} W\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\big) \to \ldots$$

Our comparison result **Theorem 3.2.14** tells us that the residue maps forming the complex above agree with those defined by Balmer i.e. those of **Definition 3.2.6**. $\square$

Chapter 4

The Chow-Witt Group

As an application of the residue maps of the previous chapter we give a definition of the Chow-Witt group of a scheme X equipped with a residual complex $\mathcal{R}$. Our construction is based on that of **Definition 1.2.16** - though we don't build an entire complex on the fiber product between the Witt complex restricted to powers of the fundamental ideal and the Gersten complex for Milnor K-theory. Instead, we focus on decribing the few terms which are relevant; looking back at the diagram following the aforementioned definition one sees that we already understand a candidate map to play the role of the boundary map $d^n_{C^*(X,J^n,\Omega)}$ - and after suitably describing the fiber products $J^1(x)$ the residue maps we already know only need slight tweaking to obtain an analogue of $d^{n+1}_{C^*(X,J^n,\Omega)}$. Our resulting notation and terminology is then reminiscent of **Definition 1.1.3** of the ordinary Chow group in terms of cycles modulo rational equivalence. Indeed, we quotient out from a group of *compatible cycles* - so named becase the coefficients of cycles in the Chow-Witt group cannot be taken independently of eachother as in **Definition 1.1.1** - a subgroup of cycles which we call *rationally equivalent to zero* for no other reason than to repeat the terminology of **Definition 1.1.2**. That cycles rationally equivalent to zero are compatible requires us to know that our residue maps for Witt groups really assemble to form a complex - i.e. we need the result of **Proposition 3.3.4**.

We hence assume throughout this final chapter that all schemes admit $1/2$ in their global sections.

Let $(X, \mathcal{R})$ be a scheme equipped with a residual complex. Then for each subvariety $j : Z \hookrightarrow X$ with generic point z that has $\mu(z) = p$, and for each point $x \in Z^1$, we have from **Definition 3.1.13** a residue map

$$\mathrm{res}^{j^\triangle(\mathcal{R})_x}_{GW} : GW\left(z, z^\natural(\mathcal{R})\right) \longrightarrow W\left(x, x^\natural(\mathcal{R}\right)$$

Notation 4.0.1. When the residual complex $\mathcal{R}$ is understood, we record in the superscript only the immediate specialisation of points $z \rightsquigarrow x$ over which this residue map is defined, that is we denote $\mathrm{res}_{GW}^{j^{\triangle}(\mathcal{R})_x}$ simply as $\mathrm{res}_{GW}^{Z,x}$ or $\mathrm{res}_{GW}^{z,x}$.

We have by **Proposition 3.3.3** that the summation of these residue maps over all $x \in Z^1$ takes values in the coproduct on the right hand side below

$$\sum_{x \in Z^1} \mathrm{res}_{GW}^{z,x} : GW\left(z, z^{\natural}(\mathcal{R})\right) \longrightarrow \bigoplus_{x \in Z^1} W\left(x, x^{\natural}(\mathcal{R})\right)$$

Definition 4.0.2. Let $(X, \mathcal{R})$ be a scheme with a residual complex. Then we write $\widetilde{Z}^p(X, \mathcal{R})$ for the group of **codimension** p **compatible cycles** on X, which is defined to be the kernel of the group homomorphism

$$\bigoplus_{\mu(z)=p} \sum_{z \rightsquigarrow x} \mathrm{res}_{GW}^{z,x} : \bigoplus_{\mu(z)=p} GW\left(z, z^{\natural}(\mathcal{R})\right) \longrightarrow \bigoplus_{\mu(x)=p+1} W\left(x, x^{\natural}(\mathcal{R})\right)$$

4.1 V-theory and Rational Equivalence

Two compatible cycles should be called *rationally equivalent* in a sense determined by **Definition 1.2.16** - namely the subgroup of cycles rationally equivalent to zero in $\widetilde{Z}^p(X, \mathcal{R})$ should be those in the image of a map

$$\bigoplus_{\mu(x)=p-1} J^1\big(x, x^{\natural}(\mathcal{R})\big) \xrightarrow{\bigoplus \mathrm{res}_{J^1}^{x,x'}} \bigoplus_{\mu(x)=p} GW\big(x', x'^{\natural}(\mathcal{R})\big)$$

fitting into the diagram concluding the first chapter. Our departure from the aforementioned definition is that we prefer to give an explicit description of the fiber products $J^1(x)$ and *define* the residue maps $\mathrm{res}_{J^1}^{x,x'}$ in terms of that description, rather than obtaining $\mathrm{res}_{J^1}^{x,x'}$ as a fiber product between residue maps for Milnor K-theory and Witt groups. Of course a possible description of the groups $J^1(x)$ is as the Milnor-Witt K-theory group $K_1^{MW}(x)$ - but given how we've taken to working directly with symmetric spaces rather than the presentations of **Propositions 1.2.3** and **1.2.12**, this description is rather unnatural for us. A description of the fiber products $J^1(x)$ in terms of symmetric space is provided by the V-theory of [8, §4.5].

Definition 4.1.1. Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Then we call a triple (M, ψ_1, ψ_2) a **1-symmetric space** in $(\mathcal{E}, *, \eta)$ when each pair (M, ψ_1) and (M, ψ_2) are ordinary symmetric spaces. We denote by $Sym_1(\mathcal{E}, *, \eta)$ the collection of 1-symmetric spaces in $(\mathcal{E}, *, \eta)$.

We call two 1-symmetric spaces (M, ψ_1, ψ_2) and (M', ψ_1', ψ_2') **isometric** when there is an isomorphism $f : M \to M'$ which gives isometries $(M, \psi_1) \to (M, \psi_1')$ and $(M, \psi_2) \to (M, \psi_2')$. We will denote the isometry class of (M, ψ_1, ψ_2) by $[\psi_1, \psi_2]$. The **orthogonal sum** of two 1-symmetric spaces is defined componentwise, that is we set

$$(M, \psi_1, \psi_2) \perp (M', \psi_1', \psi_2') := (M \oplus M', \psi_1 \oplus \psi_1', \psi_2, \oplus \psi_2')$$

Definition 4.1.2. We define the **V-theory** of $(\mathcal{E}, *, \eta)$, denoted $V(\mathcal{E}, *)$, to be the Grothendieck group of the abelian monoid of isometry classes $[M, \psi_1, \psi_2]$ of 1-symmetric spaces in $(\mathcal{E}, *, \eta)$ under the orthogonal summation above, modulo the relation

$$[M, \psi_1, \psi_2] + [M, \psi_2, \psi_3] = [M, \psi_1, \psi_3]$$

for any $M \in \mathcal{E}$ with symmetric forms $\psi_1, \psi_2, \psi_3 : M \to M^*$.

An element $[\psi_1, \psi_2] \in V(\mathcal{E}, *)$ is supposed to represent formally the difference $\psi_2 - \psi_1$ of the two symmetric spaces. As in [8, §4.5] we immediately note that in $V(\mathcal{E}, *)$ we have the relations $[\psi, \psi] = 0$ and hence $[\psi_1, \psi_2] = -[\psi_2, \psi_1]$. An arbitrary element of the Grothendieck group of the monoid of isometry classes of 1-symmetric spaces in $(\mathcal{E}, *)$ is a formal difference $[M, \psi_1, \psi_2] - [N, \phi_1, \phi_2]$. In $V(\mathcal{E}, *)$ we then have

$$[M, \psi_1, \psi_2] - [N, \phi_1, \phi_2] = [M, \psi_1, \psi_2] + [N, \phi_2, \phi_1] = [M \oplus N, \psi_1 \oplus \phi_2, \psi_2 \oplus \phi_1]$$

and hence that any element in V-theory may be represented by a single isomorphism class of something in $Sym_1(\mathcal{E}, *)$.

Definition 4.1.3. Let F be a field, and L a one-dimensional F-vector space. Then we define the **determinant map** to be the group morphism $\det : V(F, L) \to F^*$ by sending the class of a space $(M, \psi_1, \psi_2) \in Sym_1(F, L)$ to the determinant of the isomorphism $(\psi_2)^{-1}\psi_1 : M \to M$.

