On the Mod p Homology of Some Finite Simple Groups

by

Andrew S. Pollock

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1 Introduction and Notation

1.1 Introduction

This dissertation will prove vanishing theorems for the low dimensional mod p (where p is prime) group homology of three classes of finite simple groups.

Usually, group homology is introduced with integral (integer) coefficients where it measures the failure of a particular functor from $\mathbb{Z}[G]$-modules to abelian groups. Group homology with general coefficients in some module can then be defined via the previous construction with a tensor functor applied [1, §III.1]. We take a different approach and define homology with coefficients in the finite field of p elements in §2 (mod p homology) without reference to integral homology. We reach these definitions from a homological algebra perspective, rather than from a topological perspective which is briefly outlined at the end of §2.3. The end of our second chapter is spent discussing the functoriality of group homology.

In [7, §11], Quillen proved that the mod p cohomology of the general linear groups over finite fields of characteristic p vanishes in low positive dimensions. We translate this result and corresponding proof into homology and also strengthen it to obtain an analogous statement for the special linear groups. In §3 we establish some preliminaries for these results, appealing to the aforementioned Quillen paper and [6, §7]. Then §4.1 is dedicated to proving vanishing theorems for low dimensional mod p homology of the general and special linear groups.

We adapt Quillen's proof to other classical groups over our finite field. Namely, we prove statements for the low dimensional mod p homology of the symplectic and orthogonal groups (or more precisely, the even degree special orthogonal groups of plus type - for these we only look at the case $p \neq 2$) in §4.2 and §4.3 respectively. The symplectic and special orthogonal cases have been proven independently by Friedlander in [3, §4], but our proofs are more elementary and much truer to Quillen's original methods. We also go further and explore the index 2 subgroup of the special orthogonal group.

The importance of the special linear, symplectic, and special orthogonal groups is that they are a source of finite simple groups (as explained in [9, §3]) - each one of these 'classical' groups provides us with an infinite class of finite simple groups. In fact, they give us three out of the four infinite classes of Chevalley groups (these can be thought of as Lie groups over k rather than $\mathbb{R}$, an good reference for these is given by [2]). Through this correspondence we use the results of §4 to prove vanishing statements for the low dimensional mod p homology of these finite simple groups in §5.

1.2 Background and notational conventions

As is usual in homological algebra, the notation C_* will refer to a sequence of abelian groups $(C_i)_{i\geqslant 0}$. A chain complex (C_*, ∂), is a sequence of abelian groups C_*, with homomorphisms $\partial_i : C_i \to C_{i-1}$, satisfying $\partial_i\partial_{i+1} = 0$ for each $i \geqslant 0$ (the abelian group C_{-1} is set to be zero). The homology groups $H_*(C_*)$, of a chain complex (C_*, ∂) is given by the quotients $H_i(C_*) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$. We sometimes call $H_i(C_*)$ the i^{th} dimensional homology group of the chain complex.

A chain map τ, between chain complexes (C_*, ∂) and (C'_*, ∂'), is a sequence of homomorphisms $\tau_i : C_i \to C'_i$, such that $\tau_i\partial_{i+1} = \partial'_i\tau_{i+1}$ for all $i \geqslant 0$. A chain map induces homomorphisms $H_i(C_*) \to H_i(C'_*)$ for each $i \geqslant 0$.

Let (C_*, ∂) and (C'_*, ∂') be chain complexes. A chain homotopy h, between chain maps $\tau, \rho : (C_*, \partial) \to (C'_*, \partial')$, is a sequence of homomorphisms $h_i : C_i \to C'_{i+1}$ satisfying $h_{i-1}\partial_i + \partial'_{i+1}h_i = \tau - \rho$. We say that τ and ρ are chain homotopy equivalent, or simply homotopic. Homotopic chain maps induce the same homomorphisms on homology.

Throughout this dissertation p is assumed to be a fixed prime. We denote the finite field with p elements by $\mathbb{F}_p$, and also write $\mathbb{F}_p$ for the underlying abelian group. By the homology of a group G we will mean the group homology of G with coefficients in $\mathbb{F}_p$, as defined in §2. This is also called the mod p group homology of G. We write $H_*(G)$ for the homology of G, usually written $H_*(G, \mathbb{F}_p)$ in the literature. We will use the phrase topological homology if we ever refer to the usual homology groups of a topological space.

If k is any field, then k^* denotes the multiplicative group obtained from k by removing zero. We denote by k^{2*} the subgroup of k^* consisting of squares in k^*. As is usual, we write I_n for the $n \times n$ idenity matrix over k. We denote the transpose of a matrix A over k by A^τ.

2 Mod p Group Homology

Later in this essay we compute the homology of many finite groups. We first need to understand what group homology is. Much like in [1], we opt for a homological algebra point of view. None of the content in this section is new, although defining homology with coefficients in $\mathbb{F}_p$ without reference to integral coefficients first is my own approach - it is more appropriate however, as mod p homology is all we will work with.

2.1 Constructing abelian groups from $\mathbb{F}_p$

If S is a finite set we write $\mathbb{F}_p[S]$ for the abelian group underlying the vector space generated by S over $\mathbb{F}_p$, we call $\mathbb{F}_p[S]$ the abelian group generated by S over $\mathbb{F}_p$ accordingly. Explicitly, we view $\mathbb{F}_p[S]$ as formal sums of elements of S with coefficients in $\mathbb{F}_p$:

$$\mathbb{F}_p[S] = \left\{ \sum_{s \in S} a_s s : a_s \in \mathbb{F}_p \right\}.$$

Addition is defined termwise. As a group $\mathbb{F}_p[S] \cong \mathbb{F}_p^{|S|}$.

If G is a finite group then $\mathbb{F}_p[G]$ naturally has the structure of a ring with multiplication determined completely by $(1g)\cdot(1h) = 1gh$ for all $g, h \in G$. Thus in this case we may treat $\mathbb{F}_p[G]$ as a ring, allowing us to talk about $\mathbb{F}_p[G]$-modules (meaning left modules) without issue. The ring $\mathbb{F}_p[G]$ is often called the group ring of G over $\mathbb{F}_p$.

Recall that given abelian groups A and B, we can form the tensor product $A \bigotimes B$, by the following steps: Let F be the free abelian group on the elements of $A \bigoplus B$. Let R be the subgroup of F generated by elements

$$(a + a', b) - (a, b) - (a', b), \quad (a, b + b') - (a, b) - (a, b'),$$

where a, a' ranges over A and b, b' ranges over B. Then $A \bigotimes B = F/R$. If $\theta : F \to A \bigotimes B$ is the quotient map we write $a \otimes b$ for $\theta(a, b)$.

There is a universal property for tensor products. Indeed suppose A, B and C are abelian groups and there is a map of sets $\psi : A \bigoplus B \to C$ satisfying

$$\psi(a + a', b) = \psi(a, b) + \psi(a', b), \quad \psi(a, b + b') = \psi(a, b) + \psi(a, b'), \tag{2.1}$$

for all $a, a' \in A$ and $b, b' \in B$. Then there is a unique homomorphism of abelian groups $\bar{\psi} : A \bigotimes B \to C$ such that $\psi = \bar{\psi} \circ \theta$.

It is known that the tensor product is commutative and distributes over direct sum. In other words if A, A' and B are abelian groups then

$$A \bigotimes B \cong B \bigotimes A, \quad \left(A \bigoplus A'\right) \bigotimes B \cong \left(A \bigotimes B\right) \bigoplus \left(A' \bigotimes B\right).$$

Both of the isomorphisms are natural - the first is given by $a \otimes b \leftrightarrow b \otimes a$ and the second is given by $(a, a') \otimes b \leftrightarrow (a \otimes b, a' \otimes b)$. One uses the universal property to check these maps are well defined group homomorphisms.

As an example, suppose A is a finite abelian group. Let's consider the tensor product $\mathbb{F}_p \bigoplus A$. Since $|A| < \infty$, the classification of finitely generated abelian groups implies that

$$A \cong \bigoplus_{i=1}^{t} \mathbb{Z}/p_i^{a_i}\mathbb{Z},$$

where the p_i are primes, the a_i are positive integers and $t \geqslant 1$. It is known that if m, n

are positive integers then $\mathbb{Z}/m\mathbb{Z} \bigotimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m,n)$. Since $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ as a group, we deduce that $\mathbb{F}_p \bigotimes A$ is isomorphic to a direct sum of copies of $\mathbb{F}_p$, with one copy for each p_i equal to p.

If S is a finite set and A an abelian group, both $\mathbb{F}_p[S]$ and $\mathbb{F}_p \bigotimes A$ are abelian groups with each non-identity element having order p. Such groups are called elementary abelian p-groups. We will make use of the following analogue of the universal property for free abelian groups (recall that all abelian groups are $\mathbb{Z}$-modules):

Proposition 2.1 Let S be a set and $\iota : S \to \mathbb{F}_p[S]$ be the map $s \mapsto 1s$. Let E be an elementary abelian p-group. Then for any map of sets $\phi : S \to E$, there is a unique homomorphism of abelian groups $\bar{\phi} : \mathbb{F}_p[S] \to E$ such that $\bar{\phi} \circ \iota = \phi$.

Proof. Let $\mathbb{Z}[S]$ be the free abelian group generated by S. By the usual universal property for free abelian groups, the map ϕ extends uniquely to a homomorphism $\phi' : \mathbb{Z}[S] \to E$. Since E is an elementary abelian p group, $\phi'(p \cdot x) = p \cdot \phi'(x) = 0$ for each $x \in \mathbb{Z}[S]$. In other words the $\mathbb{Z}$-submodule of $\mathbb{Z}[S]$ generated by p lies in the kernel of ϕ'. We thus get a well-defined homomorphism $\mathbb{Z}[S]/p\mathbb{Z}[S] \to E$. It is easily checked that $\mathbb{Z}[S]/p\mathbb{Z}[S] \cong \mathbb{F}_p[S]$ and this homomorphism is the desired $\bar{\phi}$. Uniqueness follows from the uniqueness of ϕ'. $\square$

Suppose that S is a finite set and E is an elementary abelian p-group. The above proposition allows us to define homomorphisms from $\mathbb{F}_p[S]$ to E just by stating the images of the $1s$ for each $s \in S$. We will often do this.

2.2 The homological algbra - defining $H_*(G)$

We now begin to work towards the definition of group homology with coefficients in $\mathbb{F}_p$. To do this we develop a more general scenario. Our definition is then a special case.

Let R be a ring and let M be an R-module. A resolution P_*, of M over R is an exact sequence

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0, \tag{2.2}$$

where the P_i are all R-modules and the maps ∂_i, ε are R-module homomorphisms. An R-module P is called projective if there is an R-module Q such that $P \bigoplus Q$ is a free R-module. The above resolution is said to be projective if the P_i are all projective. It is shown in [1, §I.7] that there is exactly one projective resolution of M over R up to chain homotopy equivalence.

Let $\mathscr{C}$ be the category of $\mathbb{F}_p$-modules and $\mathscr{D}$ the category of abelian groups. Both $\mathscr{C}$ and $\mathscr{D}$ are preadditive categories. For $\mathscr{C}$ this means that if $M, N \in \mathrm{Obj}(\mathscr{C})$, then the set of R-module homomorphisms from M to N, $\mathrm{Hom}_R(M,N)$, has the structure of an abelian group, and that composition of morphisms is bilinear in the sense that

$$f \circ (g + h) = (f \circ g) + (f \circ h), \quad (f + g) \circ h = (f \circ h) + (g \circ h),$$

wherever f, g and h are morphisms and the compositions are defined. The statements for $\mathscr{D}$ being preadditive are similar. A covariant functor $F : \mathscr{C} \to \mathscr{D}$ is called additive if the maps

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}(FM, FN),$$

determined by F are homomorphisms of abelian groups for all $M, N \in \mathrm{Obj}(\mathscr{C})$.

