Burnside form rings and the K-theory of forms

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick towards my application for the degree of Doctor of Philosophy. The work presented in this thesis is my own, except where I have indicated otherwise. This thesis has not been submitted to any other university, or for any other degree.

Abstract

The Burnside form ring $\underline{\mathbb{Z}}$ is the initial object and tensor unit in the category of form rings; therefore, its Grothendieck-Witt ring $GW_0(\underline{\mathbb{Z}})$, since it acts on $GW_i(R, \Lambda)$ for any $i \geq 0$ and any form ring (R, Λ), is of fundamental importance in the study of the K-theory of forms. We show that $GW_0(\underline{\mathbb{Z}})$ is isomorphic to $\mathbb{Z}^3$ as an abelian group, and also give its ring structure.

Using an extension of scalars construction defined by a universal property, one can define a Burnside form ring $\underline{R}$ for any commutative ring R. After calculating $GW_0(\underline{\mathbb{Z}})$, the remainder of the thesis calculates $GW_0(\underline{R})$ when R is a finite field. Along the way, we calculate $GW_0(\underline{R})$ for any ring R with 2 invertible and finitely-generated projective R-modules free, and we define a determinant map which generalises the classical determinant map on symmetric bilinear forms.

Chapter 1

Introduction

1.1 Introduction

1.1.1 Background and motivation

This thesis proves some results on the *K-theory of forms*. As the terminology suggests, this is a conceptual framework which generalises Hermitian K-theory, which here means the K-theory of finitely-generated projective modules over some ring R equipped with a quadratic or symmetric bilinear form, which itself generalises the algebraic K-theory of projective modules. Here, the role of rings in algebraic K-theory and rings with involution in Hermitian K-theory is taken by *form rings*. The definition, taken from [22], is as follows:

Definition 1.1.1 ([22], Definitions 3.3, 4.1)**.** Let (R, σ) be a ring with involution. Then a *form ring* over R is a pair (R, Λ) where Λ is an abelian group equipped with the trivial action of C_2, the cyclic group with order 2, together with C_2-equivarient group homomorphisms τ and ρ

$$R \xrightarrow{\tau} \Lambda \xrightarrow{\rho} R,$$

where the C_2-action on R is given by its involution. Also, Λ is equipped with a multiplicative $\mathbb{Z}$-linear left action

$$Q : (R, \cdot, 0, 1) \to (\mathrm{End}_{\mathbb{Z}}(\Lambda), \circ, 0, 1)$$

of R which preserves 0 and 1. This data is subject to the following stipulations:

1. $\rho \circ \tau = 1 + \sigma$

2. The deviation of Q is given by the formula

$$Q(a+b)(x) - Q(a)(x) - Q(b)(x) = \tau(a \cdot \rho(x) \cdot \bar{b})$$

   for $a, b \in R$ and $x \in \Lambda$.

3. The C_2-equivariant maps ρ and τ commute with the quadratic actions of R on itself and Λ, where R acts on itself by setting $a \cdot b = ab\sigma(a)$.

Also taken over from [22], we have the definition of a Λ-quadratic form over a form ring:

Definition 1.1.2 ([22], Definition 3.8)**.** Let (R, Λ) be a form ring. Then a Λ-*quadratic form* is a triple (M, q, β) is a triple where

1. M is a left R-module

2. $q : M \to \Lambda$ is a function with $q(ax) = Q(a)(q(x))$ for all $a \in R$ and $x \in M$

3. $\beta : M \otimes_{\mathbb{Z}} M^{op} \to R$ is a symmetric bilinear form with $\beta(x, x^{op}) = \rho(q(x))$ for all $x \in M$.

4. The deviation of q, that is, $q(x+y) - q(x) - q(y)$, is equal to $\tau\beta(x, y^{op})$.

We say a module (M, q, β) over a form ring (R, Λ) is *non-degenerate* if the symmetric bilinear form β is non-degenerate, and we say that (M, q, β) and (M', q', β') are *isometric* if there exists an isomorphism $f : M \xrightarrow{\cong} M'$ such that $q = q' \circ f$ and $\beta = \beta' \circ (f \otimes f^{op})$. One can also define *orthogonal sum* of Λ-quadratic form modules: given (M, q, β) and (M', q', β'), their orthogonal sum is $(M \oplus M', q \perp q', \beta \perp \beta')$, where $\beta \perp \beta'$ is the orthogonal sum of symmetric bilinear forms and $q \perp q'$ is defined analogously, by setting $q((x, y)) = q(x) + q(y)$. With this terminology, one defines:

Definition 1.1.3 (c.f. [22], Definition 3.10)**.** Given a form ring (R, Λ), its *Grothendieck-Witt group* $GW_0(R, \Lambda)$ is the Grothendieck group of the abelian monoid whose elements are isometry classes $[P, q, \beta]$ of non-degenerate forms over (R, Λ) where P is a finitely-generated projective module. The monoid operation is orthogonal sum.

Remark 1.1.4. As is standard in the field of K-theory, given a form ring (R, Λ), one has abelian groups $GW_i(R, \Lambda)$ for all $i \geq 0$, which are defined to be the homotopy groups of a topological space $GW(R, \Lambda)$. In the present thesis we focus on the case

$i = 0$, but it is worthwhile to keep in mind that our results are part of the larger setting of the higher K-theory of forms.

Form rings generalise classical symmetric bilinear and quadratic forms. For example, writing R^σ for the fixed points of the C_2-action on R given by σ, a module over the form ring

$$R \xrightarrow{1+\sigma} R^\sigma \hookrightarrow R$$

is the same thing as a symmetric bilinear form module over the ring with involution (R, σ); if σ is the identity, then these are exactly the classical symmetric bilinear forms over R. Similarly, writing R_σ for the orbits of the same C_2-action, a module over the form ring

$$R \twoheadrightarrow R_\sigma \xrightarrow{1+\sigma} R$$

is a quadratic form module over (R, σ).

It is appropriate at this point to mention that the study of the K-theory of forms is not a recent development. Indeed, the *lower* K-theory of forms is fairly well-studied, most famously by Bak in [1]. For the objects Bak calls "form rings", Λ is always a certain subgroup of R, and ρ is always its inclusion in R. These restrictions are not technically adequate for the higher K-theory of forms. Although the results of this thesis concern GW_0 of form rings, they are nevertheless within the homotopical framework of the higher K-theory of forms given in [22]. In fact, in the following definition, which is that of our main object of study, ρ fails to be injective.

Definition 1.1.5. The *Burnside form ring* $\underline{\mathbb{Z}} = (\mathbb{Z}, \mathbb{A}(\mathbb{Z}))$ is the form ring

$$\mathbb{Z} \xrightarrow{\tau} \mathbb{A}(\mathbb{Z}) \xrightarrow{\rho} \mathbb{Z}$$

where $\mathbb{Z}$ has trivial involution, $\mathbb{A}(\mathbb{Z}) = \mathbb{Z}[C_2] = \mathbb{Z}[t]/(t^2 - 1)$ is the integral group ring over C_2, and the maps are as follows:

$$\tau(n) = n + nt, \quad \rho(a + bt) = a + b, \quad Q(n)(a + bt) = \left(\frac{n(n+1)}{2} + \frac{n(n-1)}{2}t \right)(a + bt)$$

The importance of the Burnside form ring is that form rings can be made into a symmetric monoidal category with unit given by $\underline{\mathbb{Z}}$. The tensor product of this symmetric monoidal category induces a map on modules over the form rings involved which extends the usual tensor product of modules. This implies that the Grothendieck-Witt groups of *any* form ring are modules over the ring $GW_0(\underline{\mathbb{Z}})$, where the action is

given by a cup product defined in (2.13) in [22]. Therefore, understanding this ring is of fundamental importance to the K-theory of forms. This is done in Chapter 4 of the thesis, using a trace map which generalises the ordinary trace of a matrix. Moreover, one can use an extension of scalars construction to define a Burnside form ring $\underline{R}$ for any ring with involution (R, σ). Along the way in Chapter 4, we also calculate $GW_0(\underline{R})$, where R is a commutative ring with 2 invertible, such that finitely-generated projective R-modules are free, and where R is considered to have trivial involution.

Chapter 5 is concerned with calculating $GW_0(\underline{R})$ for the case when R is a finite field, again with trivial involution. Since the odd characteristic case is covered by results from Chapter 4, the focus is on the case of characteristic 2. Along the way, we define a determinant map which generalises the classical determinant map of symmetric bilinear forms.

1.1.2 Synopsis of the thesis and statement of the main results

Chapter 1 is the present general introduction.

Chapter 2 gives a quick overview of some foundational material in the subject area: more specifically, Quillen's paper [15] which defines K-theory of exact categories and the higher K-groups, as well as the corresponding paper by Schlichting ([19]) for Hermitian K-theory of symmetric bilinear forms (without assuming 2 invertible). Strictly speaking, this chapter is not a logical foundation for the rest of the thesis. Rather, it is intended to give a sense of where our original results lie in the literature, in a more mathematically precise and detailed manner than was possible in Section 1.1.1.

Chapter 3 is also expositional. It sets out the language of the K-theory of forms, which is the foundation of our original results.

Chapter 4 is devoted to the proof of the following result:

Theorem 1.1.6. *As an abelian group, $GW_0(\underline{\mathbb{Z}})$ is isomorphic to*

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

with ring structure given by the quotient ring

$$\mathbb{Z}[X,Y]/\langle X^2 - 1 \rangle, \langle XY + Y \rangle, \langle Y^2 - 8Y \rangle.$$

We use a quadratic version of the classical trace map on matrices to define a map from forms over $\underline{\mathbb{Z}}$ to $\mathbb{Z}[X,Y]/\langle X^2-1 \rangle, \langle XY+Y \rangle, \langle Y^2-8Y \rangle$ which is well-defined on

4

isometry classes. Most of the difficulty is in the proof that this map is injective. The key is a bijection which lets us view a form over $\underline{\mathbb{Z}}$ on a finitely-generated projective $\mathbb{Z}$-module P as a point in (P, q), where q is a non-homogeneous version of a classical quadratic form over $\mathbb{Z}$. Since the elements of $GW_0(\underline{\mathbb{Z}})$ are stable equivalence classes of forms over $\underline{\mathbb{Z}}$, we can orthogonal sum with certain classical symmetric bilinear form modules to obtain the technical conditions we need to prove injectivity of the map to $\mathbb{Z}[X, Y]/\langle X^2 - 1 \rangle, \langle XY + Y \rangle, \langle Y^2 - 8Y \rangle$.

Along the way, we also prove

Theorem 1.1.7. *Let R be a commutative ring with trivial involution and 2 invertible such that finitely-generated projective R-modules are free. Then we have an isomorphism of abelian groups*

$$GW_0(\underline{R}) \cong R \oplus GW_0(R),$$

where $GW_0(R)$ is the Grothendieck-Witt group of symmetric bilinear forms over R.

Chapter 5 is mostly concerned with calculating $GW_0(\underline{R})$ when R is a finite field. Since the odd characteristic case is covered by Theorem 1.1.7, almost all of the work is to prove the following result, which says that, for $\mathbb{F}_q$ a finite field of characteristic 2, the Grothendieck-Witt group of the Burnside form ring $(\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$ is isomorphic to the Grothendieck-Witt group of symmetric bilinear forms over the commutative ring $\mathbb{A}(\mathbb{F}_q)$. Its proof is based on the fact that, subject to certain hypotheses which are fulfilled in this case, the kernel of the map $\rho : \mathbb{A}(\mathbb{F}_q) \to \mathbb{F}_q$ is a nilpotent ideal. Since one of the aforementioned hypotheses is that ρ is surjective, we have $\mathbb{A}(\mathbb{F}_q)/\ker\rho$ is isomorphic to $\mathbb{F}_q$. From that point, the proof is essentially an adaptation of an analogous result for K-theory: if $J \subset R$ is a nilpotent ideal, then $K_0(R) \to K_0(R/J)$ is an isomorphism.

Theorem 1.1.8. *Let $\mathbb{F}_q$ be a finite field of characteristic 2, and let $(\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$ be its Burnside form ring. Then we have an isomorphism*

$$GW_0(\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q)) \cong GW_0(\mathbb{A}(\mathbb{F}_q)),$$

where the right hand side is the Grothendieck-Witt group of symmetric bilinear forms over the commutative ring $\mathbb{A}(\mathbb{F}_q)$.

Combining this with Theorem 1.1.7, we immediately obtain

Corollary 1.1.9. *Let $\mathbb{F}_q$ be a finite field. Denoting its Burnside form ring by*

$\underline{\mathbb{F}_q} = (\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$, *we have*

$$GW_0(\underline{\mathbb{F}_q}) \cong \begin{cases} \mathbb{F}_q \oplus GW_0(\mathbb{F}_q) & \text{if } \mathbb{F}_q \text{ has odd characteristic} \\ GW_0(\mathbb{A}(\mathbb{F}_q)) & \text{if } \mathbb{F}_q \text{ has characteristic } 2 \end{cases}$$

Chapter 2

Preliminaries

In this chapter, we will recall some basic definitions and facts about "classical" algebraic and Hermitian K-theory, to give a sense of where our original results lie in the literature. We begin with some underlying topological definitions.

2.1 Simplicial sets and classifying spaces

Given a category $\mathcal{C}$, the goal of the section is to explain how to build $B\mathcal{C}$, a CW complex known as the *classifying space* of $\mathcal{C}$. This material is classical.

Definition 2.1.1. Let Δ be the category whose objects are the finite totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are the functions $[n] \to [m]$ that preserve the ordering. We call Δ the *simplex category*.

Definition 2.1.2. A *simplicial set* is a functor from $\Delta^{\mathrm{op}} \to \mathrm{Set}$. Given a simplicial set X, we denote $X([n])$ by X_n. Maps of simplicial sets are natural transformations of functors. We will denote the category of simplicial sets by $\Delta^{\mathrm{op}}\mathrm{Set}$.

