ON THE WITT GROUPS OF SCHEMES

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Abstract

We consider two questions about the Witt groups of schemes: the first is the question of finite generation of the shifted Witt groups of a smooth variety over a finite field; the second is the Gersten conjecture. Regarding the first, we prove that the shifted Witt groups of curves and surfaces are finite, and that finite generation of the motivic cohomology groups with mod 2 coefficients implies finite generation of the Witt groups. Regarding the second, for a discrete valuation ring Λ having an infinite residue field $\Lambda/\mathfrak{m}$, we prove the Gersten conjecture for the Witt groups in the case of a local ring that is essentially smooth over Λ, and deduce from this the case of a local ring A that is regular over Λ (*i.e.* there is a regular morphism from Λ to A).

Introduction

In Section 1.1 we explain where the mathematics in this thesis sits within the wider setting of mathematics. We then introduce the main questions that are addressed in this thesis, what is known about them, and the motivation for studying them in Section 1.2. This is followed by a detailed account of our results and methods given in a synopsis of each chapter.

1.1   Background

Over the past century, the construction and subsequent study of cohomology theories was an important development in mathematics. Cohomology theories take as input a topological space, and output an object consisting of classes equipped with an addition and a multiplication. By understanding the algebraic relations that these classes satisfy, the topological spaces can be better understood. Important examples of cohomology theories are singular cohomology, topological complex K-theory, and topological real K-theory.

However, it was only in the last two decades that mathematicians understood completely how to develop analogous cohomology theories which take as input algebraic varieties. In 2002, a fields medal was awarded to V. Voevodsky in part for his work developing motivic cohomology, which is the algebraic analogue of singular cohomology. Recently, M. Schlichting has developed the Grothendieck-Witt (*aka* hermitian K-theory) groups $GW_m(X)$ of an algebraic variety X [60]. These are abelian groups which form a bigraded cohomology theory $GW_m^n(X)$ for schemes which generalizes Knebusch's Grothendieck-Witt group $L(X)$ of a scheme X [47, Chapter 1 §4] with $L(X) \simeq GW_0^0(X)$ [60, Proposition 4.11]. They are the

algebraic analogue of real topological K-theory in the same way that algebraic K-theory is the algebraic analogue of complex topological K-theory.

Another example of a cohomology theory for algebraic varieties is the theory of Witt groups, which are closely related to the Grothendieck-Witt groups. The shifted (*aka* derived) Witt groups $W^i(X)$ were introduced by P. Balmer about a decade ago [5]. They are abelian groups which form a cohomology theory for algebraic varieties, are periodic $W^i(X) \cong W^{i+4}(X)$ of period 4, and they agree with the Grothendieck-Witt groups in negative degrees, $GW_{-i}(X) \cong W^i(X)$ for $i > 0$. Both theories are closely tied to quadratic forms. In particular, when k is a field having characteristic different from two, and $\mathrm{Spec}(k)$ denotes the variety defined by k, then $GW_0(\mathrm{Spec}(k))$ is the Grothendieck-group of the abelian monoid of isometry classes of quadratic forms over k and $W^0(\mathrm{Spec}(k))$ is the classical Witt group $W(k)$ first introduced by E. Witt in the thirties.

1.2 Motivation and Principal Results

In many respects, the Witt and Grothendieck-Witt groups follow a development very similar to algebraic K-theory, however, algebraic K-theory has been around for far longer and for this and other reasons has been studied considerably more. As a result, a major goal is to understand the Witt and Grothendieck-Witt groups as well as algebraic K-theory is understood. This thesis contributes to this goal by proving Witt and Grothendieck-Witt analogues of important theorems that are known for algebraic K-theory.

The first question we consider in this thesis is that of finite generation (*i.e.*, finite generation as an abelian group) of the shifted Witt groups $W^n(X)$ of a smooth variety X over a finite field of characteristic different from 2. This amounts to the question of finiteness since in this case the Witt groups $W^n(X)$ are known to be torsion groups (*e.g.*, see Corollary 2.23). Regarding what was known about

this question, the most important result was a theorem of J. Arason, R. Elman, and B. Jacob which states that, when X is a complete regular curve over a finite field of characteristic different from 2, the Witt group $W^0(X)$ is a finite group [4, Theorem 3.6]. For smooth varieties over finite fields, little is known in general about the shifted Witt groups so certainly one motivation for studying this finiteness question is simply to have a better understanding of them. Another motivation relates to the Grothendieck-Witt groups of schemes [60]. Before introducing it, let X be a regular finite type $\mathbb{Z}$-scheme. Recall that the Bass conjecture states that the higher algebraic K-groups $K_m(X)$ of X are finitely generated as abelian groups [42, §4.7.1 Conjecture 36]. There are two main results on this conjecture:

(a) When $\dim(X) \leq 1$, Quillen proved the conjecture [42, §4.7.1 Proposition 38 (b)];

(b) The "motivic" Bass conjecture, that is, finite generation of the motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}(n))$ [42, See §4.7.1 Conjecture 37], implies the Bass conjecture. This follows from the Atiyah-Hirzebruch spectral sequence [42, §4.3.2 Equation (4.6) and the final paragraph of §4.6].

   The second motivation for studying the finiteness question was to attempt to reproduce for the Grothendieck-Witt groups the two results above about K-theory. Regarding the Grothendieck-Witt analogue of (a), finite generation of the Grothendieck-Witt groups was known to follow (*e.g.* Karoubi induction [13, Proposition 3.5]) from finiteness of the shifted Witt groups and finite generation of the higher algebraic K-groups. So a corollary of the finiteness result for Witt groups is a finite generation result for the Grothendieck-Witt groups of curves over finite fields. Similarly, knowing that the "motivic" Bass conjecture implies finite generation of the Witt groups, then one obtains the analogue of (b). Up to the condition

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that we must assume that no residue field of X is formally real, we are successful in obtaining (a) and (b). We prove here that when X is smooth over a finite field of characteristic different from 2 and $\dim X \leq 2$, then the shifted Witt groups $W^n(X)$ are finite (see Theorem 3.33). In higher dimensions, we give conditional results demonstrating that finiteness of the Witt groups follows from finiteness of motivic cohomology with mod 2 coefficients, as well as the converse statement in low dimensions. We also consider the case of smooth schemes of finite type over the integers. These results appear in Chapter 2.4, see the synopsis of Chapter 2.4 below for a more detailed account.

The second question we consider in this thesis is called the Gersten conjecture for the Witt groups of local rings essentially smooth over a discrete valuation ring (DVR). To introduce it, let A be a regular local ring and let K denote the fraction field of A. It is a classic question to ask if $W(A) \to W(K)$ is injective. Although this question has been studied a lot, the answer is still not known for any regular local ring. This map carries on to the right as the first map in a complex of abelian groups

$$0 \to W(A) \to W(K) \to \bigoplus_{\mathrm{ht}\mathfrak{p}=1} W(k(\mathfrak{p})) \to \bigoplus_{\mathrm{ht}\mathfrak{p}=2} W(k(\mathfrak{p})) \to \cdots \to \bigoplus_{\mathrm{ht}\mathfrak{p}=d} W(k(\mathfrak{p})) \to 0$$

$$(1.1)$$

called the *Gersten complex for the Witt groups.* The Gersten complex 1.1 will be constructed in Chapter 1.6, Section 2.3. The Gersten conjecture for the Witt groups asserts that this complex is exact for any regular local ring A, in particular $W(A)$ injects into $W(K)$.

Considering what was known about this conjecture, the story starts in 1982 when a Gersten complex for the Witt group was first introduced by W. Pardon [56]. He conjectured that his Gersten complex is exact when A is any regular local ring.

At that time, the Witt group $W(A)$ was expected to admit a development into a cohomology theory $W^i(A)$ by following essentially the same lines as Quillen's development of higher K-theory. It was expected that it should be possible to prove the Gersten conjecture for the Witt groups by following Quillen's strategy that he used to prove the Gersten conjecture for K-theory in the case of local rings essentially smooth over a field.

However, it wasn't until the last decade–with Balmer's theory of triangulated Witt groups–that the Witt group indeed became a part of a cohomology theory W^i, and it became possible to construct, in essentially the same manner as for K-theory, *the* Gersten complex for the Witt groups. It is this complex, and not Pardon's complex, that is subject of this thesis. Although Pardon's complex is identical in appearance to the Gersten complex 1.1, it does not seem to be known if the differentials agree [30, second to last paragraph of introduction]. In 2005, in [31, Theorem 3.1] J. Hornbostel and S. Gille succeeded in adapting Quillen's strategy to prove the Gersten conjecture for the Witt groups of a local ring essentially smooth over a field (proofs by other means appeared earlier, c.f. [8]). In [10], it was proved for equicharacteristic regular local rings (or it is the same to say, regular local rings which contain a field) using an argument of I.A. Panin to deduce it from the essentially smooth case. The mixed characteristic case, that is, regular local rings which do not contain a field, remains open except in low dimensions: the Gersten conjecture for any regular local ring A having $\dim A \leq 4$ essentially follows from Balmer's vanishing result for the derived Witt groups of local rings [11, Corollary 10.4].

One motivation for considering this question is that the known cases of the Gersten conjecture are used in the proofs of many theorems. For example, the Gersten conjecture for the Witt groups of equicharacteristic regular local rings was

used to prove homotopy invariance of the Witt sheaf W_{Nis} by I.A. Panin [54], which was in turn used by F. Morel in his famous calculations of certain $\mathbb{A}^1$-homotopy groups of spheres in terms of the Grothendieck-Witt and Witt groups of the base field [52]. It is reasonable to think that an extension of this conjecture from the equicharacteristic case (important to the study of smooth algebraic varieties over a field) to the case of local rings essentially smooth over a DVR (important to the study of smooth algebraic schemes over the integers) may be useful for a similar extension of the theorems just mentioned.

Now a remark on terminology and the situation in K-theory. It is standard to refer to this question by the name 'the Gersten conjecture for the Witt groups' because it is the analogue for the Witt groups of a conjecture made by Gersten for K-theory, known as *the Gersten conjecture.* The Gersten conjecture asserts that for any regular local ring A the *Gersten complex* for K-theory is exact (it is a complex similar in appearance to the complex 1.1 but begins with $K_n(A)$ in place of $W(A)$ and consists of K-groups). The Gersten conjecture is known for equicharacteristic regular local rings: it was proved for local rings essentially smooth over a field by D. Quillen; I. Panin developed an argument for deducing the equicharacteristic case from the essentially smooth case [55]. Also, it is known for the K-theory with finite coefficients $K_n(A, \mathbb{Z}/p\mathbb{Z})$ of local rings A essentially smooth over a DVR [33].

In this thesis, in Chapter 3.3 we will prove the Gersten conjecture in the case of a local ring essentially smooth over a DVR Λ with infinite residue field. Additionally, we present a version of Panin's argument that allows us to deduce from this the Gersten conjecture for the Witt groups in the case that A is regular over Λ (that is, there exists a regular morphism $\Lambda \to A$). For example, this result includes the case of all unramified regular local rings. We also remark that this use of Panin's argument applies to other cohomology theories for which the Gersten conjecture

is known in the essentially smooth over a DVR case, such as K-theory with finite coefficients or motivic cohomology with finite coefficients. Hence, this argument also gives a new result on the Gersten conjecture for these theories.

1.3 Synopsis of the Results: Chapter 1.6

In this chapter there are no new results, we only recall the definition and basic properties of the Witt groups of schemes as well as some results that are essential to later chapters.

1.4 Synopsis of the Results: Chapter 2.4

In this chapter, we prove that when X is a smooth surface over a finite field of characteristic different from 2, the shifted Witt groups $W^n(X)$ are finite (see Theorem 3.33). In higher dimensions, we give conditional results. Theorem 3.34 states that, for X a finite type $\mathbb{Z}[\frac{1}{2}]$-scheme with no residue field of X formally real, if the motivic cohomology groups of X with mod 2 coefficients $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$ are finite groups, then the shifted Witt groups $W^n(X)$ are finite. Furthermore, we give partial converses to this last result. We prove that for certain arithmetic schemes of dimension less than four, finiteness of the shifted Witt groups is equivalent to finiteness of the mod 2 motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$ (see Theorem 3.36).

The argument that we use for these results is essentially that of Arason, Elman, and Jacob mentioned earlier [4], but significantly strengthened by the fact that we now can use Voevodsky's solution of the Bloch-Kato conjecture. Indeed, let X be a smooth variety over a field k of characteristic different from 2. Using Bloch-Kato, S. Gille noted that his graded Gersten-Witt spectral sequence relates étale cohomology to the Witt groups [30, §10.7]. When the base field k is the complex numbers, $k = \mathbb{C}$, B. Totaro also used this spectral sequence, noting that it easily gave Parimala's theorem, equating finiteness of $CH^2(X)/2CH^2(X)$ to finiteness

of $W^0(X)$ [67, Theorem 1.4]. Here, we adapt these ideas to the arithmetic setting (smooth schemes over $\mathbb{Z}[\frac{1}{2}]$) using Arason's Theorem (Theorem 3.26). Also, we apply Kerz and Saito's positive solution of the Kato conjecture, which we use here in the form of Proposition 3.15), to relate finiteness of motivic cohomology with mod 2 coefficients to finiteness of the Witt groups for varieties having dimension as high as 4 (see the proof of Lemma 3.21 (b) and (c), and the statement of Theorem 3.36).

1.5   Synopsis of the Results: Chapter 3.3

In this chapter we prove the Gersten conjecture for the Witt groups in the case of a local ring A that is regular over a discrete valuation ring Λ having an infinite residue field $\Lambda/\mathfrak{m}$ (Theorem 4.28). For the proof, we follow the strategy that was devised by S. Gille and J. Hornbostel for the case of local rings essentially smooth over a field whereby they deduced the Gersten conjecture from Quillen normalization. Instead of Quillen normalization, we use a normalization result due to S. Bloch. Bloch used this normalization result to prove the Gersten conjecture for K-theory in the case of the localization $A[\pi^{-1}]$ of A, where π is a uniformizing parameter for Λ, and A is essentially smooth over Λ. Using Bloch's strategy, we first prove the Gersten conjecture for the localization $A[\pi^{-1}]$ of A, where π is a uniformizing parameter for Λ (Theorem 4.19). Then, for Witt groups, the Gersten conjecture in the case of a local ring essentially smooth over Λ follows (Corollary 4.20). Note, this is not the case for K-theory. We then prove the Gersten conjecture in the case of a local ring that is regular over Λ (Theorem 4.28) by deducing it from the essentially smooth case. The proof of Theorem 4.28 adapts, in a very straightforward way, a strategy of I.A. Panin that he used to obtain the equicharacteristic case from the case of local rings essentially smooth over a field.

1.6 Synopsis of the Results: Chapter 4.3

In this last chapter we present some applications of the results from the preceding chapters.

From the finiteness result for the Witt groups we give the known consequence (suggested to the author by M. Schlichting), that the Grothendieck-Witt groups of a curve over a finite field are finitely generated (Theorem 5.3). We also prove that for smooth arithmetic schemes with no residue field formally real, that finite generation of the motivic cohomology groups implies finite generation of the Grothendieck-Witt groups (Theorem 5.4).

Next, we present a result on finiteness of certain Chow-Witt groups (Theorem 5.7). It has been observed (*e.g.* [39], [20]) that the Chow-Witt groups appear on the second page of the coniveau spectral sequence for the p-th shifted Grothendieck-Witt groups as $E_2^{p,-p} \cong \widetilde{CH}^p(X)$. For the usual Chow groups, they appear in a similar way in the coniveau spectral sequence converging to K-theory, and there is a classical finiteness result stating that the d-th Chow group $CH^d(X)$ of a smooth variety of dimension d over a finite field is finite (c.f. [46, Theorem 9.2], [44, theorem 1]). The result given here is the Chow-Witt analogue, stating that $\widetilde{CH}^d(X)$ is finite. We obtain the result as a corollary of a finiteness result from Chapter 2.4 on the subcomplexes of the Gersten complex for the Witt groups which are obtained by filtering by the powers of the fundamental ideal.

The Witt Groups of Schemes

The Witt group $W(X)$ of a scheme X was introduced by M. Knebusch [47, Chapter 1 §5] in the seventies. When k is a field having characteristic different from 2, $W(\mathrm{Spec}(k))$ is the classical Witt group $W(k)$ of quadratic forms over k which we recall in Section 2.1. More recently, the Witt group of Knebusch was revealed to be a part of a cohomology theory $W^n(X)$ for schemes. When 2 is invertible on X, each $W^n(X)$ can be constructed as a "triangular" Witt group [6, 7] of a certain triangulated category. They recover the classical Witt group of Knebusch as $W(X) \simeq W^0(X)$ [9, Theorem 1.4.11]. For a more complete overview of the Witt groups of schemes and what is known about them we refer the reader to [9].

The triangulated Witt groups W^n were introduced by P. Balmer in his thesis (c.f. [5]) and take as input a triangulated category $\mathcal{A}$ with 2 invertible together with a duality $\sharp : \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$. They output an abelian group $W^n(\mathcal{A})$. One limitation of triangulated Witt groups is that the theory only works for categories $\mathcal{A}$ with 2 invertible. Technically, this means that the morphism groups in $\mathcal{A}$ are uniquely 2-divisible, but in practice a consequence is that we cannot work with varieties over fields of characteristic 2. We recall the definition of the triangulated Witt groups in Section 2.2, and then recall the definition of the (derived) Witt groups and coherent Witt groups in Section 2.2.1. In the last two sections we recall the construction of the Gersten complex for the Witt groups and the coniveau spectral sequence, both of which are used in the later chapters.

2.1   The Witt Group of a Field

In this section, to motivate and help the reader to fix their intuition about the Witt groups, we briefly recall the definition and certain facts about the Witt group of

a field. We do not give any proofs, referring the reader to [59] for a comprehensive introduction. These classical notions have generalizations to the setting of triangulated categories with duality.

Let k be a field. Recall that a *symmetric form* (V, β) is a vector space V together with a symmetric bilinear map $\beta : V \times V \to k$, while a *quadratic form* (V, ϕ) is a vector space V together with a function $\phi : V \to k$ such that firstly $\phi(a \cdot v) = a^2 \cdot v$ for all $a \in k, v \in V$, and secondly the assignment $(v_1, v_2) \mapsto \phi(v_1 + v_2) - \phi(v_1) - \phi(v_2)$ defines a symmetric bilinear map $V \times V \to k$. When the characteristic of k is not 2, then quadratic forms and symmetric forms are essentially equivalent notions and they determine the same Witt groups (notice that $\frac{1}{2}[\phi(v_1 + v_2) - \phi(v_1) - \phi(v_2)]$ determines a symmetric form and $\beta(v, v)$ a quadratic form). Otherwise, quadratic forms and symmetric bilinear forms are no longer equivalent notions, in which case one must consider both quadratic Witt groups and symmetric Witt groups. In this thesis we will always assume that every field has characteristic not equal to 2. We next recall E. Witt's definition of the Witt group and then explain in what sense this group classifies quadratic forms on k. To begin, let $\varphi : V \to k$ be a quadratic form. A vector $v \in V$ is said to be *isotropic* if $v \neq 0$ and $\varphi(v) = 0$. The quadratic form φ is said to be *isotropic* if it admits an isotropic vector and is said to be *anistropic* otherwise. The hyperbolic form is the form with underlying vector space k^2 and form $q : k^2 \to k$ defined by $q(x, y) = xy$. It is denoted by $\mathbb{H}$. A quadratic form is said to be hyperbolic if it is an orthogonal sum (see Definition 2.2) of hyberbolic forms. Witt proved that every quadratic form splits $\varphi = \varphi_{an} \perp i\mathbb{H}$ as an orthogonal sum of an anistropic form φ_{an} and some number i of hyperbolic forms, and that furthermore, this decomposition is unique. Two quadratic forms φ and ϕ are said to be Witt-equivalent if φ_{an} equals ϕ_{an}. In 1937, E. Witt defined what is now known as the Witt group by demonstrating that the

orthogonal sum induces a well-defined operation on the set $W(k)$ of equivalence classes of quadratic forms for Witt-equivalence. Actually, using tensor product he actually defined a ring $W(k)$. The Witt group of k classifies quadratic forms on k in the sense that two quadratic forms are isometric (*i.e.* there exists an isomorphism of their underlying vector spaces which respects the forms) if and only if they have the same dimension and belong to the same class in the Witt group.

2.2   Triangulated Witt Groups

In this Section we recall the definition as well as those essential properties of the triangulated Witt groups that are used in later chapters.

Here, we follow a presentation of triangulated categories with duality which explicitly identifies the isomorphism relating the duality and the shift functor and is based upon a note of M. Schlichting.

Definition 2.1. A *triangulated category* is an additive category $\mathcal{A}$ together with an auto-equivalence

$$T : \mathcal{A} \to \mathcal{A}$$

and a class (objects of which are called distinguished triangles) of sequences of maps in $\mathcal{A}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

satisfying certain axioms TR1-TR4 (*e.g.*, see [62, A.2.1]). When, for all objects X, Y in $\mathcal{A}$, the groups $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are uniquely 2-divisible, we write that $\frac{1}{2} \in \mathcal{A}$ and say that *2 is invertible in $\mathcal{A}$*. A *triangulated category with duality* is a triangulated category equipped with an additive functor $\sharp : \mathcal{A}^{op} \to \mathcal{A}$ (for any X in $\mathcal{A}$ we write either $X^{\sharp}$ or $\sharp X$) and natural isomorphisms $\varpi : 1 \simeq \sharp\sharp$ and $\lambda : \sharp \simeq T\sharp T$ such that:

i) the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\;\varpi T\;} & \sharp\sharp T \\
\downarrow{\scriptstyle T\varpi} & & \downarrow{\scriptstyle \lambda\sharp T} \\
T\sharp\sharp & \xleftarrow[\;T\sharp\lambda\;]{} & T\sharp T\sharp T
\end{array}
$$

is commutative;

ii) for any object X in $\mathcal{A}$, the composition $X^\sharp \xrightarrow{\varpi_{X^\sharp}} X^{\sharp\sharp\sharp} \xrightarrow{\varpi_X^\sharp} X^\sharp$ equals $1_{X^\sharp}$;

iii) whenever $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is a distinguished triangle in $\mathcal{A}$, the triangle $Z^\sharp \xrightarrow{v^\sharp} Y^\sharp \xrightarrow{u^\sharp} X^\sharp \xrightarrow{(*)} TZ^\sharp$, where $(*)$ denotes the composition $X^\sharp \xrightarrow{\lambda_{X^\sharp}} T\sharp TX \xrightarrow{Tw^\sharp} TZ^\sharp$, is a distinguished triangle.

A morphism of triangulated categories with duality $(F, \rho, \varphi) : (\mathcal{A}_1, \sharp_1, \varpi_1, \lambda_1) \to (\mathcal{A}_2, \sharp_2, \varpi_2, \lambda_2)$ is an additive functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ together with natural isomorphisms $\rho : FT_1 \xrightarrow{\cong} T_2 F$ and $\varphi : F\sharp_1 \xrightarrow{\cong} \sharp_2 F$ such that:

i) the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\;F\varpi_1\;} & F\sharp_1\sharp_1 \\
\downarrow{\scriptstyle \varpi_2 F} & & \downarrow{\scriptstyle \varphi\sharp_1} \\
\sharp_2\sharp_2 F & \xrightarrow{\;\sharp_2(\varphi)\;} & \sharp_2 F\sharp_1
\end{array}
$$

is commutative;

ii) the diagram

$$
\begin{array}{ccccc}
F\sharp_1 & \xrightarrow{\;F\lambda_1\;} & FT_1\sharp_1 T_1 & \xrightarrow{\;\rho\sharp_1 T_1\;} & T_2 F\sharp_1 T_1 \\
\downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle T_2\varphi T_1} \\
\sharp_2 F & \xrightarrow{\;\lambda_2 F\;} & T_2\sharp_2 T_2 F & \xrightarrow{\;T_2\sharp_2\rho\;} & T_2\sharp_2 FT_1
\end{array}
\tag{2.1}
$$

is commutative.

Associated to every triangulated category with duality $(\mathcal{A}, \sharp, \varpi, \lambda)$ there is a *triangulated category with shifted duality* $(\mathcal{A}^{[1]}, \sharp^{[1]}, \varpi^{[1]}, \lambda^{[1]})$ having the same underlying triangulated category as $\mathcal{A}$ but equipped with the duality shifted by T,

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that is, $\sharp^{[1]} := T\sharp$, $\varpi^{[1]} := -\lambda\sharp \circ \varpi$, $\lambda^{[1]} := -T\lambda$. Similarly, we may shift in the other direction to obtain the shifted duality $(\mathcal{A}^{[-1]}, \sharp^{[-1]}, \varpi^{[-1]}, \lambda^{[-1]})$, where $\sharp^{[-1]} := \sharp T, \varpi^{[-1]} := (\lambda^\sharp)^{-1} \circ \varpi$, $\lambda^{[-1]} := -\lambda T$. Continuing in this manner, we obtain, for every $n \in \mathbb{Z}$, a triangulated category with duality $(\mathcal{A}^{[n]}, \sharp^{[n]}, \varpi^{[n]}, \lambda^{[n]})$. A morphism of triangulated categories (F, ρ, φ) induces a morphism of the associated triangulated categories with shifted dualities, $e.g.$ we obtain a morphism $(F, \rho, \varphi^{[1]})$ of the triangulated categories with the duality shifted by T by defining $\varphi^{[1]} := T\varphi \circ \rho\sharp$.