Suppose $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor. Applying F termwise to (2.2) yields a chain complex

$$\cdots \to FP_2 \xrightarrow{F\partial_2} FP_1 \xrightarrow{F\partial_1} FP_0 \xrightarrow{F\varepsilon} FM \longrightarrow 0. \tag{2.3}$$

The homology groups of this complex, in general, can be thought of as measuring the failure of F to be exact - if F were an exact functor then (2.3) would be an exact sequence and all of its homology groups would be zero.

Assume further that F is an additive functor. Then if $f, g \in \mathrm{Hom}_R(M, N)$, for objects M, N of $\mathscr{C}$, we have $F(f+g) = F(f)+F(g)$. It follows that F preserves chain homotopies and thus the uniqueness of projective resolutions up to chain homotopy passes to the chain complexes (2.3) obtained by applying F. The homology groups of (2.3) are thus uniquely determined by F and M (and R). In particular they are independent of P_*.

Now let G be any group. In the above we let $R = \mathbb{F}_p[G]$, whose ring structure is explained in §2.1. We let $M = \mathbb{F}_p$ with $\mathbb{F}_p[G]$ action determined by $(1g) \cdot x = x$ for each $g \in G, x \in \mathbb{F}_p$ (so G acts trivially on $\mathbb{F}_p$). Let F be the 'coinvariants' functor sending an $\mathbb{F}_p[G]$-module to its largest quotient on which G acts trivially. Explicitly, given an object M of $\mathscr{C}$ we have

$$FM = M_G = M/N \in \mathrm{Obj}(\mathscr{D}), \quad N = \langle m - g \cdot m : g \in G, m \in M \rangle.$$

If $f \in \mathrm{Hom}_R(M, M')$, for $\mathbb{F}_p[G]$-modules M, M', then $f(m - g \cdot m) = f(m) - g \cdot f(m)$ for any $g \in G, m \in M$. This implies that f induces a homomorphism of abelian groups $M_G \to M'_G$. This induced homomorphism is precisely Ff. It follows that F is an additive functor. We are now ready to define $H_*(G)$. Let

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{F}_p \longrightarrow 0$$

be any projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[G]$. Then the group homology of G with coefficients in $\mathbb{F}_p$, $H_*(G)$, is the homology of the chain complex obtained by applying our functor F to the above resolution and dropping the augmentation map ε. In other words it is the homology of the complex

$$\cdots \to (P_2)_G \xrightarrow{\partial_2} (P_1)_G \xrightarrow{\partial_1} (P_0)_G \longrightarrow 0.$$

Note we have slightly abused notation and written ∂_i for $F\partial_i$. We have already justified that $H_*(G)$ depends only on G and $\mathbb{F}_p$, and not P_*. Notice that each P_i is an elementary abelian p group and hence each $H_i(G)$ is as well.

2.3 The standard resolution

The purpose of this section is to construct a projective resolution for $\mathbb{F}_p$ over $\mathbb{F}_p[G]$ for a finite group G. By definition, such a resolution can then be used to compute $H_*(G)$. This resolution arises from a topological space constructed from G [1, §I.5].

Let X be the $(|G|+1)$-simplex with vertex set given by G as a set. There is a natural Δ-complex structure (see [5, §2.1]) on X with each finite subset $H \subset G$ defining a $(|H|+1)$-simplex in this structure. Let $(\Delta_*(X), \partial)$ be the ordered simplicial chain complex of X with coefficients in $\mathbb{F}_p$. In other words, for each $i \geqslant 0$, let $\Delta_i(X) = \mathbb{F}_p[G^{i+1}]$. We are viewing G^{i+1} as a set, giving no attention to its structure as a group (so we are not viewing $\Delta_i(X)$ as a ring).

We make each $\Delta_i(X)$ into a $\mathbb{F}_p[G]$ module by defining

$$(1g) \cdot 1(g_0, g_1, \ldots, g_i) = 1(gg_0, gg_1, \ldots, g_i), \quad g, g_j \in G.$$

Each $\Delta_i(X)$ is then a free (and hence projective) $\mathbb{F}_p[G]$-module with an $\mathbb{F}_p[G]$-basis given by the elements of the form $(1, g_1, \ldots, g_i), g_j \in G$. It is easily checked that these span $\Delta_i(X)$ and are $\mathbb{F}_p[G]$-linearly independent. The boundary maps $\partial_i : \Delta_i(X) \to \Delta_{i-1}(X)$, for $i \geqslant 1$, are given by

$$\partial_i = \sum_{j=0}^{i} (-1)^j d_j, \quad d_j\left(1(g_0, g_1, \ldots, g_i)\right) = 1(g_0, g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_i).$$

The augmentation map $\varepsilon : \Delta_0(X) \to \mathbb{F}_p$ is determined by $\varepsilon(1g) = 1$ for each $g \in G$. It is easily seen from these formulae that the ∂_i, ε are $\mathbb{F}_p[G]$-module homomorphisms. We thus have a sequence

$$\cdots \to \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\varepsilon} \mathbb{F}_p \longrightarrow 0. \tag{2.4}$$

Using §4.3 in [8], we see that that the homology of this chain complex coincides with the reduced topological homology of X with coefficients in $\mathbb{F}_p$. Since X is contractible (it is just a $(|G|+1)$-dimensional simplex), these homology groups are all zero implying that (2.4) is exact and therefore gives a projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[G]$. This resolution is called the standard resolution.

We now apply the coinvariants functor F, as in §2.2, to the standard resolution and perform some computations. In particular we will obtain descriptions of $H_0(G)$ and $H_1(G)$. We shall use the following proposition:

Proposition 2.2. Suppose P is a free $\mathbb{F}_p[G]$-module with $\mathbb{F}_p[G]$-basis $B = \{b_1, \ldots, b_i\}$. Let N be the subgroup of P such that $FP = P_G = P/N$. Write $\bar{b}_j$ for the coset $b_j + N$ in P_G and let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_i\}$. Then P_G is the abelian group generated by $\bar{B}$ over $\mathbb{F}_p$, i.e. $P_G \cong \mathbb{F}_p[\bar{B}]$.

Proof. Define a group homomorphism $\phi : P \to \mathbb{F}_p[\bar{B}]$ by

$$\phi\left(\sum_{j=1}^{i}\left(\sum_{g\in G} a_{jg} g\right) b_j\right) = \sum_{j=1}^{i}\left(\sum_{g\in G} a_{jg}\bar{b}_j\right).$$

It can be seen from this formula that $b_j - gb_j \in \ker(\phi)$ for all j, $1 \leqslant j \leqslant i$, and all $g \in G$. Hence as N is generated by these elements, ϕ induces a homomorphism $\bar{\phi} : P_G \to \mathbb{F}_p[\bar{B}]$. The homomorphism $\psi : \mathbb{F}_p[\bar{B}] \to P_G$ defined by $\bar{b}_j \mapsto \bar{b}_j$ is an inverse to $\bar{\phi}$, finishing the proof. $\square$

We've already pointed out that for $i \geqslant 0$, $\Delta_i(X)$ is a free $\mathbb{F}_p[G]$ module. Its basis is given by the set S_i where

$$S_i = \{(1, g_1, \ldots, g_i) : \quad g_j \in G\}.$$

Proposition 2.2 tells us that $\Delta_i(X)_G \cong \mathbb{F}_p[\bar{S}_i]$. We write $(1 : g_1 : \cdots : g_i)$ for the image of $(1, g_1, \ldots, g_i)$ in $\Delta(X)_G$ (one may notice the analogy with homogeneous coordinates in projective geometry) and identify $\mathbb{F}_p[S_0]$ with $\mathbb{F}_p$ in the obvious way. The sequence

$$\cdots \to \Delta_2(X)_G \xrightarrow{\partial_2} \Delta_1(X)_G \xrightarrow{\partial_1} \Delta_0(X)_G \longrightarrow 0$$

defining group homology becomes the sequence

$$\cdots \to \mathbb{F}_p[S_2] \xrightarrow{\partial_2} \mathbb{F}_p[S_1] \xrightarrow{\partial_1} \mathbb{F}_p \longrightarrow 0.$$

Notice that if $g \in G$ then $\partial_1(1 : g) = (g) - (1) = 0 \in \Delta_0(X)_G$, implying that $\partial_1 = 0$. This gives us $H_0(G) \cong \mathbb{F}_p$. If $g_1, g_2 \in G$ we have

$$\partial_2(1 : g_1 : g_2) = (g_1 : g_2) - (1 : g_2) + (1 : g_1) = (1 : g_1^{-1}g_2) - (1 : g_2) + (1 : g_1).$$

Recall that the abelianisation of G, G_{ab}, is the largest quotient of G which is abelian. Explicitly, define $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ for each pair $g_1, g_2 \in G$, then

$$G_{ab} = G/[G,G], \quad [G,G] = \langle [g_1, g_2] : g_1, g_2 \in G\rangle.$$

The element $[g_1, g_2]$ is called the commutator of g_1 and g_2 and $[G, G]$ is called the commutator subgroup of G.

Proposition 2.3. We have $H_1(G) \cong \mathbb{F}_p \bigotimes G_{ab}$.

Proof. By definition we have

$$H_1(G) = \mathbb{F}_p[S_1]/R, \quad R = \left\langle (1 : g_1^{-1} g_2) - (1, g_2) + (1, g_1) \right\rangle .$$

Hence in $H_1(G)$ we can deduce the following identities for each $g \in G$:

$$\begin{aligned} &(1 : g) = (1 : 1g) = (1 : g) - (1 : 1) \quad \Longrightarrow \quad (1 : 1) = 0, \\ &(1 : g) = (1 : g1) = (1 : 1) - (1 : g^{-1}) = -(1 : g^{-1}). \end{aligned}$$

These can in turn be used to obtain

$$\begin{aligned} &(1 : g_1 g_2) = (1 : g_2) - (1 : g_1^{-1}) = (1 : g_1) + (1 : g_2), \\ &(1 : g_1^{-1} g_2^{-1} g_1 g_2) = (1 : g_1 g_2) - (1 : g_2 g_1) = 0 \end{aligned} \tag{2.5}$$

whenever $g_1, g_2 \in G$. If $g_1[G, G] = g_2[G, G]$ then one can write $g_2 = c_1 \dots c_t g_1$ where the c_i are commutators. This is because $g_1 g_2^{-1}$ is in $[G, G]$, and is hence a product of commutators and their inverses, but the inverse of a commutator is another commutator. Using (2.5) we get

$$(1 : g_1) = (1 : c_1 \dots c_t g_2) = (1 : c_1) + \dots + (1 : c_t) + (1 : g_2) = (1 : g_2).$$

We define a homomorphism (using Proposition 2.1) $\phi : \mathbb{F}_p[S_1] \to \mathbb{F}_p \bigotimes G_{ab}$ by $1(1 : g) \mapsto 1 \otimes g[G, G]$. Notice that $R \subset \ker(\phi)$ since $g_1^{-1} g_2[G, G] - g_2[G, G] + g_1[G, G] = 0 \in G_{ab}$. Hence ϕ induces a homomorphism $\bar{\phi} : H_1(G) \to \mathbb{F}_p \bigotimes G_{ab}$.

Now define a map of sets $\psi : \mathbb{F}_p \bigoplus G_{ab} \to H_1(G)$ by $\psi(a, g[G, G]) = a(1 : g)$. This is well defined because we have already seen that $g_1[G, G] = g_2[G, G] \Rightarrow (1 : g_1) = (1 : g_2)$. We can also check the equations (2.1) hold for ψ and hence we get a unique group homomorphism $\bar{\psi} : \mathbb{F}_p \bigotimes G_{ab} \to H_1(G)$ satisfying $\bar{\psi}(a \otimes g[G, G]) = a(1 : g)$. One can see that $\bar{\phi}$ and $\bar{\psi}$ are inverse to each other so we are done. □

The construction of projective resolutions from topological spaces isn't a coincidence - group homology was initially motivated from topology. In fact, a topologist may define mod p homology to be the topological homology groups of a $K(G, 1)$ space with coefficients in $\mathbb{F}_p$ ([1, §II.2-4] [5, §1.B & p. 153])

2.4 Induced maps on homology - functoriality

We have seen that group homology assigns to each group G a sequence of abelian groups $H_*(G)$. We now look at how group homology assigns certain maps of groups $G \to G'$ (not just homomorphisms) to induced homomorphisms on homology $H_i(G) \to H_i(G')$. In particular group homomorphisms $G \to G'$, satisfy this construction. The map sending a group G to its i^{th} homology group $H_i(G)$ can then be viewed as a covariant functor from the category of groups to the category of abelian groups for each $i \geqslant 0$.

