The Universal Central Extension and Schur Multiplier for SL_2 Over Fields

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1 The Structure of $SL_2(k)$

We are interested in the structure of the universal central extension for $SL_2(k)$ where k denotes an infinite field. The two-dimensional special linear group is perfect over any local ring with infinite residue field (such as k) and, as such, admits a universal central extension. Its Schur Multiplier $H_2(SL_2(k))$ is then the kernel of this extension from chapter 5 of [3]. A presentation for the universal central extension and Schur multiplier were found by [6] and [4] respectively. However, these proofs are quite computational. A modern presentation for the Schur multiplier was given by [5] in 2017 for local rings by considering the induced action of k^* arising from conjugation by elements of $GL_2(k)$. On $H_2(SL_2(k))$, this is trivial for elements of $SL_2(k)$ and so factors to an action of $GL_2(k)/SL_2(k) \cong k^*$. The aim of this article is to give a modern, and more conceptual description of the universal central extension as well as to prove this new presentation for the Schur Multiplier from the old one, without making use of the modern methods in [5]. Brown's book [1] on the cohomology of groups was referred to for background reading and several basic results on the homology and cohomology of groups. All group actions of a group on another group will be by automorphisms.

This first section will briefly summarize the elements of the structure of $SL_2(k)$ that we will be using.

Definition 1.1: An **elementary matrix** in $SL_2(k)$ takes the form

$$x'(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ or } y'(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \text{ for } t \in k.$$

We will soon introduce the free group on symbols $x(t), y(t)$ so the $'$ is used to distinguish these elements. We often use $x'_r(t)$ to stand for $x'(t)$ or $y'(t)$ where $r = \pm\alpha$ with $-\alpha$ yielding $y'(t)$.

Multiplying on the left by $x'(t), y'(t)$ corresponds to adding t times the second row to the first and first to the second respectively. Multiplying on the right corresponds to adding t times the first column to the second for $x'(t)$ and adding the second to the first for $y'(t)$.

Definition 1.2: We introduce the following elements for $t \in k^*$:

$$w'_r(t) := x'_r(t)x'_{-r}(-t^{-1})x'_r(t), h'_r(t) := w'_r(t)w'_r(1)^{-1}$$

We denote $w'_{+\alpha}(t)$ by $w'(t) = x'(t)y'(-t^{-1})x'(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$ and $h'_\alpha(t)$ by $h'(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. We also write $w' = w'(1)$. We denote by U, U_σ, H the subgroups generated by the $x'(t)$, the $y'(t)$ and the $h'_r(t)$'s respectively.

It immediately follows from the definition that $x'_r(t)x'_r(s) = x'_r(t+s)$, $w'_{-\alpha}(t) = w'(t)^T = w'(-t^{-1})$ and $w'(t)^{-1} = w'(-t)$. Moreover, $h'_{-\alpha}(t) = h'(-t^{-1})$.

Theorem 1.3 (Bruhat Normal Form): [2] Every matrix in $SL_2(k)$ can be uniquely written in the form uh or $uhwu'$ for $u, u' \in U, h \in H$.

Proof. It is clear that uh uniquely expresses every upper triangular matrix. For the second form, note that

$$x'(t)h'(a)wx'(t') = \begin{pmatrix} -a^{-1}t & -a^{-1}tt' + a \\ -a^{-1} & -a^{-1}t' \end{pmatrix}.$$

So, given an arbitrary matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $z \neq 0$, we see that z determines a, x determines t, w determines t' and the determinant formula guarantees the final value as z is invertible. Clearly, the matrix can be expressed in this form and the expression is unique. □

This is actually true in any special linear group over a field and the general case follows from row and column reducing where w can be any permutation matrix. The reason the proof is written in this form is that we can see that this normal form will not hold in $SL_2(R)$ for a local ring R as it implies the lower left entry is either 0 or a unit.

Proposition 1.4: Let $(R, \mathfrak{m})$ be a local ring such that $R/\mathfrak{m}$ is infinite.

1. $SL_2(R)$ is generated by elementary matrices.
2. $SL_2(R)$ is perfect.

Proof. Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$, as $1 \notin \mathfrak{m}$, one of a, b and c, d must be a unit. If c is a unit, we can use row operations to change a to 1 and hence can obtain the identity. Else, d is a unit so we can eliminate b and similarly A is a product of elementary matrices.

To show that $SL_2(R)$ is perfect, we have that: $y'(t(1-u^2)) = [h'(u), y'(t)]$. Now, $R/\mathfrak{m}$ is infinite so we may choose some $a \in R$ such that $1 - a^2 \notin \mathfrak{m}$ as this is just anything that isn't a root of $1 - x^2 \in \frac{R}{\mathfrak{m}}[x]$. Thus, $y'(t) \in [SL_2(R), SL_2(R)]$ for each t as $1 - a^2$ is a unit and we may take $a = u$. Similarly, we can get $x'(t)$ by transposing the formula. So, by 1, $SL_2(R)$ is perfect. □

It follows from this proposition and chapter 5 of [3] that $SL_2(R)$ admits a universal central extension.

Proposition 1.5: The following relations are all true in $SL_2(R)$ for any ring R:
(A): $x'_r(t)x'_r(s) = x'_r(t+s)$.
(B'): $w'_r(t)x'_r(s)w'_r(t)^{-1} = x'_{-r}(-st^{-2})$.
(C): $h'_r(t)h'_r(s) = h'_r(ts)$

Proof. We do them for $r = +\alpha$ as it is analogous for $-\alpha$. (A) is clear. $(B)'$ is just:

$$w'(t)x'(s)w'(-t) = \begin{pmatrix} 0 & t \\ t^{-1} & -st^{-1} \end{pmatrix} \begin{pmatrix} 0 & -t \\ t^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -st^{-2} & 1 \end{pmatrix} = y'(-st^{-2}).$$

(C) is clear from the expression for $h'(t)$ following Definition 1.2. □

Definition 1.6: $SL_2(k)$ has an action of k^*, c, where c_a is conjugation by $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, we have an involution σ, conjugation by $w' = w'(1)$. Then, $w' = x'(1)\sigma(x'(1))x'(1)$. Moreover, $\sigma \circ c_a = c_{a^{-1}} \circ \sigma$, $\sigma(x'(t)) = y'(-t)$ and, writing $x = x'(1)$ $c_t(x)c_s(x) = c_{t+s}(x)$. So, $SL_2(k)$ is generated by x and the action c and involution σ.

2 A Presentation for $SL_2(k)$

We aim to show that the relations in 1.5 completely determine $SL_2(k)$ and then that the relations following 1.6 are equivalent to them.

Consider the free group on symbols $x(t), y(t)$ for $t \in k$. Call it F. We analogously use the notation $x_r(t)$ where $r = \pm\alpha$ and define $w_r(t), h_r(t)$ with $w(t) = w_\alpha(t), h(t) = h_\alpha(t), w = w(1)$.

The results and many of the ideas for proofs are taken from [6] and [2]. Where it is not indicated, I supplied the proofs myself. In all cases, the proofs have been heavily condensed from a much more general setting.

Definition 2.1: We introduce the following expressions:
(A): $x_r(t)x_r(s) = x_r(t+s)$.
(B'): $w_r(t)x_r(s)w_r(t)^{-1} = x_{-r}(-st^{-2})$.
(C): $h_r(t)h_r(s) = h_r(ts)$
We let Δ be the free group on $x(t), y(t)$ with relations (A) and (B'). Let Γ be Δ but with relation (C) as well.