Definition 2.2. Let $(\mathcal{A}, \sharp, \varpi, \lambda)$ be a triangulated category with duality. A *symmetric form* is a pair (X, υ), where X is an object in $\mathcal{A}$ and $\upsilon : X \to X^\sharp$ is a morphism in $\mathcal{A}$ such that $\upsilon^\sharp \circ \varpi = \upsilon$. When υ is an isomorphism in $\mathcal{A}$, the (X, υ) is said to be a *symmetric space*. A symmetric form with respect to the shifted duality $\sharp^{[n]}$ will be called an $\mathcal{A}^{[n]}$-*symmetric form*, $e.g.$ a $\mathcal{A}^{[-1]}$-symmetric form is a pair (X, υ) with $\upsilon : X \to (TX)^\sharp$ a morphism in $\mathcal{A}$ such that $T(\upsilon^\sharp) \circ \varpi_X^{[-1]} = \upsilon$ $(\varpi_X^{[-1]} = (\lambda^\sharp)^{-1} \circ \varpi$, see Definition 2.1), and when the morphism defining the symmetric form is an isomorphism in $\mathcal{A}$ we say it is an $\mathcal{A}^{[n]}$-*symmetric space*. Given two symmetric spaces (X, υ) and (Y, ϕ): the *orthogonal sum* is defined to be the symmetric space $(X, \upsilon) \perp (Y, \phi) := \left(X \oplus Y, \begin{pmatrix} \upsilon & 0 \\ 0 & \phi \end{pmatrix}\right)$; an *isometry* $h : (X, \upsilon) \xrightarrow{\simeq} (Y, \phi)$ is an isomorphism $h : X \xrightarrow{\simeq} Y$ in $\mathcal{A}$ such that $h^\sharp \phi h = \upsilon$; (X, υ) and (Y, ϕ) are said to be *isometric* if there exists an isometry between them.

Proposition 2.3. *[6, Theorem 1.6] Let $(\mathcal{A}, \sharp, \varpi, \lambda)$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{A}$. Let (X, υ) be a $\mathcal{A}^{[-1]}$-symmetric space (see Definition 2.2). Choose a distinguished triangle $X \xrightarrow{\upsilon} (TX)^\sharp \xrightarrow{u_1} C \xrightarrow{u_2} TX$ over υ in $\mathcal{A}$. Then, there exists an isomorphism $\psi : C \xrightarrow{\simeq} C^\sharp$ in $\mathcal{A}$ defining a symmetric space (C, ψ)*

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in $(\mathcal{A}, \sharp, \varpi, \lambda)$ and a commutative diagram

$$
\begin{array}{ccccccc}
X & \xrightarrow{\;v\;} & (TX)^\sharp & \xrightarrow{\;u_1\;} & C & \xrightarrow{\;u_2\;} & TX \\
\downarrow{\scriptstyle \varpi_X^{[-1]}} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle T\varpi_X^{[-1]}} \\
\sharp T \sharp TX & \xrightarrow{\;T(v^\sharp)\;} & (TX)^\sharp & \xrightarrow{\;u_2^\sharp\;} & C^\sharp & \xrightarrow{\;\lambda \sharp \circ u_1^\sharp\;} & T\sharp T\sharp TX
\end{array}
\qquad (2.2)
$$

in which the rows are distinguished triangles. If we choose another distinguished triangle $X \xrightarrow{v} (TX)^\sharp \xrightarrow{u_1'} C' \xrightarrow{u_2'} TX$ over v and another isomorphism ψ' defining a symmetric space (C', ψ) such that the resulting diagram 2.2 is commutative, then the symmetric spaces (C', ψ') and (C, ψ) are isometric.

Definition 2.4. Let $(\mathcal{A}, \sharp, \varpi, \lambda)$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{A}$. The isometry class of the symmetric space (C, ψ) from Proposition 2.3 is called the *cone of the $\mathcal{A}^{[-1]}$-symmetric form $v : X \to (TX)^\sharp$.* A symmetric space in $(\mathcal{A}, \sharp, \varpi, \lambda)$ which is equal to a cone is called a *metabolic space (aka neutral space)* space in $(\mathcal{A}, \sharp, \varpi, \lambda)$.

Definition 2.5. Let $(\mathcal{A}, \sharp, \varpi, \lambda)$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{A}$. The *Witt group* $W^0(\mathcal{A})$ is the free abelian group generated by isometry classes of symmetric spaces in $(\mathcal{A}, \sharp, \varpi, \lambda)$ modulo the metabolic spaces and the relations $[X \perp Y] = [X] + [Y]$ (or equivalently, the Grothendieck-group of the abelian monoid of isometry classes of symmetric spaces modulo the metabolic spaces). For every $n \in \mathbb{Z}$, the *shifted Witt group* $W^n(\mathcal{A})$ is defined to be $W^0(\mathcal{A}^{[n]})$, in other words, it is the Witt group of the triangulated category with duality $(\mathcal{A}^{[n]}, \sharp^{[n]}, \varpi^{[n]}, \lambda^{[n]})$.

Proposition 2.6. *[6, Proposition 1.14] Let $(\mathcal{A}, \sharp, \varpi, \lambda)$ be a triangulated category with duality. For every $n \in \mathbb{Z}$, there is an isomorphism of triangulated categories with duality $\mathcal{A}^{[n]} \simeq \mathcal{A}^{[n+4]}$ which induces an isomorphism of triangulated Witt groups $W^n(\mathcal{A}) \simeq W^{n+4}(\mathcal{A})$.*

Proposition 2.7. *[9, Theorem 1.4.14] Let $(\mathcal{A}, \sharp, \varpi, \lambda)$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{A}$. Let*

$$\mathcal{A}_0 \to \mathcal{A} \to S^{-1}\mathcal{A} \tag{2.3}$$

be a short exact sequence of triangulated categories with duality, that is, $\mathcal{A} \to S^{-1}\mathcal{A}$ is a localization with respect to a class S of morphisms in $\mathcal{A}$, and $\mathcal{A}_0$ is the full subcategory of $\mathcal{A}$ consisting of objects which become isomorphic to zero in $S^{-1}\mathcal{A}$, and the dualities on $\mathcal{A}_0$ and $S^{-1}\mathcal{A}$ are induced from the duality on $\mathcal{A}$. Then, the short exact sequence 2.3 induces a long exact sequence of abelian groups

$$\cdots \to W^n(\mathcal{A}_0) \to W^n(\mathcal{A}) \to W^n(S^{-1}\mathcal{A}) \to \cdots \tag{2.4}$$

2.2.1 The Derived and Coherent Witt Groups of Schemes

Now we will give an important example of a triangulated category with duality.

2.8. Let $(\mathcal{E}, \sharp)$ be an exact category with duality $\sharp$. The homotopy category $\mathcal{K}^b(\mathcal{E})$ of bounded chain complexes in $\mathcal{E}$ is a triangulated category having as objects bounded chain complexes in $\mathcal{E}$ and as morphisms the chain maps up to chain homotopy. The bounded derived category $D^b(\mathcal{E})$ is obtained from the homotopy category by formally inverting quasi-isomorphisms. The duality $\sharp$ on $\mathcal{E}$ induces a duality on the homotopy category $\mathcal{K}^b(\mathcal{E})$ and on the derived category $D^b(\mathcal{E})$. Let ϖ denote the isomorphism to the double dual $\varpi : 1 \overset{\simeq}{\to} \sharp\sharp$ in $D^b(\mathcal{E})$ that is induced from the canonical one in $\mathcal{E}$. Then, $(D^b(\mathcal{E}), \sharp, \varpi, 1)$ is a triangulated category with duality. For a reference for these facts see [62, Section 3.1.3],[7, Section 2.6].

In particular, from 2.8 we have the following.

Definition 2.9. Let X be a scheme with 2 invertible in the global sections $\mathcal{O}_X(X)$. Let $\mathrm{Vect}(X)$ denote the exact category of vector bundles on X, that is, the category of $\mathcal{O}_X$-modules which are locally free and of finite rank. For any vector bundle

$\mathcal{E}$ on X, the usual duality $\mathcal{E}^\sharp := \mathrm{Hom}_{\mathrm{Vect}(X)}(\mathcal{E}, \mathcal{O}_X)$ defines a duality on $\mathrm{Vect}(X)$. Then, $(D^b(\mathrm{Vect}(X)), \sharp, \varpi, 1)$ is a triangulated category with duality (2.8) with $\frac{1}{2} \in D^b(\mathrm{Vect}(X))$. The *(derived) Witt groups of X* are defined to be the triangulated Witt groups $W^n(D^b(\mathrm{Vect}(X)))$ of the triangulated category with duality $(D^b(\mathrm{Vect}(X)), \sharp, \varpi, 1)$, and are denoted by $W^n(X)$.

As mentioned earlier, there is the following identification.

Proposition 2.10. *[9, Theorem 1.4.11] Let X be a scheme with 2 invertible in its global sections. Then $W^0(X)$ is the Witt group $W(X)$ as defined by Knebusch. In particular, when $X = \mathrm{Spec}\, k$, where k is field of characteristic not 2, then $W^0(\mathrm{Spec}\, k)$ equals $W(k)$ the classical Witt group of k.*

The principal result on the derived Witt groups of local rings is the following theorem of P. Balmer.

Proposition 2.11. *[7, Theorem 5.6] Let A be a local ring in which 2 is invertible. Then, among the derived Witt groups of A, we have $W^1(A) = 0, W^2(A) = 0$, and $W^3(A) = 0$. That is, there is only one non-trivial Witt group, namely $W^0(A) \simeq W(A)$. This holds in particular for fields of characteristic not 2.*

Next we recall the definition of the coherent Witt groups.

Definition 2.12. Let X be a noetherian scheme with 2 invertible in its global sections. Let $\mathcal{M}(X)$ denote the category of $\mathcal{O}_X$-modules, and let $D^b_{coh}(X)$ denote the full subcategory of the bounded derived category $D^b(\mathcal{M}(X))$ consisting of complexes having coherent homology. A dualizing complex for X is defined to be a bounded complex K of injective coherent sheaves with the property that the natural morphism of complexes $\overline{\omega}^K$ (essentially the evaluation map, see [30, §1.6] for a precise description) from an object M of $D^b_{coh}(X)$ to its double dual

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(M,K),K)$ is an isomorphism (in $D^b_{coh}(X)$). The *coherent Witt groups of* X are defined to be the triangulated Witt groups of the triangulated category with duality $(D^b_{coh}(X), \mathcal{H}om_{\mathcal{O}_X}(-,K), \overline{\omega}^K, 1)$ (2.5), and are denoted by $\widetilde{W}^n(X,K)$.

Remark 2.13. Let X be a noetherian scheme with 2 invertible in its global sections.

(a) When X is regular, any injective resolution $I_\bullet$ of $\mathcal{O}_X$ yields a dualizing complex, and the quasi-isomorphism $\mathcal{O}_X \overset{\sim}{\rightarrow} I_\bullet$ induces an isomorphism $W^n(X) \overset{\sim}{\rightarrow} \widetilde{W}^n(X, I_\bullet)$ from the derived Witt groups to the coherent Witt groups [30, Example 2.4]. However, this isomorphism is not functorial.

(b) Every dualizing complex $I_\bullet$ for X yields a codimension function $\mu_I : X \to \mathbb{Z}$ [30, Lemma 1.12 and following discussion]. When X is regular and the dualizing complex is given by an injective resolution of the structure sheaf, this function is exactly the usual codimension function $x \mapsto \mathrm{codim}(x)$ [30, Example 1.13].

2.3   The Gersten Complex for the Witt Groups

We construct the Gersten complex for the Witt groups and then identify it with the 'usual' Gersten complex. Even though we work with regular schemes, we use the coherent Witt groups to construct the Gersten complex for two reasons: one is that in we use results of S. Gille on the differentials of the Gersten complex that is defined in terms of coherent Witt groups in Chapter 2.4; the other is that we use the transfer for the coherent Witt groups for an argument involving the Gersten complex in Chapter 3.3.

2.3.1 Construction

Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d, and let $I_\bullet$ denote the dualizing complex obtained by taking an injective resolution of the structure sheaf $\mathcal{O}_X$ (Remark 2.13 (a)). We will denote the coherent Witt groups of X with coefficients in $I_\bullet$ by $\widetilde{W}(X, I_\bullet)$. We briefly recall the well-known construction of *the coniveau spectral sequence* for coherent Witt groups [30, §5.8]. For any $\mathcal{O}_X$-module M, we denote by $\mathrm{supp}M$ the set of points $x \in X$ for which the localization M_x is non-zero. For any complex $M_\bullet \in D^b_c(X)$, $\mathrm{supp}H_i(M_\bullet)$ is a closed subscheme of X since $H_i(M_\bullet)$ is coherent [34, corollary 7.31]. Then

$$\mathrm{supp}M_\bullet := \bigcup_{i \in \mathbb{Z}} \mathrm{supp}H_i(M_\bullet)$$

is also a closed subspace of X as it is a finite union of closed subspaces. Recall that the codimension $\mathrm{codim}_X(Z)$ of a closed subspace Z of X is defined as

$$\mathrm{codim}(Z) = \inf_{\eta \in Z} \dim \mathcal{O}_{X,\eta}$$

where the infimum runs over the generic points $\eta \in Z$ of Z.

For any integer $q \geq 0$, let $D^q(X) \subset D^b_c(X)$, or simply D^q, consist of those complexes having codimension of support greater than or equal to q, that is

$$D^q(X) = \{M_\bullet \in D^b_c(X) | \mathrm{codim}_X(\mathrm{supp}M_\bullet) \geq q\} \tag{2.5}$$

For $q \geq 0$, we have short exact sequences [29, Section 3.3] of triangulated categories with duality
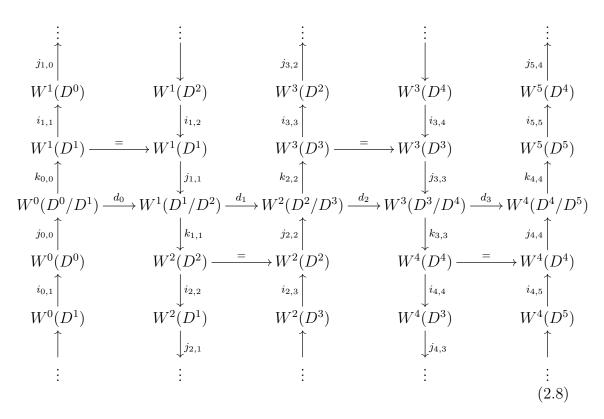
$$D^{q+1} \to D^q \to D^q/D^{q+1} \tag{2.6}$$

so we obtain from Balmer's localization theorem (Proposition 2.7) the long exact sequence

$$\cdots \to W^p(D^{q+1}) \overset{i_{p,q+1}}{\to} W^p(D^q) \overset{j_{p,q}}{\to} W^p(D^q/D^{q+1}) \overset{k_{p,q}}{\to} \cdots \tag{2.7}$$

where we index the maps based on the indices of their respective domain. The map i is induced by the inclusion $D^{q+1} \to D^q$, j by the quotient $D^q \to D^q/D^{q+1}$, and k is the connecting morphism.

Consider the commutative diagram below which is obtained by placing the long exact sequences 2.7 vertically and using the compositions $d_i := j_{i+1,i+1} \circ k_{i,i}$ to define the maps d_i.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\Big\uparrow {\scriptstyle j_{1,0}} & & \Big\downarrow & & \Big\uparrow {\scriptstyle j_{3,2}} & & \Big\downarrow & & \Big\uparrow {\scriptstyle j_{5,4}} \\
W^1(D^0) & & W^1(D^2) & & W^3(D^2) & & W^3(D^4) & & W^5(D^4) \\
\Big\uparrow {\scriptstyle i_{1,1}} & & \Big\uparrow {\scriptstyle i_{1,2}} & & \Big\uparrow {\scriptstyle i_{3,3}} & & \Big\uparrow {\scriptstyle i_{3,4}} & & \Big\uparrow {\scriptstyle i_{5,5}} \\
W^1(D^1) & \xrightarrow{=} & W^1(D^1) & & W^3(D^3) & \xrightarrow{=} & W^3(D^3) & & W^5(D^5) \\
\Big\uparrow {\scriptstyle k_{0,0}} & & \Big\uparrow {\scriptstyle j_{1,1}} & & \Big\uparrow {\scriptstyle k_{2,2}} & & \Big\uparrow {\scriptstyle j_{3,3}} & & \Big\uparrow {\scriptstyle k_{4,4}} \\
W^0(D^0/D^1) & \xrightarrow{d_0} & W^1(D^1/D^2) & \xrightarrow{d_1} & W^2(D^2/D^3) & \xrightarrow{d_2} & W^3(D^3/D^4) & \xrightarrow{d_3} & W^4(D^4/D^5) \\
\Big\uparrow {\scriptstyle j_{0,0}} & & \Big\uparrow {\scriptstyle k_{1,1}} & & \Big\uparrow {\scriptstyle j_{2,2}} & & \Big\uparrow {\scriptstyle k_{3,3}} & & \Big\uparrow {\scriptstyle j_{4,4}} \\
W^0(D^0) & & W^2(D^2) & \xrightarrow{=} & W^2(D^2) & & W^4(D^4) & \xrightarrow{=} & W^4(D^4) \\
\Big\uparrow {\scriptstyle i_{0,1}} & & \Big\uparrow {\scriptstyle i_{2,2}} & & \Big\uparrow {\scriptstyle i_{2,3}} & & \Big\uparrow {\scriptstyle i_{4,4}} & & \Big\uparrow {\scriptstyle i_{4,5}} \\
W^0(D^1) & & W^2(D^1) & & W^2(D^3) & & W^4(D^3) & & W^4(D^5) \\
\Big\uparrow & & \Big\downarrow {\scriptstyle j_{2,1}} & & \Big\uparrow & & \Big\downarrow {\scriptstyle j_{4,3}} & & \Big\uparrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$
$$\tag{2.8}$$

The diagram 2.8 continues on to the right; when $q > \dim X$ the qth-columns (*i.e.* the columns in which the quotient terms D^q/D^{q+1} appear) vanish. As the vertical sequences are complexes of abelian groups, it follows that the horizontal sequence

$$W^0(D^0/D^1) \xrightarrow{d_0} W^1(D^1/D^2) \xrightarrow{d_1} W^2(D^2/D^3) \xrightarrow{d_2} \cdots \tag{2.9}$$

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is a complex of abelian groups. When X is finite dimensional of dimension d, the complex (2.9) becomes the complex

$$W^0(D^0/D^1) \xrightarrow{d_0} W^1(D^1/D^2) \xrightarrow{d_1} \cdots \xrightarrow{d_{d-1}} W^d(D^d) \longrightarrow 0 \qquad (2.10)$$

where we have identified $W^d(D^d/D^{d+1}) = W^d(D^d)$.

As $W^0(D^0) = \widetilde{W}(X)$, the complex (2.10) augmented by the map $j_{0,0} : W^0(D^0) \to W^0(D^0/D^1)$ determines a complex

$$0 \longrightarrow \widetilde{W}(X) \xrightarrow{j_{0,0}} W^0(D^0/D^1) \xrightarrow{d_0} W^1(D^1/D^2) \xrightarrow{d_1} \cdots \xrightarrow{d_{d-1}} W^d(D^d) \longrightarrow 0.$$

$$(2.11)$$

Remark 2.14. If we use the derived Witt groups instead of the coherent Witt groups in the above construction of the Gersten complex 2.10 by using the codimension of support filtration on the bounded derived category of vector bundles $D^b(\mathrm{Vect}(X))$ on X, then this results in the complex

$$0 \longrightarrow W(X) \xrightarrow{j_{0,0}} W^0(D^0/D^1) \xrightarrow{d_0} W^1(D^1/D^2) \xrightarrow{d_1} \cdots \xrightarrow{d_{d-1}} W^d(D^d) \longrightarrow 0$$

$$(2.12)$$

begining with the derived Witt group $W(X)$ of X. However, the complex 2.12 and the complex 2.11 are isomorphic kjhsdafkjh kj[10, Section 3, Another Construction], where the isomorphism is induced by the quasi-isomorphism $\mathcal{O}_X \xrightarrow{\cong} I_\bullet$.

Definition 2.15. Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d. We index the complex 2.10 cohomologically by setting the degree p term to be $W^p(D^p/D^{p+1})$. The complex 2.10 will be denoted by $\mathrm{Ger}(X)$, and called the *Gersten complex for the Witt groups of X*, or simply the *Gersten-Witt complex of X*. Following Remark 2.14, the complexes 2.12 and 2.11 will both be called the *augmented Gersten complex for the Witt groups of X* since they are isomorphic.

Definition 2.16. Let A be a regular local ring. If the augmented Gersten complex 2.11 for the Witt groups of A is an exact complex, then we will say that the *Gersten conjecture is true for the Witt groups of A.*

The following lemma is well-known (*e.g.* [10, Lemma 4.2]).

Lemma 2.17. *Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d. The Witt sheaf $\mathcal{W}$ is the Zariski sheaf on X that is associated to the presheaf $U \mapsto W(U)$ on X. If, for all $x \in X$, the Gersten conjecture (see definition 2.16) is true for the Witt groups of $\mathcal{O}_{X,x}$, then, for all $p \geq 0$, $H^p(\mathrm{Ger}(X)) = H^p_{Zar}(X, \mathcal{W})$, that is, the cohomology of the Gersten complex agrees with the Zariski cohomology of X with coefficients in $\mathcal{W}$.*

Next, a simple diagram chase lemma which is well-known.

Lemma 2.18. *Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme. Then we have the following:*

(i) *If the morphism $i_{0,1} : W^0(D^1) \to W^0(D^0)$ (from the long exact sequence 2.7) is zero, then the augmented Gersten complex is exact at $W(X)$;*

(ii) *If the morphism $i_{1,2} : W^1(D^2) \to W^1(D^1)$ is zero, then the augmented Gersten complex is exact at $W^0(D^0/D^1)$;*

(iii) *Let $p > 0$ be an integer. If the morphisms $i_{p,p} : W^p(D^p) \to W^p(D^{p-1}), i_{p+1,p+2} :$ $W^{p+1}(D^{p+2}) \to W^{p+1}(D^{p+1})$ (2.7) are both zero, then the Gersten complex for the Witt groups (2.15) is exact in degree p.*

In particular, if, for all $p, q \in \mathbb{Z}$, the maps $i_{p,q} : W^p(D^q) \to W^p(D^{q-1})$ are zero, then the augmented Gersten complex (2.11) is exact.

Proof. To prove (i), consider the diagram (2.8). In view of the exactness of the vertical sequences we see that if $W^0(D^1) \overset{i_{0,1}}{\to} W^0(D^0)$ is zero, then $W^0(D^0) \overset{j_{0,0}}{\to} W^0(D^0/D^1)$ is injective. Similarly, to prove (ii) note that if $W^1(D^2) \overset{i_{1,2}}{\to} W^1(D^1)$ is zero, then $W^1(D^1) \overset{j_{1,1}}{\to} W^1(D^1/D^2)$ is injective. Hence, the kernel of the differential $d_0 = j_{1,1} \circ k_{0,0}$ equals the kernel of $k_{0,0}$, which is $W(X)$ the image of $j_{0,0}$. So the augmented Gersten complex (2.11) is exact in degree 0.

To prove (iii) we have the following in any positive degree p: if $i_{p,p} = 0$, then $k_{p-1,p-1}$ is surjective, so the image of the differential d_{p-1} is the image of $j_{p,p}$; if $i_{p+1,p+2} = 0$, then $j_{p+1,p+1}$ is injective, so the kernel of the differential d_p is the kernel of $k_{p,p}$. Hence, if $i_{p+1,p+2} = i_{p,p} = 0$, then the Gersten complex for the Witt groups is exact in degree p. The final statement of the Lemma immediately follows from $(i), (ii)$ and (iii). $\qquad\square$

Lemma 2.19. *[10, Lemma 3.3 and Section 4] Let A be a regular local ring and $f \in A$ a regular parameter. Then there is a short exact sequence of complexes*

$$0 \to \mathrm{Ger}(A/f)[-1] \to \mathrm{Ger}(A) \to \mathrm{Ger}(A_f) \to 0 \qquad (2.13)$$

where $\mathrm{Ger}(A/f)[-1]$ denotes the shift to the right, so that the degree 0 part of this complex is the degree -1 part of $\mathrm{Ger}(A/f)$, and in particular, $H^0(\mathrm{Ger}(A/f)[-1]) = 0$, and, for $p \geq 1$, $H^p(\mathrm{Ger}(A/f)[-1]) = H^{p-1}(\mathrm{Ger}(A/f))$. It follows that the long exact sequence of cohomology groups

$$\cdots \to H^p(\mathrm{Ger}(A/f)[-1]) \to H^p(\mathrm{Ger}(A)) \to H^p(\mathrm{Ger}(A_f)) \to \cdots \qquad (2.14)$$

which is determined by the short exact sequence 2.13 may be rewritten as the long exact sequence

$$\cdots \to H^{p-1}(\mathrm{Ger}(A/f)) \to H^p(\mathrm{Ger}(A)) \to H^p(\mathrm{Ger}(A_f)) \to \cdots \qquad (2.15)$$

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2.3.2 Identification

In this section we recall some results which explicitly describe the Witt groups of the quotient $W^p(D^q/D^{q+1})$ and lead to a description of the Gersten complex in terms of the Witt groups of the residue field. The following are all well-known.