Let G, G' be groups and $\phi : G \to G'$ a map of sets. Let X be the topological space constructed from G as in §2.3 and let X' be the equivalent space for G'. For each $i \geqslant 0$, suppose we have an abelian group homomorphism

$$\tau_i : \Delta_i(X) \longrightarrow \Delta_i(X') \text{ satisfying } \quad \tau_i(g \cdot x) = \phi(g) \cdot \tau_i(x), \quad g \in G, x \in \Delta_i(X). \tag{2.6}$$

Suppose the τ_i also make the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & \Delta_2(X) \arrow[r, "\partial_2"] \arrow[d, "\tau_2"] & \Delta_1(X) \arrow[r, "\partial_1"] \arrow[d, "\tau_1"] & \Delta_0(X) \arrow[r, "\varepsilon"] \arrow[d, "\tau_0"] & \mathbb{F}_p \arrow[r] \arrow[d, "1"] & 0 \\
\cdots \arrow[r] & \Delta_2(X') \arrow[r, "\partial_2'"'] & \Delta_1(X') \arrow[r, "\partial_1'"'] & \Delta_0(X') \arrow[r, "\varepsilon'"'] & \mathbb{F}_p \arrow[r] & 0
\end{tikzcd} . \tag{2.7}$$

The τ_i then form a chain map τ, between the standard resolution of G and the standard resolution of G', compatible with ϕ. Since the standard resolutions are projective and exact chain complexes, we can use [1, Lemma I.7.4] to state that τ is unique up to chain homotopy. The fact that $\tau_i(x - g \cdot x) = \tau_i(x) - \phi(g) \cdot \tau_i(x)$ for each $i \geqslant 0, g \in G$ and $x \in \Delta_i(X)$ implies that the τ_i induce homomorphisms $\bar{\tau}_i : \Delta_i(X)_G \to \Delta_i(X)_{G'}$. We can therefore apply the coinvariants functor F to the above diagram, discarding the augmentation maps $\varepsilon, \varepsilon'$, to yield the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \Delta_2(X)_G \arrow[r, "\partial_2"] \arrow[d, "\bar{\tau}_2"] & \Delta_1(X)_G \arrow[r, "\partial_1"] \arrow[d, "\bar{\tau}_1"] & \Delta_0(X)_G \arrow[r] \arrow[d, "\bar{\tau}_0"] & 0 \\
\cdots \arrow[r] & \Delta_2(X')_{G'} \arrow[r, "\partial_2'"'] & \Delta_1(X')_{G'} \arrow[r, "\partial_1'"'] & \Delta_0(X')_{G'} \arrow[r] & 0
\end{tikzcd} . \tag{2.8}$$

The $\bar{\tau}_i$ form a chain map $\bar{\tau}$, from the chain complex $(\Delta_*(X)_G, \partial)$, to the chain complex $(\Delta_*(X')_{G'}, \partial')$. This chain map induces homomorphisms on homology. The uniqueness of the chain map τ up to homotopy implies that these homomorphisms are independent of the choice of τ and thus depend only on ϕ. We write ϕ_* for these induced homomorphisms.

It is easy to see that
$$(\phi \circ \psi)_* = \phi_* \circ \psi_*, \quad (\mathrm{id}_G)_* = \mathrm{id}_{H_i(G)},$$
whenever $\phi : G \to G'$, $\psi : G' \to G''$ are so that ϕ_* and ψ_* are defined.

Now suppose that $\phi : G \to G'$ is a group homomorphism. Then we define our chain map τ by
$$\tau_i(1(g_0, g_1, \ldots, g_i)) = 1(\phi(g_0), \phi(g_1), \ldots, \phi(g_i)).$$
This satisfies (2.6) and makes the diagram (2.7) commute. We obtain homomorphisms $\phi_* : H_i(G) \to H_i(G')$. In light of the composition identities above, we deduce the promised functoriality of group homology.

Proposition 2.4. Suppose that G is a group and $g \in G$. The conjugation homomorphism $\phi = g^{-1}(\cdot)g : G \to G$ induces the identity homomorphisms on the homology of G.

Proof. Let X be the space constructed for the standard resolution for G. For each $i \geqslant 0$, define $\tau_i : \Delta_i(X) \to \Delta_i(X)$ by
$$\tau_i(1(g_0, g_1, \ldots, g_i)) = 1(g^{-1}g_0, g^{-1}g_1, \ldots, g^{-1}g_i).$$
One checks that (2.6) is satisfied and that these make (2.7) commute. The chain map defined by the τ_i can then be used to construct the induced homomorphisms ϕ_*. If F is the coinvariants functor then clearly $\bar{\tau}_i = F\tau_i = \mathrm{id}$ for each $i \geqslant 0$, and so the induced homomorphisms ϕ are the identity homomorphisms. □

Now suppose that G is a group and T is a group which acts on G via group homomorphisms. That is to say the map $t : G \to G$, $g \mapsto t \cdot g$ is a group homomorphism for each $t \in T$. Then each t in T induces homomorphisms $t_* : H_i(G) \to H_i(G)$. Using the functoriality, T acts on each $H_i(G)$ via group homomorphisms with $t \cdot z = t_*(z)$ for each $t \in T, z \in H_i(G)$. We can then talk of the fixed points of $H_i(G)$ under the action of T, i.e. the subgroup
$$H_i(G)^T = \{z \in H_i(G) : t \cdot z = z \text{ for all } t \in T\} \leqslant H_i(G).$$
This allows us to construct representations of T over any field. In §3 we explore these constructions in order to use them later.

3 Constructing Representations from Homology

Let G and T be groups. At the end of the previous section, we discussed how an action of T on G via group homomorphisms induced an action of T on homology. If we also have a field k then for any fixed $i \geqslant 0$, $H_i(G) \bigotimes k$ is a representation of T over k, or equivalently a $k[T]$-module. This representation is determined by

$$\lambda t \cdot (z \otimes x) = t_*(z) \otimes \lambda x, \quad \lambda, x \in k,\ t \in T,\ z \in H_i(G).$$

The purpose of this section is to prove some results for this construction in the case that T is abelian with order prime to p. None of the results in this section are my own.

Let k be the finite field of size p^d for some $d \geqslant 1$. Let $\bar{k}$ be an algebraic closure of k. The characteristic of $\bar{k}$, $\mathrm{char}(\bar{k})$, is equal to $\mathrm{char}(k) = p$. Suppose that T is a finite abelian group with $p \nmid |T|$. Let $M = M_{\bar{k}}(T)$ be the semiring of isomorphism classes of representations over $\bar{k}$. The addition operation in M is the direct sum and the product operation is the tensor product. As T is abelian with order prime to $\mathrm{char}(\bar{k})$, all irreducible representations of T over $\bar{k}$ have dimension one. Note that this uses the fact $\bar{k}$ is algebraically closed. The condition $p \nmid |T|$ is also necessary to invoke Maschke's theorem. We can then identify M with $\mathbb{Z}_{\geqslant 0}(\mathrm{Hom}(T, \bar{k}^*))$. Indeed if V is a representation of T over $\bar{k}$ we can identify

$$[V] \equiv \sum_{\chi} n_\chi(V)\chi \in \mathbb{Z}_{\geqslant 0}(\mathrm{Hom}(T, \bar{k}^*)). \tag{3.1}$$

Here $[V]$ represents the isomorphism class of V in M. The index χ ranges over the irreducible characters, and $n_\chi(V)$ is the multiplicity of the character χ in V.

Suppose G is a group on which T acts by group homomorphisms. We have already established that $H_i(G) \bigotimes \bar{k}$ is a representation of T over $\bar{k}$ for each $i \geqslant 0$. We can then define the Poincaré series of $H_*(G)$ to be

$$\sigma(H_*(G)) = \sum_{i \geqslant 0} [H_i(G) \bigotimes \bar{k}] z^i \in M[[z]].$$

This clearly depends on T and $\bar{k}$, but these will always be clear in context.

Given any homomorphism $a : T \to k^*$, let k_a denote the abelian group k with T acting via a. That is, $t \cdot x = a(t)x$ for each $x \in k, t \in T$. As $k \subset \bar{k}$, the homomorphism a defines a one-dimensional representation of T over $\bar{k}$, which we will denote by V_a. In the following, $\bigwedge(V)$ and $\Gamma(V)$ will denote the exterior and symmetric algebras of a vector space V, over $\bar{k}$ respectively.

The proof of the following lemma is the dual to Quillen's proof of Lemma 15 in [7] for the group cohomology of k_a. The method below is adapted to group homology and is written more accessibly.

Lemma 3.1 Let $a : T \to k^*$ be a homomorphism, and let k_a and V_a be defined as in the previous paragraphs. Then

$$\sigma(H_*(k_a)) = \prod_{b=0}^{d-1}(1+a^{p^b}z)\sum_{i\geqslant 0} a^{ip^b}z^{2i} = \prod_{b=0}^{d-1}\frac{1+a^{p^b}z}{1-a^{p^b}z^2}.$$

Proof. By the classification of finite fields, k_a is isomorphic as a group to $K = (\mathbb{Z}/p\mathbb{Z})^d$. It is known that the homology of K is given by

$$H_*(K) \cong \begin{cases} \bigwedge(K_p)\bigotimes\Gamma\left({}_pK\right) & \text{if } p\neq 2 \\ \Gamma\left(K_p\right) & \text{if } p=2 \end{cases}, \tag{3.2}$$

where $K_p = K\bigotimes\mathbb{Z}/p\mathbb{Z}$ is canonically isomorphic to $H_1(K)$ and ${}_pK = \{k\in K : pk=0\}$ is canonically isomorphic to a subgroup of $H_2(K)$ (see §V.6 in [1] for details). From Galois theory, there is a ring isomorphism

$$k\bigotimes_{\mathbb{F}_p}\bar{k}\cong(\bar{k})^d, \quad x\otimes y\mapsto(x^{p^b}y)_{b=0}^{d-1}.$$

This becomes an isomorphism of representations $k_a\bigotimes_{\mathbb{F}_p}\bar{k}\cong\bigoplus_{b=0}^{d-1}V_{a^{p^b}}$, since if $t\in T, x\in k_a$ and $y\in\bar{k}$ then

$$\begin{aligned} t\cdot(x\otimes y) &= (a(t)x)\otimes y \mapsto \left((a(t)x)^{p^b}y\right)_{b=0}^{d-1} \\ &= \left(a(t)^{p^b}x^{p^b}y\right)_{b=0}^{d-1} = t\cdot(x^{p^b}y)_{b=0}^{d-1}. \end{aligned}$$

Combining this with (3.2), and using that for general representations V and W we have $\bigwedge(V\bigoplus W)\cong\bigwedge(V)\bigotimes\bigwedge(W)$ and $\Gamma(V\bigoplus W)\cong\Gamma(V)\bigotimes\Gamma(W)$, we get

$$H_*(k_a)\bigotimes_{\mathbb{F}_p}\bar{k}\cong\begin{cases}\bigotimes_{b=0}^{d-1}\bigwedge\left((V_{a^{p^b}})_p\right)\bigotimes_{\bar{k}}\Gamma\left({}_p(V_{a^{p^b}})\right) & \text{if } p\neq 2\\ \bigotimes_{b=0}^{d-1}\Gamma\left((V_{a^{p^b}})_p\right) & \text{if } p=2\end{cases}.$$

If V is any one-dimensional representation over $\bar{k}$, then $\bigwedge(V)\cong\bar{k}\bigoplus V$ and $\Gamma(V)\cong\bigoplus_{i\geqslant 0}V^{\otimes i}$. Since characters multiply under the tensor product of representations, the Poincaré series of a tensor product is the product of the Poincaré series of its factors. In other words as we pass from homology to Poincaré series, the tensor products become multiplicative products in M. Recalling that $(V_{a^{p^b}})_p$ and ${}_p(V_{a^{p^b}})$ lie in $H_1(k_a)$ and $H_2(k_a)$ respectively, we obtain

$$\sigma(H_*(k_a))\bigotimes_{\mathbb{F}_p}\bar{k}\cong\begin{cases}\prod_{b=0}^{d-1}(1+a^{p^b}z)\sum_{i\geqslant 0}a^{ip^b}z^{2i} & \text{if } p\neq 2\\ \prod_{b=0}^{d-1}\sum_{i\geqslant 0}a^{ip^b}z^i & \text{if } p=2\end{cases}.$$

If $p \neq 2$ then we are done. For $p = 2$ we have

$$\prod_{b=0}^{d-1} \sum_{i \geqslant 0} a^{ip^b} z^i = \prod_{b=0}^{d-1} \frac{1}{1 - a^{p^b} z} = \prod_{b=0}^{d-1} \frac{1 + a^{p^b} z}{1 - a^{p^{b+1}} z^2} = \prod_{b=0}^{d-1} \frac{1 + a^{p^b} z}{1 - a^{p^b} z^2}.$$

Notice that we used $a^{p^d} = a$ since the order of k^* is $p^d - 1$. We have proved the lemma. $\square$

For our proofs later, we require a method of comparing Poincaré series. This is because an upper bound on the coefficient of the χz^i term in a Poincaré series $\sigma(H_*(G))$ is, by definition, an upper bound on the multiplicity of χ in $H_i(G)$. Spectral sequences are the key to enable such comparisons. We use the '$E^n_{j,k}$' notation for spectral sequences as explained by Hatcher in [4].