Consider the homomorphism $\Phi : F \to SL_2(k)$ taking $x(t) \mapsto x'(t), y(t) \mapsto y'(t)$. By proposition 1.5, Φ passes to maps $\pi : \Delta \to SL_2(k)$ and $\varphi : \Gamma \to SL_2(k)$ both taking $x(t) \mapsto x'(t), y(t) \mapsto y'(t)$. Both maps are epimorphisms by 1.4. We note that x, y are actually homomorphisms $(k, +) \to SL_2(k)$ and $h_r : k^* \to SL_2(k)$ is also a homomorphism. We also have subgroups of both Δ and Γ: U, U_σ, H. Our goal is to prove that Γ is actually a presentation for $SL_2(k)$. That is:

Theorem 2.2: φ is an isomorphism.

We first require several lemmas. Any relation in Δ will also be true in Γ. The following all holds in Δ:

Lemma 2.3: $w_r(t)^{-1} = w_r(-t)$.

Proof. $w_r(t)w_r(-t) = x_r(t)x_{-r}(-t^{-1})x_r(t)x_r(-t)x_{-r}(t^{-1})x_r(-t) = 1$ by (A). □

Lemma 2.4: 7.2 in [6]. $w_r(t)x_s(u)w_r(-t) = x_{-s}(-t^c u)$ where c is determined by s and r. If $r = s$, $c = -2$, and if $s = -r$, $c = 2$.

Proof. $s = r$ is precisely relation (B'). Else, swap t with $-t$ in (B'). This yields

$$w_r(t)x_s(-u't^{-2})w_r(-t) = x_r(u').$$

Simply take $u' = -t^2 u$ as $t \in k^*$ to yield the result. □

Lemma 2.5: (Stated in [6] with a typo in (b); proof is my own) We have the following where c is as before, and d is another integer determined by s and r:

(a) $w_r(t)w_s(u)w_r(-t) = w_{-s}(-t^c u)$.

(b) $w_r(t)h_s(u)w_r(-t) = h_{-s}(-t^c u)h_{-s}(-t^c)^{-1}$.

(c) $h_r(t)x_s(u)h_r(t)^{-1} = x_s(t^d u)$.

(d) $h_r(t)w_s(u)h_r(t)^{-1} = w_s(t^d u)$.

(e) $h_r(t)h_s(u)h_r(t)^{-1} = h_s(t^d u)h_s(t^d)^{-1}$.

Proof. The proofs are all quite similar and involve using Lemma 2.4 and the previous identities.

(a):

$$\begin{aligned} w_r(t)w_s(u)w_r(-t) &= w_r(t)x_s(u)x_{-s}(-u^{-1})x_s(u)w_r(t)^{-1} \\ &= w_r(t)x_s(u)w_r(-t)w_r(t)x_{-s}(-u^{-1})w_r(-t)w_r(t)x_s(u)w_r(t)^{-1} \\ &= x_{-s}(-t^c u)x_s(t^{-c}u^{-1})x_{-s}(-t^c u) \qquad \text{by Lemma 2.4} \\ &= w_{-s}(-t^c u) \end{aligned}$$

(b):

$$\begin{aligned} w_r(t)h_s(u)w_r(-t) &= w_r(t)w_s(u)w_s(-1)w_r(-t) \\ &= w_r(t)w_s(u)w_r(-t)w_r(t)w_s(-1)w_r(-t) \\ &= w_{-s}(-t^c u)w_r(t)w_s(-1)w_r(-t) \qquad \text{by } (a) \\ &= w_{-s}(-t^c u)w_{-s}(t^c) \\ &= w_{-s}(-t^c u)w_{-s}(-1)w_{-s}(1)w_{-s}(t^c) \\ &= h_{-s}(-t^c u)h_{-s}(-t^c)^{-1} \end{aligned}$$

(c):

$$\begin{aligned} h_r(t)x_s(u)h_r(t)^{-1} &= w_r(t)w_r(-1)x_s(u)w_r(1)w_r(-t) \\ &= w_r(t)x_{-s}(-u)w_r(-t) \qquad \text{by } (a) \\ &= x_s(t^{-c}u) = x_s(t^d u) \qquad \text{where } d = -c \end{aligned}$$

(d):

$$\begin{aligned} h_r(t)w_s(u)h_r(t)^{-1} &= h_r(t)x_s(u)x_{-s}(-u^{-1})x_s(u)h_r(t)^{-1} \\ &= h_r(t)x_s(u)h_r(t)^{-1}h_r(t)x_{-s}(-u^{-1})h_r(t)^{-1}h_r(t)x_s(u)h_r(t)^{-1} \\ &= x_s(t^d u)x_{-s}(-u^{-1}t^{-d})x_s(t^d u) \qquad \text{by } (c) \\ &= w_s(t^d u). \end{aligned}$$

(e):

$$\begin{aligned} h_r(t)h_s(u)h_r(t)^{-1} &= h_r(t)w_s(u)w_s(-1)h_r(t)^{-1} \\ &= w_s(t^d u)h_r(t)h_r(t)^{-1}w_s(-t^d) \qquad \text{by } (d) \\ &= w_s(t^d u)w_s(-1)w_s(1)w_s(-t^d) \\ &= h_s(t^d u)h_s(t^d)^{-1} \end{aligned}$$

□

These relations are what we will use to create a similar Bruhat Normal Form in Δ. This will then yield a kernel consisting entirely of elements in H which is how relation (C) gives us that φ will be an isomorphism. We first note the meaning of many of these relations. (c) implies that H acts on U by conjugation and hence $UH = HU$ is a semidirect product containing U as a normal subgroup. Similarly, elements $w(t)$ act on H by conjugation as well. This means we may freely swap the order of elements $w(t)$ and $h \in H$ up to replacing them with other elements of this form.

The proof of Theorem 2.2 will involve replacing elements of U, U_σ and H with other elements of the respective groups. The way this is done follows from the previous lemma in the following way: We may use (b) to move $w(t)h$ to the form $h'w(t)$ for $h, h' \in H$, and we may move $x(t)h$ to $h'x(t)$ for $h, h' \in H$. To condense notation, h with any additional notation will always represent an element of h using these. Finally, by the definition of $w(t)$, any element $y \in U_\sigma$ may be expressed $y = xw(t)x$ for some $x \in U$. We will be doing this in the proof. Theorem 2.2 will follow from the following analogue of Bruhat Normal Form after remarking how (C) allows us to simplify $H < \Gamma$.

Theorem 2.6 (Bruhat Normal Form): 7.6 in [6]. In Δ (and Γ), every element can uniquely be expressed in the form uh or $uhw(1)u'$ for $u, u' \in U, h \in H$.

Proof. (The idea is outlined in [6] but without any detail). Let A be all elements of this form. We show that A is closed under left multiplication with elements $x(t)$ and $w(t)$. By lemma 2.4, every element $y(t)$ can be expressed as a product of these elements and hence we may form every element of Δ as such a product. This would then imply that $A = \Delta$. As $x(t)U = U$, we need only prove it for $w(t)$. For $w(t)x(t_0)h_0$:

$$\begin{aligned} w(t)x(t_0)h_0 &= y(t_1)w(t)h_0 \\ &= y(t_1)h_1w(t) \\ &= y(t_1)h_1w(t)w(-1)w(1) \\ &= y(t_1)h_2w(1) \\ &= h_3w(1)x(t_2) \end{aligned}$$

Then, for the second case, we have a slightly longer argument:

$$\begin{aligned}
w(t)x(t_0)h_0w(1)x(t_1) &= y(t_2)h_1w(t)w(1)x(t_2 \\
&= xw(t')x'h_1w(t)w(1)x(t_2) \\
&= x'yw(t')h_1w(t)w(1)x(t_2) \\
&= x(t')yh_2w(t')w(t)w(1)x(t_2) \\
&= x'yh_2w(-1)w(1)w(t')w(-1)w(1)w(t)w(1)x(t_2) \\
&= x'yh_2w(-1)h(-t')^{-1}w(-1)h(-t)^{-1}w(1)x(t_2) \\
&= x'yh_3w(-1)x(t_2) \\
&= x'h_4w(-1)x(t_2)
\end{aligned}$$

But, $w(-1) = h(-1)w(1)$ as $h(-1) = w(-1)w(-1)$. This gives us the required form for the expression.