Proposition 4.1.4. *Let F be a field, and L a one-dimensional F-vector space. Then we have a cartesian square*

$$\begin{array}{ccc}
V(F, L) & \xrightarrow{[\psi_1, \psi_2] \mapsto [\psi_2] - [\psi_1]} & I(F, L) \\
{\scriptstyle \det} \downarrow & & \downarrow \\
F^* & \xrightarrow[\alpha \mapsto 1 - \langle \alpha \rangle]{} & I(F)/I(F)^2
\end{array}$$

where $I(F, L)$ and $I(F)$ respectively denote the fundamental ideals in $W(F)$ and $W(F, L)$.

Proof. This result is [8, Cor. 4.5.1.5] but for transparancey we note that our definition of the determinant map $\det : V(F, L) \to F^*$ is actually the negative of that defined in *loc. cit.* which further uses $F^*/(F^*)^2$ in place of $I(F)/I(F)^2$. We have applied the Milnor isomorphism $F^*/(F^*)^2 \to I(F)/I(F)^2$ defined by $\alpha \mapsto 1 - \langle \alpha \rangle$ to obtain the stated result. $\square$

Let $(\mathcal{E}_1, *_1, \eta_1)$ and $(\mathcal{E}_2, *_2, \eta_2)$ be exact categories with duality - in other words exact categories with weak equivalences and duality where the weak equivalences are the isomorphisms. In this setting we have that a non-singular exact form functor

$$(F, \psi) : (\mathcal{E}_1, *_1, \eta_1) \longrightarrow (\mathcal{E}_2, *_2, \eta_2)$$

is a functor F together with a natural isomorphism $\psi : F*_1 \to *_2 F$ fitting into the commutative diagram of **Definition 2.2.7**.

Definition 4.1.5. Let (F, ψ) be a non-singular exact form functor as above. Then we write

$$(F, \psi)_* : V(\mathcal{E}_1, *_1, \eta_1) \longrightarrow V(\mathcal{E}_2, *_2, \eta_2)$$

for the group homomorphism defined by $\big[M, \phi_1, \phi_2\big] \mapsto \big[F(M), \psi_M F(\phi_1), \psi_M F(\phi_2)\big]$.

Recalling **Proposition 2.2.8** we see that the notation $(F, \psi)_*$ now denotes all of the pushforward maps induced on the Witt, Grothendieck-Witt and V-groups of the exact categories $(\mathcal{E}_1, *_1, \eta_1)$ and $(\mathcal{E}_2, *_2, \eta_2)$. We adopt further the notation of **Definition 2.2.15** - which is the only setting we will actually need any functorality of V-theory.

Let's take now $(X, \mathcal{R})$ to be a scheme equipped with a residual complex that has codimension function μ and take a point $x \in X^p_\mu$. We can construct residue maps

$$d_V^{x,x'}(\mathcal{R}) : V\big(x, \mathcal{R}^p_x\big) \longrightarrow GW\left(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\right)$$

with the same methods as those used to obtain the residue maps of **Proposition 3.1.12**. Note the difference in the domain - on the left hand side we are using the exact category of $\kappa(x)$-vector spaces with duality $\mathrm{Hom}_{\mathcal{O}_{X,x}}(-, \mathcal{R}^p_x)$. The reason for this change is that V-theory does not have a sublagrangian reduction result and hence no devisage allowing us to use the category of finite length $\mathcal{O}_{X,x}$-modules in place of $\kappa(x)$-vector spaces. We have an isomorphism

$$\big(\mathrm{tr}_{\pi_{x,X}}(\mathcal{R}_x)^p\big)_* : V\big(x, x^\natural(\mathcal{R})\big) \longrightarrow V\big(x, \mathcal{R}^p_x\big)$$

and still ultimately want to write down a residue map whose domain is the left hand side above, but it is notationally easier to again define first the map $d_V^{x,x'}(\mathcal{R})$ above.

Lemma 4.1.6. *For any space $(V, \psi_1, \psi_2) \in Sym_1\big(x, \mathcal{R}^p_x\big)$ there exists an $\mathcal{O}_X$-lattice $\mathcal{L} \hookrightarrow \iota_x(V)$ which is self-dual with respect to both $\iota_x(\psi_1)$ and $\iota_x(\psi_2)$.*

Proof. Any sublattice of a self-dual lattice is again self-dual. So intersecting a self dual lattice for ψ_1 with another for ψ_2 will do. □

Note that by **Proposition 3.1.15** we know that the lattice of the above lemma is also non-degenerate with respect to both forms. For a space $(V, \psi_1, \psi_2) \in Sym_1\big(x, \mathcal{R}^p_x\big)$ and $\mathcal{O}_X$-lattice $\mathcal{L} \hookrightarrow \iota_x(V)$ self-dual with respect to both ψ_1 and ψ_2 we are, for any immediate specialisation $x \rightsquigarrow x'$, hence allowed to write $d^p_{\mathcal{R}}\big(L \hookrightarrow (V, \psi_1, \psi_2)\big)_{x'}$ for the element

$$d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M, \psi_2)\big)_{x'} - d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (M, \psi_1)\big)_{x'} \in GW\left(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}^{p+1}_{x'}\right)$$

which we remind the reader picks up the signs of **Definition 3.1.10**.

Proposition 4.1.7. *For a symmetric space* $(V, \psi_1, \psi_2) \in Sym_1\big(x, \mathcal{R}_x^p\big)$ *the element* $d_{\mathcal{R}}^p\big(\mathcal{L} \hookrightarrow (V, \psi_1, \psi_2)\big)_{x'} \in GW\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}_{x'}^{p+1}\big)$ *does not depend on the choice of self-dual lattice* $\mathcal{L}$.

Proof. We may suppose that $X = \mathrm{Spec}(\mathcal{O}_{X,x'})$, and take $P \leq L$ to be two $\mathcal{O}_{x,x'}$-lattices inside M self-dual with respect to both ψ_1 and ψ_2. Then we have the containments $P \leq L \leq L^\flat(\psi_1) \leq P^\flat(\psi_1)$ and may observe that $L/P \hookrightarrow P^\flat(\psi_1)/P$ is a sublagrangian of the space $d_{\mathcal{R}}^p\big(P \hookrightarrow (M, \psi_1)\big)$ with orthogonal $L^\flat(\psi_1)/P$. Hence by the sublagrangian reduction **Proposition 2.2.14** in $GW\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}_{x'}^{p+1}\big)$ we have

$$d_{\mathcal{R}}^p\big(P \hookrightarrow (M, \psi_1)\big) = d_{\mathcal{R}}^p\big(L \hookrightarrow (M, \psi_1)\big) + \mathcal{H}(L/P)$$

Similarly one obtains

$$d_{\mathcal{R}}^p\big(P \hookrightarrow (M, \psi_2)\big) = d_{\mathcal{R}}^p\big(L \hookrightarrow (M, \psi_2)\big) + \mathcal{H}(L/P)$$

and subtracting the two equations gives the result for lattices $P \leq L$. One obtains the result for general lattices L_1, L_2 since the invariance has been established for both pairs $L_1 \cap L_2 \leq L_1$ and $L_1 \cap L_2 \leq L_2$. □

Definition 4.1.8. Let $(X, \mathcal{R})$ be a scheme with a residual complex, $x \in X_\mu^p$ and $x \rightsquigarrow x'$ be an immediate specialisation. Then we write

$$d_V^{x,x'}(\mathcal{R}) : V\big(x, \mathcal{R}_x^p\big) \longrightarrow V\big(f.l.Mod_{\mathcal{O}_{X,x'}}, \mathcal{R}_{x'}^{p+1}\big)$$

for the group homomorphism defined by

$$[V, \psi_1, \psi_2] \mapsto d_{\mathcal{R}}^p\big(\mathcal{L} \hookrightarrow (V, \psi_1, \psi_2)\big)_{x'}$$

for some $\mathcal{O}_X$-lattice $\mathcal{L}$ self-dual with respect to both ψ_1 and ψ_2.