A group extension is a short exact sequence of groups

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1.$$

For a group extension as above, there is a Hochschild-Serre spectral sequence of the form

$$E^2 = H_*(N) \bigotimes H_*(Q) \Longrightarrow H_*(G). \tag{3.3}$$

This is Proposition 7 in [6] followed by an application of the universal coefficient theorem for group homology. The i^{th} abelian group in the sequence $H_*(N) \bigotimes H_*(Q)$ is defined to be

$$\bigoplus_{j+k=i} H_j(N) \bigotimes H_k(Q).$$

If G and G' are groups acted on by another group T, then a group homomorphism $\phi : G \to G'$, is called T-invariant if $\phi(t \cdot g) = t \cdot \phi(g)$ for each $t \in T, g \in G$.

Lemma 3.2. Suppose we have a group extension as above. Suppose further that T acts on N, G and Q in a way such that the group homomorphisms ι and π are T-invariant. Then the spectral sequence (3.3) for this group extension implies that

$$\sigma(H_*(N))\sigma(H_*(Q)) \gg \sigma(H_*(G)).$$

The symbol '$\gg$' means that each coefficient on the right hand side is less than or equal to the corresponding coefficient on the left hand side, working in $\mathbb{Z}_{\geqslant 0}$.

Proof. Fix a point (j, k) and consider the sequence of groups $(E^n_{j,k})_{n \geqslant 1}$ in the Hochschild-Serre spectral sequence. Each $E^{n+1}_{j,k}$ is a subquotient (denoted by $\prec$) of $E^n_{j,k}$, hence $E^\infty_{j,k} \prec E^2_{j,k}$. As such

$$H_i(G) = \bigoplus_{j+k=i} E^\infty_{j,k} \prec \bigoplus_{j+k=i} E^2_{j,k} = \bigoplus_{j+k=i} H_j(N) \bigotimes H_k(Q).$$

We know that if W and V are representations with $W \prec V$, and if χ is a (irreducible) character, then the multiplicity of χ in W is at most the multiplicity of χ in V. In terms of the notation in (3.1), $n_\chi(W) \leqslant n_\chi(V)$. The lemma follows. $\square$

4 Homology of Some Finite Classical Groups

Let k to be the field of size p^d for some $d \geqslant 1$. We use the tools developed in §3 to prove some vanishing statements for the low dimensional homology of some classical groups over k. We first look at the general linear and special linear groups over k in §4.1. Theorem 4.1 on the general linear groups is proved by Quillen. I have strengthened this result to include the special linear case. The proofs in §4.2 and §4.3 for the symplectic and orthogonal groups over k are also my own, though draw on Quillen's methods. The results of Theorems 4.5 and 4.6 themselves are not new however, and are proved independently by Friedlander in [3]. The result of Theorem 4.7 is not covered by any of these works however.

4.1 General linear and special linear groups

Fix $n \in \mathbb{N}$. As usual we write $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$ for the general linear and special linear groups of degree n over k respectively. Recall that $\mathrm{GL}_n(k)$ is the group of invertible $n \times n$ matrices in k, and $\mathrm{SL}_n(k)$ is the normal subgroup of $\mathrm{GL}_n(k)$ consisting of such matrices with determinant 1.

Theorem 4.1 [7, Theorem 6]**.** We have $H_i(\mathrm{GL}_n(k)) = 0$ whenever $0 < i < d(p-1)$.

As indicated in the introduction to this section, we prove the following, stronger, theorem.

Theorem 4.2 We have $H_i(\mathrm{GL}_n(k)) = 0 = H_i(\mathrm{SL}_n(k))$ whenever $0 < i < d(p-1)$.

To begin the proof, let Q_n be the subgroup of $\mathrm{SL}_n(k)$ (and $\mathrm{GL}_n(k)$) consisting of upper triangular matrices with 1s in each diagonal entry. Let T_n be the subgroup of $\mathrm{SL}_n(k)$ consisting of diagonal matrices. Notice that T_n acts on Q_n, $\mathrm{SL}_n(k)$ and $\mathrm{GL}_n(k)$ by conjugation. Since $|T_n| = (p^d-1)^{n-1}$ is prime to $\mathrm{char}(k) = p$ the results of §3 hold for $T = T_n$.

Fix $i \geqslant 0$. The orders of $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$ and Q_n are given respectively by

$$|\mathrm{GL}_n(k)| = p^{\frac{dn(n-1)}{2}} \prod_{j=1}^{n} (p^{dj}-1), \quad |\mathrm{SL}_n(k)| = \frac{1}{p^d-1}|\mathrm{GL}_n(k)|, \quad |Q_n| = p^{\frac{dn(n-1)}{2}}.$$

We see that Q_n is a Sylow p-subgroup of $\mathrm{GL}_n(k)$. This implies the homomorphism $H_i(Q_n) \to H_i(\mathrm{GL}_n(k))$ induced by inclusion in surjective [1, §III.9]. Taking fixed points of this homomorphism under the action of T_n, we get a surjective map

$$H_i(Q_n)^{T_n} \longrightarrow H_i(\mathrm{GL}_n(k))^{T_n} = H_i(\mathrm{GL}_n(k)).$$

The equality on the right is because T_n is a subgroup of $\mathrm{GL}_n(k)$, so we can appeal to Proposition 2.4. The order formulas above imply that Q_n is a Sylow p-subgroup of $\mathrm{SL}_n(k)$, so by the same arguement we have a surjective map $H_i(Q_n)^{T_n} \to H_i(\mathrm{SL}_n(k))$. Proving Theorem 4.2 is now just a case of showing that $H_i(Q_n)^{T_n} = 0$.

Let $\bar{k}$ be an algebraic closure of k. From §3 we know that $H_i(Q_n) \bigotimes \bar{k}$ is a representation of T_n over $\bar{k}$. Then the largest trivial subrepresentation is

$$\left(H_i(Q_n) \bigotimes \bar{k}\right)^{T_n} = H_i(Q_n)^{T_n} \bigotimes \bar{k}.$$

The right hand side of the above is zero if and only if $H_i(Q_n)^{T_n} = 0$ (this uses the fact that $H_i(Q_n)$ is an elementary abelian p group). Showing that $H_i(Q_n)^{T_n}$ is zero is equivalent to showing that the multiplicity of the trivial character ϵ, in $H_i(Q_n) \bigotimes \bar{k}$ is zero. This is the same as showing that the coefficient of ϵz^i is zero in the Poincaré series $\sigma(H_*(Q_n))$.

We want to find an upper bound for the coefficients of $\sigma(H_*(Q_n))$. To do this we make use of Lemmas 3.1 and 3.2. Let $\Delta = \{(i,j) : 1 \leqslant j < i \leqslant n\}$, and identify each $(i,j) \in \Delta$ with the homomorphism

$$T_n \longrightarrow k^*, \quad t = \mathrm{diag}(t_k)_{k=1}^n \mapsto \frac{t_i}{t_j}.$$

We switch between viewing Δ as a set of pairs and as a set of homomorphisms as appropriate. We put a total order on Δ by setting $(i,j) \leqslant (i',j')$ if either $i < i'$, or $i = i'$ and $j \leqslant j'$. Introduce the notation $m_{kl}(x)$ for the $n \times n$ matrix with (k,l)-entry $x \in k$ and all other entries equal to zero.

Recall that for a homomorphism $a : T \to k^*$, k_a denotes k as an abelian group with T acting via a. The reason why we associate the homomorphism $a : t \mapsto \frac{t_i}{t_j}$ to each (i,j) is so that the following group homomorphism is T-invariant for each $a \equiv (i,j) \in \Delta$:

$$\phi_a : k_a \longrightarrow Q_n, \quad x \mapsto I_n + m_{ji}(x).$$

Denote by Q_n^a the subgroup of Q_n generated by the images of all the ϕ_c with $c > a$.

Proposition 4.3. If $a, a', c \in \Delta$ with $a, a' > c$ and $c \equiv (i,j)$, then the (j,i)-entry of $\phi_a(x)\phi_{a'}(y)$ is zero for all $x, y \in k$.

Proof. Let $a \equiv (k,l), a' \equiv (k',l')$ and let $x,y \in k$. Then

$$\begin{aligned}\phi_a(x)\phi_{a'}(y) &= (I_n + m_{lk}(x))(I_n + m_{l'k'}(y)) \\ &= I_n + m_{lk}(x) + m_{l'k'}(y) + m_{lk}(x)m_{l'k'}(y).\end{aligned}$$

We see that the (j,i)-entry being non-zero implies that $m_{lk}(x)m_{l'k'}(y)$ has non-zero (j,i)-entry. The (j,i)-entry of $m_{lk}(x)m_{l'k'}(y)$ is given by

$$\sum_{r=1}^{n}(m_{lk}(x))_{jr}(m_{l'k'}(y))_{ri}.$$

Now if $i < k'$, this sum is just $(m_{lk}(x))_{ji} = 0$ since $(k,l) \neq (i,j)$. So if the sum is not zero we must have $i = k'$ and $j < l'$ (the latter to satisfy $c < a'$). The sum then becomes

$$(m_{lk}(x))_{jl'}\ y + (m_{lk}(x))_{jk'}.$$

If this is non-zero then $j = l$ necessarily, and either $k = l'$ or $k = k'$. If $k = l'$ then $i = k' > l' = k$ contradicts $c < a$. If $k = k'$ then $(i,j) = (k',l) = (k,l)$ contradicting $c \neq a$. We see that in all cases, the (j,i)-entry of $\phi_a(x)\phi_{a'}(y)$ is zero, as required. □

Proposition 4.3 makes clear the intuition that if $a \equiv (i,j) \in \Delta$, then Q_n^a is the subgroup of Q_n consisting of those matrices $q = (q_{j'i'})_{j'i'} \in Q_n$ with $q_{j'i'} = 0$ whenever $(i',j') \leqslant (i,j)$ in Δ. Furthermore, if $q \in Q_n$ and $t = \operatorname{diag}(t_k)_{k=1}^n \in T_n$ then $(t \cdot u)_{ji} = t_j^{-1}u_{ji}t_i$ (it is this calculation that shows the ϕ_a are T_n-invariant). This tells us that each Q_n^a is an invariant subgroup of Q_n under the action of T_n. Overall we have deduced that for each $a \in \Delta$ we get a T_n-invariant group extension

$$1 \longrightarrow k_a \longrightarrow Q_n/Q_n^a \longrightarrow Q_n/Q_n^{a'} \longrightarrow 1.$$

Here a' denotes the largest element of Δ less that a. If a is minimal in Δ we set $Q_n^{a'} = Q_n$. The first non-trivial map in the extension is induced by ϕ_a. The second is the natural map. Lemma 3.2 applied to this group extension gives us

$$\sigma(H_*(Q_n/Q_n^a)) \ll \sigma(H_*(k_a))\sigma(H_*(Q_n/Q_n^{a'}))$$

for each $a \in \Delta$. Combining these as we descend through Δ, noting that $Q_n^{(n,n-1)} = 1$, as well as appealing the Lemma 3.1, we find

$$\sigma(H_*(Q_n)) \ll \prod_{a\in\Delta}\sigma(H_*(k_a)) = \prod_{a,b}(1 + a^{p^b}z)\sum_{i\geqslant 0}a^{ip^b}z^{2i}. \tag{4.1}$$

Here and throughout the rest of §4.1, a will range over Δ whilst b ranges over the set $\{0,1,\ldots,d-1\}$.