For uniqueness, apply π (or φ) to the element to determine u, u', and the existence of w from 1.3. This then determines h. □

Definition 2.8: Denote by h_+ the subgroup of $H < \Delta$ generated by all elements $h(t)$. By (C), if we view $h_+ < \Gamma$, then $h_+ = \{h(t) \mid t \in k^*\}$.

Lemma 2.9 (7.7 in [6] – proven in much greater generality) $H = h_+$.

Proof. It suffices to show that every $h_{-\alpha}(t)$ is a product of elements of h_+. We have, by (a):

$$\begin{aligned}
h_{-\alpha}(t) &= w_{-\alpha}(t)w_{-\alpha}(-1) \\
&= w(1)w(-t)w(-1)w(1)w(1)w(-1) \\
&= w(1)w(-t)w(1)w(-1) \\
&= h(t)^{-1} \in h_+
\end{aligned}$$

□

We note that our expression agrees with the corresponding formula in $SL_2(k)$. Thus, by proposition 2.7, $\ker(\varphi) = \{h(t) \mid h'(t) = 1\} = \{1\}$. This proves Theorem 2.2.

We can describe Δ in terms of different, more natural, relations.

Definition 2.9: We let $\widetilde{SL_2(k)}$ be the free group on one generated x with an involution σ and action of k^*, c, such that $\sigma \circ c_a = c_{a^{-1}} \circ \sigma, c_t(x)c_s(x) = c_{t+s}(x)$, and such that σ is conjugation by $w = x\sigma(x)x$. From the end of section 1, we get an equivariant map $\pi : \widetilde{SL_2(k)} \to SL_2(k)$ taking $x \mapsto x'(1)$.

Δ also comes equipped with such an action. Consider the map $c_a : F \to \Delta$ taking $x(t) \mapsto x(at)$, $y(t) \mapsto y(a^{-1}t)$. This respects relations (A) clearly and, as for relation (B'), $c_a(w(t)) = w(at)$ and thus $w(at)x(sa)w(at) = y(-sa^{-1}(t^{-2})) = c_a(y(-st^{-2}))$. The case for the roots swapped just follows by replacing a with a^{-1}. Thus, this passes to a map $c_a : \Delta \to \Delta$. We also have an involution σ, conjugation by $w = w(1)$. We have that $\sigma \circ c_a = c_{a^{-1}} \circ \sigma$ by (B').

Theorem 2.10: $\Delta \cong \widetilde{SL_2(k)}$

Proof. The relations in $\widetilde{SL_2}(k)$ hold in Δ so we get a map $\Phi : \widetilde{SL_2}(k) \to \Delta$ taking $x \to x(1)$ that respects σ and the action of k^* (i.e. $\Phi\sigma = \sigma\Phi, c_a\Phi = \Phi c_a$). Now, we wish to show that Φ has an inverse. Define a map $\psi : F \to \widetilde{SL_2}(k)$ taking $x(t) \mapsto c_t(x)$, $y(t) \mapsto \sigma(c_{-t}(x))$. We show that this map respects $(A), (B')$ to give us a map $\psi : \Delta \to \widetilde{SL_2}(k)$ which will be an inverse to Φ as $\Phi(\psi(x(t))) = \Phi(c_t(x)) = c_t(x(1)) = x(t), \Phi(\psi(y(t))) = \Phi(\sigma(c_{-t}(x))) = \sigma(c_{-t}x(1)) = y(t)$, and to show it is a two-sided inverse: $\psi(\Phi(c_t(x))) = c_t(x), \psi(\Phi(\sigma(c_t(x)))) = \psi(y(-t)) = \sigma(c_t(x))$ and these elements generate $\widetilde{SL_2}(k)$ as we may interchange $\sigma c_a = c_{a^{-1}}\sigma$.

Now, we check that ψ respects the relations:

$$\psi(x(t))\psi(x(s)) = c_t(x)c_s(x) = c_{t+s}(x) = \psi(x(t+s))$$

and the same holds for y by the same argument. For (B'), we note that $\psi(w(t)) = c_t(x)\sigma(c_{t^{-1}}(x))c_t(x) = c_t(x\sigma(x)x)$. Moreover, $\psi(x(s)) = (c_t c_{t^{-1}} c_s)(x)$ and each map c_t is an automorphism of $\widetilde{SL_2}(k)$. So,

$$\begin{aligned}
\psi(w(t)x(s)w(t)^{-1}) &= c_t[x\sigma(x)x(c_{t^{-1}}c_s)(x)x^{-1}\sigma(x^{-1})x^{-1}] \\
&= c_t(\sigma(c_{t^{-1}}c_s(x))) \\
&= \sigma(c_{st^{-2}}(x)) \\
&= \psi(y(-st^{-2}))
\end{aligned}$$

The case with $w_{-\alpha}$ and $y(t)$ instead of $w(t)$ and $x(t)$ follows by applying σ and swapping t, s with their negatives since $\psi(w_{-\alpha}(t)) = \sigma\psi((w(-t)))$. Thus, we get such a map $\psi : \Delta \to \widetilde{SL_2}(k)$.

So, we can replace all our working with F being the free group on one generator with such an action. We define $h(t) = c_t(w)c_{-1}(w)$ and then $SL_2(k)$ is generated by the relations in $\widetilde{SL_2(k)}$ along with $h(t)h(s) = h(ts)$.

$\square$

3 Universal Central Extension for $SL_2(k)$

The universal central extension for a perfect group G has a lifting property. Using this, we can explain why the universal central extension of $SL_2(k)$ has all the relations in $\widetilde{SL_2(k)}$ except $c_t(x)c_s(x) = c_{t+s}(x)$. The difficult part is proving this one should be in the universal central extension. The bulk of this section will be showing this result. Let us first study lifting properties.

We start with a general set-up. Let (E, π) be a central extensions of a group G_2 and $f : G_1 \to G_2$ a homomorphism. We construct a central extension of G_1, f^*E satisfying a universal property. Define $f^*E := E \times_{G_2} G_1 = \{(e, g) \mid \pi(e) = f(g)\}$. This gives us a map $\pi^* : f^*E \to G_1$ such that the following diagram commutes:

$$\begin{array}{ccc}
f^*E & \xrightarrow{(e,g)\mapsto e} & E \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi} \\
G_1 & \xrightarrow{f} & G_2
\end{array}$$

Moreover, $\pi^*(e,g) = 1 \iff g = 1$ so $\ker(\pi^*) = \ker(\pi) \times 1$ as $\pi(e) = f(g) = 1$. Thus, they have the same kernel and we get a central extension for G_1 called the pull-back of G_2 along f. It satisfies the following universal property:

Given any group Q and maps q_2, q_1 to E, G_1 respectively such that $fq_1 = \pi q_2$, there is a unique map $u : Q \to f^*E$ such that

$$\begin{array}{ccccc} Q & & \xrightarrow{\quad q_2 \quad} & & \\ & \overset{u}{\searrow} & & & \searrow \\ & & f^*E & \xrightarrow{(e,g)\mapsto e} & E \\ {\scriptstyle q_1}\searrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi} \\ & & G_1 & \xrightarrow{\quad f \quad} & G_2 \end{array}$$

commutes. In our case, u is just the product of each map since we already have $\pi q_2 = fq_1$. Identifying $H^2(G, A)$ with central extensions of G by A up to isomorphism, $[E] \mapsto f^*E$ is the induced homomorphism on cohomology by f. Taking $G_1 = G_2 = G$, $E = \widetilde{G}$, for $(\widetilde{G}, \pi)$ the universal central extension of G, and $Q = \widetilde{G}$ with $q_1 = \pi$, we get a unique map u by the universal property of $\widetilde{G}$, and thus $q_2 =: \tilde{f}$ is determined by u and q_2 is then a unique map $\tilde{f} : \widetilde{G} \to \widetilde{G}$. such that $f\pi = \pi\tilde{f}$. It is unique as if q' were any other such map, we would get a u' but $u' = u$ by the universal property of $\widetilde{G}$ and thus $q' = \tilde{f}$. We have constructed:

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\tilde{f}} & \widetilde{G} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow{f} & G \end{array}$$

This property is enough to ascertain uniqueness of $\tilde{f}$ as $f, \tilde{f}$ give us a map $u : \widetilde{G} \to f^*\widetilde{G}$. However, by the universal property of $\widetilde{G}$, there is only one possible u we can have in the diagram and it is determined by just f. So, if we had some $\widetilde{f'}$, it must then be the same as $\tilde{f}$.