Remark 2.1.3. We can replace Set above with any category; for example, if we use Ab, we obtain the notion of a *simplicial abelian group*. If we use Top, we obtain a *simplicial space*, and so on. We can also dualize the concept of a simplicial object; given a category $\mathcal{C}$, a *cosimplicial object* in $\mathcal{C}$ is a functor from Δ to $\mathcal{C}$.

Remark 2.1.4. For each natural number n and $i \in \{0, 1, \ldots, n\}$, we call the unique map $\varepsilon_i : [n-1] \to [n]$ in Δ which "misses" i the *i-th face map*. Similarly, we call the unique map from $\eta_i : [n+1] \to [n]$ which "hits" i twice the *i-th degeneracy map*. Now, the following identities hold:

$$\varepsilon_j \varepsilon_i = \varepsilon_i \varepsilon_{j-1} \qquad \text{if} \quad i < j$$
$$\eta_j \eta_i = \eta_i \eta_{j+1} \qquad \text{if} \quad i \le j$$

$$\eta_j \varepsilon_i = \begin{cases} \varepsilon_i \eta_{j-1} & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i = j+1 \\ \varepsilon_{j-1} \eta_j & \text{if } i > j+1 \end{cases}$$

One can show that every map in Δ has a unique factorization consisting of face and degeneracy maps. Since a simplicial object in a category $\mathcal{C}$ is a functor from Δ^{op} to $\mathcal{C}$, this means that giving a simplicial object is the same as giving the data of a sequence $X_0, X_1, \ldots$ of objects in $\mathcal{C}$ together with maps $\partial_i : X_n \to X_{n-1}$ (the face maps) and $\sigma_i : X_n \to X_{n+1}$ (the degeneracy maps) satisfying the following identities:

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad \text{if} \quad i < j$$
$$\partial_i \partial_j = \partial_{j+1} \partial_i \qquad \text{if} \quad i \le j$$

$$\partial_i \sigma_j = \begin{cases} \partial_{j-1} \partial_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i = j+1 \\ \partial_j \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

This whole discussion dualizes, so that giving a cosimplicial object of $\mathcal{C}$ is the same as giving a sequence $X^0, X^1, \ldots$ of objects of $\mathcal{C}$ together with "coface" maps from X^n to X^{n+1} and "codegeneracy" maps from X^{n+1} to X^n.

Example 2.1.5. • The standard n-simplex

$$\Delta^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \ge 0, \sum x_i = 1 \}$$

is a cosimplicial topological space. The i-th coface map includes Δ^n as the i-th face of Δ^{n+1}. The i-th codegeneracy map $\Delta^{n+1} \to \Delta^n$ projects onto the i-th face of Δ^n; that is, it is the quotient map identifying x_i and x_{i+1}.

• In the process of calculating the simplicial homology of a topological space, one takes free abelian groups on the sets of n-simplices. These free abelian groups combine to form a simplicial abelian group.

- Given a group G, we can define a simplicial set BG as follows: $BG_0[\{e_g\}, BG_1 = G, \cdots, BG_n = G^n$, and so on. The face and degeneracy maps are as follows:

$$\sigma_i(g_1, \ldots, g_n) = (g_1, \cdots, g_i, 1, g_{i+1}, \ldots, g_n)$$

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

We now give a crucial definition, which also gives a very important example of a simplicial set.

Definition 2.1.6. Given a category $\mathcal{C}$ we define a simplicial set $N\mathcal{C}$, the *nerve* of $\mathcal{C}$, as follows. The set $N\mathcal{C}_n$ is the set of functors from the poset $[n]$, viewed as a category, to $\mathcal{C}$. That is to say, an element of $N\mathcal{C}_n$ looks like a composable string

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$$

of arrows in $\mathcal{C}$. The face and degeneracy maps look like those in the last part of the previous example; the i-th degeneracy map inserts an identity morphism in the i-th position, and the i-th face map gives a composition in the i-th position. That is, ∂_i sends

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$$

to

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_2}} X_{i-1} \xrightarrow{f_i f_{i-1}} X_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} X_n.$$

And ∂_i sends

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$$

to

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_1}} X_i \xrightarrow{\text{id}} X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} X_n.$$

A functor $F : \mathcal{C} \to \mathcal{D}$ induces a map of simplicial sets $N\mathcal{C} \to N\mathcal{D}$ in the obvious way. Functoriality ensures that the map induced by F commutes with the face and degeneracy maps as required.

Definition 2.1.7. Given a simplicial set X, we define a CW complex $|X|$, called

the *geometric realization* of X, as follows:

$$|X| := \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation $\sim$ is defined as follows: $(x, s) \in X^m \times \Delta^m$ and $(y, t) \in X_n \times \Delta^n$ are equivalent if we have a map $\alpha : [m] \to [n]$ in Δ such that, for the induced maps $\alpha_* : X_n \to X_m$ and $\alpha^* : \Delta^m \to \Delta^n$, we have $\alpha_*(y) = x$ and $\alpha^*(s) = t$. We view each X_n as a topological space with the discrete topology, and the coproduct denotes the topological disjoint union. We give the topological space $|X|$ the quotient topology.

A map $f : X \to Y$ of simplicial sets induces a continuous map $|X| \to |Y|$.

Definition 2.1.8. For a category $\mathcal{C}$ we call $|N\mathcal{C}|$, the geometric realization of the nerve of $\mathcal{C}$, the *classifying space* of $\mathcal{C}$. We will denote the classifying space by $B\mathcal{C}$.

This association will allow us to think of categories in homotopical terms; for example, we say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy equivalence if the induced functor $BF : B\mathcal{C} \to B\mathcal{D}$ is one. We say $\mathcal{C}$ is contractible if $B\mathcal{C}$ is, and so on.

2.2 Exact categories and Quillen K-theory

Armed with the topological concepts explained in the previous section, we will now give an account of "classical" algebraic K-theory. Throughout this section, we follow [15]. We begin with the notion of an exact category, a useful generalisation of an abelian category.

Definition 2.2.1. An *exact category* is an additive category $\mathcal{E}$ together with a class of *admissible exact sequences* in $\mathcal{E}$ which we will denote

$$X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z.$$

We call the maps i admissible monomorphisms, and the maps p admissible epimorphisms. Moreover, the following properties are satisfied:

- All split exact sequences

$$X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{(0\ 1)} Y$$

  are admissible.

- For every admissible exact sequence

$$X \xrightarrow{\ \ i\ \ } Y \xrightarrow{\ \ p\ \ } Z.$$

  we have that $i = \ker(p)$ and $p = \operatorname{coker}(s)$.

- If we have a commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\ i\ } & Y & \xrightarrow{\ p\ } & Z \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
X' & \longrightarrow & Y' & \longrightarrow & Z'
\end{array}
$$

  where the top row is an admissible exact sequence and the vertical maps are isomorphisms in $\mathcal{E}$, then the bottom row is an admissible exact sequence.

- Admissible monomorphisms and admissible epimorphisms are closed under composition.

- For every diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ s\ } & Y \\
\downarrow & & \\
Z & &
\end{array}
$$

  the pushout W exists, and the map $Z \to W$ is an admissible monomorphism. Dually, for every diagram

$$
\begin{array}{ccc}
 & & Y \\
 & & \downarrow{\scriptstyle p} \\
Z & \longrightarrow & W
\end{array}
$$

  the pullback X exists, and the map $X \to Z$ is an admissible epimorphism.

Definition 2.2.2. An exact category $\mathcal{E}$ is called *idempotent complete* if every idempotent $p = p^2$ has a kernel.

Remark 2.2.3. In [15], Quillen gives an additional axiom, which was later shown by Keller ([11], Appendix) to be a consequence of the axioms given in Definition 2.2.1. The axiom says this: if an admissible monomorphism $i : N \rightarrowtail Y$ has a factorization $N \xrightarrow{f} B \to Y$, and if the map f has a cokernel, then f is an admissible monomorphism. It turns out that, if $\mathcal{E}$ is idempotent complete, then f automatically has a cokernel, and f is thus an admissible monomorphism.

Example 2.2.4. Every abelian category $\mathcal{A}$ is an exact category: the admissible exact

sequences are simply the usual exact sequences. Moreover, any full subcategory $\mathcal{E} \subset \mathcal{A}$ which is closed under extensions is an exact category which may not be abelian.

Example 2.2.5. Given a commutative ring R, consider $P(R)$, the category of finitely-generated projective R-modules. This is a full subcategory of RMod, the abelian category of R-modules. Moreover, $P(R)$ is closed under extensions, so it is an exact category: the exact sequences are those sequences of projective R-modules which are short exact in RMod. However, it is not an abelian category; for example, take $R = \mathbb{Z}$. Then the cokernel of the map $\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}$ is $\mathbb{Z}/2\mathbb{Z}$, which is not a projective $\mathbb{Z}$-module since every projective $\mathbb{Z}$-module is free.

Example 2.2.6. Given a scheme $(X, \mathcal{O}_X)$, the category of vector bundles over X is exact. The exact structure, in the same way as Example 2.2.5, is inherited from the ambient abelian category consisting of all quasicoherent sheaves on X.

Given any exact category $\mathcal{E}$, one can define a topological space $K(\mathcal{E})$ whose homotopy groups are the K-groups of $\mathcal{E}$. When $\mathcal{E}$ is $P(R)$ for a ring R, one defines these groups to be the K-theory of R. To do this, we require the following crucial definition.

Definition 2.2.7. (Quillen's Q-construction) Let $\mathcal{E}$ be an exact category. One defines a category $Q\mathcal{E}$ as follows. The objects of $Q\mathcal{E}$ are the same as those of $\mathcal{E}$. Given X and Y, objects of $\mathcal{E}$, a map from X to Y in $Q\mathcal{E}$ is an equivalence class of diagrams of the form

$$X \twoheadleftarrow^{p} U \rightarrowtail^{i} Y$$

where two diagrams (U, p, i) and (U', p', i') are equivalent if and only if there exists an isomorphism from U to U' such that the following diagram commutes:

$$
\begin{array}{ccccc}
X & \xleftarrow{\;p\;} & U & \xrightarrow{\;i\;} & Y \\
\downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\mathrm{id}} \\
X & \xleftarrow{\;p'\;} & U' & \xrightarrow{\;i'\;} & Y
\end{array}
$$

Given maps

$$X \twoheadleftarrow^{p} U \rightarrowtail^{i} Y$$

and

$$Y \twoheadleftarrow^{q} W \rightarrowtail^{j} Z$$

we define their composition via pullbacks; precisely, form the diagram

$$
\begin{array}{ccc}
 & & Z \\
 & & \Big\uparrow{\scriptstyle j} \\
Y & \xleftarrow{\quad q \quad} & W \\
\Big\uparrow{\scriptstyle i} & & \\
X & \xleftarrow{\quad p \quad} & U
\end{array}
$$

then use the existence of pullbacks to fill in

$$
\begin{array}{ccccc}
 & & & & Z \\
 & & & & \Big\uparrow{\scriptstyle j} \\
 & Y & \xleftarrow{\quad q \quad} & W & \\
 & \Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle i'} & \\
X & \xleftarrow{\quad p \quad} U & \xleftarrow{\quad q' \quad} & U \times_Y W
\end{array}
$$

so that the composition of (U, p, i) and (W, q, j) is $(U \times_Y W, pq', ji')$.

Definition 2.2.8. Given an exact category $\mathcal{E}$, we define its *K-theory space* $K(\mathcal{E})$ to be $\Omega BQ\mathcal{E}$, the loop space of the geometric realization of the category $Q\mathcal{E}$. We define the K groups thus:

$$
K_i(\mathcal{E}) = \pi_i K(\mathcal{E}) \quad \text{for } i \geq 0.
$$

For this to be a proper definition, one would expect, for example, an isomorphism $K_0(\mathcal{E}) \to \pi_1 BQ\mathcal{E}$ of abelian groups, where $K_0(\mathcal{E})$ is the abelian group generated by symbols $[X]$ where X is an object of $\mathcal{E}$, with the relation that $[Y] = [X] + [Z]$ if there exists an admissible exact sequence

$$
X \xrightarrow{\ i\ } Y \xrightarrow{\ p\ } Z.
$$

This is indeed the case. See Theorem 1 in [15] or Proposition 2.2.4. in [20].

Definition 2.2.9. Given a commutative ring R, let $\mathcal{E}$ be the exact category $P(R)$ of finitely-generated projective R-modules. We set $K_i(R) := K_i P(R)$.

Example 2.2.10. Let R be a local ring, or any commutative ring for which all finitely-generated projective modules are free. Then $K_0(R) \cong \mathbb{Z}$, since the abelian monoid of isomorphism classes of finitely-generated projective modules is isomorphic to $(\mathbb{N}, +)$.

Remark 2.2.11. There exists another approach to higher algebraic K-theory which is via symmetric monoidal categories rather than exact categories. For a ring R, one defines the K-groups to be the homotopy groups of the space

$$K_0(R) \times (BGL(R))^+,$$

where the abelian group $K_0(R)$ is viewed as a discrete topological space, and $+$ denotes the *plus construction* first defined by Kervaire in [12].

The following theorem, due to Quillen but proved in [7], and known as the '+=Q' theorem, tells us that, when we're concerned with rings, the "exact" and "symmetric monoidal" viewpoints are equivalent.

Theorem 2.2.12 (Quillen). *For every ring R,*

$$\Omega BQP(R) \sim K_0(R) \times BGL(R)^+$$

so that the definitions of the K-groups via both spaces coincide.