For each $a \in \Delta$, define a set $\mathcal{J}_a$ by $\mathcal{J}_a = \{((m_{ab}, n_{ab}))_b \in (\{0,1\} \times \mathbb{Z}_{\geqslant 0})^d\}$. Then define

$$D_a(J) = \sum_b (m_{ab} + 2n_{ab}), \quad M_a(J) = \sum_b (m_{ab} + n_{ab})p^b,$$

for each $J = ((m_{ab}, n_{ab}))_b \in \mathcal{J}_a$. Then

$$\prod_b (1 + a^{p^b} z) \sum_{i \geqslant 0} a^{ip^b} z^{2i} = \sum_{J \in \mathcal{J}_a} a^{M_a(J)} z^{D_a(J)}. \tag{4.2}$$

To see how (4.2) works, we show there is a natural bijection between $\mathcal{J}_a$ and a way of choosing a term from the expanded product on the left hand side. Indeed suppose the b^{th} component of $J \in \mathcal{J}_a$ is (m_{ab}, n_{ab}). Then we choose '1' from the bracket $(1 + a^{p^b} z)$ if $m_{ab} = 0$, or we choose '$a^{p^b} z$' if $m_{ab} = 1$. We choose the n_{ab}^{th} term of the series $\sum_{i \geqslant 0} a^{ip^b} z^{2i}$.

In this way the component (m_{ab}, n_{ab}) contributes the term

$$a^{m_{ab} p^b} z^{m_{ab}} \cdot a^{n_{ab} p^b} z^{2n_{ab}} = a^{(m_{ab} + n_{ab})p^b} z^{m_{ab} + 2n_{ab}},$$

and hence, multiplying the contribution from each of the b components, $J = ((m_{ab}, n_{ab}))_b$ contributes $a^{M_a(J)} z^{D_a(J)}$. Now let $\mathcal{I} = \bigsqcup_a \mathcal{J}_a \equiv \{((m_{ab}, n_{ab}))_{a,b} \in (\{0,1\} \times \mathbb{Z}_{\geqslant 0})^{d|\Delta|}\}$. Using (4.2), the right hand side of (4.1) is equal to

$$\prod_a \left(\sum_{\mathcal{J}_a} a^{M_a(J)} z^{D_a(J)} \right) = \sum_{I \in \mathcal{I}} \left(\prod_a a^{M_a(I)} \right) z^{D(I)}. \tag{4.3}$$

In the above we have defined, for each $I = (J_a)_a \in \bigsqcup_a \mathcal{J}_a = \mathcal{I}$,

$$D(I) = \sum_a D_a(J_a), \quad M_a(I) = M_a(J_a).$$

Recall that we want to show there is no occurence of the term ϵz^i for $0 < i < d(p-1)$. We can now see this is just a case of showing that if $I \in \mathcal{I}$, then

$$\prod_a a^{M_a(I)} = \epsilon, \tag{4.4}$$

implies either $D(I) = 0$ or $D(I) \geqslant d(p-1)$.

Let $a_i \equiv (i+1, i) \in \Delta$ for $1 \leqslant i \leqslant n-1$. Then if $a \equiv (j,k) \in \Delta$ we have

$$a = \prod_{i=k}^{j-1} a_i = \prod_{i=1}^{n-1} a_i^{c_{ai}},$$

where $c_{ai} = 1$ for $k \leqslant i \leqslant j-1$ and $c_{ai} = 0$ otherwise. Then

$$\prod_a a^{M_a(\mathcal{I})} = \prod_{i=1}^{n-1} a_i^{e_i}, \quad e_i = \sum_{a,b} c_{ai}(m_{ab} + n_{ab})p^b. \tag{4.5}$$

The homomorphism $\gamma : T_n \to (k^*)^{n-1}$ sending t to $(a_i(t))_{i=1}^{n-1}$ is an isomorphism (remember matrices in T_n have determinant 1) - in particular it is surjective. We claim that, since k^* is cyclic of order $p^d - 1$,

$$\prod_a a^{M_a(I)} = \epsilon \quad \Longrightarrow \quad \sum_{a,b} c_{ai}(m_{ab} + n_{ab})p^b \equiv 0 \pmod{p^d - 1} \tag{4.6}$$

for each $i \in \{1, 2, \dots, n-1\}$. To prove this claim, let ζ be a cyclic generator of k^* and suppose $e_{i'} \not\equiv 0 \pmod{p^d - 1}$, where $1 \leqslant i' \leqslant n-1$. Then using surjectivity, let $t \in T_n$ be so $\gamma(t) = (x_i)_{i=1}^{n-1}$, where $x_i = \zeta$ if $i = i'$ and $x_i = 1$ otherwise. Then

$$\prod_{i=1}^{n-1} a_i^{e_i}(t) = \zeta^{e_{i'}} \neq 1 = \epsilon(t)$$

which contradicts the left hand side of the implication (4.6).

Now suppose $I = ((m_{ab}, n_{ab}))_{a,b} \in \mathcal{I}$ is such that $D(I) > 0$ and (4.4) holds. Since $D(I) > 0$ we have $m_{ab} + n_{ab} > 0$ for some a and b. Then $\sum_a c_{ai}(m_{ab} + n_{ab}) > 0$ for some b and i. We use the following lemma:

Lemma 4.4 [7, Lemma 16]**.** Suppose we have non-negative integers $(j_b)_{b=0}^{d-1}$, not all zero, then

$$\sum_{b=0}^{d-1} j_b p^b \equiv 0 \pmod{p^d - 1} \quad \Longrightarrow \quad \sum_{b=0}^{d-1} j_b \geqslant d(p-1).$$

Using implication (4.6) and applying the lemma, we must have

$$d(p-1) \leqslant \sum_{a,b} c_{ai}(m_{ab} + n_{ab}) \leqslant \sum_{a,b} (m_{ab} + 2n_{ab}) = D(I).$$

This shows that if $I \in \mathcal{I}$ satisfies (4.4), then $D(I) = 0$ or $D(I) \geqslant d(p-1)$. As remarked immediately after (4.4) this proves Theorem 4.2.

4.2 Symplectic groups

Our proof of Theorem 4.2 spanning §4.1 started by finding a Sylow p-subgroup Q of our group G (either $\mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$). We then introduced a subgroup T of G, with order prime to p, that acted upon Q via conjugation. We then broke this Sylow p-subgroup down in a sequence of group extensions. Applying Lemmas 3.1 and 3.2 to these we were able to bound the coefficients of the Poincaré series $\sigma(H_*(G))$. Using this bound, we were able to show that the ϵz^i coefficients in $\sigma(H_*(G))$ were zero for $0 < i < d(p-1)$. We now adapt this method to yield a similar statement for the symplectic groups over k.

Again, let's fix $n \in \mathbb{N}$. We write $\mathrm{Sp}_{2n}(k)$ for the group of $2n \times 2n$ symplectic matrices in k. That is,

$$\mathrm{Sp}_{2n}(k) = \{g \in \mathrm{GL}_{2n}(k) : g^\tau J g = J\}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Theorem 4.5. We have $H_i(\mathrm{Sp}_{2n}(k)) = 0$ whenever $0 < i < \frac{d(p-1)}{2}$.

As for the $\mathrm{SL}_n(k)$ and $\mathrm{GL}_n(k)$ case, we first identify a Sylow p-subgroup of $\mathrm{Sp}_{2n}(k)$. To this end let Q_n be as in §4.1, and let R_{2n} be the subset of $\mathrm{Sp}_{2n}(k)$ defined by

$$R_{2n} = \left\{ \begin{pmatrix} q & A \\ 0 & (q^\tau)^{-1} \end{pmatrix} : q \in Q_n \text{ and } (q^{-1}A)^\tau = (q^{-1}A) \right\}.$$

A simple computation tells us that R_{2n} is a subgroup of $\mathrm{Sp}_{2n}(k)$. Let T'_n be the subgroup of $\mathrm{Sp}_{2n}(k)$ defined by

$$T'_n = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathrm{GL}_n(k) \text{ is diagonal} \right\}.$$

It is often useful to identify T'_n with the subgroup of $\mathrm{GL}_n(k)$ consisting of diagonal matrices. The identification is given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \equiv t, \quad t = \mathrm{diag}(t_k)_{k=1}^n.$$

Notice that the T_n used in §4.1 is $T'_n \cap \mathrm{SL}_n(k)$. We can also check that T'_n acts on both R_{2n} and $\mathrm{Sp}_{2n}(k)$ by conjugation. We can use the results of §3 with $T = T'_n$, as $|T'_n| = (p^d-1)^n$ is prime to p.

Fix $i \geqslant 0$. The order of $\mathrm{Sp}_{2n}(k)$ is given by (for example, [9, §3.5])

$$|\mathrm{Sp}_{2n}(k)| = p^{dn^2} \prod_{j=1}^{n} (p^{2dj} - 1).$$

We claim that $|R_{2n}| = p^{dn^2}$, so that R_{2n} is a Sylow p-subgroup of $\mathrm{Sp}_{2n}(k)$. Now any element $r \in R_{2n}$ is determined uniquely by an element $q \in Q_n$ and a choice of symmetric $n \times n$ matrix B, via the bijection

$$(q, B) \longleftrightarrow \begin{pmatrix} q & qB \\ 0 & (q^\tau)^{-1} \end{pmatrix} \in R_{2n}.$$

Let $\mathrm{Sym}_n(k)$ be the abelian group of $n \times n$ symmetric matrices, whose order is $p^{\frac{dn(n+1)}{2}}$. Then $|R_{2n}| = |Q_n||\mathrm{Sym}_n(k)| = p^{dn^2}$ as required. Exactly as with the $Q_n \leqslant \mathrm{SL}_n(k)$ case, we have a surjective map $H_i(R_{2n})^{T'_n} \to H_i(\mathrm{Sp}_{2n}(k))$. We will show that $H_i(R_{2n})^{T'_n} = 0$ in a similar fashion to how we showed $H_i(Q_n)^{T'_n} = 0$. Namely, we show that the ϵz^i coefficient in the Poincaré series $\sigma(H_*(R_{2n}))$ (now with respect to $T = T'_n$) is zero for $0 < i < \frac{d(p-1)}{2}$.

In §4.1, we defined a set Δ of homomorphisms corresponding to how T_n acted on the entries of matrices in Q_n. To do a similar thing for T'_n on R_{2n}, we use the calculation

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} q & A \\ 0 & (q^\tau)^{-1} \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} t^{-1}qt & t^{-1}At^{-1} \\ 0 & t(q^\tau)^{-1}t^{-1} \end{pmatrix}. \tag{4.7}$$

In the above equation $t \in T'_n$, $q \in Q_n$ and $q^{-1}A \in \mathrm{Sym}_n(k)$ so that

$$r = \begin{pmatrix} q & A \\ 0 & (q^\tau)^{-1} \end{pmatrix}$$

is a general element of R_{2n}.