As is by now automatic, the notation of **Definition 3.1.13** continues; we write

$$\mathrm{res}_V^{x,x'}(\mathcal{R}) : V\big(x, x^\natural(\mathcal{R})\big) \longrightarrow V\big(x', x'^\natural(\mathcal{R})\big)$$

for the map $\big(\mathrm{tr}_{\pi_{x',X}}(\mathcal{R}_{x'})^{p+1}\big)_*^{-1} d_V^{x,x'}(\mathcal{R})\big(\mathrm{tr}_{\pi_{x,X}}(\mathcal{R}_x)^p\big)_*$ and usually drop the residual complex from its appearance in our notation for any of these residue maps. We again obtain, with the same arguments, the results of **Propositions 3.1.14** and **3.1.19** in this setting - for the former, we have that if $j : Z \hookrightarrow X$ is a closed embedding containing the immediate specialisation $x \rightsquigarrow x'$ then we have the commutative diagram

$$\begin{tikzcd}
V\big(x,\mathcal{R}^p_x\big) \arrow[r, "d_V^{x,x'}(\mathcal{R})"] & GW\big(f.l.Mod_{\mathcal{O}_{X,x'}},\mathcal{R}^{p+1}_{x'}\big) \\
V\big(x,j^{\triangle}(\mathcal{R})^p_x\big) \arrow[u, "\big(\mathrm{tr}_j(\mathcal{R})^p_x\big)_*"] \arrow[r, "d_V^{x,x'}\big(j^{\triangle}(\mathcal{R})\big)"'] & GW\big(f.l.Mod_{\mathcal{O}_{Z,x'}},j^{\triangle}(\mathcal{R})^{p+1}_{x'}\big) \arrow[u, "\big(\mathrm{tr}_j(\mathcal{R})^{p+1}_{x'}\big)_*"']
\end{tikzcd}$$

and for the latter we have the below result.

Proposition 4.1.9. *Let A be a semi-local domain with residual complex $\mathcal{R}$ concentrated in degrees p and $p+1$ and field of fractions F. Then after writing ζ for the zero ideal and $m_1,...,m_n$ for the maximal ideals of A we have a group homomorphism*

$$d_V(\mathcal{R}) : V\big(F,\mathcal{R}^p\big) \longrightarrow GW\big(f.l.Mod_A,\mathcal{R}^{p+1}\big)$$

defined by

$$[V,\psi_1,\psi_2] \mapsto d^p_{\mathcal{R}}\big(\mathcal{L} \hookrightarrow (V,\psi_1,\psi_2)\big) \in GW\big(f.l.Mod_A,\mathcal{R}^{p+1}\big)$$

for some A-lattice self-dual with respect to both ψ_1 and ψ_2. This map fits into a commutative diagram

$$\begin{tikzcd}
V\big(F,\mathcal{R}^p\big) \arrow[r, "d_V(\mathcal{R})"] \arrow[d, equal] & GW\big(f.l.Mod_A,\mathcal{R}^{p+1}\big) \\
V\big(F,\mathcal{R}^p\big) \arrow[r, "\sum d_V^{\zeta,m_i}(\mathcal{R})"'] & \bigoplus_{i=1}^n GW\big(f.l.Mod_{A_{m_i}},\mathcal{R}^{p+1}_{m_i}\big) \arrow[u, "\bigoplus\big(\rho(m_i)\big)_*"']
\end{tikzcd}$$

Proof. This goes exactly as **Proposition 3.1.19** - the well-definedness of $d_V(\mathcal{R})$ follows by the same argument as that of **Proposition 4.1.7** and the commutativity of the diagram comes straight from the definition of the maps $d_V^{\zeta,m_i}(\mathcal{R})$. □

Proposition 4.1.10. *Let $x \in X^p_\mu$ and $(V,\psi_1,\psi_2) \in Sym_1\big(x,x^{\natural}(\mathcal{R})\big)$. Then there are at most finitely many points $x' \in X^{p+1}_\mu$ with $\mathrm{res}_V^{x,x'}(V,\psi_1,\psi_2) \neq 0$.*

Proof. Repeating the idea of the proof of **Proposition 3.3.3**, we see that the points $x' \in X^{p+1}_\mu$ with $\mathrm{res}_V^{x,x'}(V,\psi_1,\psi_2) \neq 0$ are among the irreducible components of the supports of coherent modules $\mathcal{L}^{\flat}(\psi_2)/\mathcal{L}$ and $\mathcal{L}^{\flat}(\psi_1)/\mathcal{L}$ - hence there are finitely many. □

Definition 4.1.11. Let $(X,\mathcal{R})$ be a scheme equipped with a residual complex. Then we have a map

$$\bigoplus_{\mu(z)=p-1} \sum_{z \rightsquigarrow x} \mathrm{res}_V^{z,x} : \bigoplus_{\mu(z)=p-1} V\left(z, z^\natural(\mathcal{R})\right) \longrightarrow \bigoplus_{\mu(x)=p} GW\left(x, x^\natural(\mathcal{R})\right)$$

whose image we denote by $\widetilde{\mathrm{Rat}}^p(X, \mathcal{R})$ and call the cycles **rationally equivalent to zero**.

Lemma 4.1.12. *Cycles rationally equivalent to zero are compatible cycles on X, that is we have $\widetilde{\mathrm{Rat}}^p(X, \mathcal{R}) \leq \widetilde{Z^p}(X, \mathcal{R})$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\mu(z)=p-1} I\big(z, z^\natural(\mathcal{R})\big) \arrow[r, "d^{p-1}_{W(X,\mathcal{R})}"] & \bigoplus_{\mu(x)=p} W\big(x, x^\natural(\mathcal{R})\big) \arrow[r, "d^{p}_{W(X,\mathcal{R})}"] & \bigoplus_{\mu(y)=p+1} W\big(z, z^\natural(\mathcal{R})\big) \\
\bigoplus_{\mu(z)=p-1} V\big(z, z^\natural(\mathcal{R})\big) \arrow[u] \arrow[r, "\bigoplus_z \sum_{z \rightsquigarrow x} \mathrm{res}_V^{z,x}"'] & \bigoplus_{\mu(x)=p} GW\big(x, x^\natural(\mathcal{R})\big) \arrow[u] \arrow[r, "\bigoplus_x \sum_{x \rightsquigarrow y} \mathrm{res}_{GW}^{x,y}"'] & \bigoplus_{\mu(y)=p+1} W\big(z, z^\natural(\mathcal{R})\big) \arrow[u, equal]
\end{tikzcd}$$

where the vertical map on the left is the coproduct of the corresponding maps of **Proposition 4.1.4** - the result hence follows from **Proposition 3.3.4**. $\square$

Definition 4.1.13. For a scheme X equipped with a residual complex $\mathcal{R}$, we define the **Chow-Witt group of codimension p cycles on X** to be the quotient

$$\widetilde{CH}^p(X, \mathcal{R}) = \widetilde{Z^p}(X, \mathcal{R}) / \widetilde{\mathrm{Rat}}^p(X, \mathcal{R})$$

Note that the grading on the Chow-Witt group $\widetilde{CH}^*(X, \mathcal{R})$ is given by the codimension function of the residual complex $\mathcal{R}$.

4.2 Pushforward along Proper maps

If $f : X \to Y$ is a proper morphism between schemes which admit residual complexes, we construct in this section a pushforward

$$f_* : \widetilde{CH}(X, \mathcal{R}_X) \longrightarrow \widetilde{CH}(Y, \mathcal{R}_Y)$$

in terms of additional compatibility data between the dualities $\mathcal{H}om_X(-\mathcal{R}_X)$ and $\mathcal{H}om_Y(-, \mathcal{R}_Y)$. In the same way, one obtains pushforwards of the Witt complex of the previous chapter.

Definition 4.2.1. If $f : X \to Y$ is any morphism, we write $\mathcal{E}_f$ for the full subcategory of $\mathrm{Ch}(X)$ consisting of those complexes whose terms are acyclic for the pushforward $f_* : QCoh_X \to QCoh_Y$, precisely

$$\mathcal{E}_f = \{M \in \mathrm{Ch}(X) \mid (R^p f_*)(M^n) = 0 \text{ for any } n \in \mathbb{Z} \text{ and } p \geq 1\}$$

We further write $\mathcal{E}^b_{f,Coh}$ for the intersection of this category with $\mathrm{Ch}^b_{Coh}(X)$.