2.3 Classical Hermitian K-theory

In this section, we will give an account of the Hermitian K-theory of symmetric bilinear forms on an exact category with duality. This is due to Schlichting in [19], which we follow for our exposition. Note, however, that this is just one facet of a relatively old subject; for example, higher Hermitian K-theory for a ring with involution and 2 invertible was studied by Karoubi in [10].

Definition 2.3.1. An *exact category with duality* is a datum $(\mathcal{E}, *, \mathrm{can})$, with $\mathcal{E}$ an exact category, $* : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ an exact functor, and $\mathrm{can} : \mathrm{id} \overset{\cong}{\Rightarrow} **$ a natural isomorphism such that, for all objects A of $\mathcal{E}$ we have $1_{A^*} = \mathrm{can}_A^* \circ \mathrm{can}_{A^*}$. The exact structure on $\mathcal{E}^{\mathrm{op}}$ is the opposite of that on $\mathcal{E}$; that is to say, a sequence

$$X \xrightarrow{\ i\ } Y \xrightarrow{\ p\ } Z.$$

in $\mathcal{E}^{\mathrm{op}}$ is admissible exact if and only if

$$Z \xrightarrow{\ p^{\mathrm{op}}\ } Y \xrightarrow{\ i^{\mathrm{op}}\ } X.$$

is admissible exact in $\mathcal{E}$.

Example 2.3.2. Let R be a commutative ring with involution (i.e. we have a map $R^{\mathrm{op}} \to R$ that sends a to $\bar{a}$ and $\bar{\bar{a}} = a$). Then, if we let $P(R)$ be the category of finitely generated projective R-modules, we have the following:

- The functor $* : P(R)^{\mathrm{op}} \to P(R)$ is given by $P \mapsto \mathrm{Hom}_R(P, R)$.

- The double dual identification can is defined as follows; for a finitely generated projective R-module P, $\mathrm{can}_P : P \xrightarrow{\cong} P^{**}$ is given by $\mathrm{can}_P(x)(f) = f(\bar{x})$.

Together, all of this makes the triple $(P(R), *, \mathrm{can})$ an exact category with duality.

Definition 2.3.3. Give an exact category with duality $(\mathcal{E}, *, \mathrm{can})$, a *symmetric form* in $\mathcal{E}$ is a pair (X, φ) where X is an object of $\mathcal{E}$ and $\varphi : X \to X^*$ is a morphism in $\mathcal{E}$ satisfying the equation $\varphi^* \mathrm{can}_X = \varphi$. If φ happens to be an isomorphism, we call the form *non-degenerate*. If φ is non-degenerate, we call the datum (X, φ) a *symmetric space*.

Let (X, φ) be a symmetric form in $\mathcal{E}$ and let $f : Y \to X$ be a morphism. Then $(Y, \varphi_{|Y})$ is a symmetric form, where $\varphi_{|Y} = f^* \varphi f$ is called the restriction of φ via f. A *map of symmetric forms* $f : (Y, \psi) \to (X, \varphi)$ is a map $F : Y \to X$ in $\mathcal{E}$ with $\psi = \varphi_{|Y}$. Such a map is called an *isometry* if f is an isomorphism. Composition in $\mathcal{E}$ gives composition of maps of symmetric forms.

The *orthogonal sum* of two symmetric forms (X, φ) and (Y, ψ) is the symmetric form $(X \oplus Y, \varphi \oplus \psi)$, where we view $\varphi \oplus \psi$ as a map to $(X \oplus Y)^*$ by composing with the isomorphism $X^* \oplus Y^* \xrightarrow{\cong} (X \oplus Y)^*$. If (X, φ) and (Y, ψ) are symmetric spaces then so is their orthogonal sum.

Definition 2.3.4. Let (X, φ) be a symmetric space in an exact category with duality $(\mathcal{E}, *, \mathrm{can})$. A *totally isotropic subsapce* of X is an admissible monomorphism $i : L \rightarrowtail X$ such that $0 = \varphi_{|L} = i^* \varphi i$ and such that the induced map $L \to L^\perp \subset X$ is also an admissible monomorphism, where $L^\perp$ is called the *orthogonal* of L and is defined as $\ker(i^* \varphi)$. Furthermore, if $L \rightarrowtail X$ is a totally isotropic subspace, we call L a *Langrangian* of (X, φ) if $L = L^\perp$. In other words, $i : L \rightarrowtail X$ is a Langrangian if and only if the sequence

$$L \xrightarrow{\ \ i\ \ } X \xrightarrow{\ \ i^* \varphi\ \ } L^*$$

is admissible exact.

We call a symmetric space (X, φ) *metabolic* if it has a Langrangian L. For any object X of $\mathcal{E}$, the object $H(X) := (X \oplus X^*, \left(\begin{smallmatrix} 0 & 1 \\ \mathrm{can} & 0 \end{smallmatrix}\right))$ is a symmetric space called the *hyperbolic space* of X. It is always the case that $H(X)$ is metabolic with Langrangian $X \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} X \oplus X^*$ Also we always have that, for any objects X and Y,

the symmetric spaces $H(X \oplus Y)$ and $H(X) \perp H(Y)$ are isometric.

Given an exact category with duality $(\mathcal{E}, *, \mathrm{can})$ we can define two groups which are in some sense analogous to K_0 for exact categories. The *Witt group* $W_0(\mathcal{E})$ is constructed as follows: take the set of isometry classes $[X, \varphi]$ of symmetric spaces in $\mathcal{E}$. This is an abelian monoid with orthogonal sum. Now, quotient by the submonoid of metabolic spaces. The resulting quotient monoid is a group: indeed, the class $[X, \varphi]$ has inverse $[X, -\varphi]$, because $(X, \varphi) \perp (X, -\varphi)$ is metabolic with Lagrangian

$$X \to X \oplus X.$$
$$x \mapsto (x, x)$$

The *Grothendieck-Witt group* $GW_0(\mathcal{E})$ is the Grothendieck group of the abelian monoid of isometry classes of symmetric spaces in $\mathcal{E}$, modulo the relation that if M is metabolic with Lagrangian L, $[M] = [H(L)]$. The definitions of the Witt and Grothendieck-Witt groups give rise to the following:

- A class $[X]$ in $W_0(\mathcal{E})$ is 0 if and only if there exists a metabolic space M such that $X \perp M$ is metabolic. We call such a space X *stably metabolic.*

- An element $[X] - [Y]$ in $GW_0(\mathcal{E})$ is 0 if and only if there exist metabolic spaces M_1 and M_2 with respective associated Lagrangians L_1 and L_2, and an isometry
$$X \perp M_1 \perp H(L_2) \cong Y \perp M_2 \perp H(L_1)$$

We now give a proposition: in Chapters 4 and 5, we think of the Grothendieck-Witt groups with which we work as merely the Grothendieck groups of the appropriate abelian monoids: we do not impose the relation $[M] = [H(L)]$. The proposition shows that this is safe.

Proposition 2.3.5 ([19], Corollary 2.10)**.** *Let $\mathcal{E}$ be a split exact category with duality. Then $GW_0(\mathcal{E})$ is the Grothendieck group of the abelian monoid of isometry classes of symmetric spaces in $\mathcal{E}$.*

We have now defined two groups which can be thought of as analogous or related to K_0. We now proceed to the hermitian version of the higher K-groups; as one might expect, we begin with a hermitian version of the Q-construction (Definition 2.2.7):

Definition 2.3.6. Let $(\mathcal{E}, *, \mathrm{can})$ be an exact category with duality. We define a category $Q^h(\mathcal{E}, *, \mathrm{can})$ as follows. The objects are the symmetric spaces (X, φ) in $\mathcal{E}$. A map $(X, \varphi) \to (Y, \psi)$ is an equivalence class of diagrams

$$X \xleftarrow{\;\;p\;\;} U \rightarrowtail^{\;i\;} Y$$

where $\varphi_{|U} = \psi)_{|U}$ and i induces an isomorphism $\ker(p) \to \ker(i^*\varphi)$. Composition and what is meant by equivalence of diagrams are the same as in Definition 2.2.7.

Armed with this, analogously to the case of K-theory, we can define a space whose homotopy groups will be the higher Grothendieck-Witt groups of the exact category with duality:

Definition 2.3.7. Let $(\mathcal{E}, *, \mathrm{can})$ be an exact category with duality. The forgetful functor from $Q^h\mathcal{E} \to Q\mathcal{E}$ which sends (X, φ) to X induces a map

$$BQ^h\mathcal{E} \to BQ\mathcal{E}$$

on classifying spaces. The homotopy fibre of this map is defined to be the *Grothendieck-Witt space* $GW(\mathcal{E}, *, \mathrm{can})$ of $\mathcal{E}$.

Example 2.3.8. Analogously to the K-theory case, the Grothendieck-Witt group GW_0 of a ring with involution is π_0 of the Grothendieck-Witt space of the exact category with duality $(P(R), *, \mathrm{can})$. That is, the Grothendieck-Witt group is the Grothendieck group of the abelian monoid of isometry classes of symmetric spaces on finitely-generated projective modules over R, with operation given by the orthogonal sum. We give here some examples of calculations of these groups. In this example, we take all rings to have trivial involution.

- The Grothendieck-Witt group $GW_0(\mathbb{C})$ is isomorphic to $\mathbb{Z}$; that is to say, non-degenerate symmetric bilinear forms over $\mathbb{C}$ are characterised completely by their rank. The reason is that for any $\alpha \in \mathbb{C}$ we have an isometry $\langle 1 \rangle \to \langle \alpha \rangle$ given by multiplication with $\sqrt{\alpha}$, so that GW_0 is generated by the class of $\langle 1 \rangle$. The same argument works for any algebraically closed, or indeed, quadratically closed, field.

- Sylvester's law of inertia implies that $GW_0(\mathbb{R})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ where the summands are the rank and signature.

- $GW_0(\mathbb{Z})$ is also isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, although the argument is less simple, since it requires use of the Hasse-Minkowski Theorem, which is difficult to

prove. (Theorem II.3.1 in [14], [17] gives a proof.)

Chapter 3

The K-theory of forms

Thus far, as some background to where our results are situated in the literature, we have given an overview of "classical" K-theory, as well as the "classical" Hermitian K-theory of symmetric bilinear forms. The results themselves, however, concern the "K-theory of forms", a more general framework which encompasses symmetric, quadratic, and skew-symmetric forms; in fact, in a certain sense which we will make precise, anything which deserves the name "form".

The study of the lower K-theory of forms is not new; see for example [1]. Although our results fall within the lower K-theory of forms, we follow Schlichting's framework for the higher K-theory of forms given in [22]; we will discuss why in Remark 3.1.21. For now, we begin our exposition of the K-theory of forms, which is the subject matter of this chapter.

3.1 Rings with form parameter

In this section, we introduce the key notion of a *ring with form parameter*; these are to the K-theory of forms what rings are to K-theory and what rings with involution are to Hermitian K-theory. Our exposition follows Section 3 of [22].

Definition 3.1.1. Let R be a ring with involution $R^{op} \to R$ which sends a to $\bar{a}$. A *duality coefficient* for R is an R-bimodule I along with a bimodule homomorphism $\sigma : I^{op} \to I$ such that $\sigma^{op} \circ \sigma = 1$.

Example 3.1.2. The simplest example of a duality coefficient, and the primary one used in "classical" Hermitian K-theory, is $(I, \sigma) = (R, a \mapsto \bar{a})$. A little more gener-

ally, given any $\varepsilon \in Z(R)$ such that $\varepsilon \cdot \bar{\varepsilon} = 1$, $(I, \sigma) = (R, a \mapsto \varepsilon \cdot \bar{a})$ also gives an example.

Given a duality coefficient (I, σ), we have a category with duality

$$(R\text{Mod}, *, \text{can})$$

in the sense of Definition 2.3.1. The duality functor is given by $M \mapsto M^* := \text{Hom}(M^{op}, I)_R$, the set of right R-module homomorphisms from M^{op} to I. For each M in RMod the double dual identification $\text{can}_M : M \to M^{**}$ is defined by

$$\text{can}_M(x)(f) = \sigma(f(x^{op})), \ x \in M, \ f \in M^*$$

Given a left R-module M we will commonly identify the set $\text{Hom}_R(M, M^*)$ of left R-module maps from M to $M^* = \text{Hom}(M^{op}, I)_R$ with the set of R-bimodule maps $\text{Hom}_R(M \otimes_{\mathbb{Z}} M^{op}, I)_R$ which consists of the R-bimodule maps from $M \otimes_{\mathbb{Z}} M^{op}$ to I; we can make such an identification because of the standard tensor-hom adjunction:

$$\text{Hom}_R(M, M^*) \xrightarrow{\cong} \text{Hom}_R(M \otimes_{\mathbb{Z}} M^{op}, I)_R, \ f \mapsto (x \otimes y^{op} \mapsto f(x)(y^{op})) \qquad (3.1)$$

Given a duality coefficient (I, σ) the abelian group I is canonically equipped with the following:

- the C_2 action which sends an element x to $\sigma(x^{op})$

- the quadratic multiplicative action $Q : (R, \cdot, 0, 1) \to (\text{End}_{\mathbb{Z}}(I), \circ, 0, 1)$ defined by $Q(a)(x) = a \cdot x \cdot \bar{a}$

We say that the map Q is quadratic since its deviation $Q(a+b)(x) - Q(a)(x) - Q(b)(x) = a \cdot x \cdot \bar{b} + b \cdot x \cdot \bar{a}$ is $\mathbb{Z}$-bilinear in the variables a and b.

Almost all of the pieces are in place to define a ring with form parameter. Before we do so, we need one more definition:

Definition 3.1.3. Let C_2 denote the cyclic group of order 2 with generator σ. A C_2-*Mackey functor* is a diagram $M = (M(e), M(C_2), \tau, \rho)$ of C_2-abelian groups and C_2-equivariant group homomorphisms

$$M(e) \xrightarrow{\tau} M(C_2) \xrightarrow{\rho} M(e)$$

where the C_2-action on $M(C_2)$ is trivial, and where $\rho \circ \tau = 1 + \sigma$. The maps τ and ρ are called *transfer* and *restriction* respectively.