Proposition 3.1: If $f, g \in \mathrm{Aut}(G)$, then $\tilde{f}\tilde{g} = \widetilde{fg}$.

Proof. First apply the universal property of the pull-back to $\tilde{f}\tilde{g}$ and $\widetilde{fg}$ separately to get $u, u' : \widetilde{G} \to (fg)^*\widetilde{G}$. By the universal property of $\widetilde{G}$, $u = u'$ and thus, by uniqueness of the map above, we have our claim. $\square$

In particular, if H acts on G via $c : H \to \mathrm{Aut}(G), a \mapsto c_a$, then we get maps $\tilde{c_a} : \widetilde{G} \to \widetilde{G}$ satisfying $\tilde{c_a}\tilde{c_b} = \tilde{c_{ab}}$, and 1 is certainly a lift of 1 so we get an action $\tilde{c} : H \to \mathrm{Aut}(G)$.

Applying this to c, σ, we get lifts to the universal central extension (E, π) for $SL_2(k)$ such that $\sigma \circ c_a = c_{a^{-1}} \circ \sigma$. Moreover, we can define $f : E \to Z(E)$ via $x\sigma(x)xg(x\sigma(x)x)^{-1}f(g) = \sigma(g)$ where x is chosen such that $\pi(x) = x'(1)$. This indeed defines a homomorphism and $f(g) \in \ker(\pi)$. As E is perfect, f is trivial and so σ is conjugation by $x\sigma(x)x$.

There is a homomorphism $x : k \to SL_2(k)$ taking $t \mapsto x'(t)$. k comes equipped with an action of k^* by multiplication. To show that the final relation in $\widetilde{SL_2(k)}$ holds is the same as to say that x admits an equivariant lift to E. It suffices to say that x^*E admits an equivariant section $k \to x^*E$. To prove such a section exists, we will first find a non-equivariant section. It suffices to prove that $x^* : H^2(SL_2(k), A) \to H^2(k, A)$ takes the universal central extension to $[0] = [A \times k]$ where $A = H_2(SL_2(k))$. This is because $t \mapsto (0, t)$ is the required section.

We wish to show that, considering the classes of central group extensions of a group G by an abelian group A with trivial G-module structure (in fact the trivial structure is what makes it central), x^* takes the universal central extension to 0 via $H^2(SL_2, K) \to H^2(k, K)$ (where $K = H_2(SL_2(k)))$. To do this, we first make use of the universal coefficient theorem noting that, since $SL_2(k)$ is perfect, $\mathrm{Ext}^1(H_1(SL_2(k)), A) = 0$ since $H_1(SL_2(k)) = 0$. So, $H^2(SL_2(k), A) \cong \mathrm{Hom}(H_2(SL_2(k)), \mathbb{Z}))$ by the universal coefficient theorem's naturality with x_*, we get the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & H^2(SL_2(k), A) & \xrightarrow{\cong} & \mathrm{Hom}(H_2(SL_2(k), \mathbb{Z}), A) \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x^*} & & \downarrow{\scriptstyle (x_*)^*} \\ 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(k, A) & \longrightarrow & H^2(k, A) & \longrightarrow & \mathrm{Hom}(H_2(k), A) \end{array}$$

Let us show that x_* takes the universal central extension in $H^2(SL_2(k), H_2(SL_2(k))$ to $0 \in H^2(k, H_2(SL_2(k)))$. First we claim:

Lemma 3.2: Squares of units act trivially on $H_2(SL_2(k))$.

Proof. The map $c_u : H_p(SL_2(k))$ is induced by conjugation via any matrix in $GL_2(k)$ with determinant u. If $u = v^2$, we may take vI_2 as our matrix which lies in the centre of $GL_2(k)$ and thus the induced map is trivial. □

Proposition 3.3: Any k^*-module homomorphism $f : H_2(k) \to H_2(SL_2(k))$ is zero.

Proof. k is infinite so we may pick units $u_1, \ldots u_4$ such that all their partial sums over sets $\emptyset \neq I \subset \{1, 2, 3, 4\}, a_I := \sum_{i \in I} a_i$ are still units. Then, we will use the following element introduced in [5]

$$s := -\sum_{\emptyset \neq I \subset \{1, \ldots 4\}} (-1)^{|I|} a_I^2.$$

By 6.2, each a_I^2 acts as the identity on $H_2(SL_2(k))$. The fact that s acts as the identity then follows from the fact that $(1-1)^4 = 1 + \sum_{i>0} \binom{4}{i}$. Then, $H_2(k) = k \bigwedge k$, and k^* acts via $\sum n_i g_i(x \wedge y) = \sum n_i(g_i x \wedge g_i y)$. [5] shows that it acts as 0 on $k \bigwedge k$. We will give a more direct proof. Consider $s(x \wedge y)$. In sx, we get a sum of terms of the form either $a_i^2 x$ or $a_i a_j x$. For each of these, we may expand by bilinearity to get a sum,

$$-(a_i^2 x \wedge \sum_{i \in I} (-1)^{|I|} a_I^2 y)$$

or a sum

$$-(a_i a_j x \wedge \sum_{i,j \in I} (-1)^{|I|} a_I^2 y).$$

We just need to do this computation for $i = 1, j = 2$ to simplify notation. For the case a_1^2, the sum in the right hand term is precisely (omitting a_i^2 for $i \neq 1$ as these clearly cancel):

$$-a_1^2+3a_1^2+2a_1a_2+2a_1a_3+2a_1a_4-(3a_1^2+4[a_1a_2+a_1a_3+a_1a_4])+a_1^2+2a_1a_2+2a_1a_3+2a_1a_4 = 0.$$

For terms with both an i and a $j \neq i$, then we need only compute the right hand side for $i = 1, j = 2$ which is:

$$a_1^2+2a_1a_2+a_2^2-[2a_1^2+2a_2^2+4a_1a_2+2(a_1a_3+a_1a_4+a_2a_3+a_2a_4)+a_3^2+a_4^2]+\sum_k a_k^2+2\sum_{k\neq l} a_ka_l = 0$$

Thus, s acts as 0 on $k \bigwedge k$.

Localize the k^* modules $H_2(SL_2(k)), H_2(k)$ at s. and consider how f passes to the localizations via $\frac{m}{r} \mapsto \frac{f(m)}{r}$. This yields a commutative diagram:

$$\begin{array}{ccc} H_2(k) & \xrightarrow{f} & H_2(SL_2) \\ \downarrow & & \downarrow \\ s^{-1}H_2(k) & \longrightarrow & s^{-1}H_2(SL_2) \end{array}$$

Now, localizing $H_2(k)$ is just 0 as s^2 acts as 0. Moreover, the right vertical arrow is an isomorphism as $f - s^n f = f - f = 0$ as s acts as the identity, and $\frac{f}{s^n} \neq \frac{g}{1}$ for $g \neq f$ or else we would have $s^n(f - g) = 0$ but $s^n(f - g) = f - g$. Thus, the bottom arrow and hence the top arrow are both 0.