Lemma 4.2.2. *The category $\mathcal{E}_f$ defined above is exact. Further, if $\mathcal{R}_X$ is any residual complex on X we have that the functor $\mathcal{H}om_X(-, \mathcal{R}_X)$ preserves $\mathcal{E}_f \cap \mathrm{Ch}^-(X)$.*

Proof. That $\mathcal{E}_f$ is closed under extensions in $\mathrm{Ch}(X)$ can be checked in each degree, where it follows from the long exact sequence for the right derived functors $R^p f_*$. If $\mathcal{R}_X \in \mathrm{Res}(X)$, then appealing to [50, Ch.II, Lem.7.3.2] one learns that for any bounded above complex $M \in \mathcal{E}_f$ the dual object $\mathcal{H}om_X(M, \mathcal{R}_X)$ is a bounded complex of flasque sheaves and hence again lies in $\mathcal{E}_f$. □

We hence have an exact category with duality $(\mathcal{E}^b_{f,Coh}, \mathcal{R}_X)$ where the double dual identification is as in **Definition 2.1.6**.

Lemma 4.2.3. *If $f : X \to Y$ is proper, then the functor*

$$f_* : \mathcal{E}^b_{f,Coh} \longrightarrow \mathrm{Ch}^b_{Coh}(Y)$$

is exact.

Proof. That f_* is exact follows immediately from the definition of $\mathcal{E}_f$. Since all our schemes are Noetherian we apply [6, Ch.II, Prop.2.2] to see that if $M \in \mathcal{E}^b_{f,Coh}$ then the cohomology groups of f_*M are coherent. □

Lemma 4.2.4. *For any morphism $f : X \to Y$ there is a natural transformation*

$$\mathbf{R}f_*\mathbf{R}\operatorname{Hom}_X(-,-) \longrightarrow \mathbf{R}\operatorname{Hom}_Y\big(\mathbf{R}f_*(-), \mathbf{R}f_*(-)\big)$$

between the above bifunctors $D^-(X)^{op} \times D^+(X) \to D(Y)$.

Proof. This is [6, Ch.II, Prop.5.5] - we overview the details here. Writing $K(\mathcal{E}_f)$ for the full subcategory of the homotopy category $K\big(\mathrm{Ch}(X)\big)$ consisting of those complexes lying in $\mathcal{E}_f$, we have that the natural map $K(\mathcal{E}_f)_{qis} \to D(X)$ from the localisation of $K(\mathcal{E}_f)$ by the quasi-isomorphisms is an equivalence, and the functor

$$\mathbf{R}f_* : D(X) \longrightarrow D(Y)$$

may be constructed by first taking an inverse to this equivalence and applying f_*. To define this natural transformation then, we may assume that we have a bounded above complex $F \in \mathcal{E}_f$ and bounded below complex G of injective quasi-coherent $\mathcal{O}_X$-modules. Since both F and G lie in $\mathcal{E}_f$ we can identify, up to canonical isomorphism in $D(Y)$, $\mathbf{R}f_*F = f_*F$ and $\mathbf{R}f_*G = f_*G$. Similarly we identify $\mathbf{R}\operatorname{Hom}_X(F, G) = \mathcal{H}om_X(F, G)$

which, by the lemma above again lies in $\mathcal{E}_f$ - allowing us to, up to canonical isomorphism in $D(Y)$, write

$$\mathbf{R}f_*\mathbf{R}\operatorname{Hom}_X(F,G) = f_*\mathcal{H}om_X(F,G)$$

So we obtain the transformation by taking

$$\mathbf{R}f_*\mathbf{R}\operatorname{Hom}_X(F,G) = f_*\mathcal{H}om_X(F,G) \longrightarrow \mathcal{H}om_Y(f_*F, f_*G) \longrightarrow \mathbf{R}\operatorname{Hom}_Y\big(\mathbf{R}f_*F, \mathbf{R}f_*G\big)$$

in which the first map is the natural extension of the transformation $f_*\mathcal{H}om_{\mathcal{O}_X}(-,-) \to \mathcal{H}om_{\mathcal{O}_Y}(f_*(-), f_*(-))$ on the level of sheaves, and the second map is that of the definition of the right derived functor $\mathbf{R}\operatorname{Hom}_Y(-,-)$. □

Definition 4.2.5. Let $(X,\mathcal{R}_X)$ and $(Y,\mathcal{R}_Y)$ be schemes with residual complexes whose codimension functions we assume to satisfy

$$\mu_Y(y) = \mu_X(x) + \operatorname{tr.deg}(\kappa(x)/\kappa(y))$$

whenever $f(x) = y$. Then a **proper map**

$$(f,\tau) : (X,\mathcal{R}_X) \longrightarrow (Y,\mathcal{R}_Y)$$

consists of a proper morphism $f : X \to Y$ together with a duality compatibility map

$$\tau : f_*\mathcal{R}_X \longrightarrow \mathcal{R}_Y$$

between chain complexes of $\mathcal{O}_Y$-modules which is such that the natural transformation τ_*, defined for any $M \in D^b_{Coh}(X)$ by the below composition,

$$\begin{tikzcd}
\mathbf{R}f_*\mathbf{R}\operatorname{Hom}_X(M,\mathcal{R}_X) \arrow[r] \arrow[dr, "\tau_*(M)"'] & \mathbf{R}\operatorname{Hom}_Y\big(\mathbf{R}f_*M, \mathbf{R}f_*\mathcal{R}_X\big) \arrow[d, "\mathbf{R}\operatorname{Hom}_Y(\mathbf{R}f_*M,\tau)"] \\
 & \mathbf{R}\operatorname{Hom}_Y\big(\mathbf{R}f_*M, \mathcal{R}_Y\big)
\end{tikzcd}$$

is a natural isomorphism - where to be clear the map across the top is the transformation of **Lemma 4.2.4** and we have slightly abused notation by writing on the vertical map $\tau : \mathbf{R}f_*\mathcal{R}_X \to \mathcal{R}_Y$ again for the map obtained from τ via the canonical isomorphism $f_*\mathcal{R}_X \to \mathbf{R}f_*\mathcal{R}_X$.

Remark 4. *From the Duality Theorem [6, Ch. VII Thm3.3] we obtain the main example of such data - if $\mathcal{R}_X = f^{\triangle}(\mathcal{R}_Y)$ is the exceptional inverse image of $\mathcal{R}_Y$, then the trace map constructed in [6, Ch. VI] gives a duality compatibility as above.*

Example 4.2.6. Let $(X, \mathcal{R})$ be a scheme with a residual complex, and $j : Z \hookrightarrow X$ a closed embedding. Then the pair

$$\big(j, \mathrm{tr}_j(\mathcal{R})\big) : (Z, j^{\triangle}(\mathcal{R})) \longrightarrow (X, \mathcal{R}_X)$$

is a proper map, where $\mathrm{tr}_j(\mathcal{R})$ is as in **Definition 2.1.28**. In fact, in this case

$$\mathrm{tr}_j(\mathcal{R})_* : j_* \mathcal{H}om_Z(-, j^{\triangle}(\mathcal{R})) \longrightarrow \mathcal{H}om_X(j_*(-), \mathcal{R})$$

is a natural isomorphism.

If $(f, \tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is a proper map, then we obtain a non-singular exact form functor

$$(f_*, \tau_*) : (\mathcal{E}_f, \mathcal{R}_X) \longrightarrow \big(\mathrm{Ch}^b_{Coh}(Y), \mathcal{R}_Y\big)$$

Observe that in each degree the morphism

$$\tau^p : \bigoplus_{\mu_X(x)=p} f_* \mathcal{R}_X(x) \longrightarrow \bigoplus_{\mu_Y(y)=p} \mathcal{R}_Y(y)$$

must be given by a coproduct of maps $\tau(x) : f_* \mathcal{R}_X(x) \to \mathcal{R}_Y(y)$, where $y = f(x)$, taken over points $x \in X$ such that $\mathrm{tr.deg}(\kappa(x)/\kappa(y)) = 0$. Indeed, any component $f_* \mathcal{R}_X(x) \to \mathcal{R}_Y(y)$ of τ^p will be zero unless y lies in the closure of $f(x)$, and under our condition $\mu_Y(f(x)) = \mu_X(x) + \mathrm{tr.deg}(\kappa(x)/\kappa(f(x)))$ this is only possible if $y = f(x)$. We similarly observe that there are no chain homotopies between $f_* \mathcal{R}_X$ and $\mathcal{R}_Y$ - so while our convention is to take the map $\tau : f_* \mathcal{R}_X \to \mathcal{R}_Y$ to be a morphism of chain complexes we can equivalently give τ as a morphism in the derived category.