Remark 3.1.4. The notion of a Mackey functor is due to Dress ([6]) and Green ([8]), and can be defined for any finite group. We only use the case of C_2, and follow Appendix B of [22] for our treatment. For a more modern treatment in full generality, one may consult [4].

Definition 3.1.5 ([22], Definition 3.3)**.** Let R be a ring with involution. Then a *form parameter* for R is a pair (I, Λ) where $I = (I, \sigma)$ is a duality coefficient and Λ is an abelian group equipped with the trivial C_2-action together with C_2-equivariant group homomorphisms τ and ρ,

$$(I, \sigma) \xrightarrow{\tau} \Lambda \xrightarrow{\rho} (I, \sigma) \tag{3.2}$$

and a multiplicative $\mathbb{Z}$-linear left action

$$Q : (R, \cdot, 0, 1) \to (\mathrm{End}_{\mathbb{Z}}(\Lambda), \circ, 0, 1)$$

of R on Λ preserving 0 and 1 such that the following holds:

1. The diagram (3.2) is a C_2-Mackey functor

2. The deviation of Q is given by the formula

$$Q(a + b)(x) - Q(a)(x) - Q(b)(x) = \tau(a \cdot \rho(x) \cdot \bar{b})$$

   for $a, b \in R$ and $x \in \Lambda$.

3. The C_2-equivariant maps ρ and τ commute with the quadratic actions of R on I and Λ.

A *ring with form parameter* (R, I, Λ) is a ring with involution equipped with a form parameter (I, Λ).

Remark 3.1.6. More explicitly, conditions 1-3 from Definition 3.1.5 mean that the following equations hold:

$$\rho(\tau(x)) = x + \sigma(x^{op}) \quad \text{for all } x \in I,$$

$$Q(a + b)(x) = Q(a)(x) + Q(b)(x) + \tau(a \cdot \rho(x) \cdot \bar{b}) \text{ for all } a, b \in R, x \in \Lambda,$$

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$$\tau(a \cdot x \cdot \bar{a}) = Q(a)\tau(x) \text{ and } \rho(Q(a)(\xi)) = a \cdot \rho(\xi) \cdot \bar{a} \text{ for all } a \in R, x \in I, \xi \in \Lambda.$$

Definition 3.1.7 ([22], Definition 3.5)**.** Let R be a ring with involution. Then a *homomorphism of form parameters* $(f_1, f_0) : (I, \Lambda) \to (J, \Gamma)$ for R is a pair of abelian group homomorphisms $f_1 : I \to J, f_0 : \Lambda \to \Gamma$ such that the diagram

$$
\begin{array}{ccccc}
I & \xrightarrow{\ \tau\ } & \Lambda & \xrightarrow{\ \rho\ } & I \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\
J & \xrightarrow{\ \tau\ } & \Gamma & \xrightarrow{\ \rho\ } & J
\end{array}
$$

commutes, the map $f_1 : I \to J$ is a homomorphism of R-bimodules commuting with the involutions on I and J, and the map f_0 commutes with the quadratic actions of R on Λ and Γ.

Remark 3.1.8. Definition 3.1.5 is a generalisation of the one given by Bak in [1]. Therein, R is always equal to I, Λ is always a certain subgroup of R, and ρ is always the inclusion.

Example 3.1.9. Given $f : R \to S$, a homomorphism of rings with involution, a form parameter (J, Γ) for S defines a form parameter (J, Γ) for R by restriction of scalars along f.

We now give a very important definition, which says what the "forms" are in the K-theory of forms.

Definition 3.1.10 ([22], Definition 3.8)**.** Let (R, I, Λ) be a ring with form parameter. An (R, I, Λ)-*quadratic form*, or a *form over* (R, I, Λ) is a triple (M, q, β) where

1. M is a left R-module.

2. $q : M \to \Lambda$ is a function such that $q(ax) = Q(a)(q(x))$ for all $a \in R$ and $x \in M$.

3. $\beta : M \otimes_{\mathbb{Z}} M^{\mathrm{op}} \to I$ is a symmetric bilinear map such that $\beta(x, x^{\mathrm{op}}) = \rho(q(x))$ for all $x \in M$.

4. the deviation of q, that is, $q(x + y) - q(x) - q(y)$, is equal to $\tau\beta(x, y^{\mathrm{op}})$.

In this context, saying that β is symmetric bilinear means that $\beta((ax), (bx)^{\mathrm{op}})) = a\beta(x, y^{\mathrm{op}})\bar{b}$ and $\beta(x, y^{\mathrm{op}}) = \sigma(\beta(y, x^{\mathrm{op}}))$. Also, from (4), we have that $q : M \to \Lambda$ is quadratic in the sense that its deviation $q(x + y) - q(x) - q(y)$ is the symmetric bilinear map $(x, y) \mapsto \tau\beta(x, y^{\mathrm{op}})$. We denote the set of all (R, I, Λ)-quadratic forms

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on M by $Q(M)$. This set is an abelian group: the operation is addition of functions $M \to \Lambda$ and $M \otimes_{\mathbb{Z}} M^{\mathrm{op}} \to I$, using the abelian group structures on Λ and I.

The following example gives rings with form parameter which generalise the classical notions of symmetric bilinear and quadratic forms.

Example 3.1.11 ([22], Example 4.3). Let (R, σ) be a commutative ring with involution. Then we have a form parameter

$$R \xrightarrow{1+\sigma} R^{\sigma} \xrightarrow{1} R$$

with quadratic action given by $Q(a)(x) = ax\sigma(a) \in R^{\sigma}$, whose quadratic forms are the usual Hermitian modules over (R, σ). In particular, if $\sigma = 1$ and we restrict to finitely-generated projective modules, we get the classical notion of symmetric bilinear forms over the ring R.

To see this, first note that the assumption $\sigma = 1$ gives

$$R \xrightarrow{\cdot 2} R \xrightarrow{1} R$$

with quadratic action given by $Q(a)(x) = a^2 x$. Now, let (P, q, β) be an (R, R, R)-quadratic form, with P a finitely generated projective R-module. Then (2) from Definition 3.1.10 says that $q : P \to R$ is a function such that $q(ax) = a^2 q(x)$ for all $a \in R$ and $x \in M$. Also, (3) from Definition 3.1.10 says exactly that β is a symmetric bilinear form on P. It also says, since the restriction map $R \to R$ is the identity, that $\beta(x, x) = q(x)$ for all $x \in P$. Finally, recalling that $\sigma = 1$ implies that the transfer map $R \to R$ is multiplication by 2, (4) from Definition 3.1.10 says that we have $q(x + y) - q(x) - q(y) = 2\beta(x, y)$.

Putting all of this together, we have that (R, R, R)-quadratic forms (P, q, β) are in bijection with symmetric bilinear form modules (P, β): one obtains (P, β) by forgetting q, but, since, as we have seen, the stipulations of Definition 3.1.10 force q to be the associated quadratic form of β, we can recover (P, q, β) from (P, β).

Example 3.1.12. Let (R, σ) be a commutative ring with involution. Then we have a form parameter

$$R \twoheadrightarrow R_{\sigma} \xrightarrow{1+\sigma} R$$

with quadratic action given by $Q(a)(x) = [ax\sigma(a)] \in R_{\sigma}$, whose quadratic forms are quadratic form modules over the ring with involution (R, σ). In particular, if $\sigma = 1$ and we restrict to finitely generated projective R-modules, we get the classical

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notion of quadratic forms over R.

To see this, note that $\sigma = 1$ gives

$$R \xrightarrow{1} R \xrightarrow{\cdot 2} R$$

with quadratic action given by $Q(a)(x) = a^2 x$. Now, let (P, q, β) be an $(R, R, R)_q$ quadratic form with P a finitely generated projective R-module, and where we add the subscript q to avoid confusion with Example 3.1.11. Checking through the stipulations (2) - (4) from Definition 3.1.10 shows that, similarly to Example 3.1.11, q is exactly a classical quadratic form over the ring R, with associated bilinear form β.

At this point, we require some preliminary definitions. Once these are in place, we can define the Grothendieck-Witt group of a ring with form parameter.

Definition 3.1.13. Let (R, I, Λ) be a ring with form parameter and let (M, q, β) be an (R, I, Λ)-quadratic form. We say that (M, q, β) is *non-degenerate* if β is non-degenerate as a symmetric bilinear form; that is to say, the map $M \to M^*$ given by $x \mapsto \beta(x, -)$ is an isomorphism.

Remark 3.1.14. In this thesis, we will be exclusively concerned with non-degenerate forms. We tacitly assume non-degeneracy for the entirety of the thesis beyond this point.

Definition 3.1.15. Let (R, I, Λ) be a ring with form parameter and let (M, q, β) and (M', q', β') be two (R, I, Λ)- quadratic forms. An *isometry* between (M, q, β) and (M', q', β') is an R-module isomorphism $f : M \to M'$ such that $q = q' \circ f$ and $\beta = \beta' \circ (f \otimes f^{op})$. If an isometry exists between two forms, we say they are *isometric*.

Definition 3.1.16. Let (R, I, Λ) be a ring with form parameter and let (M, q, β) and (M', q', β') be two (R, I, Λ)-quadratic forms. Then their *orthogonal sum* $(M \oplus M', q \perp q', \beta \perp \beta')$ is defined by the formulae:

$$q \perp q'(x, x') = q(x) + q'(x'),$$
$$\beta \perp \beta'((x, x'), (y^{op}, y'^{op})) = \beta(x, y^{op}) + \beta(x', y'^{op})$$

Lemma 3.1.17. *Let (R, I, Λ) be a ring with form parameter and let (M, q, β) and (M', q', β') be two (R, I, Λ)-quadratic forms. Their orthogonal sum $(M \oplus M', q \perp q', \beta \perp \beta')$ is also a (R, I, Λ)-quadratic form.*

Proof. Using the formulae from Definition 3.1.16, we check the conditions 1-4 from Definition 3.1.10:

1. Clear.

2. For $x \in M, x' \in M'$, we have

$$
\begin{aligned}
q \perp q'(ax, ax') &= q(ax) + q'(ax') \\
&= Q(a)q(x) + Q(a)q(x') \\
&= Q(a)(q(x) + q(x')) \\
&= Q(a)(q \perp q'(x, x'))
\end{aligned}
$$

as required. The third equality comes from the $\mathbb{Z}$-linearity of $Q(a)$, as stipulated in Definition 3.1.5.

3. The map $\beta \perp \beta' : (M \otimes M') \otimes_{\mathbb{Z}} (M \otimes M')^{op} \to I$ is clearly symmetric bilinear. Also, we have

$$
\begin{aligned}
\beta \perp \beta((x, x'), (x^{op}, x'^{op})) &= \beta(x, x^{op}) + \beta'(x', x'^{op}) \\
&= \rho(q)(x) + \rho(q')(x') \\
&= \rho(q(x) + q'(x')) \\
&= \rho(q \perp q(x, x'))
\end{aligned}
$$

as required, where the third equality comes from the fact that ρ is a group homomorphism.

4. Let x and y be elements of M and let x' and y' be elements of M'. Then we have

$$
\begin{aligned}
q \perp q((x, x') + (y, y')) &- q \perp q'(x, x') - q \perp q'(y, y') \\
&= q(x + y) - q(x) - q(y) + q'(x' + y') - q'(x') - q'(y') \\
&= \tau\beta(x, y^{op}) + \tau\beta'(x', y'^{op}) \\
&= \tau\beta \perp \beta'((x, x'), (y^{op}, y'^{op})),
\end{aligned}
$$

where the third equality comes from τ being a group homomorphism.

$\square$

Remark 3.1.18. Let (R, I, Λ) be a ring with form parameter. Recall from the discussion after Example 3.1.2 that we can view RMod as a category with duality $(R\mathrm{Mod}, *, \mathrm{can})$. Assume that the R-bimodule I is finitely generated and projective as a left R-module, and that the canonical double dual identification can : $R \to R^{**}$ is an isomorphism. Then, for any finitely generated projective left R-module P, the dual $P^* = \mathrm{Hom}(P^{op}, I)_R$ is also finitely generated and projective, and $\mathrm{can}_P : P \to P^{**}$ is an isomorphism since this is true if one takes $P = R$ and the relevant properties are preserved under taking finite direct sums and direct factors.

Moreover, given two (R, I, Λ)-quadratic forms (M, q, β) and (M', q', β') we have that $(M \oplus M', q \perp q', \beta \perp \beta')$ is isometric to $(M' \oplus M, q' \perp q, \beta' \perp \beta)$ via the canonical "factor-swapping" isomorphism. Recalling that finitely generated projective left R-modules are closed under taking direct sum, we have that the set of isometry classes of (R, I, Λ)-quadratic forms forms an abelian monoid with orthogonal sum. This gives rise to the following definition.

Definition 3.1.19. Let (R, I, Λ) be a ring with form parameter. Then its *Grothendieck-Witt group* $GW_0(R, I, \Lambda)$ is the Grothendieck group of the abelian monoid of isometry classes of (R, I, Λ)-quadratic forms on finitely-generated projective R-modules.

Remark 3.1.20. The fact that we have written GW_0 rather than just GW suggests that there exist abelian groups GW_1, GW_2, and so on. This is indeed the case. In fact, in [22], a procedure akin to those we have seen for K-theory and Hermitian K-theory is set out. Roughly speaking, one starts with an object called a "form category"; this is a category $\mathcal{C}$ which comes equipped with a "quadratic" contravariant functor $Q : \mathcal{C} \to \mathrm{Ab}$. For some object C, we think of $Q(C)$ as the abelian group of forms on C. To this, using an analogue of either the plus-construction or the Q-construction, one associates a topological space whose homotopy groups are the Grothendieck-Witt groups of the form category. As one may expect, a ring with form parameter gives rise to the structure of a form category on the category RMod.