□

We want to finish the proof that the universal central extension is taken to 0 via x^* where $A = H_2(SL_2(k))$. Our diagram from the naturality of the universal coefficient theorem comes equipped with an action of k^* everywhere. The maps induced by x are all equivariant, and the remaining maps are slightly more complicated to describe. Use the bar resolution to obtain everything. Then, the map $H^2(G, A) \to \text{Hom}(H_2(G), A)$ is just the map taking $f \in \text{Hom}_G(F, A) = \text{Hom}(F_G, A)$ to the map obtained by restriction to $\ker \partial$ as $f\partial = 0$ so f vanishes on $\text{im}\partial$ and thus factors to a map $H_2(G) \to A$. Restriction is clearly equivariant so this is done. In the proof of the universal coefficient theorem, the Ext term arises from the cokernel of the dual of an inclusion map from boundaries to cycles and thus this map is equivariant.

Now, we may take k^* fixed points. The universal central extension of $SL_2(k)$ is equivariant by its lifting property studied in the beginning of this section and thus lies in the fixed points of $H^2(SL_2(k), A)$. This is because we may choose an equivariant section σ (as a map of sets) of (Δ, π) such that $\sigma(1) = 1$. Then, the cocycle corresponding to Δ, c, will satisfy $c(a^{-1}g, a^{-1}h) = a^{-1}c(g_1, g_2)$ and thus this cocycle will be invariant under the action $a \cdot \dot{c}(x) = a\, \dot{c}(a^{-1}x)$ where $\dot{c}$ is the map from the second degree group in the bar resolution corresponding to $c : G^2 \to A$.

We know that $\text{Hom}(H_2(k), A)^{k^*}$ is 0 by 3.3. We just wish to show that the term $\text{Ext}^1_{\mathbb{Z}}(k, A)$ has no fixed points. We need to take a projective resolution of k and then apply $\text{Hom}(-, A)$. We can take the usual resolution of k via taking the free abelian group on k where F_0 and F_1 is obtained by choosing a presentation for the kernel. Call

$\partial : F_1 \to F_0, p : F_0 \to k$ the maps. These carry an action just by multiplying the elements of k since $a \in \ker(p) \iff xa \in \ker(p)$ for each $x \in k^*$ and ∂, p are then clearly equivariant. Apply $\text{Hom}(-, A)$ and say δ is the map corresponding to ∂. If $a \in k^*$, then for $[f] \in \text{Ext}^1(k, A)$, $af = f \iff af(a^{-1}x) = f(x) + g\partial(x)$ for $g : F_0 \to A$. Pick a to be any square so that it acts as the identity on A. So, $f(x(a^{-1} - 1)) = \delta(g)(x)$. Set $y = x(a^{-1} - 1)$ so $f(y) = g\partial((a^{-1} - 1)^{-1}y) \in \text{im}\delta$ as we may choose a to be a square such that $a^{-1} \neq 1$. Thus, $[f] = 0$. So, this is indeed 0 and thus the central extension is mapped to 0 in $H^2(k, A)$.

Now, this yields the existence of a section from $k \to x^*[\Delta]$ of $x^*[\Delta] \to k$ and thus a homomorphism $\tilde{x} : k \to \Delta$ such that $\pi\tilde{x} = x$. However, we wish for our section to be equivariant. To accomplish this, we define a variant of $H^2(G, A)$ for A a trivial G-module where G and A are equipped with an action of a group N by automorphisms.

Definition 3.4: For a group G and abelian group A with action of a group N, we define $H^2_N(G, A)$ as follows. Its elements are central extensions of G, (E, p) with kernel A that come equipped with an action of N agreeing with the action on A and G such that p is N-equivariant, modulo equivariant isomorphisms of extensions. On the Baer sum (with its usual definition) we apply the action component-wise to get a well-defined abelian group structure and, due to the identification in the Baer sum, it doesn't matter which component we do the action on for elements of A so it will be the correct action on A. We will denote all actions by c.

Lemma 3.5: Equivariant homomorphisms $f : G \to H$ pass contravariantly to homomorphisms $f^* : H^2_N(H, A) \to H^2_N(G, A)$ and this is functorial.

Proof. Set f^*E to be the usual pull-back with the action $c_a(e, g) = (c_a(e), g)$. Since $f^*p(e, g) = g$, this clearly is an element of $H^2_N(G, A)$. Moreover, this respects the Baer sum just as in the case of $H^2(G, A)$. Functoriality is also identical to the case of cohomology since all we have actually done is just add an action to the pull-back (thus checking that it lies in $H^2_N(G, A)$). □

Now we reach the main point of this construction, it allows for an understanding of cohomology with group action in terms of regular cohomology:

Theorem 3.6: There are homomorphisms between $H^2_N(G, A)$ and $H^2(G \rtimes_N N, A)$ (where the action of $G \rtimes N$ on A is given by projection $G \rtimes N \to N \to \text{Aut}(A)$), † and ˆ going in the forward and reverse directions respectively such that $\widehat{(E^\dagger)} = E$. That is, ˆ is surjective and † is injective so $H^2_N(G, A) \cong H^2(G \rtimes_N N, A)^\dagger$. We note that the action of $G \rtimes N$ on A is just the first component acting trivially, with the action of the second component already fixed.

Proof. We will first describe the correspondence of extensions and then check they are well-defined inverse homomorphisms. Their definition is very natural which makes the many checks quite straightforward. Given an exact sequence $1 \to A \to E \to G \rtimes N \to 1$, say (E, p) is the extension, consider $\pi : G \rtimes N \to N$, the quotient map. Set $\hat{E} = \ker(\pi p) = \{e \mid p(e) = (g, 1)\}$. Then, set $\hat{p} = p\mid_{\hat{E}}$. $A \subset \hat{E}$ and $\hat{p}(e) = 1 \iff p(e) = 1$ so $\ker(\hat{p}) = A$. So, $(\hat{E}, \hat{p})$ is a central extension of G since $\{(g, 1) \mid g \in G\} \cong G$. This also comes equipped with an action of N as follows: we have an action of E on $\hat{E}$ via conjugation and this vanishes on A as elements of $G \times 1$ act trivially on A. Therefore, the action factors to an action of $E/A \cong G \rtimes N$ via p on $\hat{E}$. Hence, we have a homomorphism $G \rtimes N \to \mathrm{Aut}(\hat{E})$. Just precompose this with inclusion $N \to G \rtimes N$ via $a \mapsto (1, a)$. More concretely, we compute $c_a(e)$ by choosing some $e' \in E$ such that $p(e') = (1, a)$. Then, conjugate e by e' to yield $c_a(e)$. As this is just conjugation, we have that $\hat{p}(c_a(e)) = p(e'e(e')^{-1}) = (1, a)e(1, a)^{-1} = c_a(e)$ and thus $\hat{p}$ is N-equivariant as required. Before checking that this operation is well-defined, we will give the operation going the other direction. Let $(E, p) \in H^2_N(G, A)$. Then, set $E^\dagger := E \rtimes N$ and $p^\dagger(e, a) = (p(e), a) \in G \rtimes N$. We know that $(p(e_1), a_1)(p(e_2), a_2) = (p(e_1)c_{a_1}(p(e_2)), a_1 a_2) = p^\dagger((e_1, a_1)(e_2 a_2))$ since p is N-equivariant. The kernel of $p^\dagger$ is clearly A so this defines an element of $H^2(G \rtimes N, A)$ since conjugation by elements in $A \times a$ act as a on A so the $G \rtimes N$-module structure on A corresponds to the N-module structure. Now, we will verify that $\hat{}, \dagger$ are indeed well-defined and that they are one-sided inverses.