Notation 4.2.7. Suppose (f, τ) is a proper map as in the discussion above, and that $x \in X$ is a point of codimension p with $\mathrm{tr.deg}(\kappa(x)/\kappa(y)) = 0$, where $y = f(x)$. Then we write

$$\tau(x) : f_* \mathcal{R}_X(x) \to \mathcal{R}_Y(y)$$

for the morphism of $\mathcal{O}_Y$-modules which appears as a component of τ^p, and we write

$$\tau_x : \mathcal{R}^p_{X,x} \to \mathcal{R}^p_{Y,y}$$

for the stalk of $\tau(x)$ at y.

Lemma 4.2.8. *Let $(f, \tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a proper map, and $i : U \to Y$ be the inclusion of an open subset or localisation to a point. Let $i' : V = f^{-1}(U) \to X$ be the induced map. Then we have a proper map*

$$\big(f|_V, i'^* \tau\big) : \big(V, i'^*(\mathcal{R}_X)\big) \longrightarrow \big(U, i^* \mathcal{R}_Y\big)$$

Proof. The statement with U being an open set is clear enough, so we just consider the case $U = \mathrm{Spec}(Y_p)$ for some point p in Y; and it suffices to suppose that $Y = \mathrm{Spec}(A)$ is affine. The fact that $i'^*\big(\mathcal{R}_X\big)$ is again a residual complex over V does follow from i' being a *residually stable* morphism [6, Ch.VI] but we find it helpful to give some details establishing this part of the result explicitly.

The fibre $V = f^{-1}(Y_p)$ is covered by affine open subsets $\mathrm{Spec}(A_p \otimes_A B)$ where $\mathrm{Spec}(B)$ is an affine open subset of X. It hence becomes clear that $i'^*(\mathcal{R}_X)$ is a bounded complex of injectives with coherent cohomology on V - and further that for any point $q \in X^n_\mu$ we have that $i'^*(\mathcal{R}^n_X)_q = \mathcal{R}^n_{X,q}$ is still an $\mathcal{O}_{X,q}$-injective hull of $\kappa(q)$; hence the codimension function for $i'^*(\mathcal{R}_X)$ agrees with that of $\mathcal{R}_X$. One further with this affine description sees that the map

$$i'^*(\mathcal{R}_X) \longrightarrow \bigoplus_{x \in V^n_\mu} \iota_x\big(\mathcal{R}^n_x\big)$$

induced by localisation to the points $x \in V^n_\mu$ is an isomorphism. It only remains to understand the abuse of notation which allows us to view the map

$$i^*\tau : i^* f_* \mathcal{R}_X \longrightarrow i^* \mathcal{R}_Y$$

as a suitable duality compatibility $\big(f|_V\big)_* i'^* \mathcal{R}_X \to i^* \mathcal{R}_Y$. Let us first note that by [6, Ch.II Prop.5.12] we have that the canonical map $i^* f_* \mathcal{R}_X \to \big(f|_V\big)_* i'^* \mathcal{R}_X$ is a quasi-isomorphism. Hence there is a unique map $\big(f|_V\big)_* i'^* \mathcal{R}_X \to i^* \mathcal{R}_Y$ in $D(U)$ fitting into the diagram

$$\begin{tikzcd} i^* f_* \mathcal{R}_X \arrow[r, "i^*\tau"] \arrow[d, "\wr"'] & i^* \mathcal{R}_Y \\ \big(f|_V\big)_* i'^* \mathcal{R}_X \arrow[ur] & \end{tikzcd}$$

which we continue to denote by $i^*\tau$. To see that the induced transformation $\big(i^*\tau\big)_*$ between functors $D^b_{Coh}(V) \to D^b_{Coh}(U)$ is a natural isomorphism it suffices by the lemma on way out functors [6, Ch.I Prop.7.1] to check that $\tau_*(\mathcal{O}_V) = \tau_*(i'^*\mathcal{O}_X)$ is an isomorphism - which follows from the compatibilities of [6, Prop.5.8, 5.12]. $\square$

Lemma 4.2.9. *Let $(f, \tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a proper map of schemes with residual complexes, $j : Z \hookrightarrow X$ be a subvariety of X and write $j' : W \hookrightarrow Y$ for the subvariety $W = f(Z)$ of Y. Denoting by $g : Z \to W$ the restriction of f to Z, we set*

$$\tau|_Z : g_* j^\triangle(\mathcal{R}_X) \longrightarrow j'^\triangle(\mathcal{R}_Y)$$

to be the morphism of chain complexes of $\mathcal{O}_W$-modules with $\mathrm{tr}_{j'}(\mathcal{R}_Y) \circ j'_* \tau|_Z = \tau \circ f_* \mathrm{tr}_j(\mathcal{R}_X)$. Then the pair $(g, \tau|_Z)$ is a proper map $(Z, j^\triangle(\mathcal{R}_X)) \to (W, j'^\triangle(\mathcal{R}_Y))$.

Proof. To check that the map $\left(\tau|_Z\right)_*$ is a natural isomorphism, it suffices to suppose that $M \in \mathcal{E}_g$ is some bounded above complex and see that the map $j'_*(\tau|_Z)_*(M)$ depicted below is a quasi-isomorphism.

$$\begin{tikzcd}
j'_* g_* \mathcal{H}om_Z(M, j^\triangle(\mathcal{R}_X)) \arrow[r] \arrow[dr, "j'_*(\tau|_Z)_*(M)"'] & j'_* \mathcal{H}om_W(g_* M, g_* j^\triangle(\mathcal{R}_X)) \arrow[d, "j'_* \mathcal{H}om(g_* M, \tau|_Z)"] \\
 & j'_* \mathcal{H}om_W(g_* M, j'^\triangle(\mathcal{R}_Y))
\end{tikzcd}$$

Since j' is a closed embedding, it commutes with the internal chain complex Homs above. So we may move both of the j'_* on the right side of the above diagram on the inside of the $\mathcal{H}om_W$, and then identify $j'_* g_* = f_* j_*$ to see that our map $j'_*(\tau|_Z)_*(M)$ fits into the diagram below.

$$\begin{tikzcd}
f_* j_* \mathcal{H}om_Z(M, j^\triangle(\mathcal{R}_X)) \arrow[r] \arrow[dr, "j'_*(\tau|_Z)_*(M)"'] & \mathcal{H}om_Y(f_* j_* M, f_* j_* j^\triangle(\mathcal{R}_X)) \arrow[d, "\mathcal{H}om_Y(f_* j_* M, j'_*(\tau|_Z))"] \\
 & \mathcal{H}om_Y(j'_* g_* M, j'_* j'^\triangle(\mathcal{R}_Y))
\end{tikzcd}$$

As in **Example 4.2.6** we have an isomorphism

$$\mathcal{H}om_Y(j'_* g_* M, \mathrm{tr}_{j'}(\mathcal{R}_Y)) : \mathcal{H}om_Y(j'_* g_* M, j'_* j'^\triangle(\mathcal{R}_Y)) \xrightarrow{\simeq} \mathcal{H}om_Y(j'_* g_* M, \mathcal{R}_Y)$$

and similarly the isomorphism

$$\mathcal{H}om_X(j_* M, \mathrm{tr}_j(\mathcal{R}_X)) : \mathcal{H}om_X(j_* M, j^\triangle j_*(\mathcal{R}_X)) \xrightarrow{\simeq} \mathcal{H}om_X(j_* M, \mathcal{R}_X)$$

We use these isomorphisms to extend our above triangle for $j'_*(\tau|_Z)_*(M)$ to the diagram below, in which the two trace-map-isomorphisms we just wrote down are used to give the three horizontal isomorphisms.