In this thesis, we are largely choosing to adopt the "lower" viewpoint of rings with form parameter and their zeroth Grothendieck-Witt groups, but we feel it worthwhile at this point to indicate that, as one would expect and desire, the K-theory of forms fits into the same kind of homotopy-theoretic framework as Hermitian K-theory and K-theory.

Remark 3.1.21. Remark 3.1.20 touches on one important reason why we do not follow Bak's framework; from a homotopical point of view, the stipulation that ρ is injective is meaningless. Moreover, as we will see, the restriction in the Burnside

form ring $\underline{\mathbb{Z}}$ is not injective, so that following Bak would disqualify our main object of study from consideration.

We conclude this section by discussing homomorphisms of rings with form parameter:

Definition 3.1.22 ([22], Definition 3.12). A *homomorphism of rings with form parameter*

$$f : (R, I, \Lambda) \to (S, J, \Gamma)$$

is a homomorphism of rings with involution $f : R \to S$ together with a homomorphism of form parameters $(f_1, f_0) : (I, \Lambda) \to (J, \Gamma)$ for R, where we view (J, Γ) as a form parameter for R via restriction of scalars along the map f. Composition is simply composition of the underlying maps of sets.

Remark 3.1.23. As detailed in Section 3 of [22], and as one would expect, the Grothendieck-Witt groups we have defined are functorial: covariantly via an extension of scalars construction, and contravariantly via restriction of scalars. In this thesis we only use covariant functoriality, and the setting in which we work simplifies the extension of scalars construction considerably. We will therefore not define the extension of scalars in full generality, and delay giving the simplified version (Lemma 4.1.8) until we have given the assumptions that allow for it.

3.2   Form rings

In many cases of interest, given a form parameter ring (R, I, Λ), the duality coefficient (I, σ) is equal to the ring with involution (R, σ). All of our main results are covered by this simpler case. Therefore, we will focus on this case from now on, and make the following definition:

Definition 3.2.1 ([22], Definition 4.1). A *form ring* is a ring with form parameter with (R, I, Λ) such that the duality coefficient (I, σ) satisfies $I = R$ and the map σ is the involution on the ring R.

Since we are primarily concerned with form rings from now on, we usually omit I from the notation and write (R, Λ) for the ring with form parameter (R, R, Λ). Note that any form ring fits into the framework of Remark 3.1.18, so that $GW_0(R, \Lambda)$ is defined for all form rings (R, Λ). A homomorphism of form rings $(f, f_0) : (R, \Lambda) \to (S, \Gamma)$ is the obvious notion; namely, a homomorphism of rings with form parameter where the map of duality coefficients is equal to the homomorphism of rings with

involution.

Example 3.2.2. The rings with form parameter given in Example 3.1.11 and 3.1.12 are examples of form rings.

Remark 3.2.3. Lemma 4.5 in [22] shows that a form ring is the same as a *form category with strict duality* which has one object. This corresponds to the fact that one may view a ring as a linear category with one object.

Definition 3.2.4 ([22], Definition 4.6). A form ring (R, Λ) is called *commutative* if R and Λ are commutative rings, the restriction $\rho : \Lambda \to R$ is a ring homomorphism, $Q(x) : \Lambda \to \Lambda$ is Λ-linear for all $x \in R$, and $\tau : R \to \Lambda$ is Λ-linear where R is considered a Λ-module via ρ. A homomorphism $(A, \Lambda_A) \to (B, \Lambda_B)$ of commutative form rings is a homomorphism of form rings such that the map $\Lambda_A \to \Lambda_B$ is a ring homomorphism.

We now give a definition of the related concept of a C_2-*Tambara functor*. We will see that these give us examples of commutative form rings. Tambara functors were originally introduced in [26], although our definition is taken over from [22].

Definition 3.2.5 (From [22], Remark 4.8). A C_2-*Tambara functor* is a diagram

$$R \underset{\tau}{\overset{\eta}{\rightrightarrows}} \Lambda \xrightarrow{\rho} R$$

where (R, Λ, τ, ρ) is a C_2-Mackey functor, R and Λ are commutative rings, ρ is a ring homomorphism, the C_2-action on R is a ring homomorphism, and $\eta : (R, \cdot, 0, 1) \to (\Lambda, \cdot, 0, 1)$ is a C_2-equivariant multiplicative map preserving 0 and 1 such that

$$\tau(a) \cdot \lambda = \tau(a \cdot \rho(\lambda)), \quad \rho(\eta(a)) = a \cdot \bar{a}, \quad \eta(a + b) - \eta(a) - \eta(b) = \tau(a \cdot \bar{b})$$

for all $a, b \in R$ and $\lambda \in \Lambda$.

For more details on Tambara functors, one may consult [26] or [25]. What is relevant for us is that, per Example 1.1.4 of [5], a C_2 Tambara functor gives rise to a commutative form ring in the following way.

Let (R, Λ, τ, ρ) be the C_2-Mackey functor underlying our proposed form ring, and let the quadratic action Q of R on Λ be given by setting $Q(a)\lambda = \eta(a) \cdot \lambda$ for $a \in R$ and $\lambda \in \Lambda$. We now check that conditions 1-3 from Definition 3.1.5 hold:

1. Holds trivially, because a Tambara functor is in particular a C_2-Mackey functor.

2. We have that

$$\eta(a + b)x = (\eta(a) + \eta(b) + \tau(a\bar{b})x = \eta(a)x + \eta(b)x + \tau(a\rho(x)\bar{b}),$$

where the last equality is true because $\tau(a)\lambda = \tau(a\rho(\lambda))$ for all $a \in R$ and $\lambda \in \Lambda$. Then the condition is satisfied because we set $\eta(a)x = Q(a)(x)$.

3. is clear from the equations given at the end of Definition 3.2.5.

Remark 3.2.6. Recall that, for a C_2-Tambara functor

$$R \underset{\tau}{\overset{\eta}{\rightrightarrows}} R \overset{\rho}{\longrightarrow} R$$

the map η is C_2-equivariant; that is $\eta(x) = \eta(\bar{x})$ for all $x \in R$. On the other hand, for a commutative form ring, this need not be true. This means that not all commutative form rings come from C_2-Tambara functors. However, for a commutative form ring (R, Λ), if we have $Q(x) = Q(\bar{x})$ for all x, we have a C_2-Tambara functor with $\eta(x) = Q(x)(1_\Lambda)$. In particular, a commutative form ring (R, Λ) where R has trivial involution is given by a C_2-Tambara functor

$$R \underset{\tau}{\overset{\eta}{\rightrightarrows}} R \overset{\rho}{\longrightarrow} R.$$

We are now in position to define the main object of study of this thesis.

Definition 3.2.7 ([22], Example 4.10, [5], Example 1.1.3). The *Burnside form ring* $\underline{\mathbb{Z}} = (\mathbb{Z}, \mathbb{A}(\mathbb{Z}))$ of the integers is given by the Tambara functor

$$\mathbb{Z} \underset{\tau}{\overset{\eta}{\rightrightarrows}} \mathbb{A}(\mathbb{Z}) \overset{\rho}{\longrightarrow} \mathbb{Z}$$

where $\mathbb{A}(\mathbb{Z}) = \mathbb{Z}[C_2] = \mathbb{Z}[t]/(t^2 - 1)$ is the integral group ring over $C_2 = \langle t \rangle$ and the maps are as follows:

$$\tau(n) = n(1 + t), \quad \rho(a + bt) = a + b, \quad \eta(n) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2}t.$$

We discuss the reasons why $\underline{\mathbb{Z}}$ is of interest to us in Section 3.3. Before that, we will define a notion of Burnside form ring for any ring R. We begin with an extension of scalars construction for form rings:

Definition 3.2.8 ([22], Definition 4.13). Let (S, Λ, τ, ρ) be a form ring and let

$f : S \to R$ be a homomorphism of rings with involution. One obtains a new form ring $(R, \Lambda_R, \tau_R, \rho_R)$ called the *extension of scalars* of (S, Λ) along f. The abelian group Λ_R is generated by symbols

$$[x], [y, \lambda], \quad x, y \in R, \lambda \in \Lambda$$

with the following relations:

1. $[x \cdot f(a) \cdot \bar{x}] = [x, \tau(a)]$,

2. $[x + y] = [x] + [y]$,

3. $[x] = [\bar{x}]$,

4. $[x, \lambda_1 + \lambda_2] = [x, \lambda_1] + [x, \lambda_2]$,

5. $[x + y, \lambda] = [x, \lambda] + [y, \lambda] + [x \cdot f(\rho\lambda) \cdot \bar{y}]$

6. $[x, Q(a)\lambda] = [x \cdot f(a), \lambda]$

where $a \in S, x, y \in R, \lambda, \lambda_1, \lambda_2 \in \Lambda$. We define the maps $\rho_R : \Lambda_R \to R$ and $Q(x) : \Lambda_R \to \Lambda_R$ for $x \in R$ as follows:

$$\rho_R([x]) = x + \bar{x},$$
$$\rho_R([y, \lambda]) = y \cdot f(\rho\lambda) \cdot \bar{y},$$
$$Q(x)([y]) = [x \cdot y \cdot \bar{x}],$$
$$Q(x)([y, \lambda]) = [xy, \lambda],$$

where $y \in R$ and $\lambda \in \Lambda$. One checks by direct verification that ρ_R and $Q(x)$ are well-defined. Then, setting $\tau_R(x) = [x]$ makes the datum $(R, \Lambda_R, \tau_R, \rho_R)$ into a form ring. Moreover, setting $f_0(\lambda) = [1, \lambda]$ gives the homomorphism of form rings $(f, f_0) : (S, \Lambda) \to (R, \Lambda_R.)$

Remark 3.2.9. Checking that ρ_R and $Q(x)$ are well-defined is a simple matter of directly verifying that each of the six relations above are respected by both of the maps. To illustrate this, we give one example; taking ρ_R and the first relation, we

have

$$\rho_R([x \cdot f(s) \cdot \bar{x}]) = x \cdot f(s) \cdot \bar{x} + \overline{(x \cdot f(s) \cdot \bar{x})}$$
$$= x \cdot f(s) \cdot \bar{x} + x \cdot f(\bar{s}) \cdot \bar{x}$$
$$= x \cdot (f(s) + f(\bar{s})) \cdot \bar{x}$$
$$= x \cdot f(\rho\tau(s)) \cdot \bar{x}$$
$$= \rho_R([x, \tau(s)])$$

as required. Notice that the second equality uses the assumption that f is a map of rings with involution: $f(\bar{s}) = f(\bar{s})$.

We have the following lemma, which says that commutativity of form rings is preserved by the extension of scalars construction:

Lemma 3.2.10 ([22], Lemma 4.14). *Let (S, Λ) be a commutative form ring and let $f : S \to R$ be a homomorphism of commutative rings with involution. Then the extension of scalars (R, Λ_R) along f is a commutative form ring. The multiplication $\Lambda_R \otimes \Lambda_R \to \Lambda_R$ is defined on symbols as follows:*

$$[x] \cdot [y] = [x \cdot y] + [x \cdot \bar{y}], \quad x, y \in R$$
$$[x] \cdot [y, \lambda] = [x \cdot y \cdot f(\rho\lambda) \cdot \bar{y}], \quad x, y \in R, \ \lambda \in \Lambda$$
$$[x, \lambda] \cdot [y, \xi] = [x \cdot y, \lambda \cdot \xi], \quad x, y \in R, \ \lambda, \xi \in \Lambda$$

Proof. Direct verification. For example, the commutativity of the multiplication on Λ_R comes from commutativity of R. $\square$

We can now define the Burnside form ring for a commutative ring R:

Definition 3.2.11 ([22], Notation 4.15). Given a commutative ring R with involution, its *Burnside form ring* $\underline{R} = (R, \mathbb{A}(R))$ is the extension of scalars of $\underline{\mathbb{Z}}$ along the unique ring homomorphism $\mathbb{Z} \to R$.

The reasons one cares about or studies the form ring $\underline{\mathbb{Z}}$ are given in the following brief section.

3.3 C_2-Mackey functors and the Burnside form ring

It turns out that the category of C_2-Mackey functors, which we will denote by Mac, can be made into a symmetric monoidal category, with unit given by the Burnside

form ring $\mathbb{Z}$. In this subsection, which follows Appendix B of [22], we outline how this can be done. This material does not originate in [22], it is given, for example, in [4]. The treatment in [4] is done for any finite group G and thus generalises what we write, since we specify $G = C_2$.

Definition 3.3.1. Let M and N be C_2-Mackey functors. Then a *homomorphism of C_2-Mackey functors*, denoted $f : M \to N$ is a pair $f = (f^e, f^{C_2})$ of C_2-equivariant maps $f^e : M(e) \to N(e)$ and $f^{C_2} : M(C_2) \to N(C_2)$ which commute with the transfer and restriction.