We will do $\dagger$ first. If $\varphi : E \to E'$ is an equivariant isomorphism (that is, $[(E, p)] = [(E', p')]$ in $H^2_N(G, A)$), then define a map $\tilde{\varphi} : E \rtimes N \to E' \rtimes N, (e, a) \mapsto (\varphi(e), k)$. Since φ is equivariant, this gives a homomorphism and it is clearly bijective. Lastly, $p'^\dagger \tilde{\varphi}(e, a) = (p'(\varphi(e)), a) = (p(e), a) = p^\dagger(e, a)$. Thus, $\dagger$ is well-defined. Conversely, if φ is just an isomorphism of extensions of $G \rtimes N$, then $\varphi\mid_{\hat{E}}: \hat{E} \to \hat{E}'$ is well-defined since, if $e \mapsto (g, 0)$, then $p'(\varphi(e)) = (g, 1)$ so $\varphi(e) \in \hat{E}'$. By symmetry, φ^{-1} also restricts to such a map and thus this yields an isomorphism. Since the action comes from conjugation, $\varphi\mid_{\hat{E}}$ is equivariant and we are done.

Now, we must show that these operations respect the Baer sum. $\hat{}$ is the easier one to verify. Say $(E, p), (F, q)$ are the extensions. On $E \times_{G \rtimes N} F$, $(e, f) \mapsto (g, 1) \iff p(e) = q(f) = (g, 1)$ so $\widehat{E \times_{G \rtimes N} F} = \hat{E} \times_G \hat{F}$ as $\hat{p}, \hat{q}$ are just restrictions and this isn't affected by the identification in the Baer sum. $\dagger$ is slightly more subtle. Define a map $\psi : E \times_G F \rtimes N \to E^\dagger \times_{G \rtimes N} F^\dagger$ via $\psi((e, f), a) = ((e, a), (f, a))$. This is well-defined since $(p(e), a) = (q(f), a)$ Moreover, ψ is a homomorphism as:

$$\psi((e, f)c_a(e', f'), ab) = ((ec_a(e'), ab), (fc_a(f'), ab)) = ((e, a)(f, a))((e', b)(f', b))$$

as the actions are component-wise. ψ is clearly bijective and clearly passes to the quotient in the Baer sum. This proves the theorem.

Finally, we must check the composition: $\widehat{E \rtimes N} = \{(e, a) \mid a = 1\} \cong E$.

□

In addition to these maps, we can study how they interact with functoriality. Given an equivariant homomorphism $f : G \to H$, we get an induced homorphism $\tilde{f} : G \rtimes N \to H \rtimes N$ by setting $\tilde{f}(g, a) = (f(g), a)$. In particular, we note that $x : k \to SL_2(k)$ is N-equivariant where the action on k is left multiplication since $c_a x(t) = x(at) = x(c_a(t))$. We will quickly show that, given f as above, $(\tilde{f})^*$ corresponds to f^* via $\dagger$. $\tilde{f}^*(E) = \{(e, (g, a)) \mid p(e) = (f(g), a)\}$. The only elements mapped into $G \times 1$ are then any $(e, (g, 1))$ such that $e \in \hat{E}$. Ignoring the component with the 1, this is precisely $\hat{E} \times_H G = f^* \hat{E}$ and this projection is an isomorphism so this completes the proof. Replacing E by $E^\dagger$ since $\dagger$ is injective, we yield the following commutative diagram.

$$\begin{array}{ccc} H^2(G \rtimes N, A) & \xrightarrow{\widehat{\ }} & H^2_N(G, A) \\ {\scriptstyle \tilde{f^*}}\uparrow & & {\scriptstyle f^*}\uparrow \\ H^2(H \rtimes N, A) & \xleftarrow[\dagger]{} & H^2_N(H, A) \end{array}$$

Thus, it suffices to study $\widehat{(\tilde{f^*})} = 0$ to learn about f^*.

There is an inclusion map $\iota : N \to G \rtimes N$. We claim that $\mathrm{im}\dagger = \ker \iota^*$ as $\iota^* : H^2(G \rtimes N, A) \to H^2(N, A)$. First, we show that $E^\dagger \in \ker \iota^*$. $\iota^*(E \rtimes N) = \{((e, a), t) \mid e \in A, a = t\} \cong A \rtimes N$ and this isomorphism respects the extension maps as they both just ignore the E component.

Conversely, we have that $\iota^*(E) = \{(e, t) \mid p(e) = (1, t)\}$. Introduce the temporary notation $E' := \{e \in E \mid p(e) \in 1 \times N\}$. Thus, $\iota^*(E) \cong E'$ via $(e, t) \mapsto (e)$ since $e \in E'$ and e determines t anyways so we have the extension $e \mapsto p(e)$ where we ignore the first component of $p(e)$. Now, if $\phi : \iota^*(E) \to A \rtimes N$ is an isomorphism, then as it is an isomorphism of extensions, we have that $\phi(a) = (a, 1)$ for $a \in A$, and $\phi(e) = (a, t)$ where $p(e) = (1, t)$. E' acts on $\hat{E}$ by conjugation but this action is exactly the same as the action of N on $\hat{E}$ by our working in the proof of 6.5.

Now, $\hat{E}$ is a normal subgroup of E and thus $\hat{E}E' < E$. Moreover, $p \mid_{\hat{E}E'}$ is clearly surjective so, given $e \in E$, if $p(e) = (g, t)$ and we choose $(e_0, e') \in \hat{E} \times E'$ such that $p(e_0e') = (g, t)$, then $e = ae_0e'$ for $a \in A < Z(\hat{E})$ so $e \in \hat{E}E'$. Moreover, $\hat{E} \cap E' = A$ by their definitions. We may form $\hat{E} \rtimes E'$ and map $(e, e') \mapsto ee'$ to yield an epimorphism. Moreover, this has kernel $\{(a, a^{-1}) \mid a \in A\} =: A'$. Thus, $E \cong \frac{\hat{E} \rtimes E'}{A'}$ and this is an isomorphism of extensions if we use $p(e, e') = p(ee')$ (this clearly vanishes on A').

We must show that E is isomorphic, as an extension of $G \rtimes N$, to an extension of the form $F \rtimes N$ where F is an equivariant extension of G. We will prove that it is isomorphic to $\hat{E} \rtimes N = (\hat{E})^\dagger$. We will use the notation $\phi(e) = (\phi(e)_A, t)$. Define a map $f : \hat{E} \rtimes E' \to \hat{E} \rtimes N$ via $f(e, e') = (e(\phi(e')_A)^{-1}, p(e'))$. f is a homomorphism since the action of $p(e')$ on $\hat{E}$ as an element of N is the same as the action of e' as an element of E'. More explicitly:

$$f(e_1c_{e_1'}(e_2), e_1'e_2') = (e_1c_{e_1'}(e_2)\phi(e_1'e_2')_A^{-1}, p(e_1'e_2'))$$

but $\phi(e_1'e_2')_A = \phi(e_1')_A c_{e_1'}(\phi(e_2')_A)$ and all these elements lie in $A \subset Z(\hat{E})$. Therefore, $f((e_1, e_1')(e_2, e_2')) = (e_1\phi(e_1')_A^{-1}c_{e_1}(e_2\phi(e_2')_A^{-1}), p(e_1'e_2')$ as required. So f is a homomorphism and it is clearly surjective. Lastly, its kernel is precisely $\{(a, a^{-1})\}$ since $\phi(a)_A = a$ for $a \in A$. Thus, f factors to an isomorphism $f : E \to \hat{E} \rtimes N$. This proves:

Theorem 3.8: $H^2_N(G, A)$ is isomorphic to $\ker(\iota^*)$ via $\dagger$ and $\hat{\ }$ restricted to $\ker(\iota^*)$.