$$
\begin{tikzcd}
f_* \left[j_*M, j_*j^{\triangle}(\mathcal{R}_X)\right]_X \arrow[rr, "\simeq", "f_* \left[j_*M, \mathrm{tr}_j(\mathcal{R}_X)\right]_X"'] & & f_* \left[j_*M, \mathcal{R}_X\right]_X \arrow[d] \\
f_*j_* \left[M, j^{\triangle}(\mathcal{R}_X)\right]_Z \arrow[u, "\wr"] \arrow[r] \arrow[dr, "j'_*(\tau|_Z)_*(M)"'] & \left[f_*j_*M, f_*j_*j^{\triangle}(\mathcal{R}_X)\right]_Y \arrow[r, "\simeq"] \arrow[d, "{\left[f_*j_*M, j'_*(\tau|_Z)\right]_Y}"] & \left[f_*j_*M, f_*\mathcal{R}_X\right]_Y \arrow[d, "{\left[f_*j_*M, \tau\right]_Y}"] \\
& \left[f_*j_*M, j'_*j'^{\triangle}(\mathcal{R}_Y)\right]_Y \arrow[r, "\simeq", "{\left[f_*j_*M, \mathrm{tr}_{j'}(\mathcal{R}_Y)\right]_Y}"'] & \left[f_*j_*M, \mathcal{R}_Y\right]_Y
\end{tikzcd}
$$

On account of our condition $\mathrm{tr}_{j'}(\mathcal{R}_Y) \circ j'_*\tau|_Z = \tau \circ f_* \, \mathrm{tr}_j(\mathcal{R}_X)$ this diagram is commutative. Then since the right hand side vertical composition

$$
f_* \left[j_*M, \mathcal{R}_X\right]_X \longrightarrow \left[f_*j_*M, f_*\mathcal{R}_X\right]_Y \longrightarrow \left[f_*j_*M, \mathcal{R}_Y\right]_Y
$$

is the quasi-isomorphism $\tau_*(j_*M)$, we have as desired that $j'_*(\tau|_Z)_*(M)$ is also a quasi-isomorphism. $\square$

Note, as has become standard in our notation, that in the below we do not distinguish whether the pushforward in question is for Witt, Grothendieck-Witt groups or for V-theory. Hopefully it will in practice be clear what is meant.

Lemma 4.2.10. *Let $(f, \tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a proper morphism and suppose $w = f(z)$ are points with $\mu(z) = \mu(w) = p$. Then postcomposition with the $\mathcal{O}_{Y,w}$-linear map $\tau_z : \mathcal{R}^p_{X,z} \to \mathcal{R}^p_{Y,w}$ induces for any finite length $\mathcal{O}_{X,z}$-module M an isomorphism*

$$
\left(\tau_z\right)_* : \mathrm{Hom}_{\mathcal{O}_{X,z}}\left(M, \mathcal{R}^p_{X,z}\right) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{Y,w}}\left(M, \mathcal{R}^p_{Y,w}\right)
$$

Hence τ_z induces as in **Definition 2.2.15** *a pushforward map*

$$
\left(\tau_z\right)_* : \mathbf{T}\left(f.l.Mod_{\mathcal{O}_{X,z}}, \mathcal{R}^p_{X,z}\right) \longrightarrow \mathbf{T}\left(f.l.Mod_{\mathcal{O}_{Y,w}}, \mathcal{R}^p_{Y,w}\right)
$$

where $\mathbf{T}$ is either of GW or W and as in **Definition 4.1.5** *a map*

$$
\left(\tau_z\right)_* : V\left(z, \mathcal{R}^p_{X,z}\right) \longrightarrow V\left(w, \mathcal{R}^p_{Y,w}\right)
$$

Proof. Let's write $j : Z \hookrightarrow X$ for the subvariety of X with generic point z, and $j' : W \hookrightarrow Y$ for the image of Z under f, and finally $g : Z \to W$ for the restriction of f. Then as in **Proposition 4.2.9** we obtain a proper map $(g, \tau|_Z) : (Z, j^{\triangle}(\mathcal{R}_X)) \to (W, j'^{\triangle}(\mathcal{R}_Y))$. The stalk at w of both complexes $g_*j^{\triangle}(\mathcal{R}_X)$ and $j'^{\triangle}(\mathcal{R}_Y)$ are concentrated in degree

p, and by localising the commutative square $\mathrm{tr}_{j'}(\mathcal{R}_Y)^p \circ j'_*(\tau|_Z)^p = \tau^p \circ f_* \mathrm{tr}_j(\mathcal{R}_X)^p$ at w, we obtain the diagram

$$\begin{tikzcd}
z^\natural(\mathcal{R}_X) \arrow[r, "f_* \mathrm{tr}_j(\mathcal{R}_X)^p_w"] \arrow[d, "(\tau|_Z)^p_w"'] & \bigoplus_{\mu(x)=p, f(x)=w} \mathcal{R}^p_{X,x} \arrow[d, "\oplus\tau_x"] \\
w^\natural(\mathcal{R}_Y) \arrow[r, "\mathrm{tr}_{j'}(\mathcal{R}_Y)^p_w"'] & \mathcal{R}^p_{Y,w}
\end{tikzcd}$$

of $\mathcal{O}_{Y,w}$-modules. We have now by **Lemma 4.2.8** a proper map $(g^{-1}(w), j^\triangle(\mathcal{R}_X)_z) \to (w, j'^\triangle(\mathcal{R}_Y)_w)$, hence the map $\left(\tau|_Z\right)^p_w$ is non-zero. Hence for any finite dimensional $\kappa(z)$-vector space, postcompostion with the map $(\tau|_Z)^p_w$ induces an isomorphism

$$\mathrm{Hom}_{\kappa(z)}\left(V, j^\triangle(\mathcal{R}_X)^p_z\right) \longrightarrow \mathrm{Hom}_{\kappa(w)}\left(V, j'^\triangle(\mathcal{R}_Y)^p_w\right)$$

Note that this in particular gives the statement of the proposition when M has length 1 as an $\mathcal{O}_{X,z}$-module. The result in full follows by induction on this length. □

Notation 4.2.11. Suppose that $(f,\tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is a proper map and $y = f(x)$ are points with $\mu(x) = \mu(y) = p$ say. Then we denote the map

$$\left(\tau|_{\overline{\{x\}}}\right)^p_y : z^\natural(\mathcal{R}_X) \longrightarrow w^\natural(\mathcal{R}_Y)$$

by $\tau|^x_y$ - so we have the commutative square $\mathrm{tr}_{\pi_{y,Y}}(\mathcal{R}_{Y,y})^p \circ \tau|^x_y = \tau_x \circ \mathrm{tr}_{\pi_{x,X}}\left(\mathcal{R}_{X,x}\right)^p$.

Definition 4.2.12. Let $(f,\tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a proper map, and $\mathbf{T}$ be any of GW, W or V. Then we write

$$f_* : \bigoplus_{\mu(x)=p} \mathbf{T}\left(x, x^\natural(\mathcal{R})\right) \longrightarrow \bigoplus_{\mu(y)=p} \mathbf{T}\left(y, y^\natural(\mathcal{R})\right)$$

for the group homomorphism which on each component of the lefthand side is

$$\left(\tau|^x_{f(x)}\right)_* : \mathbf{T}\left(x, x^\natural(\mathcal{R})\right) \longrightarrow \mathbf{T}\left(f(x), f(x)^\natural(\mathcal{R})\right)$$

if $\mu(x) = \mu(f(x))$ and zero otherwise.

We will argue that this construction induces a pushforward map on Chow-Witt groups; in particular we also get a pushforward for the Witt complex. The geometric arguments reducing the problem to a few special cases are those of the proof of the *Residue Theorem* [6, Ch.VII Thm 2.1] - which is the result that the trace map for residual complexes of *loc. cit.* really is a morphism of complexes; all we need to do is watch what happens to our lattices in these cases.