Proposition 2.20. *[26, c.f. Theorem 3.12] If X is a noetherian regular scheme, then, for all $q \geq 0$, there are equivalences of triangulated categories*

$$D^q/D^{q+1} \simeq \coprod_{x \in X^q} D^b_{fl}(\mathcal{M}(\mathcal{O}_{X,x})) \tag{2.16}$$

where $D^b_{fl}(\mathcal{M}(\mathcal{O}_{X,x}))$ denotes those complexes of $\mathcal{O}_{X,x}$-modules in $D^b_{coh}(\mathcal{M}(\mathcal{O}_{X,x}))$ whose homology has finite length and X^q denotes the set of points $x \in X$ having $\dim \mathcal{O}_{X,x} = q$.

The Witt groups of the triangulated categories $D^b_{fl}(\mathcal{M}(\mathcal{O}_{X,x})$ are described in the next Lemma.

Proposition 2.21. *[26, c.f. Theorem 3.10 and Corollary 3.11] Let $\mathcal{O}$ be a regular local ring with residue field k such that $\mathrm{char}k \neq 2$. Then $W^n(D^b_{fl}(\mathcal{M}(\mathcal{O}))) \simeq W(k)$ if $n = \dim \mathcal{O} \mod 4$, and $W^n(D^b_{fl}(\mathcal{M}(\mathcal{O}))) = 0$ otherwise.*

Taking the Witt groups of both sides of the equivalence 2.16 given in the statement of Proposition 2.20 and applying Proposition 2.21 one obtains the next proposition.

Proposition 2.22. *[26, c.f. Theorem 3.14] Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of finite Krull dimension, and let $0 \leq q \leq \dim X$ be an integer. For any $p \in \mathbb{Z}$, the Witt groups $W^{p+q}(D^q/D^{q+1})$ of the quotients D^q/D^{q+1} vanish when $p + q \neq q \mod 4$, and are otherwise, when $p + q = q \mod 4$,*

$$W^{p+q}(D^q/D^{q+1}) \simeq W^q(D^q/D^{q+1}) \simeq \bigoplus_{x \in X^q} W(k(x)) \tag{2.17}$$

where the isomorphism on the left is induced by the 4-periodicity (Proposition 2.6), and the isomorphism on the right is the composition of the isomorphism induced by the equivalence of triangulated categories 2.16 given in the statement of Proposition 2.20) followed by the isomorphism of Proposition 2.21 ($x \in X^q$ means by definition that $\dim \mathcal{O}_{X,x} = q$).

In particular, when X is a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of finite Krull dimension we obtain a complex

$$C(X, W, \iota) := \bigoplus_{x \in X^0} W\left(k\left(x\right)\right) \xrightarrow{d} \bigoplus_{x \in X^1} W\left(k\left(x\right)\right) \xrightarrow{d} \ldots \ldots \xrightarrow{d} \bigoplus_{x \in X^d} W\left(k\left(x\right)\right) \quad (2.18)$$

where the differentials d_q are the composition in the diagram below

$$(2.19)$$

and where ι indicates the isomorphisms chosen, for each $q \geq 0$, in the diagram 2.19. The differentials may differ for different choices of isomorphisms.

2.4 Coniveau Spectral Sequence

Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d, and let $I_\bullet$ denote the dualizing complex obtained by taking an injective resolution of the structure sheaf $\mathcal{O}_X$ (Remark 2.13 (a)). It is well-known that the Gersten-Witt complex appears on the first page of a spectral sequence with abutment the Witt groups. Indeed, from the long exact sequences 2.7

$$\cdots \to W^p(D^{q+1}) \overset{i_{p,q+1}}{\to} W^p(D^q) \overset{j_{p,q}}{\to} W^p(D^q/D^{q+1}) \overset{k_{p,q}}{\to} \cdots \quad (2.20)$$

we obtain an exact couple by setting $E_1^{p,q} := W^{p+q}(D^p/D^{p+1})$, $D_1^{p,q} := W^{p+q}(D^p)$, and taking the differential to be $d^{p,q} := j_{p+q+1,p+1} \circ k_{p+q,p}$. By the well-known

method of Massey's exact couples, this exact couple determines a cohomological spectral sequence with abutment the coherent Witt groups. However, had we used to codimension of support filtration on the bounded derived category of vector bundles the resulting spectral sequence would be isomorphic [10, Section 3, *Another Construction*]. Therefore, we obtain the spectral sequence below

$$E_1^{p,q} := W^{p+q}(D^p/D^{p+1}) \Rightarrow W^{p+q}(X) \tag{2.21}$$

converging to the derived Witt groups of X. The differential d_r on the r-th page of this spectral sequence has bidegree $(r, 1-r)$. Since X is finite dimensional, this spectral sequence is bounded, hence converges. Furthermore, recalling the construction of the Gersten complex for the Witt groups 2.10, we have that $\mathrm{Ger}(X) = E_1^{*,0}$ by construction. Next, using Proposition 2.22 to identify the Witt groups of the quotients, we have that the groups $E_1^{p,q}$ appearing on the E_1-page, vanish for $p+q \neq p$ mod 4, and that they takes the form

$$E_1^{p,q} = \bigoplus_{x \in X^p} W(k(x))$$

when q is congruent to 0 mod 4.

One well-known general fact about the shifted Witt groups of arithmetic schemes is the following easy corollary to the coniveau spectral sequence that was alluded to in the introduction.

Corollary 2.23. *Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of Krull dimension d. If no residue field of X is formally real (a field is formally real if and only if -1 is not a sum of squares), then the Witt groups $W^n(X)$ are torsion groups.*

Proof. As no residue field of X is formally real, for each $x \in X$, the Witt group of the residue field $W(k(x))$ is a torsion group [59, Theorem 6.4 (ii)]. As arbitrary direct sums of torsion abelian groups are torsion, from the description in Equation

2.4 of the groups on the first page of coniveau spectral sequence, we have that all the groups appearing on the first page are torsion groups. Since X is finite dimensional, the first page of the spectral sequence is bounded, hence convergent, so we have that the Witt groups are torsion. $\square$

The next result we use in Chapter 3.3.

Corollary 2.24. *Let $X = \operatorname{Spec} A$ be a regular local ring with $\frac{1}{2} \in A$. If the Gersten complex $\operatorname{Ger}(X)$ (Definition 2.15) is exact in degrees greater than or equal to 4, then the Gersten conjecture is true for the Witt groups of A (Definition 2.16).*

Proof. The hypothesis on the Gersten complex is equivalent to $E_2^{p,0} = 0$ for all $p \geq 4$. Since the differential has bidegree $(r, 1 - r)$, it follows that the coniveau spectral sequence collapses on the E_2-page, and that $W^0(A) = E_2^{0,0}$, $W^1(A) = E_2^{1,0}$, $W^2(A) = E_2^{2,0}$, $W^3(A) = E_2^{3,0}$. However, as the derived Witt groups $W^1(A), W^2(A), W^3(A)$ are all zero (Proposition 2.11), this proves that $E_1^{p,0} = 0$ for all $p \geq 1$, and that $E_1^{0,0} = W^0(A)$, proving the lemma. $\square$

The Finite Generation Question

3.1 Kato Complexes, Kato Cohomology, and Motivic Cohomology

For a very general class of schemes K. Kato [43, §1] introduced complexes that generalized to higher dimensions classical exact sequences for Galois cohomology. He made some conjectures on their exactness in various situations. In this section, we recall some finiteness results about their cohomology that are easily obtained using finiteness of étale cohomology, and we explain their relation to motivic cohomology.

3.1.1 Kato Complexes

First a remark about the implications of assuming 2 is invertible.

Remark 3.1. Recall that when X is a scheme, we say that 2 *is invertible on X* when 2 is a unit in the global sections $\Gamma(X, \mathcal{O}_X)$.

(a) When X is of finite type over $\mathbb{Z}$, saying that 2 is invertible on X is the same as saying that X is of finite type over $\mathbb{Z}[\frac{1}{2}]$. Furthermore, when X is of finite type over $\mathbb{F}_p$ $(p > 2)$, it follows that 2 is invertible on X and that X is of finite type over $\mathbb{Z}$ (as $\mathbb{F}_p$ is of finite type over $\mathbb{Z}$, and compositions of finite type morphisms are of finite type), hence X is of finite type over $\mathbb{Z}[\frac{1}{2}]$.

(b) From the assumption 2 is invertible on X, it also follows that each residue field $k(x)$ of X has characteristic different from 2, and that there is an isomorphism of $\mathrm{Gal}(k(x)_s|k(x))$-modules $\mathbb{Z}/2\mathbb{Z} \simeq \mu_2 := \{a \in k(x)_s | a^2 = 1\}$, where $k(x)_s$ denotes a separable closure of $k(x)$.

(c) When 2 is invertible on X, in the global sections $-1 \neq 1$, so -1 determines an isomorphism of étale sheaves $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$, hence $\mu_2^{\otimes n}$ is isomorphic $\mathbb{Z}/2\mathbb{Z}$. Recall that on a scheme X, isomorphisms between the étale sheaf μ_2 and the

constant sheaf $\mathbb{Z}/2\mathbb{Z}$ correspond to global sections of X which have order 2 on each connected component [66, see p. 100 for definitions and details].

Next, we recall what we mean by the residue and corestriction maps.

Definition 3.2. Let A be a discrete valuation ring (DVR) with fraction field K and residue field k. Assume that $\text{char}(k) \neq 2$. By *the residue homomorphism for A*, we mean the group homomorphism from the Galois cohomology of K to the Galois cohomology of the residue field k

$$\partial_n : H^n_{Gal}(K, \mathbb{Z}/2\mathbb{Z}) \to H^{n-1}_{Gal}(k, \mathbb{Z}/2\mathbb{Z}),$$

as defined in [43, p. 149]. Note that this is the same as the definition given in [3, p. 475]. When $n = 0$, we set $H^{n-1}(k, \mathbb{Z}/2\mathbb{Z}) = 0$.

Definition 3.3. Let F be a finite extension of a field L. By *the corestriction homomorphism for the finite extension L/K*, we mean the group homomorphism

$$cor_{L/K} : H^n_{Gal}(L, \mathbb{Z}/2\mathbb{Z}) \to H^n_{Gal}(K, \mathbb{Z}/2\mathbb{Z}),$$

as defined in [3, p. 471]. This agrees with the definition used by K. Kato in [43].

Recall that a locally noetherian scheme X is said to be excellent [38, Definition 7.8.5] if for some covering of X by affine schemes $U_\alpha = \text{Spec}(A_\alpha)$, each of the rings A_α are excellent [38, Definition 7.8.2].

The next example is important for understanding the definition of the differentials in the Kato complex.

Example 3.4. Let X be a noetherian excellent scheme, let $y \in X$ be a point of X, and let $Z := \overline{\{y\}}$ denote the reduced closed subscheme with underlying topological space $\overline{\{y\}}$. Since X is excellent, every locally finite-type X-scheme X' is excellent [38, Proposition 7.8.6]. In particular, via the closed immersion $Z \to X$,

Z is excellent. Therefore, Z is an integral excellent scheme. For an integral excellent ring A, its integral closure in $\mathrm{Frac}(A)$ is a finite A-algebra [38, Scholie 7.8.3]. It follows that the normalization morphism $Z' \to Z$ is finite [48, Theorem 2.39 (d)]. In particular, the normalization is quasi-finite, so the fiber over any point $x \in Z$ has only finitely many points $x_1, \ldots, x_n$ and for each of the x_i the field extension $k(x_i)/k(x)$ is a finite extension.

Definition 3.5. Let X be a scheme. Recall that the *dimension of a point $x \in X$* is defined to be the (combinatorial) dimension $\dim(x) := \dim(\overline{\{x\}})$ of the topological space defined by the closure of x. The set of dimension p points of X is denoted by X_p. The *codimension of a point $x \in X$* is defined to be the Krull dimension $\mathrm{codim}(x):=\dim(\mathcal{O}_{X,x})$ of the local ring of X at the point $x \in X$. This is equal to the topological codimension of the closed subspace $\overline{\{x\}}$ in X [38, Proposition 5.12]. The set of codimension p points of X will be denoted by X^p.

Definition 3.6 (The yx-component ∂_{yx} of the differential)**.** Let X be a noetherian excellent scheme with 2 invertible. Recall the facts of Example 3.4. Let $x \in X^{p+1}$ and $y \in X^p$ such that $x \in \overline{\{y\}}$. Let $Z := \overline{\{y\}}$ denote the reduced closed subscheme with underlying topological space $\overline{\{y\}}$. Let $Z' \to Z$ be the normalization of Z. The field extensions $k(x_i)/k(x)$ for each of the finitely many points $x_1, \ldots, x_n \in Z'$ lying over $x \in Z$ are finite extensions, so for all non-negative integers $j \in \mathbb{Z}$, there are well-defined corestriction maps (Definition 3.3) $cor_{k(x_i)/k(x)} : H^j(k(x_i), \mathbb{Z}/2\mathbb{Z}) \to H^j(k(x), \mathbb{Z}/2\mathbb{Z})$. Each $x_i \in Z'$ is of codimension 1 in Z' (use the dimension formula [49, Theorem 15.6] together with the fact that the extension $k(x_i)/k(x)$ is finite, hence of transcendence degree 0). So the localization $\mathcal{O}_{Z',x_i}$ of the normalization at the point x_i is a DVR, with fraction field $k(y)$ and residue field $k(x_i)$.

Hence, each x_i defines residue homomorphisms (Definition 3.2)

$$\partial^{x_i} : H^j_{Gal}\left(k\left(y\right), \mathbb{Z}/2\mathbb{Z}\right) \to H^{j-1}_{Gal}\left(k\left(x_i\right), \mathbb{Z}/2\mathbb{Z}\right)$$

for all non-negative integers $j \in \mathbb{Z}$. The yx-component d_{yx} is defined (c.f. [41, §0.6]) as

$$d_{yx} := \sum_{x_i | x} cor_{k(x_i)/k(x)} \circ \partial^{x_i},$$

where the sum is taken over the finitely many points $x_i \in Z'$ lying over x.

Definition 3.7 (Cohomological Kato complexes)**.** Let X be a noetherian excellent scheme, finite dimensional of dimension d. We assume that 2 is invertible on X (this is not necessary for the definition in general). It follows from this assumption that for every $n \geq 0$ the Tate twist $\mu_2^{\otimes n}$ is isomorphic to the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ (see Remark 3.1 (b)). The *n-th Kato complex* is defined to be the complex

$$C(X, H^n) := \bigoplus_{x \in X^0} H^n(k(x), \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{x \in X^1} H^{n-1}(k(x), \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

$$\cdots \to \bigoplus_{x \in X^d} H^{n-d}(k(x), \mathbb{Z}/2\mathbb{Z}),$$

where $k\left(x\right)$ denotes the residue field of a point $x \in X$, and we set $H^m(k(x), \mathbb{Z}/2\mathbb{Z}) = 0$ for $m < 0$. Kato complexes are often indexed homologically, but here we will always use cohomological indexing by placing the term summing over the codimension p points in degree p. The m-th cohomology of the n-th Kato complex $C(X, H^n)$ will be denoted by $H^m(C(X, H^n))$. The differential is defined componentwise. The yx-component $\partial_i^{yx} : H^i\left(k\left(y\right), \mathbb{Z}/2\mathbb{Z}\right) \to H^{i-1}\left(k\left(x\right), \mathbb{Z}/2\mathbb{Z}\right)$ of the i-th differential is defined as follows: If $x \notin \overline{\{y\}}$, then set $d_{yx} = 0$. If $x \in \overline{\{y\}}$, then d_{yx} is defined as in Definition 3.6.

Remark 3.8. When X is a variety over a field, by definition the Kato complexes are the same as Rost's cycle complexes for the cycle module defined by Galois

cohomology [58, see §2.10 for the definition of the differential, as well as Remark (2.5)].

Finally, we recall some conditions under which the dimension can be replaced by the codimension in the definition of the Kato complexes.

Definition 3.9. Let X be a scheme. The scheme X is said to be *biequidimensional* [37, §14 p. 11] if it is finite dimensional, equidimensional (*aka* pure, *i.e.* the dimension of each irreducible component is the same), equicodimensional (the codimension of each minimal closed irreducible set in X is the same), and catenary (see [37, §14 p. 11]).

Lemma 3.10 (Corollaire 14.3.5 EGA IV Premiére Partie). *For any noetherian biequidimensional scheme X of dimension d and for any point $x \in X$, the dimension and codimension of x are related as follows:* $\dim(x) = d - \operatorname{codim}(x)$. *That is for any p, the set of dimension p points of X is equal to the set of codimension $d - p$ points $X_p = X^{d-p}$.*

Remark 3.11. There are examples of finite dimensional schemes which are regular (hence catenary) and integral (hence equidimensional) possessing points x for which $\dim(x) + \operatorname{codim}(x)$ is not equal to the dimension of the scheme [38, Remark 5.2.5(i)]. However, when X is a variety over a field, pure of dimension d, it is biequidimensional [38, follows from Proposition 5.2.1].

3.1.2 Relation to Étale Cohomology

Jannsen, Saito, and Sato showed that for very general schemes, the Kato complexes appear on the first page of the étale niveau spectral sequence. As we restate their result slightly to suit our purposes, we recall briefly their proof.

Proposition 3.12. *(See [41, Section 1.5 and Theorem 1.5.3].) Let X be a noetherian regular excellent $\mathbb{Z}[\frac{1}{2}]$-scheme, pure of dimension d. Filtering by codimension*

of support gives a convergent spectral sequence

$$E_1^{p,q}(X, \mathbb{Z}/2\mathbb{Z}) := \bigoplus_{x \in X^p} H_{\acute{e}t}^{q-p}(k(x), \mathbb{Z}/2\mathbb{Z}) \Longrightarrow H_{\acute{e}t}^{p+q}(X, \mathbb{Z}/2\mathbb{Z}),$$

with differential d_r of bidegree $(r, 1-r)$. Furthermore, the complexes appearing on the first page of the spectral sequence $E_{*,q}^1(X, \mathbb{Z}/2\mathbb{Z})$ agree up to signs with the Kato complexes $C(X, H^q)$, hence the second page of the spectral sequence consists of Kato cohomology $E_2^{p,q} = H^p(C(X, H^q))$.

Proof. For any noetherian scheme, pure of dimension d, one may construct a cohomological spectral sequence of the form (*e.g.*, [17, Section 1])

$$E_1^{p,q}(X, \mathbb{Z}/2\mathbb{Z}) := \bigoplus_{x \in X^p} H_x^{p+q}(X, \mathbb{Z}/2\mathbb{Z}) \Longrightarrow H_{\acute{e}t}^{p+q}(X, \mathbb{Z}/2\mathbb{Z}).$$

where $H_x^{p+q}(X, \mathbb{Z}/2\mathbb{Z})$ is defined to be the colimit, over all non-empty open subsets $U \subset X$ containing x, of the groups $H_{\overline{\{x\}} \cap U}^{p+q}(U, \mathbb{Z}/2\mathbb{Z})$. Since X, and hence $\overline{\{x\}}$, is excellent, there exists an open $U_0 \subset X$ such that $\overline{\{x\}} \cap U$ is regular for $U \subset U_0$. In this situation, if $x \in X^p$, then $\overline{\{x\}} \cap U$ is a codimension p embedding in U, hence by absolute purity [22, Theorem 2.1]

$$H_{\acute{e}t}^{p+q-2p}\left(\overline{\{x\}} \cap U, \mathbb{Z}/2\mathbb{Z}\right) \simeq H_{\overline{\{x\}} \cap U}^{p+q}(U, \mathbb{Z}/2\mathbb{Z})$$

so it follows that

$$E_1^{p,q}(X, \mathbb{Z}/2\mathbb{Z}) \simeq \bigoplus_{x \in X^p} H_{\acute{e}t}^{q-p}(k(x), \mathbb{Z}/2\mathbb{Z})$$

For any $y \in X^p$, $x \in X^{p+1}$, the yx-components of the differentials

$$H_{\acute{e}t}^{q-p}(k(y), \mathbb{Z}/2\mathbb{Z}) \to H_{\acute{e}t}^{q-p-1}(k(x), \mathbb{Z}/2\mathbb{Z})$$

commute, up to signs, with those of the Kato complex [41, Theorem 1.1.1]. This completes the proof. □

Corollary 3.13. *Let X be a separated scheme that is smooth (i.e. formally smooth and of finite type) over $\mathbb{Z}[\frac{1}{2}]$, pure of dimension d. The spectral sequence of Proposition 3.12 takes the form*

$$E_2^{p,q}(X, \mathbb{Z}/2\mathbb{Z}) = H_{Zar}^p(X, \mathcal{H}^q) \Longrightarrow H_{ét}^{p+q}(X, \mathbb{Z}/2\mathbb{Z}),$$

where $H_{Zar}^p(X, \mathcal{H}^q)$ denotes the Zariski cohomology of the Zariski sheaf $\mathcal{H}^q$ on X associated to the presheaf $U \mapsto H_{ét}^q(U, \mathbb{Z}/2\mathbb{Z})$. Hence, for all $p, q \in \mathbb{Z}$ the Kato cohomology groups $H^p(C(X, H^q))$ and the Zariski cohomology groups $H_{Zar}^p(X, \mathcal{H}^q)$ agree.

Proof. To prove the corollary, it suffices to show that the sheaf of complexes associated to the presheaves $U \mapsto E_1^{*,q}(U, \mathbb{Z}/2\mathbb{Z})$ is a flasque resolution of $\mathcal{H}^q$. A complex of sheaves is exact if and only if it is exact on stalks. So, it suffices to demonstrate that, for every point $x \in X$, the complex $E_1^{*,q}(\mathcal{O}_{X,x}, \mathbb{Z}/2\mathbb{Z})$ is exact in positive degrees and in degree zero $E_{0,q}^2 \simeq H_{ét}^q(\mathcal{O}_{X,x}, \mathbb{Z}/2\mathbb{Z})$. This is known as the Gersten conjecture. Since the morphism $X \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ is smooth, the local ring $\mathcal{O}_{X,x}$ is formally smooth and essentially of finite type over $\mathcal{O}_{\mathbb{Z}[\frac{1}{2}],y}$. The ring $\mathcal{O}_{\mathbb{Z}[\frac{1}{2}],y}$ is either a DVR or a field. In both cases, the Gersten conjecture is known. For the field case see, for example [15], and in the DVR case it was proved by Gillet [32]. $\qquad\square$

Lemma 3.14. *Let X be a smooth variety (i.e. separated, formally smooth and of finite type) over a finite field $\mathbb{F}_p$ ($p > 2$), pure of dimension d. For any codimension p point $x \in X^p$ of X, we have $\mathrm{cd}_2(k(x)) \leq 1 + d - p$, where $\mathrm{cd}_2(k(x))$ denotes the étale cohomological 2-dimension of the residue field of x. Considering the E_1 entries of the coniveau spectral sequence, if $q > d + 1$, then $E_1^{p,q} = 0$ for all $p \in \mathbb{Z}$, and hence the Kato complex $C(X, H^q)$ vanishes.*

Proof. Let $x \in X^p$ be a codimension p point of X. By [1, Exposé X, Theorem 2.1] $\mathrm{cd}_2(k(x)) \leq 1 + \mathrm{tr.deg}_{\mathbb{F}_p} k(x)$, where $\mathrm{cd}_2(k(x))$ denotes the étale cohomological 2-dimension of the residue field of $k(x)$ of x. As X is of finite type over a field, one has [38, Corollaire 5.2.3] that $\dim_x(X) = \dim(\mathcal{O}_{X,x}) + \mathrm{tr.deg}_{\mathbb{F}_p} k(x)$. It results from [37, Proposition 14.1.4] that $d = \dim(X) \geq \dim_x(X)$ for all $x \in X$. Hence, $d - p \geq \mathrm{tr.deg}_{\mathbb{F}_p} k(x)$, proving that $\mathrm{cd}_2(k(x)) \leq 1 + d - p$. This proves the lemma. $\square$

3.1.3   Finiteness Results for Kato Cohomology

The Kato conjecture is the assertion of the next proposition. It was recently proved by M. Kerz and S. Saito. We write it down for later reference.

Proposition 3.15. *(See [45, Theorem 8.1]). Let X be a regular connected scheme of Krull dimension d, proper over a finite field $\mathbb{F}_p$ $(p > 2)$. The Kato cohomology groups $H^p(C(X, H^{d+1}))$ vanish except when $p = d$, in which case $H^d(C(X, H^{d+1})) \simeq \mathbb{Z}/2\mathbb{Z}$.*

Next, recall the following fact about étale cohomology.

Lemma 3.16. *Let X be a finite dimensional regular separated scheme of finite type over $\mathbb{Z}[\frac{1}{2}]$. In this situation, the étale cohomology groups $H^m_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$ are finite groups for all $m \geq 0$.*

Proof. Let f denote the structural morphism $X \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. From the finiteness theorem [18, Théorèmes de Finitude, §1, Theorem 1.1] we have that, for all $q \geq 0$, the étale sheaves $R^q f_* \mathbb{Z}/2\mathbb{Z}$ are constructible. Using the Leray spectral sequence $E_1^{p,q} = H^p_{\acute{e}t}(\mathbb{Z}[\frac{1}{2}], R^q f_* \mathbb{Z}/2\mathbb{Z}) \Rightarrow H^{p+q}_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$ [18, Cohomologie étale, §2, p. 6], we reduce to proving that the étale cohomology groups of $\mathbb{Z}[\frac{1}{2}]$ with coefficients in a constructible sheaf are finite, which is known [50, Chapter 2, §3, Theorem 3.1 and following discussion]. $\square$

Finally, we recall the following well known finiteness results for Kato cohomology, which use the finiteness of étale cohomology together with the coniveau spectral sequence.