Define Δ to be the disjoint union $\Delta_1 \bigsqcup \Delta_2$, where

$$\Delta_1 = \{(k,l) : 1 \leqslant l \leqslant k \leqslant n\}, \quad \Delta_2 = \{(i,j)' : 1 \leqslant j < i \leqslant n\}.$$

With each $(i,j)' \in \Delta_2$ we identify the homomorphism $T'_n \to k^*, t \mapsto \frac{t_i}{t_j}$. This corresponds to the role of Δ in §4.1 with Q_n, except now identifying

$$Q_n \cong Q'_n = \left\{ \begin{pmatrix} q & 0 \\ 0 & (q^\tau)^{-1} \end{pmatrix} : q \in Q_n \right\} \leqslant R_{2n}.$$

Indeed if a is the homomorphism associated with $(i,j)' \in \Delta_2$, we define a group homomorphism ϕ_a by

$$\phi_a : k_a \longrightarrow R_{2n}, \quad x \longmapsto \begin{pmatrix} I_n + m_{ji}(x) & 0 \\ 0 & ((I_n + m_{ji}(x))^\tau)^{-1} \end{pmatrix}.$$

This is T'_n-invariant by calculation (4.7).

We want a similar construction for Δ_1. If $(k,l) \in \Delta_1$ and $x \in k$, we let $s_{lk}(x)$ be the $n \times n$ symmetric matrix with (l,k) and (k,l) entries equal to x and all other entries equal to zero. We then have group homomorphisms for each $(k,l) \in \Delta_1$ given by

$$\phi_{(k,l)} : k \longrightarrow R_{2n}, \quad x \longmapsto \begin{pmatrix} I_n & s_{lk}(x) \\ 0 & I_n \end{pmatrix}.$$

We want to identify (k,l) with a homomorphism $a : T'_n \to k$, such that the above map ϕ_a :

$k \equiv k_a \to R_{2n}$, is T_n'-invariant. Looking at calculation (4.7) we see that the appropriate homomorphism is $t \to \frac{1}{t_l t_k}$.

We order both Δ_1 and Δ_2 individually as in §4.1 - the first component of two pairs are compared first, and then the second. We can then put a total order on Δ by setting the elements in Δ_1 to be larger than elements in Δ_2. For each $a \in \Delta$ we define R_{2n}^a to be the subgroup of R_{2n} generated by the images of all ϕ_c with $c > a$ in Δ. Much like for Q_n and T_n, we obtain T_n'-invariant group extensions

$$1 \longrightarrow k_a \longrightarrow R_{2n}/R_{2n}^a \longrightarrow R_{2n}/R_{2n}^{a'} \longrightarrow 1,$$

for each $a \in \Delta$. Again, a' is the largest element of Δ less than a and if this is not possible we set $R_{2n}^{a'} = R_{2n}$.

For $a = (i,j)' \in \Delta_2$ these exact sequences are the same as the ones in §4.1 via the isomorphisms $R_{2n}/R_{2n}^a \cong Q_n/Q_n^a$. Of course in the Q_n case, we must restrict a to $T_n \leqslant T_n'$. We therefore don't need to check the exactness and T_n'-invariance of these sequences for $a \in \Delta_2$.

Notice that the map

$$\mathrm{Sym}_n(k) \longrightarrow R_{2n}, \quad B \longmapsto \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$$

embeds $\mathrm{Sym}_n(k)$ as an abelian subgroup of R_{2n}. This description makes it straightforward to check the exactness and T_n'-invariance of the above sequences for $a \in \Delta_1$.

Using Lemmas 3.1 and 3.2 in conjunction, these group extensions provide the estimate (see how we obtained (4.1) in §4.1 first)

$$\sigma(H_*(R_{2n})) \ll \prod_{a\in\Delta} \sigma(H_*(k_a)) = \prod_{a\in\Delta,b} (1 + a^{p^b} z) \sum_{i\geqslant 0} a^{ip^b} z^{2i}. \tag{4.8}$$

In the above, and throughout the rest of §4.2, b ranges over $\{0, 1, \ldots, d-1\}$. Again, we define a set $\mathcal{I} = \{(m_{ab}, n_{ab})_{a\in\Delta,b} \in (\{0,1\} \times \mathbb{Z}_{\geqslant 0})^{d|\Delta|}\}$. With the same justification used to obtain (4.2) and then (4.3) we find that

$$\sigma(H_*(R_{2n})) \ll \sum_{I\in\mathcal{I}} \left(\prod_{a\in\Delta} a^{M_a(I)} \right) z^{D(I)},$$

$$D(I) = \sum_{a\in\Delta,b} (m_{ab} + 2n_{ab}), \quad M_a(I) = \sum_b (m_{ab} + n_{ab}) p^b.$$

Hence to prove Theorem 4.5 we need to show that if $I \in \mathcal{I}$, then

$$\prod_{a\in\Delta} a^{M_a(I)} = \epsilon \tag{4.9}$$

implies either $D(I) = 0$ or $D(I) \geqslant \frac{d(p-1)}{2}$ (compare this to the text surrounding (4.4)).

Let $a_i \equiv (i, i+1)' \in \Delta_2 \subset \Delta$. Then if $a \equiv (j,k)' \in \Delta_2$, we have $a = \prod_{i=1}^{n-1} a_i^{c_{ai}}$ where $c_{ai} = 1$ for $k \leqslant i \leqslant j-1$ and $c_{ai} = 0$ otherwise. If instead $a \equiv (k,l) \in \Delta_1$, we have

$$a(t) = \frac{1}{t_l t_k} = \frac{1}{t_k^2}\frac{t_k}{t_l} = \frac{1}{t_n^2}\left(\frac{t_n}{t_k}\right)^2 \frac{t_k}{t_l} = a_n a_{(n,k)'}^2 a_{(k,l)'}(t)$$

where $a_n \equiv (n,n)$ and $a_{(i,j)'}$ is set to be the trivial character ϵ if $i = j$. Hence for each $a \equiv (k,l) \in \Delta_1$, we have

$$a = \prod_{i=1}^{n} a_i^{f_{ai}},$$

where $f_{an} = 1$; $f_{ai} = 2$ if $k \leqslant i \leqslant n-1$; $f_{ai} = 1$ if $l \leqslant i \leqslant k-1$; and $f_{ai} = 0$ otherwise. Equation (4.9) becomes

$$\prod_{i=1}^{n} a_i^{g_i} = \epsilon, \quad g_i = \sum_{a \in \Delta_2, b} c_{ai}(m_{ab} + n_{ab})p^b + \sum_{a \in \Delta_1, b} f_{ai}(m_{ab} + n_{ab})p^b. \tag{4.10}$$

Here, all the c_{an} for $a \in \Delta_2$ are set to be zero. Each g_i is a sum of the form

$$g_i = \sum_{a \in \Delta, b} g_{ai}(m_{ab} + n_{ab})p^b,$$

with each $g_{ai} \in \{0,1,2\}$ and, for any fixed $a \in \Delta$, the $(g_{ai})_{i=1}^n$ are not all zero. We appeal to the surjectivity of the group homomorphism

$$\gamma' : T_n' \longrightarrow (k^*)^{n-1} \times k^{2*}, \quad t \longmapsto (a_i(t))_{i=1}^n \tag{4.11}$$

to show that (4.10) implies (by the same justification as for implication (4.6))

$$g_i \equiv 0 \pmod{p^d - 1} \text{ if } 1 \leqslant i \leqslant n-1, \quad g_n \equiv 0 \quad \left(\bmod\ \frac{p^d-1}{2*}\right), \tag{4.12}$$

where $2* = 1$ if $p = 2$; $2* = 2$ if $p \neq 2$. This is because k^{2*} is cyclic of order $\frac{p^d-1}{2*}$.

Now suppose $I = (m_{ab}, n_{ab})_{a \in \Delta, b} \in \mathcal{I}$ is so that $D(I) > 0$ and (4.9) holds. One possibility is that $m_{ab} + n_{ab} > 0$ for some $a \neq a_n$ and some b. Then $\sum_{a \in \Delta} g_{ai}(m_{ab} + n_{ab}) > 0$ for some b and some i with $1 \leqslant i \leqslant n-1$. We then apply Lemma 4.4 for this g_i to write

$$d(p-1) \leqslant \sum_{a,b} g_{ai}(m_{ab} + n_{ab}) \leqslant \sum_{a,b} 2(m_{ab} + 2n_{ab}) = 2D(I).$$

The other possibility is that $m_{ab} = n_{ab} = 0$ for each $a \neq a_n$ and each b, but $m_{a_n b} + n_{a_n b} > 0$

for some b. We then have (multiplying the g_n congruence in (4.12) by $2*$)

$$(2*)g_n = (2*)M_{a_n}(I) = \sum_b (2*)(m_{a_nb} + n_{a_nb})p^b \equiv 0 \pmod{p^d - 1}.$$

Applying Lemma 4.4 to this equation, we must have

$$d(p-1) \leqslant \sum_b (2*)(m_{a_nb} + n_{a_nb}) \leqslant \sum_{a\in\Delta,b} 2(m_{ab} + 2n_{ab}) = 2D(I).$$

In any case we have shown that if (4.9) holds for some $I \in \mathcal{I}$ then either $D(I) = 0$ or $D(I) \geqslant \frac{d(p-1)}{2}$. As remarked on the line below (4.9) this proves Theorem 4.5.

4.3 Orthogonal groups of even degree in odd characteristic

We first looked at the vanishing low dimensional homology of the groups $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$, strengthening a theorem proved by Quillen. Next, we adapted this method to yield analogous results for the groups $\mathrm{Sp}_{2n}(k)$. This in turn generalises to the orthogonal groups. We will only consider the case $p \neq 2$ here.

Fix $n \in \mathbb{N}$. The orthogonal group $\mathrm{O}_{2n}(k)$, over k is defined by

$$\mathrm{O}_{2n}(k) = \{g \in \mathrm{GL}_{2n}(k) : g^\tau J g = J\}, \quad J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The special orthogonal group $\mathrm{SO}_{2n}(k)$, is then the normal subgroup of $\mathrm{O}_{2n}(k)$ consisting of such matrices with determinant 1, i.e. $\mathrm{SO}_{2n}(k) = \mathrm{O}_{2n}(k) \cap \mathrm{SL}_{2n}(k)$.

Theorem 4.6. We have $H_i(\mathrm{O}_{2n}(k)) = 0 = H_i(\mathrm{SO}_{2n}(k))$ whenever $0 < i < \frac{d(p-1)}{2}$.

Let $Q_n \leqslant \mathrm{GL}_n(k)$ be as defined in §4.1 and define R_{2n} now to be the subset of $\mathrm{SO}_{2n}(k)$ (and $\mathrm{O}_{2n}(k)$) defined by

$$R_{2n} = \left\{ \begin{pmatrix} q & A \\ 0 & (q^\tau)^{-1} \end{pmatrix} : q \in Q_n \text{ and } (q^{-1}A)^\tau = -(q^{-1}A) \right\}.$$

Note the similarity with the R_{2n} defined in §4.2 - the matrix $q^{-1}A$ is now skew-symmetric, rather than symmetric. Again, R_{2n} is easily checked to be a subgroup of $\mathrm{SO}_{2n}(k)$. We now show that it is a Sylow p-subgroup of both $\mathrm{SO}_{2n}(k)$ and $\mathrm{O}_{2n}(k)$. It is known that ([9, §3.7.2])

$$|\mathrm{O}_{2n}(k)| = 2p^{dn(n-1)}(p^{dn} - 1)\prod_{j=1}^{n-1}(p^{2dj} - 1), \quad |\mathrm{SO}_{2n}(k)| = \frac{1}{2}|\mathrm{O}_{2n}(k)|.$$

Much like each $r \in R_{2n}$ in §4.2 was determined uniquely by an element of Q_n and a symmetric $n \times n$ matrix, an element of R_{2n} here is uniquely determined by an element of Q_n and a skew-symmetric $n \times n$ matrix. This implies that $|R_{2n}| = |Q_n| p^{\frac{dn(n-1)}{2}} = p^{dn(n-1)}$ and we can see that R_{2n} is indeed a Sylow p-subgroup of both $\mathrm{SO}_{2n}(k)$ and $\mathrm{O}_{2n}(k)$.