Proposition 4.2.13. *Let $(f,\tau) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a proper map between schemes with residual complexes. Then the map*

$$f_* : \bigoplus_{\mu(x)=p} GW\left(x, x^\natural(\mathcal{R})\right) \longrightarrow \bigoplus_{\mu(y)=p} GW\left(y, y^\natural(\mathcal{R})\right)$$

induces a morphism

$$f_* : \widetilde{CH}^p(X, \mathcal{R}_X) \longrightarrow \widetilde{CH}^p(Y, \mathcal{R}_Y)$$

Proof. If $\mathbf{T}$ is V then let $\mathbf{T}-1$ denote GW, while if $\mathbf{T}$ is GW, let $\mathbf{T}-1$ denote W. We aim to show that in either case the square

$$\begin{tikzcd}
\bigoplus_{\mu(x)=p} \mathbf{T}\left(x, x^\natural(\mathcal{R})\right) \arrow[r, "\bigoplus_{\mu(x)=p}\sum_{x\rightsquigarrow x'} \mathrm{res}_{\mathbf{T}}^{x,x'}"] \arrow[d, "f_*"'] & \bigoplus_{\mu(x')=p+1} (\mathbf{T}-1)\left(x', x'^\natural(\mathcal{R})\right) \arrow[d, "f_*"] \\
\bigoplus_{\mu(y)=p} \mathbf{T}\left(y, y^\natural(\mathcal{R})\right) \arrow[r, "\bigoplus_{\mu(y)=p}\sum_{y\rightsquigarrow y'} \mathrm{res}_{\mathbf{T}}^{y,y'}"'] & \bigoplus_{\mu(y')=p+1} (\mathbf{T}-1)\left(y'^\natural(\mathcal{R})\right)
\end{tikzcd}$$

commutes. It suffices to focus on a single point $x \in X$ with $\mu(x) = p$, so we will show that the below square commutes

$$\begin{tikzcd}
\mathbf{T}\left(x, x^\natural(\mathcal{R})\right) \arrow[r, "\sum_{x\rightsquigarrow x'} \mathrm{res}_{\mathbf{T}}^{x,x'}"] \arrow[d, "f_*"'] & \bigoplus_{x\rightsquigarrow x'} (\mathbf{T}-1)\left(x', x'^\natural(\mathcal{R})\right) \arrow[d, "f_*"] \\
\bigoplus_{\mu(y)=p} \mathbf{T}\left(y, y^\natural(\mathcal{R})\right) \arrow[r, "\bigoplus_{\mu(y)=p}\sum_{y\rightsquigarrow y'} \mathrm{res}_{\mathbf{T}}^{y,y'}"'] & \bigoplus_{\mu(y')=p+1} (\mathbf{T}-1)\left(y', y'^\natural(\mathcal{R})\right)
\end{tikzcd}$$

where we have written again f_* on each of the vertical maps for the restriction of the pushforwards in **Definition 4.2.12**. If $\operatorname{tr.deg}\left(\kappa(x)/\kappa(f(x))\right) \geq 2$ then there is nothing to show as both the maps f_* above will be zero.

Case 1. Setting $y = f(x)$, we have $\operatorname{tr.deg}(\kappa(x)/\kappa(y)) = 0$ - hence $\mu(x) = \mu(y)$.

Replacing X with the closure of the point x, Y by the closure of the point y, and the proper map (f,τ) by its restriction as in **Proposition 4.2.9** we see that we must in this case show that the diagram

$$
\begin{tikzcd}
\mathbf{T}\left(x, \mathcal{R}^p_{X,x}\right) \arrow[r, "\sum\limits_{x' \in X^1} \mathrm{res}^{x,x'}_{\mathbf{T}}"] \arrow[d, "(\tau_x)_*"'] & \bigoplus\limits_{x' \in X^1} (\mathbf{T}-1)\left(x', x'^{\natural}(\mathcal{R})\right) \arrow[d, "f_*"] \\
\mathbf{T}\left(y, \mathcal{R}^p_{Y,y}\right) \arrow[r, "\sum\limits_{y' \in Y^1} \mathrm{res}^{y,y'}_{\mathbf{T}}"'] & \bigoplus\limits_{y' \in Y^1} (\mathbf{T}-1)\left(y', y'^{\natural}(\mathcal{R})\right)
\end{tikzcd}
$$

commutes, where x and y are now the generic points of the varieties X and Y respectively. We consider each y' separately, so let's set $i : U \to X$ to be $f^{-1}(\mathrm{Spec}(\mathcal{O}_{Y,y'}))$. Let us note that since f is surjective the points of X mapping to y' are of codimension at least 1 in X, and since $\dim(X) = \dim(Y)$ they must indeed be of codimension 1. Further, as f is proper, $f|_U$ is a finite morphism by [51, III 4.4.11] so we may write $U = \mathrm{Spec}(A)$ where A is the one-dimensional semi-local ring of points mapping to y'.

We have by now a proper map $(f|_U, i^*(\tau)) : (U, i^*\mathcal{R}_X) \to (\mathrm{Spec}(\mathcal{O}_{Y,y}), \mathcal{R}_{Y,y})$, with f finite and $i^*(\tau)$ being the morphism between chain complexes concentrated in degrees p and $p+1$ appearing below

$$
\begin{tikzcd}
\mathcal{R}^p_A \arrow[r, "d^p_{\mathcal{R}_A}"] \arrow[d, "\tau_x"'] & \mathcal{R}^{p+1}_A \arrow[d, "(\tau|_U)^{p+1}"] & \bigoplus\limits_{f(x')=y'} x'^{\natural}(\mathcal{R}_X) \arrow[l, "\bigoplus \mathrm{tr}_{\pi_{x',X}}(\mathcal{R}_{X,x'})^{p+1}"'] \arrow[d, "\bigoplus \tau|^{x'}_{y'}"] \\
\mathcal{R}^p_{Y,y} \arrow[r] & \mathcal{R}^{p+1}_{Y,y'} & y'^{\natural}(\mathcal{R}_Y) \arrow[l, "\mathrm{tr}_{\pi_{y',Y}}\left(\mathcal{R}_{Y,y'}\right)^{p+1}"]
\end{tikzcd}
$$

as a map between residual complexes over one-dimensional (semi) local rings. Here, $\mathcal{R}_A$ denotes the residual complex $i^*(\mathcal{R}_X)$ over the semi local ring A - so we have $\mathcal{R}^p_A = \mathcal{R}^p_{X,x}$ while $\mathcal{R}^{p+1}_A = \bigoplus_{f(x')=y'} \mathcal{R}^{p+1}_{X,x'}$. We require the commutativity of the diagram

$$
\begin{tikzcd}
\mathbf{T}\left(x, \mathcal{R}^p_{X,x}\right) \arrow[r, "\sum\limits_{x' \in f^{-1}(y')} d^{x,x'}_{\mathbf{T}}"] \arrow[d, "(\tau_x)_*"'] & \bigoplus\limits_{x' \in f^{-1}(y')} (\mathbf{T}-1)\left(x', x'^{\natural}(\mathcal{R})\right) \arrow[d, "f_*"] \\
\mathbf{T}\left(y, \mathcal{R}^p_{Y,y}\right) \arrow[r, "d^{y,y'}_{\mathbf{T}}"'] & (\mathbf{T}-1)\left(y', y'^{\natural}(\mathcal{R})\right)
\end{tikzcd}
$$

Since

$$\left(\tau_{y'}\right)^{p+1} = \bigoplus_{f(x')=y'} \tau_{x'}$$

one may check by localising at each of the x' that $\left(\tau|_{y'}\right)^{p+1}$ induces a pushforward

$$\left(\tau_{y'}^{p+1}\right)_* : (\mathbf{T}-1)\left(f.l.Mod_A, \mathcal{R}_A^{p+1}\right) \longrightarrow (\mathbf{T}-1)\left(f.l.Mod_{\mathcal{O}_{Y,y'}}, \mathcal{R}_{Y,y'}^{p+1}\right)$$

By applying **Proposition 3.1.19** to the semi local ring A we see that when $\mathbf{T} = GW$ it suffices to show that the below diagram commutes.