Now, it suffices to prove that $H^2_{k^*}(k, A) = 0$ for $A = H_2(SL_2(k))$ to get our equivariant section. Here $H_2(SL_2(k))$ has the action induced by $c : k^* \to \mathrm{Aut}(SL_2(k))$. This is because the universal central extension has the required lifts and thus x^* will take it to an element of $H^2_{k^*}(k, A)$. This fact follows from a short exact sequence arising from the Hochschild-Serre spectral sequence for cohomology arising from the short exact sequence $G \to G \rtimes N \to N$. We have a filtration of the kernel of the map $H^2(G \rtimes N, A) \to H^2(N, A)$ (precisely $H^2_N(G, A)$). This yields the short exact sequence

$$0 \to H^1(N, H^1(G, A)) \to H^2_N(G, A) \to H^0(N, H^2(G, A)).$$

We claim that the exterior terms are 0 to prove that the middle is 0 when $N = k^*, G = k, A = H_2(SL_2(k))$. For the right hand term, we have already seen that the k^* fixed points of $H^2(k, A)$ are zero.

Now we treat the left hand term. By the universal coefficient theorem applied to calculate $H^1(G, A)$, at $n = 1$. Since $H_0(k) \cong \mathbb{Z}$, we get an exact:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, A) \to H^1(G, A) \to \mathrm{Hom}(k, A) \to 0.$$

The left term is 0 using the free resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$. Applying $\mathrm{Hom}(-, A)$ preserves exactness here so we get 0 since all maps are 0 or the identity. Now, $H^1(N, \mathrm{Hom}(k, A)) = \mathrm{Ext}^1_{\mathbb{Z}[k^*]}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}}(k, A))$. The arguments here are both k^*-modules and so the whole term becomes a k^*-module where a acts via $\mathrm{Ext}(a, 1) = \mathrm{Ext}(1, a)$. Take $a = s$ for s as in 3.2. s acts trivially on $\mathbb{Z}$ (as this has trivial module structure) but also acts as 0 on $\mathrm{Hom}(k, A)$ since it acts as 0 on k (this is just an easier version of the calculation in 3.2). Therefore, the whole term is 0 and $H^2_{k^*}(k, A) = 0$. It follows that there is an equivariant isomorphism $x^*E \to k \times A$ for (E, p) the universal central extension of $SL_2(k)$ and so we have an equivariant homomorphism $\tilde{x} : k \to E$ lifting $x : k \to SL_2(k)$. So, the universal central extension for $SL_2(k)$ satisfies all the relations in $\widetilde{SL_2(k)}$. This gives rise to an equivariant homomorphism $\phi : \widetilde{SL_2(k)} \to E$ such that $p \circ \phi = \pi$ where π is the map taking $x \mapsto x'(1)$. $(\widetilde{SL_2(k)}, \pi)$ is an extension of $SL_2(k)$. If we show that it is central and perfect, we will be done.

Proposition 3.9: [6] $(\widetilde{SL_2(k)}, \pi)$ is a central extension of $SL_2(k)$.

Proof. We need only show $\pi(g) = 1 \implies x \in Z(\widetilde{SL_2(k)})$. It suffices to show that g commutes with all elements $c_t(x), \sigma(c_t(x))$ as these elements generate the group. In the notation of section 2.7, these all have the form $x_r(t)$ and we will stick with this notation to easily use the relations in this section. By proposition 2.7, $g = h \in H$. g is a product of elements $\Pi h(t_i)$. We know that $h(t)x_r(u)h(t)^{-1} = x_r(t^d u)$. Thus, $gx_r(u)g^{-1} = x_r(\Pi t_i^d u)$. So, as $\pi(g) = 1$, we see $\pi g x_r(u) g^{-1} = \pi(x_r(u)$. Now, $\pi(x_r(u) = x'_r(u)$ and so we see that $\Pi t_i^d = 1$ so $gx_r(u)g^{-1} = x_r(u)$. Therefore, g is central. □

Proposition 3.10: [6] $\widetilde{SL_2(k)}$ is perfect.

Proof. We assumed k was an infinite field so we may pick $u \in k^*$ such that $u^2 - 1 \neq 0$. Then, we examine the commutator $[h(u), c_t(x)]$.

$$\begin{aligned}[h(u), c_t(x)] &= h(u)c_t(x)h(u)^{-1}c_t(x)^{-1} \\ &= c_{u^2 t}(x)c_{-t}(x) \text{ by } (c) \\ &= c_{t(u^2-1)}(x)\end{aligned}$$

Since $u^2 - 1$ is a unit, we see that every $c_t(x)$ is a commutator and hence commutators generate $\widetilde{SL_2(k)}$ since the commutator subgroup is characteristic and thus invariant under σ. $\square$

Putting these together, and precomposing any map from (E, p) to a central extension over G with ϕ, we get a unique homomorphism from $\widetilde{SL_2(k)}$ to any central extension over G (uniqueness follows from 3.10 since any two homomorphisms differ by central element and so will agree on commutators). Thus, $(\widetilde{SL_2(k)}, \pi)$ is the universal central extension of $SL_2(k)$.

4 Modern Presentation for $H_2(SL_2(k))$

In this section, we will use the notation in Δ as this is what [4] uses in calculating the old presentation for the Schur multiplier.

Definition 4.1: We define the following element of $\mathbb{Z}[k^*]$: for $r \in k^*$, $[r] = \langle r \rangle - 1$.

The goal of this section is to, from the presentation of $H_2(SL_2(k))$ given by Moore, prove that, as a $\mathbb{Z}[k^*]-$module via the action induced by conjugation from $GL_2(k)/SL_2(k) \cong k^*$, $H_2(SL_2(k)) \cong_{k^*} \frac{I_{k^*} \bigotimes_{\mathbb{Z}[k^*]} I_{k^*}}{[a]\otimes[a-1] \text{ for } a, 1-a \in k^*}$ where I_{k^*} is the augmentation ideal, for $\epsilon : \mathbb{Z}[k^*] \to \mathbb{Z}, \langle a \rangle \mapsto 1$ for each $a \in k^*$, $I_{k^*} = \ker(\epsilon)$. For now call this module A to simplify notation.

Definition 4.2: We define a map $b : k^* \times k^* \to \Delta, b(s,t) = h(s)h(t)h(st)^{-1} = h(st)^{-1}h(s)h(t)$. So, each individual $b(s,t)$ lies in $\ker(\pi)$.

Lemma 4.3: The $b(s,t)$ generate $K := \ker(\pi)$.

Proof. (8.1 in [4]) Each element of K is of the form $h = \Pi_{i=1}^n h(t_i)$ for $t_i \in k^*$ by 2.7 and 2.9. We proceed by induction on n. If $n = 1$, then $\pi(h(t)) = 1 \iff t = 1 \iff h = 1$. Then, we note that $h(t)h(t') = b(t,t')h(tt')$. So, given h as above, $b(t_1,t_2)^{-1}h = h(t_1t_2)\Pi_{i=3}^n h(t_i)$ and the right hand side lies in the group generated by the $b(t,s)$'s and hence so does h. $\square$

One can expand on the relations holding among the $b(s,t)$'s but eventually it leads to a very long computation, all the details of which are done in the appendix to [4] and it is just a straightforward calculation. The result which we will use is the following:

Theorem 4.4: K is generated by the $b(s,t)$ subject to the relations

(a) $b(s,tr)b(t,r) = b(st,r)b(s,t), b(s,1) = b(1,s) = 1$

(b) $b(s,t) = b(t^{-1},s)$

(c) $b(s,t) = b(s,-st)$

(d) $b(s,t) = b(s,(1-s)t)$

We define a map $\varphi : K \to A$ taking $b(s,t) \to [s] \otimes [r]$. To show such a map exists, we must show it satisfies these relations. (d) is clear as in A

$$[s] \otimes [t] - [s] \otimes [t(1-s)] = [s](\otimes[t] - \langle t\rangle[1-s] + [t]) = 0.$$

(a) is also fairly straightforward to show:

$$\begin{aligned}[st] \otimes [r] &= \langle t\rangle[s] \otimes [r] + [t] \otimes [r] \\ &= [s] \otimes ([rt] - [t])\end{aligned}$$

and we get (a) by just adding $[s] \otimes [t]$ to each side.