Lemma 3.17 (Absolute finiteness). *Let X be a pure regular separated scheme of finite type over a base scheme S. Consider the following situations:*

(a) $dim(X) \leq 1$, and $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$.

(b) $dim(X) \leq 2$, X is quasi-projective over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$.

(c) $dim(X) = d$, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$.

(d) $dim(X) = d$, X is quasi-projective, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$.

In situations (a) and (b), all the Kato cohomology groups of X are finite. In situation (c), the Kato cohomology group $H^d(C(X, H^{d+1}))$ is finite, and in situation (d), the Kato cohomology group $H^d(C(X, H^d))$ is finite

Proof. In all situations, X satisfies the hypotheses of Lemma 3.16, hence the étale cohomology of X is finite. Now consider the coniveau spectral sequence for étale cohomology of Proposition 3.12. In situation (a), all differentials on the second page of the spectral sequence are zero because $\dim(X) \leq 1$. So, the spectral sequence collapses on the second page. Hence, as the abutment is finite (Lemma 3.16), the Kato cohomology groups are finite.

For (b), using Lemma 3.14 we see that there is only one possibly non-zero differential $d_2 : H^0(C(X, H^3)) \to H^2(C(X, H^2))$ on the second page of the spectral sequence. It follows that all the other Kato cohomology groups appear on the stable page of the spectral sequence, hence are quotients of the induced filtration on étale cohomology, so they are finite. The kernel and cokernel of d_2 appear on

the stable page, hence are finite. Therefore $H^0(C(X, H^3))$ is finite if and only if $H^2(C(X, H^2))$ is finite. The group $H^2(C(X, H^2))$, that is, $E_2^{2,2}$, is isomorphic to the mod 2 Chow group $CH^2(X)/2$ of codimension 2 cycles [15, Theorem 7.7], which is finite for X quasi-projective over a finite field [45, Corollary 9.4(1)].

Now, assume we are in situation (c). From Lemma 3.14, it follows that the differentials on the second page of the spectral sequence that are entering and leaving the group $H^d(C(X, H^{d+1}))$ are zero. So, $H^d(C(X, H^{d+1}))$ appears on the stable page, hence is finite since the abutment is finite.

Finally, assume we are in situation (d). As in the proof of case (b), the group $E_2^{d,d} = H^d(C(X, H^d))$ is isomorphic to $CH^d(X)/2$ [15, Theorem 7.7], hence is finite for X quasi-projective over a finite field [45, Corollary 9.4(1)]. This completes the proof of the lemma. □

3.1.4 Relation to Motivic Cohomology

First recall the definition of the motivic cohomology of a smooth scheme over a Dedekind domain. Let X be a scheme that is separated and smooth over a Dedekind domain D. The standard algebraic m-simplex will be denoted by

$$\triangle_D^m := \mathrm{Spec}(D[t_0, t_1, \ldots, t_m]/\Sigma_i t^i - 1)$$

and the free abelian group on closed integral subschemes of codimension n in $X \times_D \triangle_D^m$, which intersect all faces properly, will be denoted by $z^n(X, m)$. Placing $z^n(X, 2n - m)$ in degree m, the associated complex of presheaves is denoted by $\mathbb{Z}(n)$, and we set $\mathbb{Z}/2\mathbb{Z}(n) := \mathbb{Z}(n) \otimes^{\mathbb{L}} \mathbb{Z}/2\mathbb{Z}$. The complex $\mathbb{Z}/2\mathbb{Z}(n)$ is in fact a complex of sheaves for the étale topology [23, Lemma 3.1], and when considered as a complex of sheaves for the Zariski topology it will be denoted by $\mathbb{Z}/2\mathbb{Z}(n)_{Zar}$.

Definition 3.18. The *motivic cohomology groups of X with mod 2 coefficients* $H^m_{mot}(X, \mathbb{Z}/2Z(n))$ are defined to be the hypercohomology groups of the complex of Zariski sheaves $\mathbb{Z}/2Z(n)_{Zar}$.

Remark 3.19. In this remark we explain an observation of Totaro's [67, Theorem 1.3 and surrounding discussion], that the Beilinson-Lichtenbaum conjecture leads to the long exact sequence (3.2) below. Let X be a separated scheme that is smooth over $D := \mathbb{Z}[\frac{1}{2}]$. Let $\pi : (Sm/D)_{\acute{e}t} \to (Sm/D)_{Zar}$ denote the natural morphism of sites.

(a) By the *Beilinson-Lichtenbaum conjecture with $\mathbb{Z}/2\mathbb{Z}$-coefficients*, we mean that there is a quasi-isomorphism $(\mathbb{Z}/2\mathbb{Z}(n))_{Zar} \simeq \tau_{\leq n} R\pi_* \mathbb{Z}/2\mathbb{Z}$ of complexes of Zariski sheaves on X. Recall that $R\pi_* \mathbb{Z}/2\mathbb{Z}$ is the complex of Zariski sheaves obtained by first taking an injective resolution $I^\bullet$ of the étale sheaf $\mathbb{Z}/2\mathbb{Z}$, from which we obtain an exact complex of étale sheaves. Then, apply π_* to this complex to obtain a complex of Zariski sheaves (no longer exact). The cohomology of this complex in degree i is the right derived functor $R^i \pi_* \mathbb{Z}/2\mathbb{Z}$, which is isomorphic to the Zariski sheaf $\mathcal{H}^i$ associated to the presheaf $U \mapsto H^i_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$ [66, I, Proposition 3.7.1]. The complex $\tau_{\leq n} R\pi_* \mathbb{Z}/2\mathbb{Z}$ is a complex of Zariski sheaves with cohomology in degree i equal to $R^i \pi_* \mathbb{Z}/2\mathbb{Z}$ when $i \leq n$ and zero otherwise. It follows that there is a distinguished triangle in the derived category of Zariski sheaves on X

$$\tau_{\leq n-1} R\pi_* \mathbb{Z}/2\mathbb{Z} \to \tau_{\leq n} R\pi_* \mathbb{Z}/2\mathbb{Z} \to \mathcal{H}^n[-n] \qquad (3.1)$$

then from the associated long exact sequence in hypercohomology, if the Beilinson-Lichtenbaum conjecture with $\mathbb{Z}/2\mathbb{Z}$-coefficients holds, we obtain

the long exact sequence (c.f. [67, Theorem 10.3])

$$\cdots \to H^{m+n}_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n-1)) \to H^{m+n}_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n)) \to \quad (3.2)$$

$$H^m_{Zar}(X, \mathcal{H}^n) \to H^{m+n+1}_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n-1)) \to \cdots$$

where $H^m_{Zar}(X, \mathcal{H}^n)$ denotes the Zariski cohomology of the Zariski sheaf $\mathcal{H}^n$ associated to the presheaf $U \mapsto H^n_{\acute{e}t}(U, \mathbb{Z}/2\mathbb{Z})$.

(b) For smooth schemes over fields, the Beilinson-Lichtenbaum conjecture is known, since it is equivalent to the Bloch-Kato conjecture [42, Theorem 19], and the Bloch-Kato conjecture is known [42, see Theorem 21 and surrounding discussion for an overview].

Lemma 3.20. *Let X be a pure separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$. Recall that $\mathcal{H}^q$ denotes the Zariski sheaf associated to the presheaf $U \mapsto H^q_{\acute{e}t}(U, \mathbb{Z}/2Z)$. Consider the following statements:*

(a) The Zariski cohomology groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite for all $p, q \in \mathbb{Z}$;

(b) The motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$;

(c) The Beilinson-Lichtenbaum conjecture with $\mathbb{Z}/2\mathbb{Z}$-coefficients is true (see Remark 3.19).

We have the implication (a) implies (b). Furthermore if we assume (c), then (a) is equivalent to (b). Hence, by Corollary 3.13, (b) is equivalent to finiteness of the Kato cohomology groups $H^p(C(X, H^q))$ for all $p, q \in \mathbb{Z}$.

Proof. To prove that (a) implies (b), recall that there is a coniveau spectral sequence [23, see §4 for integral coefficients version]

$$E^{p,q}_1(X, \mathbb{Z}/2\mathbb{Z}(n)_{Zar}) := \bigoplus_{x \in X^p} H^{2n-p+q}_{mot}(k(x), \mathbb{Z}/2\mathbb{Z}(n-p)) \Longrightarrow H^{p+q}_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n),$$

and the sheaf of complexes associated to the presheaf $U \mapsto E_1^{*,q}\left(U, \mathbb{Z}/2\mathbb{Z}(n)_{Zar}\right)$ gives a flasque resolution of the sheaf $\mathcal{H}^q$ [23, Theorem 1.2 (2),(4) and (5), also see remark at start of page 775], hence the Zariski cohomology groups $H^p_{Zar}(X, \mathcal{H}^q)$ are the only groups on the E_2 page of the above spectral sequence, which converges to the motivic cohomology groups, and it follows that (a) implies (b).

Now assume (c), from which we obtain the long exact sequence 3.2 (see Remark 3.19), from which it follows that (b) implies (a). $\qquad\square$

Next we recall some finiteness theorems relating the Kato cohomology to motivic cohomology in the cases that finiteness of these groups is only partially known.

Lemma 3.21 (Relative finiteness)**.** *Let X be a pure separated scheme that is smooth over a base scheme S. Consider the following situations:*

(a) *$dim(X) \leq 2$, no residue field of X is formally real, $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;*

(b) *$dim(X) \leq 3$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$;*

(c) *$dim(X) \leq 4$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$.*

In situation (a):

(i) *The groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^0_{Zar}(X, \mathcal{H}^5)$ are finite if and only if the groups $H^2_{Zar}(X, \mathcal{H}^2)$, $H^2_{Zar}(X, \mathcal{H}^3)$, and $H^2_{Zar}(X, \mathcal{H}^4)$ are finite. Furthermore, all the other groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite.*

(ii) *If all the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^0_{Zar}(X, \mathcal{H}^5)$ are finite, then the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(p))$ are finite for all $p, q \in \mathbb{Z}$. Assuming the Beilinson-Lichtenbaum conjecture (see Remark 3.19 for an explanation of what we mean by this), the converse is true.*

In situation (b):

(i) The group $H^0_{Zar}(X, \mathcal{H}^3)$ is finite if and only if the group $H^2_{Zar}(X, \mathcal{H}^2)$ is finite. Furthermore, all the other cohomology groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite.

(ii) The motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(p))$ are finite for all $p, q \in \mathbb{Z}$ if and only if the group $H^0_{Zar}(X, \mathcal{H}^3)$ is finite.

In situation (c):

(i) The groups $H^2_{Zar}(X, \mathcal{H}^2)$, $H^2_{Zar}(X, \mathcal{H}^3)$, and $H^3_{Zar}(X, \mathcal{H}^3)$ are finite if and only if all the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^1_{Zar}(X, \mathcal{H}^4)$ are finite. Furthermore, all the other groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite.

(ii) The motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(p))$ are finite for all $p, q \in \mathbb{Z}$ if and only if all the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^1_{Zar}(X, \mathcal{H}^4)$ are finite.

Proof. In all situations, for (ii), finiteness of motivic cohomology implies finiteness of the Zariski cohomology groups by Lemma 3.20. Also, in each situation, to prove (ii), it suffices to prove (i), for if groups named in (i) are finite then all the groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite, hence the motivic cohomology groups are finite by Lemma 3.20. To prove (i), we work with the coniveau spectral sequence for étale cohomology of Proposition 3.12.

First assume that we are in situation (a). The étale cohomological 2-dimension of X is less than or equal to $2\dim(X) + 1$ [1, Exposé 5, §6, Theorem 6.2], from which it follows that whenever $q > 2\dim(X) + 1$, we obtain vanishing of the Zariski sheaf $\mathcal{H}^q$, and hence vanishing of $H^p_{Zar}(X, \mathcal{H}^q)$. This, together with the fact that $\dim(X) \leq 2$, gives that there are only three possibly non-zero differentials on the second page of the spectral sequence, each having domain and codomain one of the groups named in (i). This proves the second statement of (i). To prove

the first statement of (i), we prove finiteness of the kernel and cokernel of these differentials. To prove this claim, observe that the kernel and cokernel of these differentials appear on the third page of the spectral sequence, and the spectral sequence collapses on the third page. As the abutment is finite (Lemma 3.16) this proves the claim, finishing the proof of (i).

Assume that we are in situation (b). Then Lemma 3.14, Proposition 3.15, and the fact that $\dim(X) \leq 3$, give that there is only one possible non-zero differential on the second page of the spectral sequence. The domain of this differential is the group $H^0_{Zar}(X, \mathcal{H}^3)$. By the same argument used in the previous situation, this differential has finite kernel and cokernel, which concludes the proof in situation (b).

Finally, assume that we are in situation (c). Again, Lemma 3.14, Proposition 3.15, and the fact that $\dim(X) \leq 4$, give that there are only three possibly non-zero differentials on the second page of the spectral sequence, each having domain one of the groups named in the lemma. As before, the kernel and cokernel of these differentials are finite. Hence, this concludes the proof in the case of situation (c). $\square$

3.2 Arason's Theorem

In this section, for an excellent scheme X with 2 invertible, we recall the definition of the complex of abelian groups $C(X, \overline{I}^n)$. Arason essentially showed in [3] that if the Bloch-Kato conjecture is true, then $C(X, \overline{I}^n)$ is isomorphic to the Kato complex $C(X, H^n)$. We call this result Arason's theorem (see Theorem 3.26).

We first recall the definitions of the maps e^n, s^n, and h^n, relating Galois cohomology, Witt groups, and Milnor K-theory.

3.2.1 Galois Cohomology: Definition of h^1

Let k be a field having char $(k) \neq 2$ and let G denote the absolute Galois group $G :=$ Gal (k_s/k), where k_s denotes a separable closure of k. Let $\mu_2 := \{a \in k_s | a^2 = 1\}$ denote the group of square roots of unity in k_s. Let $\mathbf{G}_m$ denote the multiplicative group k_s^* of units of k_s. The exact sequence of G-modules

$$1 \to \mu_2 \to \mathbf{G}_m \overset{2}{\to} \mathbf{G}_m \to 1,$$

where 2 denotes the endomorphism $x \mapsto x^2$, induces a long exact sequence in cohomology, from which we obtain the exact sequence (as G acts by evaluation on elements of $\mathbf{G}_m$, $H^0_{Gal}(k, \mathbf{G}_m) = k^*$)

$$k^* \overset{2}{\to} k^* \to H^1(k, \mu_2) \to H^1_{Gal}(k, \mathbf{G}_m). \tag{3.3}$$

Since $H^1_{Gal}(k, \mathbf{G}_m) = 0$ [65, Chapter 1, §1.2, Proposition 1], the exact sequence (3.3) induces the isomorphism

$$k^*/k^{*2} \overset{\cong}{\to} H^1_{Gal}(k, \mu_2). \tag{3.4}$$

After identifying μ_2 with $\mathbb{Z}/2\mathbb{Z}$, the isomorphism (3.4) is denoted by

$$h^1_k : k^*/k^{*2} \overset{\cong}{\to} H^1_{Gal}(k, \mathbb{Z}/2\mathbb{Z}), \tag{3.5}$$

and is said to be the *norm-residue homomorphism in degree one*.

3.2.2 Witt Groups: Definition of s^1

Let k be a field having char $(k) \neq 2$. The fundamental ideal $I(k)$ is defined to be the kernel of the mod 2 rank map $W(k) \to \mathbb{Z}/2\mathbb{Z}$. The q-th quotients $I^q(k)/I^{q+1}(k)$ of the powers of the fundamental ideal will be denoted by $\overline{I}^q(k)$. Taking the quotient by the kernel of the surjective dimension homomorphism induces an isomorphism $\overline{I}^0(k) = W(k)/I(k) \cong \mathbb{Z}/2\mathbb{Z}$.

Every unit $a \in k^*$ determines a non-degenerate symmetric bilinear form on k given by $\mathfrak{b}(k_1, k_2) := ak_1 k_2$, and this form is denoted by $\langle a \rangle$, and the orthogonal sum of n such forms $\langle a_i \rangle$, where $a_i \in k^*$, is denoted by $\langle a_1, \ldots, a_n \rangle$. The diagonal forms $\langle 1, -a \rangle$, where $a \in k^*$, are denoted by $\langle\langle a \rangle\rangle$, and are said to be Pfister forms. The n-fold products of Pfister forms $\langle\langle a_i \rangle\rangle$ are denoted by $\langle\langle a_1, \ldots, a_n \rangle\rangle$. It is a classic theorem that, for any $p \geq 0$, the p-th power $I^p(k)$ of the fundamental ideal is generated by p-fold Pfister forms $\langle\langle a_1, \ldots, a_p \rangle\rangle$.

Define $s_1 : k^*/k^{*2} \to \overline{I}^1(k)$ by the assignment sending the class of a unit $a \in k^*$ to the class of the Pfister form $\langle\langle a \rangle\rangle$ in $\overline{I}^1(k)$. This is a well-defined isomorphism [19, Proof of Proposition 4.13], with inverse the signed determinant $\mathfrak{b} \mapsto (-1)^{\frac{\dim(\mathfrak{b})}{2}} \det(\mathfrak{b})$.

3.2.3 Milnor K-theory

Let k be a field. The n-th Milnor K-group $K_n^M(k)$ of k is defined to the abelian group defined by the following generators and relations: The generators are length n sequences $\{a_1, \ldots, a_n\}$ of units $a_i \in k^*$ (called symbols), and the relations are *multilinearity*

$$\{a_1, \ldots, a_{j-1}, xy, a_{j+1}, \ldots, a_n\} = \{a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n\}$$
$$+ \{a_1, \ldots, a_{j-1}, y, a_{j+1}, \ldots, a_n\} \text{ for all } a_i, x, y \in k^* \text{ and } 1 \leq j \leq n;$$

and the *Steinberg relation* $\{a_1, \ldots, x, \ldots, 1-x, \ldots, a_n\} = 0$ for all $a_i \in k^*$, and $x \in k - \{0, 1\}$.

3.2.4 The Maps s^n and h^n

Assume char $(k) \neq 2$. Consider the assignments

$$\{a_1, \ldots, a_n\} \mapsto \langle a_1, \ldots, a_n \rangle := s_1(a_1) \otimes \ldots \otimes s_1(a_n)$$

and

$$\{a_1, \ldots, a_n\} \mapsto (a_1, \ldots, a_n) := h_1(a_1) \cup \ldots \cup h_1(a_n)$$

that send the class of the symbol $\{a_1, \ldots, a_n\}$ to the class of the n-fold Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ and the class of the symbol $\{a_1, \ldots, a_n\}$ to the Galois cohomology class $(a_1, \ldots, a_n)$, respectively. It is a classic fact that these maps respect the Steinberg and multilinearity relations, and send 2 to 0 [51]. It follows from the definition of the Milnor K-groups by generators and relations, that for all $n \geq 0$, the assignments above induce unique group homomorphisms

$$s^n : K_n^M(k)/2K_n^M(k) \to \overline{I}^n(k)$$

and

$$h^n : K_n^M(k)/2K_n^M(k) \to H_{Gal}^n(k, \mathbb{Z}/2).$$

We know that the homomorphism s^n is an isomorphism [53], and from the work of V. Voevodsky [68, Corollary 7.5], we know that h_n is an isomorphism.

Definition 3.22. Define $e_k^n : \overline{I}^n(k) \to H_{Gal}^n(k, \mathbb{Z}/2)$ to be the composition

$$\overline{I}^n(k) \xrightarrow{s_n^{-1}} K_n^M/2K_n^M(k) \xrightarrow{h^n} H^n(k, \mathbb{Z}/2\mathbb{Z}).$$

The homomorphism e_k^n is an isomorphism, and from the definition of s_n and h_n, the homomorphism e_k^n sends the class of a Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ to the Galois cohomology class $(a_1, \ldots, a_n)$, hence agrees with e^n as defined by Arason in [3, p. 456].

3.2.5 Cycle Complexes with Coefficients in $\overline{I}^n$

We start by recalling what we mean by the residue and corestriction maps in the setting of Witt groups.

Definition 3.23. Let A be a DVR with fraction field K and residue field k, with $\text{char}(k) \neq 2$. For every uniformizing element $\pi \in A$, there is an associated group homomorphism

$$\partial_\pi : W(K) \to W(k),$$

satisfying

$$\partial_\pi \left(I^n \left(K \right) \right) \subset I^{n-1} \left(k \right),$$

and the induced homomorphism of abelian groups

$$\partial_\pi : \overline{I}^n \left(K \right) \to \overline{I}^{n-1} \left(k \right)$$

is independent of the choice of uniformizing element π [3, Satz 3.1], hence is said to be *the residue homomorphism.*

Definition 3.24. For any finite field extension L/K and any non-trivial K-linear morphism $s : L \to K$ (see first sentence of the proof of Satz 3.3 for the fact that such a non-trivial K-linear morphism exists), the induced homomorphism on Witt groups $s_* : W(L) \to W(K)$ induces a homomorphism of groups

$$cor_{L/K} : \overline{I}^n \left(L \right) \to \overline{I}^n \left(K \right),$$

which is independent of s [3, Satz 3.3], hence is defined to be *the corestriction for the finite field extension L/K.*

We proceed, as we did with the Kato complexes, by simply defining the yx-component of the differential.

Definition 3.25. Let X be an excellent scheme, finite dimensional of dimension d, with 2 invertible on X. Recall the notation of Definition 3.6. We define a sequence (one way to see that it is a complex is to use Arason's theorem below) of abelian groups

$$C^*(X, \overline{I}, n) := \bigoplus_{x \in X^0} \overline{I}^n \left(k \left(x \right) \right) \xrightarrow{d} \bigoplus_{x \in X^1} \overline{I}^{n-1} \left(k \left(x \right) \right) \xrightarrow{d} \cdots \cdots \xrightarrow{d} \bigoplus_{x \in X^d} \overline{I}^{n-d} \left(k \left(x \right) \right)$$

by defining the differentials componentwise. Set $\overline{I}^m \left(k \left(x \right) \right) = 0$ for $m < 0$. For $y \in X^p$, $x \in X^{p+1}$, define the yx-component

$$d_{yx} : \overline{I}^i \left(k \left(y \right) \right) \longrightarrow \overline{I}^{i-1} \left(k \left(x \right) \right)$$

as follows: If $x \notin \overline{\{y\}}$, then define $d_{yx} = 0$. If $x \in \overline{\{y\}}$, then define

$$d_{yx} := \sum_{x_i | x} cor_{k(x_i)/k(x)} \circ \partial^{x_i},$$

where ∂^{x_i} denotes the residue map of Definition 3.23, and $cor_{k(x_i)/k(x)}$ the core-striction of Definition 3.24. These complexes are called the *cycle complexes with coefficients in $\overline{I}^n$*.

Now we are able to state and prove Arason's theorem.

Theorem 3.26 (Arason's theorem)*. Let X be a noetherian excellent scheme with 2 invertible in the global sections of X. If the Bloch-Kato conjecture is true, then the maps $e^n_{k(x)}$ define, for all $n \geq 0$, an isomorphism of complexes $e^n : C(X, \overline{I}^n) \xrightarrow{\widetilde{\to}} C(X, H^n)$, from the cycle complex with coefficients in $\overline{I}^n$ to the Kato complex.*

Proof. Fix $n \geq 0$. The map $e^n : C(X, \overline{I}^n) \to C(X, H^n)$ is defined in the obvious way. On the degree i terms, it is

$$\bigoplus_{x \in X_i} \overline{I}^{n-i} (k(x)) \overset{\oplus e^{n-i}_{k(x)}}{\to} \bigoplus_{x \in X_i} H^{n-i}_{Gal} (k(x), \mathbb{Z}/2\mathbb{Z})$$

where $\oplus e^{n-i}_{k(x)}$ sums over the set X_i. To prove the theorem, we must prove that e^n defines a map of complexes.