The same T'_n used in §4.2 is a subgroup of $\mathrm{SO}_{2n}(k)$ acting on R_{2n}, $\mathrm{SO}_{2n}(k)$ and $\mathrm{O}_{2n}(k)$ by conjugation. By similar arguements to the $\mathrm{SL}_n(k)$ and $\mathrm{Sp}_{2n}(k)$ cases, to prove Theorem 4.6 it suffices to show that $H_i(R_{2n})^{T'_n} = 0$ whenever $0 < i < \frac{d(p-1)}{2}$.

We define $\Delta = \Delta_1 \bigsqcup \Delta_2$, where

$$\Delta_1 = \{(k,l) : 1 \leqslant l < k \leqslant n\}, \quad \Delta_2 = \{(i,j)' : 1 \leqslant j < i \leqslant n\}.$$

In fact Δ_2 here plays an identical role to the Δ_2 in §4.2. Indeed we identify each $(i,j)' \in \Delta_2$ with the homomorphism $T'_n \to k^*, t \mapsto \frac{t_i}{t_j}$. If $a \equiv (i,j)' \in \Delta_2$ we define a T'_n-invariant (calculation (4.7) still holds) homomorphism ϕ_a by

$$\phi_a : k_a \longrightarrow R_{2n}, \quad x \longmapsto \begin{pmatrix} I_n + m_{ji}(x) & 0 \\ 0 & ((I_n + m_{ij}(x))^\tau)^{-1} \end{pmatrix}.$$

If $(k,l) \in \Delta_1$ and $x \in k$, we let $s_{lk}(x)$ be the $n \times n$ matrix with (l,k)-entry equal to x, (k,l)-entry equal to $-x$, and all other entries equal to zero. We also associate $(k,l) \in \Delta_1$ with the homomorphism $T'_n \to k^*, t \mapsto \frac{1}{t_l t_k}$. Then for $a \equiv (k,l) \in \Delta_1$ we define a T'_n-invariant homomorphism ϕ_a by

$$k_a \longrightarrow R_{2n}, \quad x \longmapsto \begin{pmatrix} I_n & s_{lk}(x) \\ 0 & I_n \end{pmatrix}.$$

We order Δ as in §4.2, with elements of Δ_1 larger than those in Δ_2. The individual sets Δ_1 and Δ_2 are each ordered by comparing first components first, then the second components. For each $a \in \Delta$ we define R^a_{2n} to be the subgroup of R_{2n} generated by the images of all the ϕ_c with $c > a$ in Δ. We get T'_n-invariant group extensions

$$1 \longrightarrow k_a \longrightarrow R_{2n}/R^a_{2n} \longrightarrow R_{2n}/R^{a'}_{2n} \longrightarrow 1,$$

for each $a \in \Delta$. As usual, a' is the largest element of Δ less than a and we set $R^{a'}_{2n} = R_{2n}$ if this is not possible. Applying Lemmas 3.1 and 3.2 to these extensions we get the estimate

$$\sigma(H_*(R_{2n})) \ll \prod_{a\in\Delta} \sigma(H_*(k_a)) = \prod_{a\in\Delta,b} (1 + a^{p^b} z) \sum_{i\geqslant 0} a^{ip^b} z^{2i}.$$

The rest of the proof is identical to the text following (4.8). The surjective homomorphism γ' defined by (4.11) implies the same modular congruences. The only seeming difference is that the homomorphism $a_n : t \mapsto \frac{1}{t_n^2}$ is not in Δ anymore, but this doesn't matter - the proof still applies.

Now that we have proved Theorem 4.6, we introduce the index 2 subgroup $\Omega_{2n}(k)$ of

$\mathrm{SO}_{2n}(k)$. A more precise definition is given by [9, §3.7 & §3.9]. Let V be the vector space k^{2n} and define a symmetric bilinear form $\langle \cdot, \cdot \rangle : V \times V \to k$, by $\langle u, v \rangle = u^\tau J v$, where J is the symmetric matrix in the definition of $\mathrm{O}_{2n}(k)$ at the start of this section. If $v \in V$, we call $N(v) = \langle v, v \rangle$ the norm of v. Two vectors $u, v \in V$ are called orthogonal to each other if $\langle u, v \rangle = 0$.

A reflection r_u, of V in the plane orthogonal to a vector u with $N(u) \neq 0$, is characterised by sending u to $-u$ and fixing all vectors orthogonal to u. In other words it can be defined by the formula

$$r_u : v \longmapsto v - 2\frac{\langle v, u \rangle}{\langle u, u \rangle} u.$$

There are two types of these reflections - either $N(u)$ is a square in k, so $N(u) \in k^{2*}$, or $N(u)$ is not a square in k. Note that if $\lambda \neq 0$ in k, then $r_{\lambda u} = r_u$. We then have $N(\lambda u) = \lambda^2 N(u)$, thus we have a well-defined map

$$\{r_u : N(u) \neq 0\} \longrightarrow k^*/k^{2*}, \quad r_u \longmapsto [N(u)]$$

It is known that $\mathrm{O}_{2n}(k)$ is generated by reflections r_u with $N(u) \neq 0$. Since a reflection matrix has determinant -1, a general element of $\mathrm{SO}_{2n}(k)$ is the product of an even number of these reflections. Now if $g \in \mathrm{SO}_{2n}(k)$, we can write $g = r_{u_1} r_{u_2} \dots r_{u_{2t}}$ for some reflections r_{u_i}, with each $N(u_i) \neq 0$. We can then define $\theta(g) = [N(u_1)N(u_2)\dots N(u_{2t})] \in k^*/k^{2*}$. The map $\theta : \mathrm{SO}_{2n}(k) \to k^*/k^{2*}$ is a well-defined surjective group homomorphism called the spinor norm. Its kernel is an index 2 subgroup of $\mathrm{SO}_{2n}(k)$ denoted by $\Omega_{2n}(k)$. We can now look to prove the following.

Theorem 4.7. We have $H_i(\Omega_{2n}(k)) = 0$ whenever $0 < i < \frac{d(p-1)}{4}$.

We already know that R_{2n} is a Sylow p-subgroup of $\mathrm{SO}_{2n}(k)$. We claim it is also a Sylow p-subgroup of $\Omega_{2n}(k)$. This is a particular case of the next proposition with $G = \mathrm{SO}_{2n}(k)$, $N = \Omega_{2n}(k)$ and $U = R_{2n}$.

Proposition 4.8. Let G be a group and U a Sylow p-subgroup. Let N be a normal subgroup of G such that $p \nmid [G : N]$. Then $U \leqslant N$. In particular U is a Sylow p-subgroup of N.

Proof. Let $\pi : G \to G/N$ be the quotient map, and denote by ϕ its restriction to U. By the first isomorphism theorem $|U| = |\ker(\phi)||\mathrm{im}(\phi)|$. Since $\mathrm{im}(\phi) \leqslant G/N$, and by assumption $p \nmid G/N$, we must have $p \nmid |\mathrm{im}(\phi)|$. Since $|U|$ is just a power of p, the above equation, by primality of p, now implies that $|\ker(\phi)| = |U|$, and thus $\ker(\phi) = U$. Of course, $\ker(\phi)$ is $U \cap N$, so $U \leqslant N$ as required. ☐

Since R_{2n} is a Sylow p-subgroup of $\Omega_{2n}(k)$, we have, for each $i \geqslant 0$, a surjective map $H_i(R_{2n}) \to H_i(\Omega_{2n}(k))$. We can't just take fixed points under the action of T'_n as in previous cases however, because we don't know whether T'_n is a subgroup of $\Omega_{2n}(k)$. In fact, we have the following:

Proposition 4.9. An element $t \equiv \mathrm{diag}(t_j)_{j=1}^n$ in T_n' lies in $\Omega_{2n}(k)$ if and only if $\prod_{j=1}^n t_j \in k^{2*}$.

Proof (sketch). Let $\{\mathbf{e}_j\}_{j=1}^{2n}$ be the standard basis of $V = k^{2n}$. Write

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = JA, \quad A = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}.$$

Consider a new basis $\{\mathbf{f}_j\}_{j=1}^{2n}$ of V given by

$$\mathbf{f}_j = \begin{cases} \mathbf{e}_j - t_j \mathbf{e}_{j+n} & \text{if } j \leqslant n \\ \mathbf{e}_j + t_j \mathbf{e}_{j+n} & \text{if } j \geqslant n+1 \end{cases}.$$

One can check the $\mathbf{f}_j$ form an orthogonal basis with respect to $\langle \cdot, \cdot \rangle$, in the sense that $\langle \mathbf{f}_j, \mathbf{f}_{j'} \rangle$ whenever $j \neq j'$. Now for $j \leqslant n$, we have $N(\mathbf{f}_j) = -2t_j$ and $A\mathbf{f}_j = -\mathbf{f}_j$. On the other hand, for $j \geqslant n+1$ we have $A\mathbf{f}_j = \mathbf{f}_j$. This implies that $A = r_{\mathbf{f}_1} r_{\mathbf{f}_2} \ldots r_{\mathbf{f}_n}$ as a product of reflections. Now J is a special case of A with $t = I_n$, so $J = r_{\mathbf{f}_1'} r_{\mathbf{f}_2'} \ldots r_{\mathbf{f}_n'}$, where $\mathbf{f}_j'$ is given by $\mathbf{e}_j - \mathbf{e}_{j+n}$ for each $j \leqslant n$. Notice each $N(\mathbf{f}_j')$ is equal to -2. By definition we have

$$\theta(t) = \left[\prod_{j=1}^n N(\mathbf{f}_j') \prod_{j=1}^n N(\mathbf{f}_j) \right] = \left[(-2)^{2n} \prod_{j=1}^n t_j \right] = \left[\prod_{j=1}^n t_j \right],$$

giving the proposition. □

Let S_n' be the intersection of T_n' with $\Omega_{2n}(k)$. Now we follow the proof for $H_i(R_{2n})^{T_n'} = 0$, except with T_n' replaced by S_n'. Since S_n' is an index 2 subgroup of T_n', the results in §3 for $T = S_n'$ are valid. The homomorphisms in Δ are replaced by their restrictions to S_n'. The only issue we find is that $\gamma'' = \gamma'|_{S_n'}$ is not surjective.

We claim the image of γ'' contains the set

$$\{(x_1, x_2, \ldots, x_{n-1}, y^2) \in (k^*)^{n-1} \times k^{2*} : x_1 x_2^2 \ldots x_{n-1}^{n-1} y^n \in k^{2*}\}.$$

Indeed suppose $(x_1, x_2, \ldots, x_{n-1}, y^2)$ lies in this set. Then

$$\gamma'' \left(\frac{1}{x_1 x_2 \ldots x_{n-1} y}, \frac{1}{x_2 \ldots x_{n-1} y}, \ldots, \frac{1}{x_{n-1} y}, \frac{1}{y} \right) = (x_1, x_2, \ldots, x_{n-1}, y^2).$$

This means that, for each j with $1 \leqslant j \leqslant n$, the element $(x_j')_{j'=1}^n$ with $x_j = \zeta^2$ and $x_{j'} = 1$ if $j' \neq j$ is in the image of γ''. Here, ζ is a cyclic generator of k^*, so that ζ^2 has order $\frac{p^d-1}{2}$ in k^*. Instead of (4.12) we get the congruences

$$2g_i \equiv 0 \pmod{p^d - 1} \text{ if } 1 \leqslant i \leqslant n-1, \quad 2g_n \equiv 0 \quad \left(\text{mod } \frac{p^d-1}{2*} \right).$$

The rest of the method is the same as before, except the extra factor of 2 means we get $H_i(\Omega_{2n}(k)) = 0$ only for the interval $0 < i < \frac{d(p-1)}{4}$. We have thus proven Theorem 4.7.

5 Homology of Some Finite Simple Groups

In this section we use our results from §4 to prove vanishing theorems for the low dimensional homology of some finite simple groups. All the results and methods in this section are my own, although we use statements from [9, §3].

Recall that a group G is a said to be simple if for any normal subgroup N of G we have $N = 1$ or $N = G$. We also use the conventional notation $Z(G)$ for the centre of a group G. That is, for any group G,

$$Z(G) = \{g \in G : gh = hg \text{ for any } h \in G\} \leqslant G.$$

Once more k is the finite field with p^d elements for some $d \geqslant 1$.