Before continuing, let us examine why the action in K corresponds to this action on A. We wish to compute $c_a(b(s,t))$. We have already shown that $c_a(w(t)) = w(at)$ and hence $c_a(h(t)) = w(at)w(-a)$ Thus,

$$\begin{aligned}c_a(b(t,s)) &= w(at)w(-a)w(sa)w(-a)w(a)w(-tsa) \\ &= h(at)w(1)w(-a)w(sa)w(-tsa) \\ &= h(at)h(s)h(s)^{-1}w(1)w(-a)w(sa)w(-tsa) \\ &= h(at)h(s)h(s)^{-1}h(a)^{-1}h(sa)h(tsa)^{-1} \text{ but } h(s)^{-1}h(a)^{-1}h(sa) = b(a,s)^{-1} \in Z(\Delta) \\ &= b(at,s)b(a,s)^{-1}\end{aligned}$$

but, in A, $\langle a\rangle[t] \otimes [s] = [at] \otimes [s] - [a] \otimes [s]$ and precisely the same argument works if we bring the $\langle a\rangle$ to the second argument.

To prove the last two, we use a result from [5] (Lemma 4.3 and 4.4). We prove all the necessary statements here. Note that all expressions we write down except (1) will lie in the degree 2 part of the graded tensor algebra. For instance, $[s][r] \otimes [t] = (\langle sr\rangle - \langle s\rangle - \langle r\rangle + 1) \otimes [t]$. Besides this lemma, the argument is mine.

Lemma 4.5: All the following hold in A:

1. $[\frac{a}{b}] = [a] - \langle \frac{a}{b}\rangle[b]$ in $\mathbb{Z}[k^*]$
2. $[a][b] \otimes [c] = [a][c] \otimes [b]$
3. $[a] \otimes [-a] = 0$
4. $[a] \otimes [b] = -\langle -1\rangle[b] \otimes [a]$
5. $([a](1 + \langle -1\rangle))[b] \otimes [c] = 0$
6. $[a^2] \otimes [b] = (1 + \langle -1\rangle)[a] \otimes [b]$
7. $[a^2]([b] \otimes [c]) = 0$ Or, equivalently, $\langle a^2\rangle[b] \otimes [c] = [b] \otimes [c]$

Proof. [5]

(1) We have that $[a] = [\frac{a}{b}b] = \langle \frac{a}{b}\rangle[b] + [\frac{a}{b}]$ which is sufficient.

(2) Immediate from the $\mathbb{Z}[k^*]$ module structure as all three elements $[a], [b], [c]$ lie in I_{k^*}.

(3) If $a = 1$ then it is immediate as $[1] = 0$. Else, $1-a, 1-a^{-1}$ are both units. Moreover, $-a(1-a^{-1}) = 1-a$ so $-a = \frac{1-a}{1-a^{-1}}$. Thus, by (1), $[a]\otimes[-a] = [a]\otimes([1-a]-\langle -a\rangle[1-a^{-1}]) = 0 - \langle -a\rangle[a] \otimes [1-a^{-1}] = \langle -a\rangle\langle a\rangle[a^{-1}] \otimes [1-a^{-1}] = 0$ as $[a] = -\langle a\rangle[a^{-1}]$.

(4) By (3), $0 = [ab] \otimes [-ab] = ([a] + \langle a\rangle[b]) \otimes [-ab] = [a] \otimes [-ab] + \langle a\rangle[b] \otimes [-ab] = [a] \otimes ([-a] + \langle -a\rangle[b]) + \langle a\rangle[b] \otimes ([a] + \langle a\rangle[-b]) = \langle -a\rangle[a] \otimes [b] + \langle a\rangle[b] \otimes [a]$ Then, multiply by $\langle a^{-1}\rangle$.

(5) Applying (4) to $[a]([b] \otimes [c])$, $[a]([b] \otimes [c]) = -\langle -1\rangle[a]([c] \otimes [b])$ which, by (2) is precisely $-\langle -1\rangle[a]([b] \otimes [c]$. the LHS to yield the identity.

(6) $[a^2] \otimes [b] = [a] \otimes [b] + \langle a\rangle[a] \otimes [b] = 2[a] \otimes [b] + [a]([a] \otimes [b]) = 2[a] \otimes [b] + [b][a] \otimes [a]$. But, $[a] \otimes [a] = [(-1)(-a)] \otimes [a] = ([-1] + \langle -1\rangle[-a]) \otimes [a] = [-1] \otimes [a]$ by (3). So, $[a^2] \otimes [b] = 2[a] \otimes [b] + [b]([-1] \otimes [a]) = (1 + \langle -1\rangle)[a] \otimes [b]$.

(7) Finally, $[a^2]([b] \otimes [c]) = [b]([a^2 \otimes [c]) = [b](1 + \langle -1\rangle)[a] \otimes [c]$ by (6) which is just 0 by (5).

□

Now we go on to finish our proof that φ satisfies the relations. We have just (b) and (c) left. For (c), $[s] \otimes [-st] - [s] \otimes [t] = [s] \otimes \langle t\rangle[-s] = \langle t\rangle([s] \otimes [-s]) = 0$. (b) is the hardest argument:

$$\begin{aligned}
[s] \otimes [t] &= \langle t^{-2}\rangle[s] \otimes [t] \\
&= [s] \otimes ([t^{-1}] - [t^{-2}] \\
&= [s] \otimes [t^{-1}] - [s] \otimes [t^{-2}] \\
&= [s] \otimes [t^{-1}] - (1 + \langle -1\rangle)[s] \otimes [t^{-1}] \text{ by (6)} \\
&= -\langle -1\rangle[s] \otimes [t^{-1}] \\
&= [t^{-1}] \otimes [s] \text{ by (4).}
\end{aligned}$$

So, we have a homomorphism $\varphi : K \to A$ sending $b(s,t) \mapsto [s] \otimes [t]$. To show that it is an isomorphism, we construct a two-sided inverse. That is, a map $\psi : A \to K$ sending $[s] \otimes [t] \mapsto b(s,t)$. First, I_{k^*} has the set of all $[a]$ for $a \in k^* \backslash \{1\}$ as a $\mathbb{Z}$ - basis. So, let $\overline{\psi}$ be the unique $\mathbb{Z}-$bilinear mapping $I_{k^*} \times I_{k^*} \to K$ sending $([s],[t]) \mapsto b(s,t)$. We show that this is indeed $\mathbb{Z}[k^*]$-bilinear and the induced map from $I_{k^*} \bigotimes I_{k^*}$ respects the relation in A to give us such a map ψ. From our calculation of $c_a(b(s,t))$ it descends to a map from the tensor product by the universal mapping property of tensors. Then, to show it satisfies the relation, by (d), $\psi(a, 1-a) = b(a, 1-a) = b(a, 1) = h(a)h(1)h(a)^{-1} = 1$. So, we have proven:

Theorem 4.6: $\varphi : K \to A$ is an isomorphism and hence $H_2(SL_2(k)) \cong_{k^*} \frac{I_{k^*} \bigotimes_{\mathbb{Z}[k^*]} I_{k^*}}{[a]\otimes[a-1] \text{ for } a, 1-a \in k^*}$.

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