Definition 3.3.2 ([22], B.1)**.** Let M and N be C_2-Mackey functors. Then the *internal homomorphism Mackey functor*, denoted $\underline{\mathrm{Mac}}(M, N)$, is defined as follows:

$$\underline{\mathrm{Mac}}(M, N)(e) = \mathrm{Hom}_{\mathrm{Ab}}(M(e), N(e)), \text{ with action } f \mapsto \sigma \circ f \circ \sigma := \bar{f},$$
$$\underline{\mathrm{Mac}}(M, N)(C_2) = \mathrm{Hom}_{\mathrm{Mac}}(M, N)$$

with structure maps

$$\tau : \underline{\mathrm{Mac}}(M, N)(e) \to \underline{\mathrm{Mac}}(M, N)(C_2), \quad f \mapsto (f + \bar{f}, \tau \circ f \circ \rho),$$
$$\rho : \underline{\mathrm{Mac}}(M, N)(C_2) \to \underline{\mathrm{Mac}}(M, N)(e), \quad (f^e, f^{C_2}) \mapsto f^e$$

That the internal hom Mackey functor is indeed a C_2-Mackey functor can be checked easily by direct verification. For example, we have that $\tau : \underline{\mathrm{Mac}}(M, N)(e) \to \underline{\mathrm{Mac}}(M, N)(C_2)$ is C_2-equivariant because $f + \bar{f}$ is a fixed point of the C_2-action on $\underline{\mathrm{Mac}}(M, N)(e)$, and because the transfers and restrictions from M and N are themselves C_2-equivariant.

Definition 3.3.3 ([22], B.2)**.** Let M and N be two C_2-Mackey functors. Then their *tensor product*, denoted $M \hat{\otimes} N$ is defined as follows. First, we have

$$(M \hat{\otimes} N)(e) = M(e) \otimes N(e), \quad \text{with action } \sigma \otimes \sigma,$$

and $(M \hat{\otimes} N)(C_2)$ is the quotient of the abelian group

$$M(C_2) \otimes N(C_2) \oplus (M(e) \otimes N(e))/(1 - \sigma \otimes \sigma)$$

by the two relations

$$[\rho(\xi) \otimes y] = \xi \otimes \tau(y), \quad [x \otimes \rho(\zeta)] = \tau(x) \otimes \zeta$$

for $x \in M(e), y \in N(e), \xi \in M(C_2)$, and $\zeta \in N(C_2)$. Transfer and restriction are defined by

$$(M \hat{\otimes} N)(e) \xrightarrow{\tau} (M \hat{\otimes} N)(C_2) : x \otimes y \mapsto [x \otimes y]$$
$$(M \hat{\otimes} N)(C_2) \xrightarrow{\rho} (M \hat{\otimes} N)(e) : \xi \otimes \eta + [x \otimes y] \mapsto \rho(\xi) \otimes \rho(\zeta) + x \otimes y + \sigma(x) \otimes \sigma(y).$$

As one would expect, the tensor product and internal hom form an adjoint pair, where the tensor product is the left adjoint and the internal hom is the right adjoint. The unit and counit are defined as follows. The counit ε, is the map

$$\varepsilon = (\varepsilon^e, \varepsilon^{C_2}) : \mathrm{Hom}_{\mathrm{Mac}}(M, N) \otimes M \to N$$

where ε^e is the usual evaluation map, and ε^{C_2} is

$$(\mathrm{Hom}_{\mathrm{Mac}}(M, N) \hat{\otimes} M)(C_2) \to N(C_2) : (f^e, f^{C_2}) \otimes \xi + [g \otimes x] \mapsto f^{C_2}(\xi) + \tau(g(x)).$$

The unit map
$$\nabla : M \to \mathrm{Hom}_{\mathrm{Mac}}(N, M \hat{\otimes} N)$$

is the usual coevaluation map at e:

$$M(e) \to \mathrm{Hom}_{\mathrm{Mac}}(N, M \hat{\otimes} N)(e) = \mathrm{Hom}_{\mathrm{Ab}}(N(e), M(e) \otimes N(e)) : x \mapsto (y \mapsto x \otimes y)$$

and, at C_2, it is given by

$$M(C_2) \to \mathrm{Hom}_{\mathrm{Mac}}(N, M \hat{\otimes} N)(C_2) = \mathrm{Hom}_{\mathrm{Mac}}(N, M \hat{\otimes} N) : \xi \mapsto (\nabla_\xi^e, \nabla_\xi^{C_2})$$

where
$$\nabla_\xi^e : N(e) \to M(e) \otimes N(e) : y \mapsto \rho(\xi) \otimes y$$

and
$$\nabla_\xi^{C_2} : N(C_2) \to (M \otimes N)(C_2) : \zeta \mapsto \xi \otimes \zeta.$$

Checking that these two functors do indeed form an adjoint pair is a matter of directly checking the unit-counit equations. This is trivial at e since all we have there are the usual evaluation and coevaluation, but a little more involved at C_2. For illustration, and denoting the functor $M \mapsto M \hat{\otimes} N$ by F for brevity, we check that $\varepsilon_{F(M)}^{C_2} \circ F(\nabla_M^{C_2}) = 1_{F(M)(C_2)}$.

Now, by definition,

$$F(M)(C_2) = M(C_2) \otimes N(C_2) \oplus (M(e) \otimes N(e))/(1 - \sigma \otimes \sigma),$$

and a general element of this group is of the form $\xi \otimes \zeta + [x \otimes y]$. The map $F(\nabla_M^{C_2})$ has codomain

$$\mathrm{Hom}_{\mathrm{Mac}}(N, M \hat{\otimes} N) \otimes N(C_2) \oplus (\mathrm{Hom}(N(e), M(e) \otimes N(e)) \otimes N(e))/1 - \sigma \otimes \sigma$$

and sends $\xi \otimes \zeta + [x \otimes y]$ to $(\nabla_\xi^e, \nabla_\xi^{C_2}) \otimes \zeta + [\eta(x) \otimes y]$, where

$$\eta : M(e) \to \mathrm{Hom}(N(e), M(e) \otimes N(e))$$

is the usual unit from the tensor-hom adjunction in the category of abelian groups. Applying ε_{C_2} gives $\nabla_\xi^{C_2}(\zeta) + \tau(\eta(x)(y))$ which is equal to $\xi \otimes \zeta + [x \otimes y]$, as required.

Now, the reason for the importance of $\underline{\mathbb{Z}}$, the Burnside form ring, is that it is the unit of the tensor product we have defined above. The unit isomorphism $u : \underline{\mathbb{Z}} \hat{\otimes} M \to M$ for a C_2-Mackey functor M is given by the following diagram,

$$
\begin{array}{ccc}
(\underline{\mathbb{Z}} \hat{\otimes} M)(e) & \xrightarrow{\cong} & M(e) \\
\rho \big\uparrow \big\downarrow \tau & & \rho \big\uparrow \big\downarrow \tau \\
(\underline{\mathbb{Z}} \hat{\otimes} M)(C_2) & \xrightarrow{\cong} & M(C_2)
\end{array}
$$

where the top horizontal map is given by $m \otimes x \mapsto m \cdot x$ (that is, it is the usual unit isomorphism for the tensor product of abelian groups) and the bottom horizontal map is given by

$$(m + nt) \otimes \xi + [r \otimes y] \mapsto (m - n) \cdot \xi + n \cdot \tau\rho(\xi) + r \cdot \tau(y).$$

Surjectivity of this bottom map is clear, since $1 \otimes \xi$ where $1 \in \mathbb{Z}[C_2]$, is a lift of $\xi \in M(C_2)$. Injectivity depends on the relations

$$[\rho(\xi) \otimes y] = \xi \otimes \tau(y), \quad [x \otimes \rho(\zeta)] = \tau(x) \otimes \zeta.$$

For example, say that we had $\tau(x) = \tau(y)$ in $M(C_2)$. Then, injectivity requires that we have $[1 \otimes x] = [1 \otimes y]$ in $(\underline{\mathbb{Z}} \hat{\otimes} M)(C_2)$. But this is true since the relations

give

$$[1 \otimes x] = [\rho(1) \otimes x]$$
$$= 1 \otimes \tau(x)$$
$$= 1 \otimes \tau(y)$$
$$= [1 \otimes y]$$

as required.

The fact that $\underline{\mathbb{Z}}$ is the tensor unit of C_2-Mackey functors means, in particular, that it is the tensor unit for rings with form parameter. This turns out to imply that the higher Grothendieck-Witt groups of any ring with form parameter (for which these groups are defined) are modules over the ring $GW_0(\underline{\mathbb{Z}})$, where the multiplication is induced by the cup product given in (2.13) in [22]. This means that understanding this ring is of significant importance to the K-theory of forms. In the next chapter, we will do just that.

Chapter 4

The Burnside form ring of the integers

4.1 Preliminaries

4.1.1 Viewing forms as matrices

We begin with some psychologically helpful lemmas. These are not specific to the Burnside form ring $\underline{\mathbb{Z}}$: indeed, in this section, our standing assumption is merely that (R, Λ) is a commutative form ring where R has trivial involution. In particular, per Remark 3.2.6, a commutative form ring (R, Λ) where R has trivial involution is always given by a C_2-Tambara functor over R. For the sake of convenience, this is the viewpoint we tend to adopt for the remainder of the thesis. Much of what we prove in this section concerns free modules, but, since projective modules over $\mathbb{Z}$ are free, we can apply it to our calculation of $GW_0(\underline{\mathbb{Z}})$.

We begin with the following lemma, which lets us view all (R, Λ)-quadratic forms on free R-modules as forms on R^n:

Lemma 4.1.1. *Let (R, Λ) be a commutative form ring where R has trivial involution. Fix a free R-module M of rank n and a free symmetric bilinear form module (R^n, β). We have a bijection φ between the sets*

$$\{q \in Q(M) \mid (M, \rho(q)) \cong (R^n, \beta)\}/\sim$$

and

$$\{q \in Q(R^n) \mid \rho(q) = \beta\}/\sim$$

where $\sim$ denotes isometry of (R, Λ)-quadratic forms, and where, in an abuse of notation, we write $\rho(q)$ for the symmetric bilinear form such that $\rho(q(x)) = \beta(x, x)$.

Proof. We define φ on the level of (R, Λ)-quadratic forms, then show that it is well-defined on isometry classes of forms. Let $(M, q, \rho(q))$ be an element of $Q(M)$. Choose an isomorphism $f : M \xrightarrow{\cong} R^n$, such that $\rho(q) \circ (f^{-1} \otimes f^{-1}) = \beta$, so that $q \circ f^{-1}$ is an element of $Q(R^n)$ with restriction equal to β. We define the map φ by choosing such an f then precomposing with f^{-1}.

First, note that φ does not depend on the choice of f. Indeed, choose another f' satisfying the same conditions as f. Then $q \circ f^{-1}$ is isometric to $q \circ f'^{-1}$ via the map $f'f^{-1}$.

Similarly, the map φ is well-defined on isometry classes of (R, Λ)-quadratic forms. Indeed, let $(M, q_1, \rho(q_1))$ and $(M, q_2, \rho(q_2))$ be (R, Λ)-quadratic forms such that there exist isomorphisms f_1, f_2 such that

$$\rho(q_1) \circ (f_1^{-1} \otimes f_1^{-1}) = \rho(q_2) \circ (f_2^{-1} \otimes f_2^{-1}) = \beta,$$

and suppose there exists an isomorphism $\alpha : M \to M$ such that $q_1 = q_2 \circ \alpha$. Then $q_1 \circ f_1^{-1}$ is isometric to $q_2 \circ f_2^{-1}$ via the isomorphism $f_2 \alpha f_1^{-1}$.

Finally, φ is a bijection. To see this, fix a rank n free R-module M and a $q \in Q(R^n)$ with $\rho(q) = \beta$. Choose an isomorphism $g : R^n \to M$, so that $q \circ g^{-1}$ is in $Q(M)$, with $\rho(q \circ g^{-1}) = \rho(q) \circ g^{-1}$ is isometric to (R^n, β) via the map g^{-1}. Different choices of g produce isometric results. Since the choice does not matter, the composition of the two maps in both directions is the identity, since we can choose any map and its inverse to define φ and its inverse.

$\square$

We can make a further identification, which lets us treat $Q(R^n)$ as a certain set of matrices. This material does not originate here. For example, it is given in Definition 1.5 and Lemma 1.6 of [5].

Definition 4.1.2. Given a commutative form ring (R, Λ) where R has trivial involution, define the abelian group $Q_n(R)$

$$Q_n(R) := \{B \in (\bigoplus_{1 \leq i < j \leq n} R \oplus \bigoplus_{1 \leq i = j \leq n} \Lambda) \mid B_{ij} = B_{ji} \ \forall i \neq j\}.$$

and the following maps and actions:

- the transfer $T : M_n(R) \to Q_n(R)$ is defined entrywise as follows:

$$T(A)_{ij} = \begin{cases} A_{ij} + A_{ji} & \text{if } i < j \\ \tau(A_{ii}) & \text{if } i = j \end{cases}$$

- the restriction $R : Q_n(R) \to M_n(R)$ is defined:

$$R(B)_{ij} = \begin{cases} B_{ij} & \text{if } i \neq j \\ \rho(B_{ii}) & \text{if } i = j \end{cases}$$

- given $A \in M_n(R)$ and $B \in Q_n(R)$, we have an action of $M_n(R)$ on $Q_n(R)$ defined as follows:

$$(Q(A)(B))_{ij} = \begin{cases} (A^T R(B)A)_{ij} & \text{if } i < j \\ \displaystyle\sum_{1 \leq k \leq n} \eta(A_{ki})B_{kk} + \tau\left(\sum_{1 \leq k < l \leq n} A_{ki}B_{kl}A_{li} \right) & \text{if } i = j \end{cases}$$

- finally, for $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$, and $B \in Q_n(R)$, we have a map $\psi : Q_n(R) \times R^n \to \Lambda$ defined as follows:

$$\psi(B, x) := \sum_{i=1}^{n} \eta(x_i)(B_{ii}) + \tau \sum_{i < j} x_i x_j B_{ij}$$

As the notation and nomenclature suggest, the maps T, R, and Q defined immediately previous make $(M_n(R, Q_n(R))$ into a form ring: this is proven in Lemma 1.6 of [5]. The restriction R and quadratic action $Q(-)$ of matrices are chosen such that $R(Q(A)(B)) = A^T R(B)A$; in other words, restricting the quadratic action gives the standard conjugation action. This is clear by definition for the off-diagonal entries; we illustrate it for the diagonal entries. For example, let $B = \left(\begin{smallmatrix} q_1 & \alpha \\ \alpha & q_2 \end{smallmatrix} \right) \in Q_2(R)$ and

let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R)$. Then:

$$
\begin{aligned}
A^T R(B) A &= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \rho(q_1) & \alpha \\ \alpha & \rho(q_2) \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}^2 \rho(q_1) + 2(a_{11}a_{21}\alpha) + a_{21}^2 \rho(q_2) & (A^T R(B) A)_{12} \\ (A^T R(B) A)_{21} & a_{12}^2 \rho(q_1) + 2(a_{12}a_{22}\alpha) + a_{21}^2 \rho(q_2) \end{pmatrix} \\
&= R(Q(A)(B)),
\end{aligned}
$$

where the last equality follows from the facts that $\rho\tau = 1 + 1 = 2$ and $a^2 \rho(\xi) = \rho(Q(a)(\xi))$, $\forall \xi \in \Lambda$ and $a \in R$.