Since the differentials are defined componentwise, it suffices to prove that the diagram below commutes

$$
\begin{array}{ccc}
\overline{I}^i (k(y)) & \xrightarrow{\;\;\;d_{yx}\;\;\;} & \overline{I}^{i-1} (k(x)) \\
{\scriptstyle e^i_{k(y)}} \downarrow & & \downarrow {\scriptstyle e^{i-1}_{k(x)}} \\
H^i_{Gal} (k(y), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\;\;\;d_{yx}\;\;\;} & H^{i-1}_{Gal} (k(x), \mathbb{Z}/2\mathbb{Z})
\end{array}
\tag{3.6}
$$

for every pair of integers i, p, and every $x \in X^p$, $y \in X^{p+1}$. If $x \notin \overline{\{y\}}$, then both d_{yx} components are zero by definition, so the diagram commutes. If $x \in \overline{\{y\}}$, then,

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by definition,

$$d_{yx} := \sum_{x_i | x} cor_{k(x_i)/k(x)} \circ \partial^{x_i}$$

so

$$e^{i-1}_{k(x)} \circ d_{yx} = \sum_{x_i | x} e^{i-1}_{k(x)} \circ cor_{k(x_i)/k(x)} \circ \partial^{x_i}$$

because $e^i_{k(x)}$ is a group homomorphism. Now we explain why, to prove that diagram 3.6 commutes, it suffices to show that both squares of the diagram below commute

$$
\begin{array}{ccccc}
\overline{I}^i(k(y)) & \xrightarrow{\partial^{x_i}} & \overline{I}^{i-1}(k(x_i)) & \xrightarrow{cor_{k(x_i)/k(x)}} & \overline{I}^{i-1}(k(x)) \\
\downarrow{\scriptstyle e^i_{k(y)}} & & \downarrow{\scriptstyle e^{i-1}_{k(x_i)}} & & \downarrow{\scriptstyle e^{i-1}_{k(x)}} \\
H^i_{Gal}(k(y), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial^{x_i}} & H^{i-1}_{Gal}(k(x_i), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{cor_{k(x_i)/k(x)}} & H^{i-1}_{Gal}(k(x), \mathbb{Z}/2\mathbb{Z})
\end{array}
\tag{3.7}
$$

for every x_i lying over x. Assume they do, that is,

$$e^{i-1}_{k(x)} \circ cor_{k(x_i)/k(x)} \circ \partial^{x_i} = cor_{k(x_i)/k(x)} \circ \partial^{x_i} \circ e^i_{k(y)},$$

for every x_i lying over x. Hence,

$$
\begin{aligned}
d_{yx} \circ e^i_{k(y)} &= \sum_{x_i | x} cor_{k(x_i)/k(x)} \circ \partial^{x_i} \circ e^i_{k(y)} \\
&= \sum_{x_i | x} e^{i-1}_{k(x)} \circ cor_{k(x_i)/k(x)} \circ \partial^{x_i} \\
&= e^{i-1}_{k(x)} \circ d_{yx}.
\end{aligned}
$$

To finish the proof, recall the following results of Arason. For A a DVR with fraction field K and residue field k with char$(k) \neq 2$, the diagram

$$
\begin{array}{ccc}
\overline{I}^n(K) & \xrightarrow{\partial} & \overline{I}^{n-1}(k) \\
\downarrow{\scriptstyle e^n_K} & & \downarrow{\scriptstyle e^{n-1}_k} \\
H^n(K, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial} & H^{n-1}(k, \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

is commutative [3, Satz 4.11]. Additionally, when L/K is a finite field extension with $\mathrm{char}(K) \neq 2$, the diagram

$$
\begin{array}{ccc}
\overline{I}^n\,(L) & \xrightarrow{\;\;cor_{L/K}\;\;} & \overline{I}^n\,(K) \\
\downarrow{\scriptstyle e_L^n} & & \downarrow{\scriptstyle e_K^n} \\
H^n\,(L, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\;cor_{L/K}\;} & H^n\,(K, \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

is commutative [3, Satz 4.18]. It follows that both squares of Diagram 3.7 commute, which concludes the proof. $\qquad\square$

3.3   Finiteness Theorems for the Shifted Witt Groups

In this section, Arason's theorem is applied to Gille's graded Gersten-Witt spectral sequence. For more general schemes than for smooth varieties over fields, this allows the Witt groups to be related to the Zariski cohomology groups $H^p_{Zar}(X, \mathcal{H}^q)$, and hence, to the motivic cohomology groups. The following proposition was proved by S. Gille [30, §10], although (b) doesn't explicitly appear in [30], so it requires proof.

Proposition 3.27 (Gille's Graded Gersten-Witt Spectral Sequence)**.** *Let X be a noetherian regular excellent $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d.*

 (a) *There is a spectral sequence (not necessarily convergent)*

$$
E_1^{p,q} := H^{p+q}(C(X, I^p, \iota)/C(X, I^{p+1}, \iota)) \Rightarrow H^{p+q}(C(X, W, \iota))
$$

 where the abutment $H^{p+q}(C(X, W, \iota))$ is the cohomology of the Gersten-Witt complex, and the differential d_r has bidegree $(r, 1 - r)$.

 (b) *The complexes $C(X, I^p, \iota)/C(X, I^{p+1}, \iota)$ are isomorphic to the cycle complexes $C(X, \overline{I}^p)$, hence $E_1^{p,q} = H^{p+q}(C(X, \overline{I}^p))$.*

Proof. First we recall briefly the construction of the spectral sequence. The differentials of the complex $C(X, W, \iota)$ respect the filtration by powers of the funda-

mental ideal [30, Theorem 6.6], hence we obtain a filtered complex

$$C(X, I^n, \iota) := \bigoplus_{x \in X^0} I^n\left(k\left(x\right)\right) \xrightarrow{d} \bigoplus_{x \in X^1} I^{n-1}\left(k\left(x\right)\right) \xrightarrow{d} \dots \dots \xrightarrow{d} \bigoplus_{x \in X^d} I^{n-d}\left(k\left(x\right)\right),$$

(3.8)

where we set $I^m\left(k\left(x\right)\right) = W\left(k\left(x\right)\right)$ when $m \leq 0$. The exact sequence of complexes

$$0 \to C(X, I^{n+1}, \iota) \to C(X, I^n, \iota) \to C(X, I^n, \iota)/C(X, I^{n+1}, \iota) \to 0$$

determines a long exact sequence in cohomology

$$\to H^{p+q}(C(X, I^{n+1}, \iota)) \to H^{p+q}(C(X, I^n, \iota)) \to H^{p+q}(C(X, I^p, \iota)/C(X, I^{p+1}, \iota)) \to$$

and setting $E_1^{p,q} := H^{p+q}(C(X, I^p, \iota)/C(X, I^{p+1}, \iota))$, we obtain an exact couple which gives the spectral sequence of the proposition.

It remains to prove (b), that the quotient complexes obtained from $C(X, W, \iota)$ agree with the cycle complexes (note that the quotient complexes do not depend on the choices of isomorphisms ι [30, Definition 7.4 and Lemma 7.5]). For a smooth variety over a field, the cycle complexes are exactly Rost's cycle complexes for the cycle module $\overline{I}^*$, so in this situation, the assertion of (b) is exactly [30, §10.7]. Nevertheless, in the general case the proof is identical. First, recall that for integral excellent rings, the integral closure is finite in the fraction field. The components

$$d^{yx} : W\left(k\left(y\right)\right) \longrightarrow W\left(k\left(x\right)\right)$$

of the differentials of the complex $C(X, W, \iota)$ may be described as follows: If $y \notin \overline{\{x\}}$, then $d^{yx} = 0$. If $x \in \overline{\{y\}}$, then

$$d^{yx} := \sum_{x_i | x} cor_{k(x_i)/k(x)} \circ \partial^{x_i},$$

where ∂^{x_i} denotes the residue map of Definition 3.23 and $cor_{k(x_i)/k(x)}$ the corestriction of Definition 3.24 [30, conjugate Proposition 6.10 (taking $L = K$ and B to

be the integral closure of A), and Proposition 6.5]. From this description of the differential, together with Lemma 3.10, it follows that the n-th quotient complexes $C(X, I^n, \iota)/C(X, I^{n+1}, \iota)$ of the filtered complex $C(X, W, \iota)$ agree with the cycle complex $C(X, \overline{I}^n)$ of Definition 3.25. $\qquad\square$

Applying Arason's theorem (Theorem 3.26), we obtain the following corollary.

Corollary 3.28. *Maintain the hypotheses of Proposition 3.27. The spectral sequence of Proposition 3.27 (not necessarily convergent) takes the form*

$$E_1^{p,q} := H^{p+q}(C(X, H^p)) \Rightarrow H^{p+q}(C(X, W, \iota)),$$

where $H^{p+q}(C(X, H^p))$ is the Kato cohomology of the p-th Kato complex.

Corollary 3.29. *Let X be a pure separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$, and suppose that no residue field of X is formally real. In this case, the spectral sequence of Proposition 3.27 is convergent and takes the form*

$$E_1^{p,q} := H_{Zar}^{p+q}(X, \mathcal{H}^q) \Rightarrow H^{p+q}(C(X, W, \iota)),$$

where $\mathcal{H}^q$ denotes the Zariski sheaf associated to the presheaf $U \mapsto H_{\text{ét}}^q(U, \mathbb{Z}/2\mathbb{Z})$.

Proof. Applying Corollary 3.13, we have that $H^{p+q}(C(X, H^p)) = H_{Zar}^{p+q}(X, \mathcal{H}^q)$. Together with the previous Corollary 3.28, this yields the description of the E_1-terms. To prove convergence, note that as no residue field of X is formally real, the cohomological dimension of X is $2\dim(X) + 1$ [1, Exposé 5, §6, Theorem 6.2]. Hence, the groups $H_{Zar}^{p+q}(X, \mathcal{H}^q)$ vanish for $p > 2\dim(X) + 1$, from which it follows that the first page of the spectral sequence is bounded, and therefore the spectral sequence strongly converges. $\qquad\square$

Proposition 3.30. *Let X be a separated scheme that is pure and smooth over a scheme S. Consider the following situations:*

(a) $dim(X) \leq 1$, no residue field of X is formally real, and $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;

(b) $dim(X) \leq 2$, X is quasi-projective, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$.

In either situation, the Witt groups $W^n(X)$ of X are finite.

Proof. In cases (a) and (b), the Kato cohomology is finite by Lemma 3.17 (a) and (b), respectively. Hence, applying the convergent spectral sequence of Corollary 3.29, we obtain finiteness of the Gersten-Witt complex $C(X, W, \iota)$. To finish the proof, use the convergent coniveau spectral sequence (Eq. (2.4)). $\qquad\square$

Next we note the following consequence of Mayer-Vietoris.

Lemma 3.31. *Let $\mathcal{S}$ denote the category of noetherian regular separated $\mathbb{Z}[\frac{1}{2}]$-schemes.*

(a) *If, for any X in $\mathcal{S}$, $W^n(X)$ is finite for all n in $\mathbb{Z}$, then, for any X in $\mathcal{S}$ and any line bundle L on X, $W^n(X, L)$ is finite for all n in $\mathbb{Z}$.*

(b) *If, for every connected X in $\mathcal{S}$, $W^n(X)$ is finite for all n in $\mathbb{Z}$, then, for every X in $\mathcal{S}$, $W^n(X)$ is finite for all n in $\mathbb{Z}$.*

(c) *If, for every affine X in $\mathcal{S}$, $W^n(X)$ is finite for all n in $\mathbb{Z}$, then, for every X in $\mathcal{S}$, $W^n(X)$ is finite for all n in $\mathbb{Z}$.*

Furthermore, the same statements are true with the Grothendieck-Witt groups in place of Witt groups.

Proof. For noetherian regular separated schemes with 2 invertible, Mayer-Vietoris holds for the Witt groups [8, Theorem 2.5], and for the Grothendieck-Witt groups [61, Theorem 1.1]. For (a), recall that as line bundles are locally free, an open cover of X on which L is trivial may be chosen. The lemma then follows by using Mayer-Vietoris and inducting on the number of open sets in the cover.

Next recall that the connected components of any locally noetherian X are open in X, and their intersection is empty. To prove (b), use Mayer-Vietoris, and proceed by induction on the number of connected components of X.

Recall that for any separated scheme, the intersection of any two affine subschemes is affine. To prove (c), use Mayer-Vietoris, and induct on the number of affine open sets necessary to cover X. □

The following well known lemma will be used together with the previous one to reduce to X integral.

Lemma 3.32. *If X is a noetherian regular connected scheme, then X is integral.*

Proof. Let X be a noetherian regular connected scheme. As X is noetherian, it has only a finite number of irreducible components and every local ring $\mathcal{O}_{X,x}$ of X is also noetherian [48, Chapter 2, Proposition 3.46(a)]. Since X has only a finite number of irreducible components, it is integral if and only if it is connected and integral at every point $(i.e.\,\mathcal{O}_{X,x}$ is an integral domain for every $x \in X)$ [48, Chapter 2, Exercise 4.4]. To finish the proof, recall that every regular noetherian local ring is a domain [48, Chapter 4, Proposition 2.11]. □

Theorem 3.33 (Absolute finiteness)**.** *Let X be a smooth variety over $\mathbb{F}_p$ $(p > 2)$, with $\dim(X) \leq 2$, and let L be a line bundle on X. In this situation, the Witt groups $W^n(X, L)$ are finite for all $n \in \mathbb{Z}$.*

Proof. We may assume that X is connected using Lemma 3.31 (b), hence, integral, using Lemma 3.32. Using 3.31 (c), we may assume X is affine, hence $X \to \mathbb{F}_p$ is quasi-projective (any finite-type morphism between affine schemes is quasi-projective). Proposition 3.30 (b), and Lemma 3.31 (a) finish the proof. □

Theorem 3.34. *Let X be a separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$, with no residue field of X formally real. Assume the Beilinson-Lichtenbaum conjecture (see Remark 3.19. Note that this is known for varieties over fields). If the motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$ are finite for all $m, n \in \mathbb{Z}$, then the Witt groups $W^n(X)$ are finite for all $n \in \mathbb{Z}$.*

Proof. We may assume that X is connected using Lemma 3.31 (b), hence, integral, using Lemma 3.32. Using that the Beilinson-Lichtenbaum conjecture holds and that the motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$ are all finite, we apply Lemma 3.20 to obtain that the Kato cohomology groups $H^m_{Zar}(X, \mathcal{H}^n)$ are all finite. Since the spectral sequence of Corollary 3.29 is convergent, the cohomology groups of the Gersten-Witt complex $C(X, W, \iota)$ are finite. To finish the proof, use the coniveau spectral sequence converging to the Witt groups of X (Eq. (2.4)). $\square$

For ease of reference we include the following corollary.

Corollary 3.35. *Let X be a smooth variety over a finite field $\mathbb{F}_p$ $(p > 2)$, and let L be a line bundle on X. If the mod 2 motivic cohomology groups of X, $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$, are finite for all $m, n \in \mathbb{Z}$, then the Witt groups $W^n(X, L)$ are finite for all $n \in \mathbb{Z}$.*

Finally, we note some partial converses to Theorem 3.34.

Theorem 3.36 (Relative finiteness)**.** *Let X be a regular separated scheme that is of finite type over a base scheme S. Consider the following situations (for (a), assume Beilinson-Lichtenbaum):*

(a) *$\dim(X) \leq 2$, no residue field of X is formally real, and $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;*

(b) *$\dim(X) \leq 3$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p \geq 2)$;*

(c) $dim(X) \leq 4$, X is connected and proper over S, and $S = \text{Spec}(\mathbb{F}_p)$ ($p \geq 2$).

In situations (a) and (b):

(i) The Witt groups $W^1(X)$ and $W^3(X)$ are finite;

(ii) Finiteness of $W^0(X)$ is equivalent to finiteness of $W^2(X)$;

(iii) $W^0(X)$ is finite if and only if the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$.

In situation (c):

(i) The groups $W^0(X)$ and $CH^3(X)/2CH^3(X)$ are both finite if and only if the motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$.

Proof. In all situations, finiteness of the Witt groups follows from finiteness of motivic cohomology by Theorem 3.34, so we will only prove the other direction.

Assume that we are in situation (a). First we prove (i). To prove that $W^1(X)$ is finite, we will prove that the group $H^1(C(X, W, \iota))$ is finite, and then use the coniveau spectral sequence (Eq. (2.4)). To prove that $H^1(C(X, W, \iota))$ is finite, considering the spectral sequence of Corollary 3.29, it suffices to prove that the groups on the on the $p + q = 1$ diagonal of the first page of the spectral sequence, $H^1_{Zar}(X, \mathcal{H}^p)$ for $p \geq 0$, are finite. This was shown in Lemma 3.21 (a) (i). The proof of finiteness of $W^3(X)$ is identical.

Now we prove (ii). Assume that $W^0(X)$ is finite. Considering the shape of the coniveau spectral sequence (Eq. (2.4)), this implies $H^0(C(X, W, \iota))$ is finite. Considering the spectral sequence of Corollary 3.29, if we prove that all the groups $H^2_{Zar}(X, \mathcal{H}^p)$ are finite, this will prove that $H^2(C(X, W, \iota))$ is finite, hence prove that $W^2(X)$ is finite. To accomplish this, using Lemma 3.21 (a) (i), it suffices

to prove that the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^0_{Zar}(X, \mathcal{H}^5)$ are finite. Note that once we prove this, it will also finish the proof of (iii). Consider the spectral sequence of Corollary 3.29. Since $H^0_{Zar}(X, \mathcal{H}^5)$ has no non-zero differentials entering or leaving it, it is stable, hence finite by finiteness of $W^0(X)$. There is one possibly non-zero differential leaving the group $H^0_{Zar}(X, \mathcal{H}^4)$. It is the differential $d^{4,-4}_1 : E^{4,-4}_1 = H^0_{Zar}(X, \mathcal{H}^4) \to E^{5,-4}_1 = H^1_{Zar}(X, \mathcal{H}^5)$. Since the kernel of $d^{0,4}_1$ is stable, and is on the 0-th diagonal, it is finite by finiteness of $W^0(X)$. So finiteness of $H^0_{Zar}(X, \mathcal{H}^4)$ follows from finiteness of $H^1_{Zar}(X, \mathcal{H}^5)$ (Lemma 3.21 (a) (i)). Next, we will prove that $H^0_{Zar}(X, \mathcal{H}^3)$ is finite. First, consider the differential $d^{3,-3}_1 : E^{3,-3}_1 = H^0_{Zar}(X, \mathcal{H}^3) \to E^{4,-3}_1 = H^1_{Zar}(X, \mathcal{H}^4)$. Since $H^1_{Zar}(X, \mathcal{H}^4)$ is finite, $H^0_{Zar}(X, \mathcal{H}^3)$ is finite if and only if the kernel of $d^{3,-3}_1$ is finite. The kernel of $d^{3,-3}_1$ equals $E^{3,-3}_2$. Consider the differential $d^{3,-3}_2 : E^{3,-3}_2 \to E^{5,-4}_2$. Since $H^1_{Zar}(X, \mathcal{H}^5)$ is finite, its quotient $E^{5,-4}_2$ is also finite. As the kernel of $d^{3,-3}_2$ is on the 0-th diagonal of the stable page, and $W^0(X)$ is finite, we obtain finiteness of $E^{3,-3}_2$. Thus, proving that $W^2(X)$ is finite. The proof that finiteness of $W^2(X)$ implies finiteness of $W^0(X)$ is identical.

Assume that we are in situation (b). First we prove prove (i), finiteness of $W^1(X)$. As in situation (a), it suffices to prove that the groups $H^1_{Zar}(X, \mathcal{H}^p)$ are finite, for $p \geq 0$. This was shown in Lemma 3.21 (b) (i). Similarly, we have that $W^3(X)$ is finite. Next, to prove (ii), assume $W^0(X)$ is finite. Consider the spectral sequence of Corollary 3.29. As $E^{3,-3}_1 = H^0_{Zar}(X, \mathcal{H}^3)$ is on the 0-th diagonal of the stable page, it is finite. So using Lemma 3.21 (b) (i) and (ii), this proves (iii), and we have that all the groups $H^2_{Zar}(X, \mathcal{H}^p)$, for $p \geq 0$, are finite, which proves finiteness of $W^2(X)$. The other direction is identical.

Finally, assume we are in situation (c). Consider the spectral sequence of Corollary 3.29. By hypothesis $W^0(X)$ is finite, so the stable term $E^{4,-4}_1 = H^0_{Zar}(X, \mathcal{H}^4)$

is finite. Additionally, by hypothesis $CH^3(X)/2CH^3(X) = H^3_{Zar}(X, \mathcal{H}^3)$ is finite, hence $H^1_{Zar}(X, \mathcal{H}^4)$ is finite (Lemma 3.21 (c) (i)). Now consider the differential $d_1^{3,-3} : H^0_{Zar}(X, \mathcal{H}^3) \to H^1_{Zar}(X, \mathcal{H}^4)$. As the kernel of $d_1^{3,-3}$ is on the 0-th diagonal of the stable page, it is finite by finiteness of $W^1(X)$. Therefore, $H^0_{Zar}(X, \mathcal{H}^3)$ is finite, which is enough to finish the proof using Lemma 3.21 (c) (i). $\square$

The Gersten Conjecture

Let Λ be a DVR with infinite residue field. The purpose of this chapter is to prove the Gersten conjecture for the Witt groups in the case of a local ring A that is regular over Λ (Theorem 4.28). We first prove exactness of the augmented Gersten complex in the case of $A[\pi^{-1}]$ (Theorem 4.19), where π is a uniformizing parameter for Λ, A is essentially smooth over Λ, and $A[\pi^{-1}]$ is the localization at the element π. For Witt groups, it follows from this result that the Gersten conjecture is also true for A. As mentioned in the introduction, we use an adaptation of an argument of S. Bloch which he used to prove exactness of the augmented Gersten complex for K-theory in the case of $A[\pi^{-1}]$. Also crucial is the work [31] of S. Gille and J. Hornbostel. They developed an argument which replaces the usual concluding argument of Quillen in the proof of the geometric case of the Gersten conjecture. Finally, we adapt an argument of I. Panin to get from the essentially smooth case to the regular over Λ case. In the following section we recall the definition and essential properties that we need of Gille's transfer map for the coherent Witt groups as well as S. Gille's revised zero-theorem. The remaining sections are devoted to the proof.

4.1 The Transfer Map

We recall some results from from S. Gille's paper [28].

Definition 4.1. Let $\pi : R \to S$ be a finite morphism of rings, in other words, a morphism of rings such that S is finitely generated as an R-module. Given any complex $M_\bullet$ of S-modules, by restriction of scalars we obtain a complex $\pi_* M_\bullet$ of R-modules (*i.e.*, by considering each S-module M_i as an R-module via the composition $R \xrightarrow{\pi} S \to M_i$). As restriction of scalars is functorial, we obtain a

functor

$$D^b_{coh}(\mathcal{M}(S)) \to D^b_{coh}(\mathcal{M}(R)) \tag{4.1}$$

where $D^b_{coh}(\mathcal{M}(S))$ is the full triangulated subcategory consisting of complexes with coherent homology within the bounded derived category $D^b(\mathcal{M}(S))$ of S-modules. When S is equipped with a dualizing complex $I_\bullet$, the finite map π determines a dualizing complex on R which is denoted by $\pi^\natural(I_\bullet)$ [28, Theorem 4.1]. Furthermore [28, Definition 4.2 and preceeding material], there exists an isomorphism of dualities $\eta : \pi_* \chi^{\pi^\natural(I)} \overset{\simeq}{\to} \chi^I \pi_*$ which makes the restriction of scalars functor 4.1 a morphism of triangulated categories with duality

$$(\pi_*, \eta) : (D^b_{coh}(\mathcal{M}(S)), \chi^{\pi^\natural(I)}, 1, \varpi^{\pi^\natural(I)}) \to (D^b_{coh}(\mathcal{M}(R)), \chi^I, 1, \varpi^I) \tag{4.2}$$

and the morphism 4.2 induces on Witt groups a morphism

$$\text{Tr}_{S/R} : \widetilde{W}^i(S, \pi^\natural(I_\bullet)) \to \widetilde{W}^i(R, I_\bullet)$$

called the *transfer morphism* for the finite morphism $\pi : R \to S$.

Definition 4.2. Let R be a Gorenstein ring of finite Krull dimension, and $t \in R$ a regular element, *i.e.* a non-zero divisor in R. Let $j : R \to I_\bullet$ be a finite injective resolution of the R-module R. Let $\pi : R \to R/t$ be the quotient map. The restriction of scalars functor respects the filtration by codimension of support, that is,

$$\pi_*(D^j(R/tR)) \subset D^{j+1}(R)$$

and so, by restricting the morphism of triangulated categories with duality 4.2 to $D^j(R/tR)$, we have a morphism

$$(\pi_*, \eta) : D^j(R/tR) \to D^{j+1}(R)$$

and a morphism [26, Theorem 4.2] of triangulated Witt groups

$$\operatorname{Tr}_{(R/tR)/R} : W^i(D^j(R/tR)) \to W^{i+1}(D^{j+1}(R)) \qquad (4.3)$$

called the *transfer morphism*. The composition of duality preserving functors

$$D^j(R/tR) \stackrel{(\pi_*, \eta)}{\to} D^{j+1}(R) \to D^j(R),$$

where $D^{j+1}(R) \to D^j(R)$ is the inclusion, will be denoted by (π_*, ξ). The duality preserving functor (π_*, ξ) induces a morphism [26, Theorem 4.2] of triangulated Witt groups

$$\operatorname{Tr}_{(R/tR)/R} : W^i(D^j(R/tR)) \to W^{i+1}(D^j(R)) \qquad (4.4)$$

which will also be called the *transfer morphism*. Since $\pi_* M_\bullet \in D^{j+1}_{R/tR}(R)$, where $D^b_{R/tR}(\mathcal{M}(R))$ denotes those complexes in $D^b_{coh}(\mathcal{M}(R))$ having homology vanishing outside $\operatorname{Spec} R/tR$ and $D^j_{R/tR}(R)$ those complexes in $D^b_{R/tR}(\mathcal{M}(R))$ with codimension of support less than or equal to j, the transfer morphism 4.3 factors in the commutative diagram below

$$W^i(D^j(R/tR)) \xrightarrow{\operatorname{Tr}} W^{i+1}(D^{j+1}(R)) \qquad (4.5)$$

$$\Big\downarrow{\alpha_*}$$

$$W^{i+1}(D^{j+1}_{R/tR}(R))$$

where α_* is the devissage isomorphism [29, 3.1 and Theorem 3.2].