5.1 Finite simple groups obtained from classical groups

We have seen some important examples of classical groups over k. We further explore the special linear, symplectic and the index 2 subgroup of the special orthogonal groups. In general these are not simple, but almost all of them are 'nearly' simple in the sense that a relatively large quotient is simple. Let us now introduce these simple quotients.

Fix $n \in \mathbb{N}$ and consider the special linear group $\mathrm{SL}_n(k)$. The centre of $\mathrm{SL}_n(k)$ consists of the scalar matrices in $\mathrm{SL}_n(k)$ - as a set we have

$$Z(\mathrm{SL}_n(k)) = \{\lambda I_n : \lambda^n = 1\}.$$

The projective special linear group of degree n, $\mathrm{PSL}_n(k)$, is then the quotient of $\mathrm{SL}_n(k)$ by its centre, i.e $\mathrm{SL}_n(k)/Z(\mathrm{SL}_n(k))$. To work out its order we need to know the order of $Z(\mathrm{SL}_n(k))$. The expression above tells us this is equal to the number of $\lambda \in k^*$ satisfying $x^n = 1$. Since k^* is cyclic with order $p^d - 1$ this number is given by $\gcd(p^d - 1, n)$. The order of $\mathrm{PSL}_n(k)$ is then given by

$$|\mathrm{PSL}_n(k)| = \frac{1}{\gcd(p^d - 1, n)} |\mathrm{SL}_n(k)|.$$

Theorem 5.1 [9, §3.3.2]**.** The groups $\mathrm{PSL}_n(k)$ are simple whenever $n > 2$ or $p^d > 3$.

We now turn to the symplectic groups $\mathrm{Sp}_{2n}(k)$. It is known that the centre $Z(\mathrm{Sp}_{2n}(k))$, of $\mathrm{Sp}_{2n}(k)$ is formed by just the two matrices I_n and $-I_n$, and hence has size 1 or 2 - the former case only when $p = 2$. The projective symplectic group $\mathrm{PSp}_{2n}(k)$ is the quotient $\mathrm{Sp}_{2n}(k)/Z(\mathrm{Sp}_{2n}(k))$. Its order is given by

$$|\mathrm{PSp}_{2n}(k)| = \frac{1}{\gcd(p,2)}|\mathrm{Sp}_{2n}(k)|.$$

Theorem 5.2 [9, §3.5.2]**.** The groups $\mathrm{PSp}_{2n}(k)$ are simple apart from the cases $\mathrm{PSp}_2(\mathbb{F}_2)$, $\mathrm{PSp}_2(\mathbb{F}_3)$ and $\mathrm{PSp}_4(\mathbb{F}_2)$.

Finally, we do the same for the groups $\Omega_{2n}(k)$ defined in §4.3. Here, as in §4.3, we only consider the case $p \neq 2$. The groups $\mathrm{P}\Omega_{2n}(k)$ are defined to be the quotients $\Omega_{2n}(k)/Z(\Omega_{2n}(k))$. The order of $\mathrm{P}\Omega_{2n}(k)$ is not as simple to calculate as the other two cases. In [2] via §8.6, Theorems 9.4.10 and 11.3.2, it is shown that

$$|\mathrm{P}\Omega_{2n}(k)| = \frac{1}{\gcd(p^d-1,2)^2}|\Omega_{2n}(k)|.$$

Theorem 5.3 [9, §3.7.3]**.** The groups $\mathrm{P}\Omega_{2n}(k)$ are simple whenever $n > 2$.

5.2 The homology of these simple groups.

In §5.1, we introduced three classes of finite simple groups. We are almost ready to explore the low dimensional homology of these groups - the end goal of this dissertation. Beforehand, we need one final proposition.

Proposition 5.4. Let G be a group and U a Sylow p-subgroup. Let N be a normal subgroup of G with $p \nmid N$. Let $\pi : G \to G/N$ be the natural map, then $\pi(U)$ is a Sylow p-subgroup isomorphic to U.

Proof. Let $\phi : U \to G/N$ be the restriction of the natural map to U. Its kernel $K = U \cap N$ is both a subgroup of U and a subgroup of N. The order of K must then divide both that of U and that of N. By assumption the orders of U and N are coprime, so $|K| = 1$ and K is trivial. By the first isomorphism theorem for groups, U is an isomorphism onto its image $\pi(U)$. The order of G/N divides that of G, so if U is a Sylow p-subgroup of G, then $|\pi(U)| = |U|$ implies that $\pi(U)$ is a Sylow p-subgroup of G/N. □

Let's start with the special linear case first, i.e. the groups $\mathrm{PSL}_n(k)$. The symplectic and orthogonal cases are very similar.

Theorem 5.5. We have $H_i(\mathrm{PSL}_n(k)) = 0$ whenever $0 < i < \frac{d(p-1)}{2}$.

Proof. Recall that Q_n is the Sylow p-subgroup of $\mathrm{SL}_n(k)$ consisting of upper triangular matrices with 1s on the diagonal. If $\pi : \mathrm{SL}_n(k) \to \mathrm{PSL}_n(k)$ is the quotient homomorphism, then Proposition 5.4 and the formula for $|\mathrm{PSL}_n(k)|$ in §5.1 tells us that $\pi(Q_n)$ is a Sylow p-subgroup of $\mathrm{PSL}_n(k)$ isomorphic to Q_n via π. In §4.1 we had the diagonal subgroup T_n acting on Q_n and $\mathrm{SL}_n(k)$ via conjugation. In the same way $\pi(T_n)$ acts on $\pi(Q_n)$ and

$\mathrm{PSL}_n(k)$ via conjugation. For each $t \in T_n$ we get a commutative diagram

$$\begin{tikzcd} Q_n \arrow[r, "t"] \arrow[d, "\pi"', "\cong"] & Q_n \arrow[d, "\cong"', "\pi"] \\ \pi(Q_n) \arrow[r, "\pi(t)"'] & \pi(Q_n) \end{tikzcd} \tag{5.1}$$

.

Here, $t : Q_n \to Q_n$ is the homomorphism $t(u) = t \cdot u = t^{-1}ut$, and the homomorphism $\pi(t)$, is defined similarly. The commutativity of (5.1) and the functoriality of group homology (recall §2.4) yields the commutative diagrams

$$\begin{tikzcd} H_i(Q_n) \arrow[r, "t_*"] \arrow[d, "\pi_*"', "\cong"] & H_i(Q_n) \arrow[d, "\cong"', "\pi_*"] \\ H_i(\pi(Q_n)) \arrow[r, "\pi(t)_*"'] & H_i(\pi(Q_n)) \end{tikzcd} \tag{5.2}$$

,

for each $i \geqslant 0$. Of course, π_* is an isomorphism since π is. The commutativity of these diagrams imply that π_* induces isomorphisms on the fixed point subgroups of homology, i.e. $H_i(Q_n)^{T_n} \cong H_i(\pi(Q_n))^{\pi(T_n)}$ for each $i \geqslant 0$. Using Theorem 4.2 we must have $H_i(\pi(Q_n))^{\pi(T_n)} = 0$ for $0 < i < d(p-1)$. Since $\pi(Q_n)$ is a Sylow p-subgroup of $\mathrm{PSL}_n(k)$ and $\pi(T_n)$ is a subgroup of $\mathrm{PSL}_n(k)$, the map

$$H_i(\pi(Q_n))^{\pi(T_n)} \longrightarrow H_i(\mathrm{PSL}_n(k))^{\pi(T_n)} = H_i(\mathrm{PSL}_n(k))$$

is surjective for each $i \geqslant 0$. We must therefore have $H_i(\mathrm{PSL}_n(k)) = 0$ whenever $0 < i < d(p-1)$. $\square$

The method used to prove Theorem 5.5 adapts easily to $\mathrm{PSp}_{2n}(k)$ and $\mathrm{P\Omega}_{2n}(k)$. For $\mathrm{PSp}_{2n}(k)$, replace the groups Q_n and T_n in the above with the groups R_{2n} and T_n' used in §4.2 respectively. We get analogous versions of the commutative squares (5.1) and (5.2). The order formula in §5.1 implies that R_{2n} projects to a Sylow p-subgroup of $\mathrm{PSL}_n(k)$, and thus we obtain the following:

Theorem 5.6. We have $H_i(\mathrm{PSp}_{2n}(k)) = 0$ whenever $0 < i < \frac{d(p-1)}{2}$.

For $\mathrm{P\Omega}_{2n}(k)$, instead replace the groups Q_n and T_n in the proof of Theorem 5.5 with R_{2n} and S_n' as defined in §4.3 respectively. By similar arguements we obtain:

Theorem 5.7. We have $H_i(\mathrm{P\Omega}_{2n}(k)) = 0$ whenever $0 < i < \frac{d(p-1)}{4}$.

6 Final Remarks

The first part of this essay was spent defining mod p group homology via projective resolutions of $\mathbb{F}_p$ over $\mathbb{F}_p[G]$. One may have wondered why mod p homology was the natural homology theory to use. The convenience of the $\mathbb{F}_p$ coefficients was that a Sylow p-subgroup U, of a group G, induced surjective maps $H_i(U) \to H_i(G)$, on mod p homology. Now the groups we looked at naturally had significant Sylow p-subgroups that were easier to write down explicitly than the groups themselves, using coefficients in $\mathbb{F}_p$ allowed us to restrict to exploring just these subgroups.

The results in §4 can be thought of as an investigation into the remark concluding §11 in [7]. Here Quillen states that the mod p cohomology of certain linear algebraic groups (matrix groups defined by polynomials in the entries) over finite fields of characteristic p vanishes in dimensions i with $0 < i < Cd$, for some constant C. Our results explicitly give such vanishing ranges in the homology setting (rather than cohomology) for the special linear, symplectic and orthogonal groups (of plus type in odd characteristic).

In §5, we showed that the vanishing ranges for the homology of the classical groups in §4 naturally project to vanishing ranges for the closely related finite simple groups. How expected are these results? In §2.3 we gave an explicit description of $H_1(G)$, where G is a group, in terms of the abelianisation $G_{ab} = G/[G, G]$, of G. Since $[G, G]$ is a normal subgroup of G, and trivial if and only if G is abelian, it is clear that $H_1(G) = 0$ whenever G is simple and non-abelian. For $i > 1$, however, simplicity of G does not give any reason for $H_i(G)$ to vanish. For example if $p = 2$ and k is the field of four elements, then $\mathrm{PSL}_2(k)$ is simple (Theorem 5.1) and isomorphic to the alternating group on five elements A_5. It is known that $H_2(A_5) \cong \mathbb{Z}/2\mathbb{Z}$, so in particular $H_2(\mathrm{PSL}_2(k)) \neq 0$. The vanishing ranges of our simple groups seem only a consequence of the vanishing ranges for the corresponding classical groups, and not a consequence of simplicity itself.

As mentioned earlier, the groups $\mathrm{PSL}_n(k)$, $\mathrm{PSp}_{2n}(k)$ and $\mathrm{P}\Omega_{2n}(k)$ nearly give a complete dictionary of all the non-exceptional Chevalley groups. The only such groups not considered here are the groups $\mathrm{P}\Omega_{2n+1}(k)$ and the groups $\mathrm{P}\Omega_{2n}(k)$ in even characteristic. The definition of the orthogonal groups in even characteristic is much more complex than in odd characteristic ([9, §3.4.7 & §3.8]), however the group $\mathrm{P}\Omega_{2n+1}(k)$, $n \in \mathbb{N}$, in even characteristic is isomorphic to the group $\mathrm{PSp}_{2n}(k)$ [2, §1.6], so in this case we refer to Theorem 5.7. One would expect similar vanishing statements for the homology of the $\mathrm{P}\Omega_{2n}(k)$ in even characteristic and the $\mathrm{P}\Omega_{2n+1}(k)$ in odd characteristic due to Quillen's remark in the second paragraph of this section. Sylow p subgroups of these last two cases are harder to write down explicitly, hence their omission from this work.

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