Lemma 4.1.3. *We have an isomorphism of abelian groups $Q(R^n) \xrightarrow{\cong} Q_n(R)$. Moreover, denoting the image of $(q, \beta) \in Q(R^n)$ by B_q, we have $\psi(B_q, x) = q(x)$ for all $x \in R^n$.*

Proof. Given $(q, \beta) \in Q(R^n)$, we define B_q. Denoting the standard basis of R^n by $\{e_i\}_{1 \leq i \leq n}$, B_q is defined as follows:

$$
B_{q_{ij}} = \begin{cases} q(e_i) & \text{if } i = j \\ \beta(e_i, e_j) & \text{if } i \neq j \end{cases}
$$

To see that this map is a group homomorphism, note:

$$
\begin{aligned}
B_{q+q'} &= \begin{cases} (q + q')(e_i) & \text{if } i = j \\ (\beta + \beta')(e_i, e_j) & \text{if } i \neq j \end{cases} \\
&= \begin{cases} q(e_i) + q'(e_i) & \text{if } i = j \\ \beta(e_i, e_j) + \beta'(e_i, e_j) & \text{if } i \neq j \end{cases} \\
&= B_q + B_{q'}.
\end{aligned}
$$

To see that the map is a bijection, note that any (q, β) in $Q(R^n)$ is defined entirely by where it sends the standard basis $\{e_1, \ldots, e_n\}$; indeed, for any vector $x \in R^n$, we have

$$
q(x) = \sum_{i=1}^n \eta(x_i) q(e_i) + \tau \sum_{i<j} x_i x_j \beta(e_i, e_j).
$$

This is proven by a simple induction argument, where the base case $n = 2$ is from the axioms governing the behaviour of q and η from Definition 3.1.10. With this in mind, given $B \in Q_n(R)$, the inverse map is given by $B \mapsto (q_B, \beta_B)$, where $q_B(e_i) = B_{ii}$

and $\beta_B(e_i, e_j) = B_{ij}$. The second assertion of the lemma follows by definition of ϕ. □

Lemma 4.1.4. *Let (R, Λ) be a commutative form ring where R has trivial involution, and fix $(q, \beta) \in Q(R^n)$. As in Lemma 4.1.3, denote the image of (q, β) under the bijection $Q(R^n) \to Q_n(R)$ by B_q. Let $f : R^n \to R^n$ be an isomorphism, and let A_f be the matrix of f in the standard basis $e_1, \dots, e_n$. Then the (R, Λ)-quadratic form $(q \circ f, \beta \circ (f \otimes f))$ has matrix $Q(A_f)(B_q)$ in the basis $e_1, \dots e_n$.*

Proof. Fix some $1 \leq i \leq n$. Then

$$f(e_i) = \sum_{j=1}^{n} A_{f_{ji}} e_j,$$

so that

$$q \circ f(e_i) = \sum_{j=1}^{n} \eta(A_{f_{ji}}) q(e_j) + \sum_{k \neq j} A_{f_{ji}} A_{f_{ki}} \beta(e_j, e_k) = \sum_{j=1}^{n} \eta(A_{f_{ji}}) B_{q_{jj}} + \sum_{k \neq j} A_{f_{ji}} A_{f_{ki}} B_{q_{jk}}$$

which, from Definition 4.1.2, is the i-th diagonal entry of the matrix $Q(A_f)(B_q)$ as required. Agreement on the non-diagonal entries follows from the classical fact that the matrix of $\beta \circ (f \otimes f)$ in our basis is given by $A_f^T R(B_q) A_f$, which, again from Definition 4.1.2, agrees with $Q(A_f)(B_q)$ on non-diagonal entries. □

Lemma 4.1.5. *Let (R, Λ) be a commutative form ring where R has trivial involution. Then, given two (R, Λ)-quadratic forms (R^n, q, β) and (R^m, q', β'), recall their orthogonal sum $(R^n \oplus R^m, q \perp q', \beta \perp \beta')$ from Definition 3.1.16.*
Choosing standard bases $e_1, \dots, e_n$ and $e_{n+1}, \dots, e_{n+m}$ for R^n and R^m respectively, under the correspondence $Q(R^{n+m}) \xrightarrow{\cong} Q_{n+m}(R)$, $(q \perp q', \beta \perp \beta')$ is mapped to the $(n+m) \times (n+m)$ block-diagonal matrix

$$\begin{bmatrix} B_q & 0 \\ 0 & B_{q'} \end{bmatrix}$$

where B_q is defined as in the proof of Lemma 4.1.3 and where R^{n+m} has basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{n+m}$.

Proof. For the diagonal entries of the matrix, the assertion is clear from Definition 3.1.16. For the non-diagonal entries, it follows from the classical fact that, if two

symmetric bilinear forms β and β' have matrices B and B' in the bases $e_1, \ldots, e_n$ and $e_{n+1}, \ldots, e_{n+m}$ respectively, then their orthogonal sum $\beta \perp \beta'$ has matrix

$$\begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix}$$

in the basis $e_1, \ldots, e_{n+m}$. $\qquad \square$

Proposition 4.1.6. *Let (R, Λ) be a commutative form ring where R has trivial involution. Let (R^n, β) be a free symmetric bilinear form module over the ring R. Suppose β has matrix B' in the standard basis $e_1, \ldots, e_n$. Define*

$$O(B') := \{A \in GL_n(R) \mid A^T B' A = B'\}$$

and

$$Q_{B'}(R) := \{B \in Q_n(R) \mid R(B) = B'\}$$

Then for any rank n free R-module M, the set

$$\{q \in Q(M) \mid (M, \rho(q)) \cong (R^n, \beta)\}/\sim$$

as given in Lemma 4.1.1 is in bijection with the orbit space of the action of $O(B')$ on $Q_{B'}$, where $O(B')$ acts via the quadratic action defined in Definition 4.1.2.

Proof. Combine Lemmas 4.1.1, 4.1.3, and 4.1.4. $\qquad \square$

Proposition 4.1.7. *Let (R, Λ) be a commutative form ring where R has trivial involution and where R is a ring such that finitely generated projective R-modules are free. Let $M_{(R,\Lambda)}$ be the set of symmetric matrices with diagonal entries in Λ and non-diagonal entries in R. Define an equivalence relation $\sim$ by setting $B \sim B'$ if there exists $A \in GL(R)$ such that $B' = Q(A)(B)$, where the quadratic action Q is given in Definition 4.1.2. Denote by $\mathcal{P}(R, \Lambda)$ the abelian monoid of isometry classes of (R, Λ)-quadratic forms. Then we have an isomorphism of abelian monoids*

$$\mathcal{P}(R, \Lambda) \xrightarrow{\cong} M_{(R,\Lambda)}/\sim$$

where the operation on $M_{(R,\Lambda)}/\sim$ is block sum of matrices as given in Lemma 4.1.5.

Proof. Combine Lemmas 4.1.5, 4.1.3, and 4.1.4, noting that the choice of basis does not matter after quotienting by $\sim$. $\qquad \square$

The moral of Proposition 4.1.7, and of this section, is that for a commutative form ring (R, Λ) such that R has trivial involution and such that finitely generated projective R-modules are free, we can view $GW_0(R, \Lambda)$ as the Grothendieck group of the abelian monoid $M_{(R,\Lambda)}/ \sim$. Moreover, Proposition 4.1.6 says that, if the isometry class of β is fixed, we can choose a representative for it and consider the orbits of the action of $O(\beta)$ on Q_β.

We conclude the section with a lemma which interprets covariant functoriality (cf. the discussion at the end of Section 3 of [22]) in the language of matrices.

Lemma 4.1.8. *Let $(f, f_0) : (R, \Lambda) \to (S, \Gamma)$ be a homomorphism of commutative form rings, where R and S are such that finitely generated projective modules are free, and both equipped with trivial involution. Then the map $(f, f_0)_* : (M_{(R,\Lambda)}/ \sim) \to (M_{(S,\Gamma)}/ \sim)$ given by entrywise application of f and f_0 is a well-defined homomorphism of abelian monoids. Moreover, if (f, f_0) is an isomorphism of form rings, then the induced map $(f, f_0)_*$ is an isomorphism of abelian monoids, so that $GW_0(R, \Lambda)$ is isomorphic to $GW_0(S, \Gamma)$.*

Proof. To show well-definedness, let B and B' be elements of $M_{(R,\Lambda)}$, and suppose $[B] = [B']$. Then there exists $A \in O(R(B))$ such that $Q(A)(B) = B'$. The fact that (f, f_0) is a map of form rings implies that $Q(f(A))((f, f_0)_*(B)) = (f, f_0)_*(B')$ where $f(A)$ is the matrix with entries in S given by entrywise application of f. Similarly, the fact that (f, f_0) is a map of form rings implies that $f(A) \in O(R((f, f_0)_*(B))$, so that $(f, f_0)_*(B)$ and $(f, f_0)_*(B')$ are in the same equivalence class as required.
That $(f, f_0)_*$ respects block sum of matrices is clear, since $f(0) = 0$.
Finally, for bijectivity, note that $(f, f_0)_*$ has inverse given by $(f^{-1}, f_0^{-1})_*$, which is well-defined for the same reason $(f, f_0)_*$ is. $\qquad\square$

Remark 4.1.9. Lemma 4.1.8 is merely an interpretation in the language of matrices of an extension of scalars notion, given in [22], which extends the usual extension of scalars of modules over rings. This can be defined for any homomorphism of rings with form parameter. When we assume that finitely-generated projective modules are free, it is usually because doing so is convenient for the present thesis, and not because the assumption is conceptually or technically fundamental.

4.1.2 The 2 invertible case

In this section, we establish some notation while investigating what happens for the Burnside form ring $\underline{R}$, where $1/2 \in R$. As we will see, this gives a model for what

to do in the integer case, although some alterations are necessary. We begin with a lemma which gives a more concrete way to view the Burnside form ring $\underline{R}$:

Lemma 4.1.10. *Let R be a commutative ring with $1/2 \in R$ and trivial involution. We have a form ring*

$$R \underset{\tau}{\overset{\eta}{\rightrightarrows}} R[C_2] \overset{\rho}{\longrightarrow} R \tag{4.1}$$

where $R[C_2] = R[t]/(t^2 - 1)$ and with maps defined as follows

$$\tau(a) = a(1 + t), \quad \rho(a + bt) = a + b, \quad \eta(a) = \frac{a(a+1)}{2} + \frac{a(a-1)}{2}t.$$

This form ring is isomorphic to the Burnside form ring $\underline{R} = (R, \mathbb{A}(R))$ given in Definition 3.2.11.

Proof. We have a map of form rings

$$\begin{array}{ccccc}
R & \rightrightarrows & \mathbb{A}(R) & \longrightarrow & R \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle 1} \\
R & \underset{\tau}{\overset{\eta}{\rightrightarrows}} & R[C_2] & \overset{\rho}{\longrightarrow} & R
\end{array} \tag{4.2}$$

where the middle vertical map sends $[a, \lambda]$ to $\eta(a)(f_0(\lambda))$ and $[b]$ to $\tau(b)$ for $a, b \in R$, $\lambda \in \mathbb{Z}[C_2]$, and where f_0 is from the map of form rings

$$\begin{array}{ccccc}
\mathbb{Z} & \underset{\tau}{\overset{\eta}{\rightrightarrows}} & \mathbb{Z}[C_2] & \overset{\rho}{\longrightarrow} & \mathbb{Z} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
R & \underset{\tau}{\overset{\eta}{\rightrightarrows}} & R[C_2] & \overset{\rho}{\longrightarrow} & R
\end{array} \tag{4.3}$$

Checking that the map (4.2) is well-defined is a matter of checking directly that the relations 1-6 from Definition 3.2.8 are respected by it. For illustration, we will show what happens for relation (1).

Recall that relation 1 says, for all integers n and $x \in R$, we have

$$[x^2 f(n)] = [x, \tau(n)].$$

The map (4.2) sends $[x, \tau(n)]$ to $\eta(x) f_0(\tau(n))$. Then commutativity of the diagram (4.3) implies $f_0(\tau(n)) = \tau(f(n))$. But then, per Definition 3.2.5,

$$\eta(x) \cdot \tau(f(n)) = \tau(f(n) \cdot \rho\eta(x))$$
$$= \tau(f(n)x^2),$$

which, as required, is the image of $[x^2 f(n)]$ under (4.2). Relations 2-6 are handled similarly.