Lemma 4.3. *Let R be a regular ring and $t \in R$ a non-zero divisor. Then, if $y \in W^{i+1}(D^{j+1}(R))$ is such that the restriction of y to the open subset $\operatorname{Spec} R_t \subset \operatorname{Spec} R$ is zero, then there exists $z \in W^i(D^j(R/tR))$ such that $y = \operatorname{Tr}_{(R/tR)/R}(z)$.*

Proof. Let $y \in W^{i+1}(D^{j+1}(R))$ is such that the restriction of y to the open subset $\operatorname{Spec} R_t \subset \operatorname{Spec} R$ is zero. The short exact sequence of triangulated categories with duality [26, Section 2.6]

$$D^b_{R/tR}(\mathcal{M}(R)) \to D^b_{coh}(\mathcal{M}(R)) \to D^b_{coh}(\mathcal{M}(R_t)) \qquad (4.6)$$

determines, using the localization result for triangulated Witt groups (Proposition 2.7), the long exact sequence

$$\ldots \to W^{i+1}(D^b_{R/tR}(\mathcal{M}(R))) \to W^{i+1}(D^b(\mathcal{M}(R))) \to W^{i+1}(D^b(\mathcal{M}(R_t))) \to \ldots$$

Since the transfer morphism 4.3 factors through the devissage isomorphism (See Definition 4.2) and y vanishes in $W^{i+1}(D^b(\mathcal{M}(R_t)))$, there exists $z \in W^i(D^b(\mathcal{M}(R/tR)))$ such that $y = \text{Tr}_{(R/tR)/R}(z)$. However, then $z \in W^i(D^j(R/tR))$ follows using the definition of the transfer map and the fact that $y \in W^{i+1}(D^{j+1}(R))$, finishing the proof of the lemma. $\qquad\square$

Lemma 4.4. *[28, Section 4 (i)] Maintain the hypotheses of Lemma 4.3. Then the transfer for the Witt groups commutes with localization, in particular, for any element $f \in R$, the diagram below commutes.*

$$
\begin{array}{ccc}
W^i(D^p(R/tR)) & \xrightarrow{\ \text{Tr}\ } & W^{i+1}(D^p(R)) \\
\downarrow{\scriptstyle \text{loc}} & & \downarrow{\scriptstyle \text{loc}} \\
W^i(D^p(R_f/tR_f)) & \xrightarrow{\ \text{Tr}\ } & W^{i+1}(D^p(R_f))
\end{array}
$$

Lemma 4.5. *[28, Section 4 (ii)] Let R, S, T be Gorenstein rings, and let $q : R \to S$, $r : S \to T$, and $s : R \to T$ be finite morphisms such that $s = r \circ q$. Then, $\text{Tr}_{T/R} = \text{Tr}_{S/R} \cdot \text{Tr}_{T/S}$.*

Next we recall S. Gille's new zero-theorem.

Proposition 4.6. *[25, 24, Theorem 5.4 and Theorem 1.4, respectively] Let $S, R, R^{'}$ be Gorenstein rings, and let $\iota : S \to J_\bullet$, $\gamma : R \to I_\bullet$ be injective resolutions of S and R, respectively. Consider the commutative diagram*

$$
\begin{array}{ccc}
R & & \\
\downarrow{\scriptstyle s} \ \ \nwarrow & \!\!\!\!\!\psi & \\
R' & \xleftarrow{\ \omega\ } & S
\end{array}
$$

where ω, ψ are flat morphisms, s is surjective and $\ker s = tR$ for some non-zero divisor $t \in R$. If x is an i-symmetric space in $D^j(S)$ with respect to the shifted duality $T^{-1}\chi^J$, then $(s_*, \xi_*)_*(\omega^*(x))$ is a neutral i-symmetric space in $D^j(R)$ with respect to the duality χ^I.

For ease of reference we write down the corollary below. In view of the definition of the triangulated Witt groups, it immediately follows from the previous proposition.

Corollary 4.7. *Maintain the hypotheses of Proposition 4.6. Note that $R' \simeq R/tR$, and $(s_*, \xi_*)_*(\omega^*(x)) = \text{Tr}_{(R/tR)/R}(\omega^*(x))$. Then, the morphism of Witt groups*

$$W^{i-1}(D^j(S)) \overset{\omega^*}{\to} W^{i-1}(D^j(R/tR)) \overset{\text{Tr}_{(R/tR)/R}}{\to} W^i(D^j(R))$$

is zero.

4.2 Proof of the Gersten Conjecture: Essentially Smooth Case

First we recall some terminology, then we prove a series of lemmas leading to the statement and proof of Theorem 4.19.

4.8. Recall that, given any local ring Λ, we say that a local Λ-algebra A is *essentially smooth* over Λ if there exists a smooth Λ-algebra R and a prime ideal $\mathfrak{n}$ of R such that A is Λ-isomorphic to $R_\mathfrak{n}$ and the composition homomorphism $\Lambda \to R \to R_\mathfrak{n}$ is local.

4.9. Let Λ be a DVR, π a uniformizing parameter for Λ. The generic point η is open in $\text{Spec}\,\Lambda$ as it is the complement of the unique closed point. For any Λ-scheme X we will denote the fiber over the closed point of Λ by X_0 (we will also say the *closed fiber*) and the fiber over the generic point by X_η (we will say the *generic fiber*). Recall that when $X = \text{Spec}\,A$ is affine, then $X_0 = \text{Spec}\,A/\pi$, and

the generic fiber of Spec A is then equal to Spec $A \otimes_\Lambda \operatorname{Frac}\Lambda$. Moreover, $A \otimes_\Lambda \operatorname{Frac}\Lambda$ is also equal to the localization $A_\pi = A[\pi^{-1}]$ of A at the element $\pi \in A$.

Proposition 4.10. *[38, Proposition 2.8.5] Let Y be a scheme that is locally noetherian, irreducible, regular and of dimension 1, η the generic point, $f : X \to Y$ a morphism, $X_\eta = f^{-1}(\eta)$ the fiber over the generic point, $i : X_\eta \to X$ the canonical morphism, and Z a closed subscheme of X_η. Then the schematic closure $\overline{Z}$ (see [34, Remark 10.31]) of the inclusion Z in X is the unique closed subscheme of X which is flat as a Y-scheme and has $i^{-1}(\overline{Z}) = Z$, that is, the generic fiber $\overline{Z}_\eta$ of $\overline{Z}$ is Z.*

Corollary 4.11. *Maintain the hypotheses of Proposition 4.10. Additionally, assume that $Y = \Lambda$ is a DVR. The Λ-scheme X is flat over Λ if and only if X equals the schematic closure $\overline{X_\eta}$ in X of the generic fiber X_η. If X is flat over Λ, then every irreducible component of X dominates Λ (i.e. the image is dense, or equivalently, the generic point of the irreducible component maps to the generic point of Y). If X is reduced, X is flat over Λ if and only if every every irreducible component dominates Λ.*

Proof. The first assertion follows immediately from Proposition 4.10. To prove the second assertion, we begin by applying Proposition 4.10 to obtain that if X is flat over Λ, then X equals the schematic closure $\overline{X_\eta}$. To prove that the irreducible components dominate we will prove that every generic point of X is in X_η. As the underlying topological space of $\overline{X_\eta}$ is the topological closure of X_η, which is equal as a set to the union $\bigcup_{x \in X_\eta} \{\overline{x}\}$, it follows that every generic point ξ_i of X is in the closure $\{\overline{x}\}$ of a point $x \in X_\eta$. That is, every generic point ξ_i is the specialization of a point $x \in X_\eta$. Since generic points are by definition the unique points which specialize only to themselves, in other words, $\xi_i \in \{\overline{x}\}$ implies $x = \xi_i$, we obtain

that every generic point is in the generic fiber X_η. To prove the final assertion, suppose that X is reduced and every irreducible component dominates Λ. Then, as every generic point ξ_i is in X_η, the schematic closure of X_η has underlying topological space X. A closed subscheme of a reduced scheme X that has the same underlying topological space is necessarily equal as a scheme to X. Therefore X equals the schematic closure of X_η and so X is flat. $\qquad\square$

Lemma 4.12. *Let Λ be a DVR, π a uniformizing parameter for Λ, R a smooth Λ-algebra, and $\mathfrak{p} \in \operatorname{Spec} R$ a prime ideal in R such that the composition $\Lambda \to R \to R_\mathfrak{p}$ is a local morphism. Then, the $R_\mathfrak{p}[\pi^{-1}]$ equals the filtered colimit of the rings $R_f[\pi^{-1}]$,*

$$\operatorname{colim}_{f \notin \mathfrak{p}} R_f[\pi^{-1}] \to R_\mathfrak{p}[\pi^{-1}] \tag{4.7}$$

where the colimit is taken over the set of elements $f \in R$ such that $f \notin \mathfrak{p}$.

Proof. For each $f \notin \mathfrak{p}$ we have the localization map $R_f \to R_\mathfrak{p}$. A partial ordering on $R - \mathfrak{p}$ is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f' \notin \mathfrak{p}$, in which case $R_{f'} \to R_f$ is given by $r/(f')^e \mapsto r(f'')^e/f^e$. So the colimit over the $f \notin \mathfrak{p}$ is filtered, and using the universal property of the localization $R_\mathfrak{p}$ it follows that the canonical map from the filtered colimit

$$\operatorname{colim}_{f \notin \mathfrak{p}} R_f \to R_\mathfrak{p} \tag{4.8}$$

is an isomorphism. As each map $R_{f'} \to R_f$, and $R_f \to R_\mathfrak{p}$, is a morphism of Λ-algebras, they induce maps on the localizations $R_{f'}[\pi^{-1}] \to R_f[\pi^{-1}]$, $R_f[\pi^{-1}] \to R_\mathfrak{p}[\pi^{-1}]$. Using the universal property of the localization $R_\mathfrak{p}[\pi^{-1}]$ of $R_\mathfrak{p}$ with respect to the element π, it follows that the canonical map 4.7 is an isomorphism. $\qquad\square$

Lemma 4.13. *Let Λ be a mixed $(0, p)$-characteristic $(p \neq 2)$ DVR (see Appendix 5.11), π a uniformizing parameter for Λ, and A a local ring essentially smooth*

over Λ. Let $i, j \geq 0 \in \mathbb{Z}$ be integers, and let $0 \neq x \in W^i(D^{j+1}A[\pi^{-1}])$. Then, we have the following:

(i) there exists a smooth integral Λ-algebra R with smooth integral closed fiber R/π such that $A \simeq R_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Spec}\,(R)$;

(ii) there exists an element in the Witt group $y \in W^i(D^{j+1}R[\pi^{-1}])$ such that y maps to x via the morphism $W^i(D^{j+1}R[\pi^{-1}]) \to W^i(D^{j+1}A[\pi^{-1}])$ induced by the localization map $R[\pi^{-1}] \to A[\pi^{-1}]$;

(iii) if A does not contain a field, then $\mathfrak{p}$ lies in the closed fiber of $\mathrm{Spec}\,R$;

(iv) there exists a non-zero-divisor $t \in R$ such that $\mathrm{Spec}\,R/t$ is reduced and is flat over Λ, and $\mathfrak{p}$ lies in every irreducible component of $\mathrm{Spec}\,R/t$;

(v) there exists an element in the Witt group $z \in W^{i-1}(D^j(R[\pi^{-1}]/t))$ such that z maps to y under the transfer map $\mathrm{Tr}_{(R[\pi^{-1}]/t)/R[\pi^{-1}]} : W^{i-1}(D^j R[\pi^{-1}]/t) \to W^i(D^{j+1}R[\pi^{-1}])$ for the finite morphism $R[\pi^{-1}] \to R[\pi^{-1}]/t$.

Proof. First recall that as A is essentially smooth over Λ, there exists a smooth Λ-algebra R' and a prime ideal $\mathfrak{p}$ of R' such that A is Λ-isomorphic to $R'_{\mathfrak{p}}$ and the composition homomorphism $A \to R' \to R'_{\mathfrak{p}}$ is local (Definition 4.8). Furthermore, as $x \in W^i(D^{j+1}A[\pi^{-1}])$ and since Witt groups commute with filtered colimits [27, Theorem 1.6], by Lemma 4.12 we have that $x \in W^i(D^{j+1}R'_g[\pi^{-1}])$ for some $g \in R'$ such that $g \notin \mathfrak{p}$. To prove (i), we will find a smaller basic open neighborhood R'_f of $\mathfrak{p}$ such that R'_f satisfies the assertions of (i). As R' is smooth over Λ, the closed fiber R'/π is smooth over Λ/π, from which it follows that $(R'/\pi)_{\mathfrak{q}}$ is a regular local ring at every prime ideal $\mathfrak{q}$ in R'/π, hence R'/π is a domain at every point. The connected components of $\mathrm{Spec}\,R$ and $\mathrm{Spec}\,R'/\pi$ are open (any locally noetherian space is locally connected [36, Corollary 6.1.9], which implies that the

connected components are open), so we may find $f \notin \mathfrak{p}$ such that $\operatorname{Spec} R'_f$ and $\operatorname{Spec} R'_f / \pi R'_f$ are neighborhoods of $\mathfrak{p}$ in $\operatorname{Spec} R'$ and $\operatorname{Spec} R'/\pi$, respectively, which are each contained in a connected component. Furthermore, we may additionally choose $f \notin \mathfrak{p}$ such that $\operatorname{Spec} R'_f$ is contained in the open subscheme $\operatorname{Spec} R'_g$. Then, both R'_f and $R'_f / \pi R'_f$ are connected and each is a domain at every point, hence they are each integral [36, Corollary 6.1.12], smooth, and $R'_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$, proving (i). To prove (ii), we take for $y \in W^i(D^{j+1}(R'_f)[\pi^{-1}])$ the image of x under the map induced by localization $R'_g[\pi^{-1}] \to R'_f[\pi^{-1}]$. By functoriality of the Witt groups, y maps to $x \in W^i(D^{j+1}A[\pi^{-1}])$. Taking for R the localization R'_f proves (i) and (ii). However, we note that for every basic open neighborhood R'_h of $\mathfrak{p}$ contained in R'_f, the assertions made in (i) and (ii) about R'_f remain true for R'_h. We will need to choose smaller open neighborhoods of $\mathfrak{p}$ to prove (iv) and (v), so we will not take R'_f to be the R that appears in the statement of the lemma. However, until we say otherwise, let $R := R'_f$ for the remainder of the proof.

Now, to prove (iii) assume that A does not contain a field. To prove that $\mathfrak{p} \in \operatorname{Spec} R$ is in the closed fiber $\mathfrak{p} \in \operatorname{Spec} R/\pi$ (hence in the closed fiber of any smaller neighborhood $\operatorname{Spec} R_g$) suppose for the purpose of obtaining a contradiction that $\mathfrak{p}$ lies in the generic fiber $\operatorname{Spec} R \otimes_\Lambda \operatorname{Frac}\Lambda$. In other words, suppose that the prime ideal in Λ that is obtained by taking the inverse image of $\mathfrak{p}$ under the structure morphism $\Lambda \to R$ is the zero ideal. Then, after localizing, we obtain the local homomorphism $\operatorname{Frac}\Lambda \to R_{\mathfrak{p}}$: this homomorphism is necessarily injective (otherwise $1 \in R_{\mathfrak{p}}$ would be zero and A would be trivial) contradicting the assumption that A does not contain a field.

We begin proving (iv) and (v) by choosing a representative $M_\bullet \in D^b_{coh}(\mathcal{M}(R[\pi^{-1}]))$ for the underlying space of the element $y \in W^i(D^{j+1}R[\pi^{-1}])$. By definition, $M_\bullet$ is a bounded chain complex of $R[\pi^{-1}]$-modules with finitely generated homology H_i.

As each H_i is finitely generated, the support $\operatorname{Supp} H_i(M_\bullet)$ is a closed subspace in $\operatorname{Spec} R[\pi^{-1}]$ [34, Corollary 7.31]. Therefore, the support $\operatorname{Supp} M_\bullet$ of $M_\bullet$ is a closed subspace in $\operatorname{Spec} R[\pi^{-1}]$ as it is a finite union

$$\operatorname{Supp} M_\bullet = \operatorname{Supp} \bigcup_i H_i(M_\bullet),$$

of closed subspaces. It follows from Proposition 4.10 that the closed subscheme $\overline{\operatorname{Supp} M_\bullet}$ defined by taking the schematic closure (see Proposition 4.10) of $\operatorname{Supp} M_\bullet$ in $\operatorname{Spec} R$ is flat over Λ and has generic fiber $(\overline{\operatorname{Supp} M_\bullet})_\eta = \operatorname{Supp} M_\bullet$. To prove (iv), we first prove that the open complement $\operatorname{Spec} R - \overline{\operatorname{Supp} M_\bullet}$ has non-trivial intersection with the closed fiber $\operatorname{Spec} R_0 = \operatorname{Spec} R/\pi$, and from this we deduce (iv). Suppose, for the purpose of obtaining a contradiction, that $\operatorname{Spec} R_0 \subset \overline{\operatorname{Supp} M_\bullet}$. In particular, we have an inclusion of closed fibers $\operatorname{Spec} R_0 \subset \overline{\operatorname{Supp} M_\bullet}_0$. Then,

$$\dim R - 1 = \dim R_0 \leq \dim \overline{\operatorname{Supp} M_\bullet}_0 \leq \dim \overline{\operatorname{Supp} M_\bullet} \leq \dim R \qquad (4.9)$$

where $\dim R_0 = \dim R - 1$ since $R_0 = R/\pi$ with R regular and π a non-zero divisor. We will now prove that the center and the rightmost inequalities of 4.9 are strict inequalities, which in view of 4.9, will result in the contradiction $\dim R - 1 \leq \dim R - 2$. If $\dim \overline{\operatorname{Supp} M_\bullet} = \dim R$, then from the inequality

$$\dim \overline{\operatorname{Supp} M_\bullet} \leq \dim \overline{\operatorname{Supp} M_\bullet} + \operatorname{codim}(\overline{\operatorname{Supp} M_\bullet}, \operatorname{Spec} R) \leq \dim R$$

we have that $\operatorname{codim}(\overline{\operatorname{Supp} M_\bullet}, \operatorname{Spec} R) = 0$, or equivalently, that $\overline{\operatorname{Supp} M_\bullet}$ contains an irreducible component of $\operatorname{Spec} R$ [37, Proposition 14.2.2 (iv)]. As $\operatorname{Spec} R$ is irreducible, this implies that $\overline{\operatorname{Supp} M_\bullet} = \operatorname{Spec} R$, and in particular, that they have the same generic fibers. Hence, $\operatorname{Supp} M_\bullet = \overline{\operatorname{Supp} M_\bullet}_\eta = \operatorname{Spec} R_\eta$, however, this equality is not possible since

$$\operatorname{codim}(\operatorname{Supp} M_\bullet, \operatorname{Spec} R[\pi^{-1}]) \neq 0$$

as $M_\bullet \in D^{j+1}R[\pi^{-1}]$, and $j + 1$ is not equal to 0. This proves that $\dim \overline{\mathrm{Supp}\, M_\bullet} <$ $\dim R$. If $\dim \overline{\mathrm{Supp}\, M_{\bullet 0}} = \dim \overline{\mathrm{Supp}\, M_\bullet}$, then by the same reasoning as above, it follows that $\overline{\mathrm{Supp}\, M_{\bullet 0}}$ contains an irreducible component of $\overline{\mathrm{Supp}\, M_\bullet}$. Yet this is impossible: since $\overline{\mathrm{Supp}\, M_\bullet}$ is flat over Λ, every irreducible component of $\overline{\mathrm{Supp}\, M_\bullet}$ dominates Λ (Corollary 4.11), hence the generic points of $\overline{\mathrm{Supp}\, M_\bullet}$ are contained in the generic fiber $\overline{\mathrm{Supp}\, M_\bullet}$, and thus cannot be contained in the closed fiber $\overline{\mathrm{Supp}\, M_{\bullet 0}}$. This concludes the proof of the claim that $(\mathrm{Spec}\, R - \overline{\mathrm{Supp}\, M_\bullet}) \cap \mathrm{Spec}\, R_0 \neq \varnothing$.

Applying what we have just proved, let $\mathfrak{q} \in (\mathrm{Spec}\, R - \overline{\mathrm{Supp}\, M_\bullet}) \cap \mathrm{Spec}\, R_0$. Choose a principal open neighborhood $\mathrm{Spec}\, R_t$ of $\mathfrak{q}$ which is contained in the open complement $\mathrm{Spec}\, R - \overline{\mathrm{Supp}\, M_\bullet}$. That is, $\mathrm{Spec}\, R_t \cap \overline{\mathrm{Supp}\, M_\bullet} = \varnothing$, so $\mathrm{Spec}\, R_t[\pi^{-1}] \cap \mathrm{Supp}\, M_\bullet = \varnothing$. As $\mathfrak{q} \in \mathrm{Spec}\, R_t \cap \mathrm{Spec}\, R_0 = \mathrm{Spec}\, (R/\pi)_t$, it follows that $t \in R$ is not nilpotent in R/π, in particular, it is non-zero in R/π. We will now prove that R/t is flat over Λ. To do so, it suffices to prove that R/t is torsion free as a Λ-module [48, Chapter 1, Corollary 2.5]. Since Λ is a DVR, if R/t has no π-torsion, then R/t is torsion free over Λ. Assume that R/t has π-torsion for the purpose of obtaining a contradiction. Then, there exists $0 \neq a \in R/t$ such that $\pi a = 0 \in R/t$, that is, $\pi a = tb$ for some $b \in R$ and $a \notin tR$. Using the hypothesis that R/π is a domain, it follows that $t \in R/\pi$ is not a zero-divisor in R/π, hence $\pi a = tb$ implies $b \in \pi R$, that is, $b = \pi c$ for some $c \in R$. Since R is a domain, from $\pi a = t\pi c$ we obtain by cancelation that $a = tc$, a contradiction. This proves that R/t is flat over Λ. Then, all irreducible components of $\mathrm{Spec}\, R/t$ dominate (Lemma 4.14) so it it follows that $\mathrm{Spec}\, (R/t)_{red}$ is also flat over Λ (Corollary 4.11). As the localization $R_\mathfrak{p}$ is a unique factorization domain, we may take $t_\mathfrak{p} = t_1^{a_1} \cdots t_n^{a_n}$ to be a factorization of $t_\mathfrak{p}$. Then $\mathrm{Spec}\, (R_\mathfrak{p}/t_\mathfrak{p})_{red} = \mathrm{Spec}\, R_\mathfrak{p}/t_1 \cdots t_n$. We may choose a sufficiently small open neighborhood $\mathrm{Spec}\, R_f$ of $\mathfrak{p}$ such that $\mathrm{Spec}\, (R_f/t)_{red} = \mathrm{Spec}\, R_f/t_1 \cdots t_n$. Then, set $R := R_f$ and $t := t_1 \cdots t_n$. Restricting y to R, with $M_\bullet$ denoting the

complex representing y, we again have that $R_t[\pi^{-1}] \cap \operatorname{Supp} M_\bullet = \varnothing$. Therefore, $y \in W^i(D^{j+1} R_t[\pi^{-1}])$ is zero. From this, we have that if $\mathfrak{p} \in \operatorname{Spec} R$ is not in the image of the closed subscheme $\operatorname{Spec} R/tR$, then it must be in the image of the complement $\mathfrak{p} \in \operatorname{Spec} R_t$, that is, $t \notin \mathfrak{p}$. However, in this case it would follow that the element $x \in W^i(D^{j+1} A[\pi^{-1}])$ is zero (using the fact that $W^i(D^{j+1} A[\pi^{-1}])$ is a colimit of $W^i(D^{j+1} R_t[\pi^{-1}])$ over $t \notin \mathfrak{p}$). This would contradict the hypothesis that $x \neq 0$. So, $\mathfrak{p} \in \operatorname{Spec} R/t$. Now we prove the assertions about the irreducible components: the irreducible components of $\operatorname{Spec} R/t$ are closed and finite in number; the union of the irreducible components which do not contain $\mathfrak{p}$ is closed and does not contain $\mathfrak{p}$, so its complement is an open neighborhood U of $\mathfrak{p}$; choosing a principal open neighborhood $\operatorname{Spec} R_f/t$ contained in U we have that the irreducible components of $\operatorname{Spec} R_f/tR_f$ all meet $\mathfrak{p}$. This completes the proof of (iv). To prove (v), let y again denote the restriction of y to R_f. Then, $y \in W^i(D^{j+1} R_t[\pi^{-1}])$ is zero, so the map of Witt groups $W^i(D^{j+1} R[\pi^{-1}]) \to W^i(D^{j+1}(R[\pi^{-1}])_t)$ that is induced by the open immersion $\operatorname{Spec} R_t[\pi^{-1}] \to \operatorname{Spec} R[\pi^{-1}]$ sends y to 0. Therefore, there exists $z \in W^i(D^j(R[\pi^{-1}]/tR[\pi^{-1}]))$ such that $y = \operatorname{Tr}_{(R[\pi^{-1}]/tR[\pi^{-1}])/R[\pi^{-1}]}(z)$ (Lemma 4.3). This completes the proof of the lemma. $\qquad\square$

Lemma 4.14. *Let Λ be a DVR, and let $Y \subset Z$ be a quasi-compact morphism of Λ-schemes, with Y a reduced scheme which is flat over Λ. If $\overline{Y}$ is the schematic closure of Y in Z, then $\overline{Y}$ is flat over Λ.*

Proof. As $Y \subset Z$ is a quasi-compact morphism the schematic closure $\overline{Y}$ of Y in Z exists and is reduced whenever Y is reduced [36, Proposition 9.5.9]. As $\overline{Y}$ is reduced, to prove that $\overline{Y}$ is flat over Λ, it suffices to demonstrate that the irreducible components of $\overline{Y}$ dominate Λ (Corollary 4.11). The irreducible components of $\overline{Y}$ are the closures $\overline{Y_i}$ in Z of the irreducible components Y_i of Y. If Y is flat over Λ,

then the irreducible components Y_i of Y dominate Λ (Corollary 4.11). Since the inclusion $Y_i \hookrightarrow \overline{Y_i}$ is also dominant, it follows that $\overline{Y_i} \to \Lambda$ is dominant, and hence the irreducible components $\overline{Y_i}$ dominate Λ. $\qquad\square$

Proposition 4.15. *[34, Proposition 14.107] Let Y be a noetherian scheme, and let $f : X \to Y$ be a morphism of finite type. Assume in addition that X and Y are irreducible, that Y is universally catenary (e.g. a regular ring is universally catenary), and that f is surjective. Then $\dim X = \dim Y + X_\eta$, where η is the generic point of Y.*

Lemma 4.16. *Let Λ be a mixed $(0, p)$-characteristic $(p \neq 2)$ DVR (see Appendix 5.11), and R a smooth integral Λ-algebra, $t \in R$ a non-zero divisor such that $\operatorname{Spec} R/t$ is reduced and flat over Λ, and $x \in \operatorname{Spec} R/t$. Suppose that $\dim R/\pi = d$. As R is of finite type over Λ, by definition $R := \Lambda[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$. The surjection $\Lambda[X_1, \ldots, X_n] \to R$ induces a closed immersion $\operatorname{Spec} R \hookrightarrow \mathbb{A}^n_\Lambda$. Embed $\mathbb{A}^n_\Lambda \hookrightarrow \mathbb{P}^n_\Lambda$ as the complement of a hyperplane. Let Y, X denote the schemes obtained taking the schematic closures of $\operatorname{Spec} R/t, \operatorname{Spec} R$ in $\mathbb{P}^n_\Lambda$, respectively (equivalently, take their topological closures with the reduced, induced, subscheme structure). Then, Y is locally a principal divisor on X at x (i.e. Y is at x a regular immersion of codimension 1), Y and X are flat over Λ, and $\dim X = d + 1$, $\dim X_0 = \dim X_\eta = d$.*

Proof. Since $\operatorname{Spec} R/t$ is a principal divisor on $\operatorname{Spec} R$, it follows that Y is locally a principal divisor on X at x. Flatness of Y, X, follow from Lemma 4.14. Now we prove the assertions about the dimensions. As $\pi \in R$ is a non-zero divisor, it follows that $\dim R = d + 1$. Applying Proposition 4.15 to $\operatorname{Spec} R \to \Lambda$ (which we may use since x is in the closed fiber $\operatorname{Spec} R/\pi$ and $\operatorname{Spec} R$ dominates Λ, hence $\operatorname{Spec} R$ surjects onto Λ) we have that $\dim \operatorname{Spec} R_\eta = d$. Since $\operatorname{Spec} R_\eta \to X_\eta$ is

an open dense inclusion of FracΛ-varieties, it follows that $\dim \operatorname{Spec} R_\eta = \dim X_\eta$. Then, applying Proposition 4.15 to X, we have that $\dim X = d + 1$. Since X dominates, the closed fiber cannot contain an irreducible component, equivalently, $\operatorname{codim}(X_0, X) \geq 1$, and hence $\dim X_0 < \dim X$. Therefore, from the inclusions $\operatorname{Spec} R/\pi \subset X_0 \subset X$ we have that $d = \dim \operatorname{Spec} R/\pi \leq \dim X_0 < \dim X = d + 1$, hence $\dim X_0 = d$. $\qquad\square$

The next proposition is Bloch's geometric Lemma and it is crucial for the proof of Theorem 4.19.