$$\begin{tikzcd} GW\left(x, \mathcal{R}_A^p\right) \arrow[r, "d_{GW}(\mathcal{R}_A)"] \arrow[d, "(\tau_x)_*"'] & W\left(f.l.Mod_A, \mathcal{R}_A^{p+1}\right) \arrow[d, "(\tau_{y'}^{p+1})_*"] \\ GW\left(y, \mathcal{R}_{Y,y}^p\right) \arrow[r, "d_{GW}^{y,y'}(\mathcal{R}_{Y,y'})"'] & W\left(f.l.Mod_{\mathcal{O}_{Y,y'}}, \mathcal{R}_{Y,y'}^{p+1}\right) \end{tikzcd} \qquad (*)$$

Let's consider then a symmetric space $\big(V, \langle\cdot,\cdot\rangle\big) \in Sym\big(\kappa(x), \mathcal{R}_A^p\big)$. Since the ring map $\mathcal{O}_{Y,y'} \to A$ is finite, any A-lattice $L \hookrightarrow V$ self-dual in the sense of **Definition 3.1.4** is also an $\mathcal{O}_{Y,y'}$-lattice self dual with respect to the inner product of the pushforward $\tau_x\langle\cdot,\cdot\rangle$. Let's fix such a lattice $L \hookrightarrow V$, and observe that the inclusion between dual lattices below

$$\begin{tikzcd} L^\flat\big(\langle\cdot,\cdot\rangle\big) \arrow[r, hook] \arrow[d, "\wr\|"'] & L^\flat\big(\tau_x\langle\cdot,\cdot\rangle\big) \arrow[d, "\wr\|"'] \\ \mathrm{Hom}_A(L, Z^p(\mathcal{R}_A)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{O}_{Y,y'}}(L, Z^p(\mathcal{R}_{Y,y'})) \arrow[d, equal] \\ H^p\big(\mathcal{H}om_A(L[0], \mathcal{R}_A)\big) \arrow[r, "\cong"] & H^p\big(\mathcal{H}om_{\mathcal{O}_{Y,y'}}(L[0], \mathcal{R}_{Y,y'})\big) \end{tikzcd}$$

is an isomorphism for the reasons depicted above. The vertical arrows are isomorphisms by **Proposition 3.1.5** and the lowest horizontal map is an isomorphism since

$$\left(\tau_{y'}\right)_*\big(L[0]\big) : \mathcal{H}om_A(L[0], \mathcal{R}_A) \longrightarrow \mathcal{H}om_{\mathcal{O}_{Y,y'}}(L[0], \mathcal{R}_{Y,y'})$$

is a quasi-isomorphism. Writing simply $L^\flat$ for this common dual lattice, we see that

$$(\tau|_U)_* d_{GW}(\mathcal{R}_A)\big(V, \langle\cdot,\cdot\rangle\big) = \left[L^\flat/L,\ (-1)^{p+1}(\tau_{y'})^{p+1} d^p_{\mathcal{R}_A}\left(\langle\cdot,\cdot\rangle^L\right)\right]$$

while

$$d_{GW}^{\mathcal{R}_{Y,y'}}(\tau_x)_*\big(V, \langle\cdot,\cdot\rangle\big) = \left[L^\flat/L,\ (-1)^{p+1} d^p_{\mathcal{R}_{Y,y'}}\big(\tau_x\langle\cdot,\cdot\rangle\big)^L\right]$$

which are equal so that we have the required commutativity of $(*)$. If $\mathbf{T} = V$ then commutativity follows by taking applying for each generator $[\psi_0, \psi_1] \in V\big(x, \mathcal{R}^p_{X,x}\big)$ this argument to each of ψ_0 and ψ_1 separately.

Case 2. Setting $y = f(x)$, we have $\operatorname{tr.deg}(\kappa(x)/\kappa(y)) = 1$ - hence $\mu(x) + 1 = \mu(y)$.

As before we replace X by the closure of the point x and Y by the closure of the point y. We have to show in this case that the triangle

$$\begin{tikzcd}
\mathbf{T}\left(x, x^{\natural}(X)\right) \arrow[r, "\sum_{x' \in X^1} \operatorname{res}^{x,x'}_{\mathbf{T}}"] \arrow[dr, "0"'] & \bigoplus_{x' \in X^1} (\mathbf{T}-1)\left(x', x'^{\natural}(\mathcal{R})\right) \arrow[d, "f_*"] \\
 & (\mathbf{T}-1)\left(y, \mathcal{R}^{p+1}_{Y,y}\right)
\end{tikzcd}$$

commutes. The map f_* above is zero on any component of the coproduct with $f(x') \neq y$. As in the previous case it suffices to localise at the point $y \in Y$ - that is we now replace Y by the point $\operatorname{Spec}(\kappa(y))$ and X by the preimage of y; so that X is now a proper curve over $\kappa(y)$.

Now let $(V, \psi) \in \mathit{Sym}\big(x, \mathcal{R}^p_{X,x}\big)$ be any symmetric space, and take an $\mathcal{O}_X$-lattice $\mathcal{L}$ self-dual with respect to ψ. Then because the support of $\mathcal{L}^\flat/\mathcal{L}$ is zero dimensional we have that the complex $\mathcal{L}^\flat/\mathcal{L}$ lies in $\mathcal{E}_f$ - hence we obtain a symmetric space

$$(--1)^{p+1}\tau^{p+1} \circ d^p_{\mathcal{R}_X} \circ \left(\langle\cdot,\cdot\rangle^{\mathcal{L}}_{\psi}\right) : \mathcal{L}^\flat/\mathcal{L} \times \mathcal{L}^\flat/\mathcal{L} \longrightarrow \mathcal{R}^{p+1}_Y$$

over $\kappa(y)$. Further, the below sum of localisation maps

$$\mathcal{L}^\flat/\mathcal{L} \longrightarrow \bigoplus_{x' \in X^1} \left(\mathcal{L}^\flat/\mathcal{L}\right)_{x'}$$

is an isometry between this space and $\sum_{x' \in X^1} (\tau_{x'})_* d^{x,x'}_{GW}(\mathcal{R}_X)([V, \psi])$. Since $\tau^{p+1} \circ d^p_{\mathcal{R}_X} = 0$ this isometry completes our argument when $\mathbf{T} = GW$. Again, for $\mathbf{T} = V$ one takes a generator $[V, \psi_0, \psi_1]$ and repeats this argument for each of ψ_1 and ψ_2 separately. $\square$

Chapter 5

Suggestions for Further Developments

5.1 Connection with Milnor-Witt K-theory

Perhaps the easiest way to see whether our construction of residue maps for Witt groups can be extended to describe residue maps for Milnor-Witt K-theory would be to first establish that our boundary maps for the Witt complex respect the filtration by powers of the fundamental ideal; precisely one should try to prove a version of [2, Thm 6.6]. It appears that Gille's argument for *loc. cit.* can be written down almost exactly as it is to give the result in our setting. In charactersitic different from 2 then, our residue maps do define some kind of Gersten complex for Milnor-Witt K-theory. The hardest part in seeing that it agrees with that of [9] might well be in understanding how our twisting by residual complexes agrees with Schmid's Ω-twisting. Assuming that this new Gersten complex for Milnor-Witt K-theory did agree with the Rost-Schmid complex of [9], one would have agreement between our definition of a Chow-Witt group and Fasel's Chow-Witt groups appearing in [52].

5.2 Follow Fulton's Intersection Theory

The quickest way to establish new properties that our Chow-Witt groups enjoy could be to see how to lift some of Fulton's constructions [10] to the Chow-Witt group. For example by defining an exterior product and Gysin homomorphisms for closed embeddings, Fasel has constructed a ring structure on the "usual" Chow-Witt group of a smooth scheme [53]. The constructions of *loc. cit.* involve the Witt groups of some derived categories; it may be possible to continue the spirit of this thesis and find some way of describing these constructions without having any derived categories intervening.

5.3 Characteristic 2

The main result we would've liked to have found is that **Definition 4.1.13** actually makes sense even in the case when $1/2 \in \Gamma(X, \mathcal{O}_X)$. This will be the case if we can show that one can remove the assumption $1/2 \notin \Gamma(X, \mathcal{O}_X)$ in **Proposition 3.3.4**; which is perhaps best done by exploiting the fact that **Proposition 3.2.10** does not require any assumption on the characteristic.

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