Proposition 4.17. *[14, SubLemma A.4] Let Λ be a mixed $(0, p)$-characteristic $(p \neq 2)$ DVR (see Appendix 5.11) with infinite residue field, and A a local ring essentially smooth over Λ. Let $X \hookrightarrow \mathbb{P}^n_\Lambda$ be a projective variety over Λ, $x \in X_0$ an element in the closed fiber of X such that $\mathcal{O}_{X,x} \simeq A$, and assume that X is smooth over Λ at x. Let d denote the dimension of X_0. Let $Y \overset{j}{\hookrightarrow} X$ be locally at x a principal divisor with Y flat over Λ, and let $W_\eta \hookrightarrow Y_\eta$ be a closed subscheme of the generic fiber Y_η of Y such that $\dim W_\eta < \dim Y_\eta$. Let W be the schematic closure of W_η in X (the notation is reasonable since the generic fiber of W is exactly W_η by Lemma 4.14) and assume that $x \notin W$. Then, there exists a commutative diagram*

$$\begin{array}{ccc} & U & \\ {\scriptstyle j}\nearrow & & \downarrow{\scriptstyle \pi} \\ Y \xrightarrow{\;\pi|_Y\;} & \mathbb{P}^{d-1}_\Lambda & \end{array} \qquad (4.10)$$

where U is an open neighborhood of x in X, π is smooth at x over $\mathbb{P}^{d-1}_\Lambda$ of relative dimension 1, $\pi|_Y$ is a finite morphism, and $\pi(x) \notin \pi(W)$.

Corollary 4.18. *Maintain the hypotheses of Lemma 4.16. Then, there exists an open affine subscheme $\operatorname{Spec} C \subset Y$ containing x, for some $f \in R$ an open immer-*

sion $\operatorname{Spec}(R_f/tR_f)[\pi^{-1}] \overset{\omega}{\hookrightarrow} \operatorname{Spec} C[\pi^{-1}]$, *and a commutative diagram*

$$
\begin{array}{ccc}
& C'[\pi^{-1}] \xleftarrow{\quad p \quad} R_f[\pi^{-1}] & \qquad (4.11) \\
\psi \nearrow & \downarrow s \quad \swarrow j & \\
C[\pi^{-1}] \xrightarrow{\quad \omega \quad} R_f/tR_f[\pi^{-1}] &
\end{array}
$$

where ψ is a smooth morphism, p is a finite morphism, j is the canonical surjection, and s is a surjection with $\ker s = t'C'[\pi^{-1}]$ for some non-zero divisor $t' \in C'[\pi^{-1}]$.

Proof. As $\operatorname{Spec}(R/tR)_\eta$ is open in Y_η, the complement $W_\eta := Y_\eta \setminus \operatorname{Spec}(R/tR)_\eta$ is a closed subspace of Y_η, and hence a closed subscheme of Y_η when equipped with the reduced induced subscheme structure. We now verify that W_η satisfies the hypotheses of Proposition 4.17. The schematic closure of W_η in X defines a closed subscheme $W \hookrightarrow X$ which is flat over Λ, and has generic fiber $W_\eta = Y_\eta \setminus \operatorname{Spec}(R/tR)_\eta$ (Lemma 4.14). Additionally, $\mathfrak{p} \notin W$: the underlying topological space of W is the topological closure of W_η in X; the subspace $Y \setminus \operatorname{Spec} R/tR$ is closed in Y, hence in X, and contains W_η; then, by definition of the topological closure $W \subset Y \setminus \operatorname{Spec} R/t$, so $\mathfrak{p} \in W$ is not possible as $\mathfrak{p} \in \operatorname{Spec} R/t$. Finally, we verify that $\dim W_\eta < Y_\eta$: $\operatorname{Spec}(R/tR)_\eta$ contains the generic points of Y_η; it follows that W_η cannot contain any irreducible component of Y_η, or equivalently, $\operatorname{codim}(W_\eta, Y_\eta) \geq 1$; from the definition of dimension we have that $\dim W_\eta + \operatorname{codim}(W_\eta, Y_\eta) \leq \dim Y_\eta$, hence $\dim W_\eta < \dim Y_\eta$. Therefore W_η satisfies the hypotheses of Proposition 4.17. The schemes Y, X also satisfy the hypotheses Proposition 4.17, so we may apply Proposition 4.17 to obtain a commutative diagram

$$
\begin{array}{ccc}
& U & \qquad (4.12) \\
j \nearrow & \downarrow \pi & \\
Y \xrightarrow{\quad \pi|_Y \quad} \mathbb{P}^{d-1}_\Lambda &
\end{array}
$$

with U an open subscheme of X containing x, $\pi|_Y$ finite, $j : Y \hookrightarrow U$ a closed immersion, and π is smooth at x over $\mathbb{P}^{d-1}_\Lambda$ of relative dimension 1, and $\pi(x) \notin$

$\pi(W)$. A finite morphism is a closed morphism, so $\pi|_Y (W)$ is closed in $\mathbb{P}_\Lambda^{d-1}$. The complement $\mathbb{P}_\Lambda^{d-1} - \pi(W)$ is then an open neighborhood of $\mathfrak{p}$. Choosing an open affine neighborhood $\operatorname{Spec} B \subset \mathbb{P}_\Lambda^{d-1} - \pi(W)$ of $\pi(x)$, then pulling back the diagram 4.12 over $\operatorname{Spec} B$, we obtain the diagram below.

$$
\begin{array}{ccc}
 & & \pi^{-1}(\operatorname{Spec} B) \\
 & \overset{j}{\nearrow} & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec} C & \xrightarrow{\ \pi|_C\ } & \operatorname{Spec} B
\end{array}
\qquad (4.13)
$$

Indeed, as the properties of being finite and smooth of relative dimension 1 are preserved by base change, in the diagram 4.13, π is smooth at x of relative dimension 1, $\pi|_C$ is a finite morphism, and $\operatorname{Spec} C = \pi|_Y^{-1}(\operatorname{Spec} B)$ is an open affine subscheme because finite morphisms are affine. Additionally, $\pi|_Y^{-1}(\operatorname{Spec} B) \subset Y \setminus W$, so in fact $\operatorname{Spec} C \subset Y \setminus W$. In particular, we have an inclusion of generic fibers $\operatorname{Spec} C_\eta \subset Y_\eta \setminus (W)_\eta = Y_\eta \setminus (W_\eta)$. Recall that $W_\eta = Y_\eta \setminus \operatorname{Spec}(R/tR)_\eta)$, so $Y_\eta \setminus (W_\eta) = \operatorname{Spec}(R/tR)_\eta$, and hence we have an open immersion $\operatorname{Spec} C_\eta \subset \operatorname{Spec}(R/tR)_\eta$. Let $\omega : (R/tR)_\eta \to C$ denote the associated ring map.

Taking the fiber product of $\operatorname{Spec} C$ and $\pi^{-1}(\operatorname{Spec} B)$ over $\operatorname{Spec} B$ in the diagram 4.13, we obtain the diagram below.

$$
\begin{array}{ccc}
Z & \xrightarrow{\ p\ } & \pi^{-1}(\operatorname{Spec} B) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec} C & \xrightarrow{\ \pi|_C\ } & \operatorname{Spec} B
\end{array}
\qquad (4.14)
$$

In the diagram 4.14, the projection q is smooth of relative dimension 1 at the points in $p^{-1}(x)$, and p is a finite morphism, as these are properties preserved by base change. By applying the universal property of the fiber product, the closed immersion $j : \operatorname{Spec} C \hookrightarrow \pi^{-1}(\operatorname{Spec} B)$ induces a closed immersion $s : \operatorname{Spec} C \hookrightarrow Z$

which is a section of $q : Z \to \operatorname{Spec} C$ such that the diagram below commutes.

$$
\begin{array}{ccc}
Z & \xrightarrow{\;p\;} & \pi^{-1}(\operatorname{Spec} B) \\
{\scriptstyle s}\big\uparrow & \nearrow{\scriptstyle j} & \\
\operatorname{Spec} C & &
\end{array}
\qquad (4.15)
$$

We have then [2, 4.15] that the sheaf of ideals in $\mathcal{O}_Z$ defining the image of s is principal, defined by a regular element (that is, a non-zero divisor) at the points $p^{-1}(x)$, hence principal in a neighborhood U of $p^{-1}(x)$. Since p is a finite morphism, it is a closed morphism, and then $p(Z - U)$ is closed in $\pi^{-1}(\operatorname{Spec} B)$ and does not contain x. Choose an open neighborhood $\operatorname{Spec} R_f$ (recall that $\operatorname{Spec} R$ is open in X) of x such that: $\operatorname{Spec} R_f$ is contained in $\pi^{-1}(\operatorname{Spec} B) - p(Z - U)$; $\operatorname{Spec} R_f$ is smooth over $\operatorname{Spec} B$. Then, as p is an affine morphism, $p^{-1}(\operatorname{Spec} R_f)$ is affine and will be denoted by $\operatorname{Spec} C'$.

Now, we have by pulling back the smooth morphism $\pi : \operatorname{Spec} R_f \hookrightarrow \pi^{-1}(\operatorname{Spec} B) \xrightarrow{\pi} \operatorname{Spec} B$ along $\pi|_C : \operatorname{Spec} C \to \operatorname{Spec} B$ the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} C' & \xrightarrow{\;p\;} & \operatorname{Spec} R_f \\
\big\downarrow & & \big\downarrow \\
Z & \xrightarrow{\;p\;} & \pi^{-1}(\operatorname{Spec} B) \\
{\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\operatorname{Spec} C & \xrightarrow{\;\pi|_C\;} & \operatorname{Spec} B
\end{array}
$$

with $\operatorname{Spec} C' \hookrightarrow Z \xrightarrow{q} \operatorname{Spec} C$ a smooth morphism as it is a base change of a smooth morphism. Denote this smooth morphism by $\psi : \operatorname{Spec} C' \to \operatorname{Spec} C$. Then, pulling back the diagram 4.15 along the open immersion $\operatorname{Spec} R_f \to \pi^{-1}(\operatorname{Spec} B)$ we obtain the commutative diagram below (using the definition of the fiber product we identify $j^{-1}(\operatorname{Spec} R_f)$ with $\operatorname{Spec} R_f/tR_f$).

$$\text{Spec}\, C' \xrightarrow{\quad p \quad} \text{Spec}\, R_f \qquad\qquad (4.16)$$

$$s \uparrow \quad \nearrow j$$

$$\text{Spec}\, R_f/tR_f$$

Rewriting diagram 4.16, we have a commutative diagram of rings

$$C' \xleftarrow{\quad p \quad} R_f \qquad\qquad (4.17)$$

$$s \downarrow \quad \swarrow j$$

$$R_f/tR_f$$

where $\ker s = t'C'$ and s is surjective, j is the canonical quotient map, and p is a finite morphism. Base changing the diagram 4.17 over the generic point, or equivalently, localizing with respect to π, we obtain the diagram

$$C'[\pi^{-1}] \xleftarrow{\quad p \quad} R_f[\pi^{-1}]$$

$$s \downarrow \quad \swarrow j$$

$$R_f/tR_f[\pi^{-1}]$$

and as localization respects quotients, we have that $\ker s = t'C'[\pi^{-1}]$. Also, t' is a non-zero divisor in $C'[\pi^{-1}]$: C' is flat over Λ as $\text{Spec}\, C'$ is smooth over $\text{Spec}\, C$, and $\text{Spec}\, C$ is flat over Λ as it is open in the flat scheme Y; then C' is flat over Λ, so it is π-torsion free; it follows that since t' is not a zero-divisor in C', t' is not a zero-divisor in $C'[\pi^{-1}]$. Also, s is surjective, j is the canonical quotient map, and p is a finite morphism. This completes the proof of the corollary. $\qquad\square$

Theorem 4.19. *Let Λ be DVR (see Appendix 5.11) having an infinite residue field, and let A be a local ring essentially smooth over Λ. Then, the augmented Gersten commplex for the Witt groups of $A[\pi^{-1}]$ is exact.*

Proof. We may assume that Λ is a mixed $(0,p)$-characteristic DVR: if Λ is an equicharacteristic DVR, then A is an equicharacteristic regular local ring; using

the long exact sequence of Lemma 2.19 (taking f to be the regular parameter $\pi \in A$) together with the geometric case of the Gersten conjecture (for A and A/π) we obtain for $i \geq 1$, $H^i(\mathrm{Ger}(A[\pi^{-1}])) = 0$. This implies the augmented Gersten complex is exact for $A[\pi^{-1}]$ (Corollary 2.24). Now, to prove the theorem, using Lemma 2.18 it suffices to prove that for all integers $j \geq 0$, $i \in \mathbb{Z}$, the morphisms $W^i(D^{j+1}A[\pi^{-1}]) \to W^i(D^j A[\pi^{-1}])$ are zero. To prove this, let $x \in W^i(D^{j+1}A[\pi^{-1}])$ be a non-zero element. By Lemma 4.13 $(i) - (v)$, we have an element $z \in W^i(D^j(R/tR)[\pi^{-1}])$ such that the transfer of z is sent to an element which maps to x, where R is a smooth integral Λ-algebra such that $A \simeq R_{\mathfrak{p}}$ and $t \in R$ is a non-zero divisor such that $\mathrm{Spec}\, R/t$ is reduced and flat over Λ, and $\mathfrak{p}$ is in the closed fiber of $\mathrm{Spec}\, R$. Then, we apply Bloch's version of Quillen normalization by using Corollary 4.18 to obtain an open immersion $\mathrm{Spec}\,(R_f/tR_f)_\eta \xrightarrow{\omega} \mathrm{Spec}\, C_\eta$, and a commutative diagram

$$C'[\pi^{-1}] \xleftarrow{\quad p \quad} R_f[\pi^{-1}] \qquad\qquad (4.18)$$

$$\psi \nearrow \quad \downarrow s \quad \swarrow j$$

$$C[\pi^{-1}] \xrightarrow{\;\omega\;} R_f/tR_f[\pi^{-1}]$$

where ψ is a smooth morphism, p is a finite morphism, j is the canonical surjection, and s is a surjection with $\ker \pi = t'C'[\pi^{-1}]$ for some non-zero divisor $t' \in C'[\pi^{-1}]$. The left hand triangle in the diagram 4.18 we claim satisfies the hypotheses of Gille's new zero theorem taking $S = C[\pi^{-1}], R = C'[\pi^{-1}], R' = R_f/tR_f[\pi^{-1}] \simeq C'/t'C'[\pi^{-1}]$ To check this claim we check that all the rings involved are Gorenstein: we have that $\mathrm{Spec}\, R/t$ and $\mathrm{Spec}\, R_f/tR_f$ are Gorenstein as they are quotients of a regular ring by a non-zero divisor; as $\mathrm{Spec}\, C$ is open in $\mathrm{Spec}\, R/t$ it too is Gorenstein; as C' is smooth over the Gorenstein ring C it has regular fibers from which it follows that C' is Gorenstein [16, Corollary 3.3.15]; finally the localization of any Gorenstein ring is Gorenstein as it is open. We apply Gille's zero theorem

using Corollary 4.7. From this result we have that the morphism of Witt groups
$W^{i-1}(D^jC[\pi^{-1}]) \xrightarrow{\omega^*} W^{i-1}(D^jC'[\pi^{-1}]/t'C'[\pi^{-1}]) \xrightarrow{\text{Tr}} W^i(D^jC'[\pi^{-1}])$ is zero. Since
the transfer commutes with the composition of finite morphisms (Lemma 4.5),
from the commutative diagram appearing in the righthand triangle of diagram
4.18 we obtain that the composition

$$W^{i-1}(D^jC[\pi^{-1}]) \xrightarrow{\omega^*} W^{i-1}(D^jR_f/tR_f[\pi^{-1}]) \xrightarrow{\text{Tr}} W^i(D^jR_f[\pi^{-1}])$$

is zero. As the transfer commutes with localization (Lemma 4.4), the diagram
below commutes

$$
\begin{array}{ccc}
W^{i-1}(D^jR/tR[\pi^{-1}]) & \xrightarrow{\text{Tr}} & W^i(D^jR[\pi^{-1}]) \\
\downarrow & & \downarrow \\
W^{i-1}(D^jC[\pi^{-1}]) \xrightarrow{\omega^*} W^{i-1}(D^jR_f/tR_f[\pi^{-1}]) & \xrightarrow{\text{Tr}} & W^i(D^jR_f[\pi^{-1}])
\end{array}
$$

where the maps in the left hand triangle are induced from open immersions. This
finishes the proof since then the image of x in $W^i(D^jA[\pi^{-1}])$ is zero as it is the
image of z and we just proved that z maps to zero in $W^i(D^jR_f[\pi^{-1}])$. $\qquad\square$

Corollary 4.20. *Let Λ be a DVR having an infinite residue field, π a uniformizing
parameter for Λ, and A a local ring essentially smooth over Λ. Then, the Gersten
conjecture is true for the Witt groups of A.*

Proof. If Λ is an equicharacteristic DVR, then it follows that A is equicharacteristic
regular local ring, in which case the Gersten conjecture is known (*e.g.* [31, Theorem
3.1]). So we may assume that Λ is a mixed $(0, p)$-characteristic $(p \neq 2)$ DVR.
Then, using the long exact sequence of Lemma 2.19 (taking f to be the regular
parameter $\pi \in A$) together with Theorem 4.19 and the geometric case applied to
the equicharacteristic regular local ring A/π (*e.g.* [31, Theorem 3.1]) of the Gersten
conjecture for the Witt group we have that, for $i \geq 2$, $H^i(\text{Ger}(A)) = 0$. This implies
the Gersten conjecture for A (Corollary 2.24). $\qquad\square$

Corollary 4.21. *Let Λ be a DVR having an infinite residue field, π a uniformizing parameter for Λ, and A a local ring essentially smooth over Λ. Then, for $p \geq 1$, $H^p_{\mathrm{Zar}}(A[\pi^{-1}], \mathcal{W}) = 0$, that is, the Zariski cohomology of $A[\pi^{-1}]$ with coefficients in the Witt sheaf (the Zariski sheaf on $\mathrm{Spec}\, A$ associated to the presheaf $U \mapsto W(U)$) vanishes.*

Proof. Using Theorem 4.19 we have that, for $p \geq 1$, $H^p(\mathrm{Ger}(A[\pi^{-1}])) = 0$. Since the Gersten conjecture is also true for every local ring of $A[\pi^{-1}]$ by the geometric case of the Gersten conjecture, we then have that $H^p(\mathrm{Ger}(A[\pi^{-1}])) = H^p_{\mathrm{Zar}}(A[\pi^{-1}], \mathcal{W})$ (Lemma 2.17), thus proving the corollary. $\qquad\square$

4.3 Proof of the Gersten Conjecture: Local Rings Regular over a DVR

4.22. Let k be a field and R a noetherian k-algebra, that is, R is a noetherian ring together with a ring morphism $k \to R$ from the field k to R. For every field extension $k \to k'$ of k, we may consider the k'-algebra $R \otimes_k k'$ obtained by taking the tensor product of R and k' over k; if, for every finite field extension k', $R \otimes_k k'$ is a regular ring, then R is said to be *geometrically regular over k*. When, additionally, R is local, then R is geometrically regular over k if and only if R is formally smooth (with the $\mathfrak{m}_R$-adic topology) over k [37, 22.5.8].

4.23. A morphism $A' \to A$ of noetherian rings is *regular* if it is flat and for every $\mathfrak{p} \in \mathrm{Spec}\, A$, the fiber $A \otimes_{A'} k(\mathfrak{p})$ is geometrically regular over $k(\mathfrak{p})$.

Lemma 4.24. *Let Λ be a DVR, π a uniformizing parameter for Λ, and A a regular local ring of mixed $(0, p)$-characteristic which is a Λ-algebra. If A is regular over Λ, then π is a regular parameter for A, and additionally:*

   (i) the closed fiber A/π of A over Λ is a regular local ring of equicharacteristic p;

(ii) the localizations $(A[\pi^{-1}])_{\mathfrak{p}}$ of $A[\pi^{-1}]$ at prime ideals $\mathfrak{p} \in \operatorname{Spec} A[\pi^{-1}]$ are regular local rings of equicharacteristic 0.

Proof. Since A is regular over Λ, from the definition we have that the closed fiber A/π is geometrically regular over Λ/π, hence is regular. As the quotient A/π is regular of dimension $\dim A - 1$, it follows that π is a regular parameter for A.

As A is a mixed $(0, p)$-characteristic regular local ring that is a Λ-algebra, Λ is contained in A and is also of mixed $(0, p)$-characteristic (Lemma 5.12). In particular the prime p is an element $p \in \mathfrak{m}_A$ of the maximal ideal $\mathfrak{m}_A$ in A. Hence, the closed fiber A/π is a regular local ring of equicharacteristic p, proving (i).

To prove (ii), in view of the fact that for any prime $\mathfrak{p} \in \operatorname{Spec} A[\pi^{-1}]$ the localizations commute $(A[\pi^{-1}])_{\mathfrak{p}} = A_{\mathfrak{p}}$, it follows that they are regular local rings, and as they are $\mathbb{Q}$-algebras via the composition $\mathbb{Q} \hookrightarrow \operatorname{Frac}\Lambda \to A[\pi^{-1}] \to (A[\pi^{-1}])_{\mathfrak{p}}$, they are regular local of equicharacteristic 0. $\qquad\qquad\square$

Next we recall Popescu's theorem below, and apply it in Lemma 4.26.

Proposition 4.25. *[57, Theorem 1.8] Let $A' \to A$ be a homomorphism of noetherian rings. The homomorphism $A' \to A$ is a regular morphism if and only if A is a filtered colimit of smooth A'-algebras.*

In the following Lemma (i) is immediate and the proof of (ii) follows since in this case the localization at an element commutes with the filtered colimit.

Lemma 4.26. *[55, c.f. Lemma 3.2] Let Λ be a DVR, π a uniformizing parameter for Λ, and A a local ring regular over Λ, that is, A is a filtered colimit of smooth Λ-algebras S_α (Proposition 4.25). Let $\phi_\alpha : S_\alpha \to A$ denote the canonical homomorphisms into the colimit. Denote by $\mathfrak{p}_\alpha$ the prime ideal in S_α defined by the inverse*

image of the maximal ideal $\phi_\alpha^{-1}(\mathfrak{m}_A)$. *Let* $S_{\mathfrak{p}_\alpha}$ *denote the localization of* S_α *at the prime ideal* $\mathfrak{p}_\alpha$. *Then:*

(i) *the ring* A *is a filtered colimit* $A \simeq \varinjlim S_{\mathfrak{p}_\alpha}$, *where the* $S_{\mathfrak{p}_\alpha}$ *are regular local rings which are essentially smooth over* Λ;

(ii) *the localization* $A[\pi^{-1}]$ *of* A *at the element* π *is a filtered colimit* $A[\pi^{-1}] \simeq \varinjlim (S_{\mathfrak{p}_\alpha})[\pi^{-1}]$, *where* $(S_{\mathfrak{p}_\alpha})[\pi^{-1}]$ *is the localization of* $S_{\mathfrak{p}_\alpha}$ *at the element* π.

Lemma 4.27. *Let* A *be a regular local ring, and* $f \in A$ *be a regular parameter on* A. *Suppose that the following two conditions are satisfied:*

i) *the Gersten conjecture is true for the Witt groups of the quotient ring* A/fA;

ii) *for every prime* $\mathfrak{p} \in \operatorname{Spec} A_f$, *the Gersten conjecture is true for the Witt groups of the localization* $(A_f)_{\mathfrak{p}} = A_{\mathfrak{p}}$.

Under these conditions, if the Zariski cohomology groups $H^p_{\mathrm{Zar}}(A_f, \mathcal{W})$ *of* A_f *with coefficients in the Witt sheaf* $\mathcal{W}$ *vanish for* $p \geq 2$, *then the Gersten conjecture is true for the Witt groups of* A.

Proof. Let A be a regular local ring, and let $f \in A$ be a regular parameter on A. The quotient ring A/fA is again a regular local ring, of Krull dimension equal to $\dim A - 1$. Suppose that $i)$ holds, that is, the Gersten conjecture is true for the Witt groups of A/fA. In particular, for $p \geq 1$, the cohomology of the Gersten-Witt complex for A/fA vanishes, $H^p(\operatorname{Ger}(A/fA)) = 0$. Then, in view of the localization long exact sequence 2.15 of Lemma 2.19, when $p \geq 2$ the cohomology of the Gersten-Witt complex $H^p(\operatorname{Ger}(A))$ of A injects into the cohomology of the Gersten-Witt complex $H^p(\operatorname{Ger}(A_f))$ of A_f. It follows that if $H^p(\operatorname{Ger}(A_f)) = 0$ for $p \geq 2$, then $H^p(\operatorname{Ger}(A)) = 0$ for $p \geq 2$, or equivalently, the Gersten conjecture is true for

the Witt groups of A. If we additionally suppose $ii)$, that is, for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A_f$ of A_f, the Gersten conjecture is true for the Witt groups of $(A_f)_{\mathfrak{p}} = A_{\mathfrak{p}}$, then we have by Lemma 2.17 that the Zariski cohomology groups $H^p_{\mathrm{Zar}}(A_f, \mathcal{W})$ of A_f with coefficients in the Witt sheaf $\mathcal{W}$ are equal to the cohomology of the Gersten-Witt complex $H^p(\operatorname{Ger}(A_f))$ of A_f. From these observations, the lemma immediately follows. $\qquad\square$

To conclude this chapter, we prove the Gersten conjecture for the Witt groups of a local ring regular over a DVR having infinite residue field.

Theorem 4.28. *Let Λ be a DVR having an infinite residue field. If A is a local ring that is regular over Λ, then the Gersten conjecture is true for the Witt groups of A.*

Proof. Let A be a local ring that is regular over Λ, and let π denote a uniformizing parameter for Λ. As the Gersten conjecture is known when A is of equicharacteristic, we may assume that both Λ and A are of mixed $(0, p)$-characteristic (Lemma 5.12). To prove the Gersten conjecture for the Witt groups of A, we first demonstrate that it is sufficient to prove that the Zariski cohomology groups $H^p_{\mathrm{Zar}}(A[\pi^{-1}], \mathcal{W})$ of $A[\pi^{-1}]$ with coefficients in the Witt sheaf $\mathcal{W}$ vanish when $p \geq 2$. To prove this sufficiency, we will verify that π is a regular parameter satisfying the conditions of Lemma 4.27. Applying Lemma 4.24, we have that $\pi \in A$ is a regular parameter of A such that: $i)$ the closed fiber A/π of $\Lambda \to A$ is a regular local ring of equicharacteristic p; $ii)$ the localizations $(A[\pi^{-1}])_{\mathfrak{p}}$ of $A[\pi^{-1}]$ at prime ideals $\mathfrak{p} \in \operatorname{Spec} A[\pi^{-1}]$ are regular local rings of equicharacteristic 0. Using the equicharacteristic case of the Gersten conjecture, we then have that the Gersten conjecture is true for A/π and for the local rings $(A[\pi^{-1}])_{\mathfrak{p}}$. Hence π satisfies the desired conditions.

To prove the vanishing of the Zariski cohomology groups $H_{\mathrm{Zar}}^p(A[\pi^{-1}], \mathcal{W})$, we begin by writing $A[\pi^{-1}]$ as a filtered colimit of rings $(S_{\mathfrak{p}_\alpha})[\pi^{-1}]$ (Lemma 4.26), where each $(S_{\mathfrak{p}_\alpha})[\pi^{-1}]$ is the localization at π of a local ring $S_{\mathfrak{p}_\alpha}$ which is essentially smooth over Λ. Zariski cohomology commutes with filtered colimits (a theorem due to Grothendieck, however see [55, Theorem 6.6] for an alternative proof), hence this finishes the proof since the Zariski cohomology groups $H_{\mathrm{Zar}}^p((S_{\mathfrak{p}_\alpha})[\pi^{-1}], \mathcal{W})$ vanish when $p \geq 2$ (Corollary 4.21). □

Applications

5.1 Finite Generation Theorems for Grothendieck-Witt Groups

In this section, we prove finite generation theorems for the Grothendieck-Witt groups of arithmetic schemes. Let X be a separated noetherian scheme over $\mathbb{Z}[\frac{1}{2}]$, and let L be a line bundle on X. Let $\mathrm{Ch}^b\mathrm{Vect}(X)$ denote the category of bounded chain complexes of vector bundles on X. By shifting L, for each $n \in \mathbb{Z}$, we obtain a duality $Hom(-, L[n])$ on $\mathrm{Ch}^b\mathrm{Vect}(X)$. We work with Schlichting's Grothendieck-Witt spectrum $GW^n(X, L)$ [63] associated to the category $\mathrm{Ch}^b\mathrm{Vect}(X)$, equipped with the duality $Hom(-, L[n])$ and with quasi-isomorphisms as weak equivalences. Its m-th homotopy groups are denoted by $GW^n_m(X, L)$, and are said to be the *Grothendieck-Witt groups of X with coefficients in L.* These groups are 4-periodic in n, $GW^n_m(X, L) \simeq GW^{n+4}_m(X, L)$. The negative Grothendieck-Witt groups, that is, the negative homotopy groups of the Grothendieck-Witt spectrum, agree with the Witt groups $GW^0_{-m}(X) \simeq W^m(X)$, for $m > 0$. For these facts, see [63].

Proposition 5.1. *Let X be a separated noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme. For every $n \in \mathbb{Z}$, there is a long exact sequence of abelian groups*

$$\ldots \to GW^n_m(X) \to GW^{n-1}_{m-1}(X) \xrightarrow{F} K_{m-1}(X) \xrightarrow{H} GW^n_{m-1}(X) \to \ldots$$

which may be completed to end in

$$\ldots GW^{n-1}_0(X) \to K_0(X) \xrightarrow{H} GW^n_0(X) \to W^n(X) \to 0.$$

Proof. It is proved in [63] that the sequence of spectra

$$GW^{n-1}(X) \xrightarrow{F} K(X) \xrightarrow{H} GW^n(X)$$

is a homotopy fibration, where $K(X)$ is the algebraic K-theory spectrum whose homotopy groups are the higher algebraic K-groups. Hence, it determines the

long exact sequence of the proposition, with one exception. To complete the long exact sequence to the form stated in the proposition, we use the isomorphism $GW^0_{-m}(X) \simeq W^m(X)$ $(m > 0)$ [63], which gives $GW^0_{-1}(X) \simeq W^1(X)$. Shifting the duality on both sides $(n-1)$-times, we obtain $GW^{n-1}_{-1}(X) \simeq W^n(X)$. Since there are no negative K-groups of X, the map $GW^n_0(X) \to W^n(X)$ is surjective as asserted. □

Karoubi induction is a well known means of proving the corollary below. We give the corollary the name "Schlichting" induction because the argument is different than the usual Karoubi induction argument (*i.e.* it uses the fibration above), and it was suggested to the author by Schlichting.

Corollary 5.2 ("Schlichting" Induction). *Maintain the hypothesis of the previous proposition. Assume that the groups $K_m(X)$ are finitely generated for all $m \in \mathbb{Z}$. If the Witt groups $W^n(X)$ are finitely generated for all $n \in \mathbb{Z}$, then the Grothendieck-Witt groups $GW^n_m(X)$ are finitely generated for all $m, n \in \mathbb{Z}$.*

Proof. We will prove the result by induction on m. The base case is $m = -1$, finite generation of the Witt groups $W^n(X) \simeq GW^{n-1}_{-1}$, for all $n \in \mathbb{Z}$. For the induction step, suppose that $GW^{n-1}_{m-1}(X)$ is finitely generated, for all $n \in \mathbb{Z}$. Using the fibration sequence of Proposition 5.1, and finite generation of the algebraic K-theory groups $K_m(X)$, we obtain that $GW^n_m(X)$ is also finitely generated, for all $n \in \mathbb{Z}$. □

Theorem 5.3. *Let X be a separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$ with no residue field of X formally real (e.g., a smooth variety over a finite field $\mathbb{F}_p$ ($p > 2$)), and let L be a line bundle on X. If $\dim(X) \leq 1$, then the Grothendieck-Witt groups $GW^n_m(X, L)$ are finitely generated groups.*

Proof. We may assume that X is connected using Lemma 3.31 (b), hence, integral using Lemma 3.32. Under the hypotheses on X, the algebraic K-groups $K_m(X)$ are finitely generated, for all $m \in \mathbb{Z}$ [35, §(4.71), Proposition 38(b)]. So, the result follows from Corollary 5.2 and Theorem 3.30 (a) (use Lemma 3.31 to get the result for any line bundle L on X). □

We have the following conditional result.

Theorem 5.4. *Let X be a separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$ with no residue field of X formally real (e.g., a smooth variety over a finite field $\mathbb{F}_p$ ($p > 2$)), and let L be a line bundle on X. Assume the Beilinson-Lichtenbaum conjecture holds (see Remark 3.19, note this is known for smooth varieties over fields). If the motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}(n))$ are finitely generated for all $m, n \in \mathbb{Z}$, then the Grothendieck-Witt groups $GW^n_m(X, L)$ are finitely generated for all $m, n \in \mathbb{Z}$.*

Proof. We may assume that X is connected using Lemma 3.31 (b), hence, integral using Lemma 3.32. After applying the Atiyah-Hirzebruch spectral sequence converging to K-theory [42, 4.3.2, Eq. (4.6) and the final paragraph of §(4.6)], we obtain that the K-theory of $K_m(X)$ is also finitely generated, for all $m \in \mathbb{Z}$. Multiplication by 2 defines a short exact sequence of motivic sheaves

$$0 \to \mathbb{Z}(n) \xrightarrow{2} \mathbb{Z}(n) \to \mathbb{Z}/2\mathbb{Z}(n) \to 0,$$

for every $n \in \mathbb{Z}$. This induces a long exact sequence

$$\ldots \to H^m_{mot}(X, \mathbb{Z}(n)) \to H^m_{mot}(X, \mathbb{Z}(n)) \to H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n)) \to \ldots$$

of motivic cohomology groups. Using the hypothesis that the motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}(n))$ are finitely generated, it follows that the groups $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$

are also finitely generated, hence finite, as they are torsion. By Theorem 3.34(a), the Witt groups $W^n(X)$ are finite. Therefore, Corollary 5.2 finishes the proof (use Lemma 3.31 to get the result for any line bundle on X). $\qquad\square$

5.2 Finiteness of the d-th Chow-Witt Group

Throughout this section, X will denote a variety (*i.e.* separated and of finite type) that is smooth over a finite field $\mathbb{F}_p$ ($p > 2$). First, we recall the definition of the Chow-Witt groups (*aka* Chow groups of oriented cycles). The n-th cycle complex with coefficients in Milnor K-theory [43] is a complex consisting of Milnor K-groups

$$C(X, K_n^M) := \bigoplus_{x \in X^0} K_n^M (\kappa(x)) \xrightarrow{d} \bigoplus_{x \in X^1} K_{n-1}^M (\kappa(x)) \xrightarrow{d} \ldots \ldots \xrightarrow{d} \bigoplus_{x \in X^d} K_{n-d}^M (\kappa(x))$$

with differential defined componentwise, exactly as was done in Definition 3.7, however using the the residue morphism for Milnor K-theory. The natural map $s^n : K_n^M(k) \to \overline{I}^n(k)$ (see Section 3.2.4), defined for every field k with char$(k) \neq 2$, induces a map of complexes $s^n : C(X, K_n^M) \to C(X, \overline{I}^n)$ [21, Theorem 10.2.6], where $C(X, \overline{I}^n)$ is the complex of Definition 3.25. To obtain the complex

$$C(X, I^n, \omega_{X/k}) := \bigoplus_{x \in X^0} I^n (\kappa(x); \Lambda^0) \xrightarrow{d} \ldots \xrightarrow{d} \bigoplus_{x \in X^d} I^{n-d} (\kappa(x); \Lambda^d), \qquad (5.1)$$

that is also needed to define the Chow-Witt groups, where $\Lambda^i := \Lambda^i((m_x/m_x^2)^*)$, one begins with Schmid's Gersten-Witt complex [64, Satz 3.3.2]

$$C(X, W, \omega_{X/k}) := \bigoplus_{x \in X^0} W (\kappa(x); \Lambda^0) \xrightarrow{d} \ldots \xrightarrow{d} \bigoplus_{x \in X^d} W (\kappa(x); \Lambda^d), \qquad (5.2)$$

and filters it by the powers of the fundamental ideal, *e.g.* see [12]. Recall that for any field k, and any one-dimensional k-vector space $\mathcal{L}$, a choice of generator for $\mathcal{L}$ defines an isomorphism $W(k) \to W(k, \mathcal{L})$, and by definition $I^n (\kappa(x); \mathcal{L}) := I^n (\kappa(x)) \cdot W (\kappa(x); \mathcal{L})$, as $I^n(k; \mathcal{L})$ does not depend on the choice of isomorphism (e.g., see [21, Lemma A.1.2]). The quotient complexes $C(X, I^n, \omega_{X/k})/C(X, I^{n+1}, \omega_{X/k})$

will be denoted simply by $C(X, \overline{I}^n)$, as they are in fact isomorphic (e.g., see [21, Lemma A.1.3]).

Definition 5.5. Define the complex $C(X, J^n)$ to be the fiber product of the complexes $C(X, I^n, \omega_{X/k})$ and $C(X, K_n^M)$ over $C(X, \overline{I}^n)$. Hence, $C(X, J^n)$ lives in a diagram

$$
\begin{array}{ccc}
C(X, J^n) & \longrightarrow & C(X, I^n, \omega_{X/k}) \\
\downarrow & & \downarrow \\
C(X, K_n^M) & \xrightarrow{\ s^n\ } & C(X, \overline{I}^n)
\end{array}
$$

where the map from $C(X, I^n, \omega_{X/k})$ to $C(X, \overline{I}^n)$ is the quotient map. For any $n \geq 0$, the *n-th Chow-Witt group* $\widetilde{CH}^n(X)$ is defined to be the n-th cohomology group of the complex $C(X, J^n)$.

The following lemma is a slight variation on an argument of Gille.

Lemma 5.6. *(See [30, Proof of Proposition 10.3]). For all $j \geq 0$, the complex $C(X, I^{j+d+2}, \omega_{X/k})$ vanishes, and the quotient map $C(X, I^{d+1}, \omega_{X/k}) \xrightarrow{\cong} C(X, \overline{I}^{d+1})$ is an isomorphism of complexes.*

Proof. Let $x \in X^p$ be a codimension p point of X. By Lemma 1.14, $\mathrm{cd}_2(k(x) \leq 1 + d - p$. Since the map $e_k^i : \overline{I}^i(k) \to H_{Gal}^i(k, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism (see Definition 3.22) for every field k, $\overline{I}^{2+d-p}(k(x)) = 0$. It follows that $I^{2+d-p}(k(x)) = \cap_{n \geq 2+d-p} I^n(k(x))$. By the Arason-Pfister Haupsatz, $0 = \cap_{n \geq 0} I^n(k(x))$. Therefore, $I^{2+d-p}(k(x)) = 0$, hence, by definition,

$$
I^{2+d-p}(k(x); \lambda^p((m_x/m_x^2)^*)) := I^{2+d-p}(k(x)) \cdot W(k(x); \lambda^p((m_x/m_x^2)^*)) = 0,
$$

and from this, for all $j \geq 0$, $C(X, \overline{I}^{j+d+2}, \omega_{X/k}) = 0$ follows. Then, the exact sequence of complexes

$$
0 \to C(X, I^{d+2}, \omega_{X/k}) \to C(X, I^{d+1}, \omega_{X/k}) \to C(X, \overline{I}^{d+1}) \to 0
$$

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degenerates into the desired isomorphism, finishing the proof of the lemma. □

Now we are ready to state and prove the finiteness theorem.

Theorem 5.7. *Let X be a smooth and quasi-projective variety over a finite field $\mathbb{F}_p$ ($p > 2$), pure dimensional of dimension d. Then the d-th Chow-Witt group $\widetilde{CH}^d(X)$ is finite.*

Proof. Recall that there is always an exact sequence

$$CH^d(X) \to \widetilde{CH}^d(X) \to H^d(X, I^d) \to 0,$$

For any quasi-projective variety over a finite field, the group $CH^d(X)$ is finite (c.f. [46, Theorem 9.2], [44, theorem 1]). So, the proof reduces to proving that $H^d(C(X, I^d, \omega_{X/k}))$ is finite. From the short exact sequence of complexes

$$0 \to C(X, I^{d+1}, \omega_{X/k}) \to C(X, I^d, \omega_{X/k}) \to C(X, \overline{I}^d) \to 0,$$

we obtain the long exact sequence in cohomology

$$\cdots \to H^d(C(X, I^{d+1}, \omega_{X/k})) \to H^d(C(X, I^d, \omega_{X/k})) \to H^d(C(X, \overline{I}^d)) \to \cdots$$

From Arason's theorem (Theorem 3.26), it follows that both $H^d(C(X, \overline{I}^d))$ and $H^d(C(X, \overline{I}^{d+1}))$ are isomorphic to the Kato cohomology groups $H^d(C(X, H^d))$ and $H^d(X, H^{d+1})$, respectively. The latter are finite by Lemma 3.17 (c) and (d). We conclude the proof by identifying $H^d(C(X, I^{d+1}, \omega_{X/k}))$ with $H^d(C(X, \overline{I}^{d+1}))$ using Lemma 5.6. □

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Appendix

We recall some facts and definitions from commutative algebra and then include a copy of the Elsevier retained author's rights.

5.8. Let A be a commutative ring with a unit element 1. A maximal ideal $\mathfrak{m}$ of A is a proper ideal not contained in any other. It is a consequence of Zorn's lemma that every commutative ring with unit which is not trivial has a maximal ideal; if A has only one maximal ideal $\mathfrak{m}$, then A is called a *local ring*; if A has only finitely many, then A is called *semi-local*. The quotient ring $k(\mathfrak{m}) := A/\mathfrak{m}$, as it is a quotient by a maximal ideal, is a field, called the residue field of A. A homomorphism of local rings $\phi : A \to B$ is called a *local homomorphism* if $\phi(\mathfrak{m}_A) \subset \mathfrak{m}_B$. This is the same as saying that $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, for $\phi^{-1}(\mathfrak{m}_B)$ is an ideal containing $\mathfrak{m}_A$ and not containing 1, hence is equal to $\mathfrak{m}_A$. It follows that ϕ induces an injective homomorphism between the residue fields $\phi : A/\mathfrak{m}_A \to B/\mathfrak{m}_B$.

5.9. Suppose now that A is a ring which is noetherian (every ascending chain of ideals stops) and local with maximal ideal $\mathfrak{m}$. As A is noetherian, every ideal is finitely generated. In particular, the maximal ideal $\mathfrak{m}$ is finitely generated, that is to say, there exist a finite number of elements $x_1, \ldots, x_n$ of the maximal ideal $\mathfrak{m}$ such that the ideal they generate in A equals $\mathfrak{m}$; if $n = \dim A$, then A is called a *regular local ring*. Any set of $\dim A$-elements in $\mathfrak{m}$ generating the maximal ideal is said to be a regular system of parameters for A, and any element $f \in \mathfrak{m}$ that belongs to a system of parameters is said to be a regular parameter for A. A regular local ring is an integral domain, integrally closed in its field of fractions (*aka* normal) [49, Theorem 14.3, Theorem 19.4].

Example 5.10.   (i) A regular local ring of dimension 0, as it is a domain, is necessarily a field, and vice versa.

   (ii) For a noetherian local ring of dimension 1 to be regular it is necessary and sufficient that it be a discrete valuation ring [49, Theorem 11.2, equivalence of (i) and (iii)].

We first recall the notions of equicharacteristic and mixed characteristic regular local rings.

5.11. Let A be a regular local ring. Since A is an integral domain, the image of $\mathbb{Z}$ in A, as a subring of A, is also an integral domain, hence is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}/p\mathbb{Z}$, where $p \in \mathbb{Z}$ is a prime; its field of fractions is isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. In the first case, one says that A *is of characteristic* 0; in the second case, that A *is of characteristic* p. Now we consider the characteristic of the residue field $k(\mathfrak{m})$ of A. When A is of characteristic $p > 0$, the image of $\mathbb{Z}$ in $k(\mathfrak{m})$ equals the image of the composition $\mathbb{Z}/p\mathbb{Z} \to A \to k(\mathfrak{m})$; since $1 \notin \mathfrak{m}$, the composition $\mathbb{Z}/p\mathbb{Z} \to A \to k(\mathfrak{m})$ is injective and $k(\mathfrak{m})$ is of characteristic $p > 0$, in which case A is said to be a *regular local ring of equicharacteristic* p. When A is of characteristic

0, consider the prime ideal $p\mathbb{Z}$ in $\mathbb{Z}$ that is defined by taking the inverse image of the maximal ideal $\mathfrak{m} \subset A$; localizing $\mathbb{Z}$ with respect to this prime ideal we obtain from the inclusion $\mathbb{Z} \to A$ an injective local homomorphism $\mathbb{Z}_{p\mathbb{Z}} \to A_\mathfrak{m} = A$; if $p = 0$, then the field $\mathbb{Z}_{p\mathbb{Z}} = \mathbb{Q}$ injects into A and A is said to be a *regular local ring of equicharacteristic 0*; if $p > 0$, then the DVR $\mathbb{Z}_{p\mathbb{Z}}$ injects into A, $p \in \mathfrak{m}$, the residue field $k(\mathfrak{m})$ is of characteristic p containing the finite field $\mathbb{Z}/p\mathbb{Z}$ as its prime field, and A is said to be a *regular local ring of mixed $(0,p)$-characteristic.* Furthermore, we see that if a regular local ring A contains a field k, then it contains the prime field of k, either $\mathbb{Q}$ or $\mathbb{F}_p$; in the former case A is of equicharacteristic 0 (as $\mathbb{Q} \subset A$ implies p is invertible in A for every prime p, so $p \notin \mathfrak{m}$), in the latter case A is of equicharacteristic p.

From the discussion above (5.11) we see that a regular local ring A contains a field if and only if A is an equicharacteristic regular local ring.

Let A be a regular local ring. Recall that a ring morphism $A \to B$ is flat if B is flat as an A-module, that is, the tensor product $(-) \otimes_A B$ with B over A preserves exact sequences. If A is of mixed characteristic, then A is flat over $\mathbb{Z}_p$ via the natural inclusion map: more generally, we have the following lemma.

Lemma 5.12. *Let Λ be a discrete valuation ring. Let A be a regular local ring of mixed $(0,p)$-characteristic. If A is a Λ-algebra, then A contains Λ, is flat over Λ, and Λ is also of mixed $(0,p)$-characteristic.*

Proof. To prove that the structure map $\Lambda \to A$ is injective, consider its kernel: as a prime ideal in Λ, it is either 0 or is equal to the maximal ideal $\mathfrak{m} \subset \Lambda$; in the latter case the image of the structure map would be the residue field $\Lambda/\mathfrak{m}$, contradicting the fact that A cannot contain a field as it is of mixed characteristic. The statement about flatness follows from the fact that any domain which is torsion-free over a principal ideal domain is flat over Λ (*e.g.* [48, Chapter 1, Corollary 2.5]). Finally, as A contains Λ, it follows that Λ must also be of mixed $(0,p)$-characteristic otherwise λ, and hence A, would contain a field. $\qquad\square$

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