We claim the map (4.2) has inverse given by

$$a + bt \mapsto [a, 1] + \left[\frac{a(1-a)}{2}\right] + [-b, 1] + \left[\frac{b(1-b)}{2}\right]$$

For the composition $R[C_2] \to \mathbb{A}(R) \to R[C_2]$, we have

$$a + bt \mapsto [a, 1] + \left[\frac{a(1-a)}{2}\right] + [-b, 1] + \left[\frac{b(1-b)}{2}\right]$$
$$\mapsto \eta(a) + \tau\left(\frac{a(1-a)}{2}\right) + t\eta(b) + \tau\left(\frac{b(1-b)}{2}\right)$$
$$= \frac{a(a+1)}{2} + \frac{a(a-1)}{2}t + \tau\left(\frac{a(1-a)}{2}\right)$$
$$+ \frac{b(b+1)}{2}t + \frac{b(b-1)}{2} + \tau\left(\frac{b(1-b)}{2}\right),$$

which is equal to $a + bt$ as required.

We check the other direction on the symbols $[a, \lambda]$ and $[b]$ which generate $\mathbb{A}(R)$. We have, for all $b \in R$:

$$[b] \mapsto \tau(b) \mapsto [b, 1] + \left[\frac{b(1-b)}{2}\right] + [-b, 1] + \left[\frac{b(1-b)}{2}\right]$$
$$= [b(1-b)] + [0, 1] + [b^2],$$

where $[b, 1] + [-b, 1] = [0, 1] + [b^2]$ by relation 5 in Definition 3.2.8.

Moreover, $[0, 1] + [b^2] = [b^2]$. To see this, first note that relation 6 in Definition 3.2.8 implies $[0, 1] = [1, 0]$. Then relation 1 implies $[1, 0] = [0]$, and $[0] + [b^2] = [b^2]$ by relation 2. This means that

$$[b(1-b)] + [0, 1] + [b^2] = [b(1-b)] + [b^2] = [b]$$

as required.

For symbols of the form $[a, \lambda]$, write $\lambda = \lambda_1 + t\lambda_2$. Then, by relations 4 and 6 in Definition 3.2.8, we have $[a, \lambda] = [a, \lambda_1] + [a, \lambda_2 t] = [a, \lambda_1] + [-a, \lambda_2]$. Since λ_1 and λ_2 are integers, and applying relation 4 again, we have $[a, \lambda] = \lambda_1[a, 1] + \lambda_2[-a, 1]$. It is therefore sufficient to check symbols of the form $[a, 1]$. To that end:

$$[a, 1] \mapsto \eta(a) = \frac{a(a+1)}{2} + \frac{a(a-1)}{2}t,$$

which is mapped to

$$\left[\frac{a^2+a}{2}, 1\right] + \left[\frac{\frac{a^2+a}{2}\left(1 - \frac{a^2+a}{2}\right)}{2}\right] + \left[\frac{a - a^2}{2}, 1\right] + \left[\frac{\frac{a^2-a}{2}\left(1 - \frac{a^2-a}{2}\right)}{2}\right].$$

Relation 5 implies

$$\left[\frac{a^2+a}{2}, 1\right] + \left[\frac{a - a^2}{2}, 1\right] = [a, 1] - \left[\frac{(a^2+a)(a-a^2)}{4}\right],$$

and it can be shown by direct verification that

$$\left[\frac{\frac{a^2+a}{2}\left(1 - \frac{a^2+a}{2}\right)}{2}\right] + \left[\frac{\frac{a^2-a}{2}\left(1 - \frac{a^2-a}{2}\right)}{2}\right] = \left[\frac{(a^2+a)(a-a^2)}{4}\right],$$

so that we have the required result. $\qquad\square$

In view of Lemma 4.1.10, if R has trivial involution and $1/2 \in R$, we can think of $GW_0(\underline{R})$ as the Grothendieck-Witt group of the form ring (4.1). Reflecting this, in an abuse of terminology, we write $\underline{R}$ for the form ring (4.1) and call it the Burnside form ring.

Let R be a ring in which 2 is invertible and over which finitely generated projective modules are free, so that Proposition 4.1.7 applies. Let β be a symmetric matrix with entries in R, and fix any $B \in Q_\beta$. We have a bijection $Q_\beta \to Q_0$ which is given by subtracting B in the sense of ordinary matrix subtraction. Any $B \in Q_\beta$ defines a bijection in this way, but over a ring R where 2 is invertible, the natural choice for our purposes is $T\frac{\beta}{2}$, where T is the transfer from Definition 4.1.2. The reason for this is shown in the following proposition, which relates the quadratic action of $GL_n(R)$ on $Q_n(R)$ to its classical action by matrix multiplication on R^n. First, we establish some notation.

Notation. Given a vector λ with entries $\lambda_1, \ldots, \lambda_n \in R^n$, denote the diagonal matrix

with non-zero entries $\lambda_1(1-t), \ldots, \lambda_n(1-t)$ by M_λ.

Proposition 4.1.11. *Let R be a commutative ring with trivial involution and $1/2 \in R$, such that finitely-generated projective R-modules are free. Then, for the projective R-module R^n, a symmetric $n \times n$ matrix β with entries in R, and an element $A \in GL_n(R)$, we have the following commutative diagram:*

$$
\begin{array}{ccc}
Q_\beta & \xrightarrow{Q(A)} & Q_{A^T\beta A} \\
{\scriptstyle -T\frac{\beta}{2}}\big\downarrow & & \big\downarrow{\scriptstyle -T\frac{A^T\beta A}{2}} \\
Q_0 & \xrightarrow{Q(A)} & Q_0 \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
R^n & \xrightarrow{A^T\cdot} & R^n
\end{array}
$$

where Q_β and Q_0 are defined as in Proposition 4.1.6 over the form ring $\underline{R}$, and where, noting that every element of Q_0 is of the form M_λ for some $\lambda \in R^n$, f is the bijection which sends M_λ to λ.

Proof. Take $B_0 \in Q_0$. Commutativity of the top square is equivalent to $Q(A)(B_0) = Q(A)(B_0 + T\frac{\beta}{2}) - T\frac{A^T\beta A}{2}$. This is true since $Q(A)$ is linear on forms and $Q(A)(T(\frac{\beta}{2})) = T\frac{A^T\beta A}{2}$.

Commutativity of the bottom square is by direct calculation; we show the case $n = 2$ for illustration. First, note that for any $a \in R$:

$$
\eta(a) = \frac{a(a+1)}{2} + t\frac{a(a-1)}{2} = (1+t)\frac{a^2}{2} + (1-t)\frac{a}{2}.
$$

Now, by definition $Q(A)$ applied to the matrix

$$
\begin{bmatrix}
\lambda_1(1-t) & 0 \\
0 & \lambda_2(1-t)
\end{bmatrix}
$$

is

$$
\begin{bmatrix}
(\eta(A_{11})\lambda_1 + \eta(A_{21})\lambda_2)(1-t) & 0 \\
0 & (\eta(A_{12})\lambda_1 + \eta(A_{22})\lambda_2)(1-t)
\end{bmatrix}
$$

Now, applying the above formula for η together with the identities

$$(1 + t)(1 - t) = 0,$$
$$(1 - t)^2 = 2(1 - t)$$

we obtain

$$\begin{bmatrix} (A_{11}\lambda_1 + A_{21}\lambda_2)(1 - t) & 0 \\ 0 & (A_{12}\lambda_1 + A_{22}\lambda_2)(1 - t) \end{bmatrix}$$

as required. All other ranks are completely analogous. $\qquad \square$

Remark 4.1.12. Proposition 4.1.11 uses the assumption that projective modules are free. Indeed, if P is a projective module which is not free, then the bijection f does not exist in general. In fact, there is no *a priori* reason why $Q_0(P)$ should be an R-module at all. It may, however, be possible to generalise Proposition 4.1.11 by assuming that R is a ring for which the bijection f exists. This may be true for a larger class of rings than those for which projective modules are free. Our calculation of $GW_0(\underline{R})$ for $1/2 \in R$ where we assume projective modules are free uses the classical theory of symmetric bilinear forms on the free module R^n. However, this classical theory only requires modules to be projective. It therefore may be possible to use analogous arguments on the projective R-module $Q_0(P)$ and thereby remove the assumption that projective modules are free over R.

Proposition 4.1.13. *Let R be a commutative ring with trivial involution and $1/2 \in R$, such that finitely generated projective R-modules are free. In accordance with Lemma 4.1.10, view the Burnside form ring $\underline{R}$ as the form ring*

$$R \underset{\tau}{\overset{\eta}{\rightrightarrows}} R[C_2] \overset{\rho}{\longrightarrow} R$$

with maps as in Lemma 4.1.10. Fix an $n \times n$ symmetric matrix β with entries in R. In accordance with Proposition 4.1.7, view $GW_0(\underline{R})$ as the Grothendieck group of the abelian monoid $M_{\underline{R}}/ \sim$. Finally, recalling Proposition 4.1.11, write x_B for the element $f(B - T\frac{\beta}{2}) \in R^n$. Then the map

$$GW_0(\underline{R}) \to R \oplus GW_0(R)$$
$$[B, \beta] \mapsto (x_B^T \beta^{-1} x_B, [\beta])$$

is a well-defined group homomorphism, where $GW_0(R)$ is the Grothendieck-Witt group of symmetric bilinear forms over R and where $\beta = R(B)$, where R is the map

given in Definition 4.1.2.

Proof. Let A be in $GL_n(R)$. To show well-definedness, it is sufficient to show that, for any $(R, \mathbb{A}(R))$-quadratic form (B, β),

$$[Q(A)(B), A^T \beta A] \mapsto (x_{Q(A)(B)}^T (A^T \beta A)^{-1} x_{Q(A)(B)}, [A^T \beta A])$$
$$= (x_B^T \beta^{-1} x_B, [\beta]).$$

That $[\beta] = [A^T \beta A]$ in $GW_0(R)$ is clear. Proposition 4.1.11 implies that $x_{Q(A)(B)} = A^T x_B$. Substituting this into the expression

$$x_{Q(A)(B)}^T (A^T \beta A)^{-1} x_{Q(A)(B)}$$

and applying properties of matrix transposes and inverses gives the required equality. To show the map is a group homomorphism, consider the block-diagonal matrix $(B \perp B', \beta \perp \beta')$. This is mapped to

$$x_{B \perp B'}^T (\beta \perp \beta')^{-1} x_{B \perp B'}$$

We have the following facts:

- The inverse of a block sum $(\beta \perp \beta')^{-1}$ is the block sum of β^{-1} and β'^{-1}.

- The vector $x_{B \perp B'}$ is equal to $x_B \perp x_{B'}$, where, if we have vectors $x \in R^n$ and $y \in R^m$, then $x \perp y$ means the vector in R^{n+m} whose entries are the concatenation of the entries of x and the entries of y. This is clear from the definition of the vectors x_B and $x_{B'}$.

Combining these facts, we have

$$x_{B \perp B'}^T (\beta \perp \beta')^{-1} x_{B \perp B'} = (x_B \perp x_{B'})^T (\beta^{-1} \perp \beta'^{-1})(x_B \perp x_{B'})$$
$$= x_B^T \beta^{-1} x_B + x_{B'}^T \beta'^{-1} x_{B'},$$

where the second equality follows because the centre matrix is the block-diagonal matrix

$$\begin{bmatrix} \beta^{-1} & 0 \\ 0 & \beta'^{-1} \end{bmatrix},$$

so that the map is a group homomorphism as required. $\qquad\square$

The rest of this section is devoted to proving that the map given in Proposition

4.1.13 is injective and surjective. We begin with a remark which clarifies how we use the identification of Proposition 4.1.11 in our proofs.

Remark 4.1.14. For any form ring (R, Λ), recall that $GW_0(R, \Lambda)$ is the Grothendieck group of the abelian monoid whose elements are isometry classes of (R, Λ)-quadratic forms (P, q, β), where P is a finitely generated projective R-module, with operation orthogonal sum. One way of explicitly realising this is to say that the elements of $GW_0(R, \Lambda)$ are (R, Λ)-quadratic forms considered up to stable equivalence; that is, we say (P_1, q_1, β_1) and (P_2, q_2, β_2) are equivalent if there exists an (R, Λ)-quadratic form (P, q, β) with P a finitely generated projective R-module such that

$$(P_1, q_1, \beta_1) \perp (P, q, \beta) \cong (P_2, q_2, \beta_2) \perp (P, q, \beta)$$

where $\cong$ denotes isometry.

Applying this to the specific situation of Proposition 4.1.11, we have that two (R, Λ)-quadratic forms (B_1, β_1) and (B_2, β_2) given by $n \times n$ matrices are stably equivalent if and only if:

- β_1 and β_2 are stably equivalent as symmetric bilinear forms; that is, there exists an $m \times m$ symmetric matrix β and $A \in GL_{n+m}(R)$ such that $A^T(\beta_1 \perp \beta)A = \beta_2 \perp \beta$.

- there exists a vector v in (R^m, β) such that $A^T(x_{B_1} \perp v) = x_{B_2} \perp v$, where $\perp$ means concatenation of vectors as given in the proof of Proposition 4.1.13.

To prove injectivity, we first prove some preliminary general lemmas about symmetric bilinear forms, then apply these to our situation.

Lemma 4.1.15. *Let R be a commutative ring with 2 invertible. Let (P, β) be a symmetric bilinear form module with P a finitely generated projective R-module. Let (R^2, H) be the hyperbolic plane symmetric bilinear form module, which is defined by the symmetric matrix*

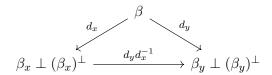$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let x and y be in P such that $\beta(x, x) = \beta(y, y) = b$. Then there exists $v \in (R^2, H)$ such that $(\beta \perp H)(x \perp v, x \perp v) = (\beta \perp H)(y \perp v, y \perp v) = 1$.

Proof. For a vector $v = (v_1, v_2)$, we have that $H(v, v) = 2v_1v_2$. Therefore take

$$v = \begin{bmatrix} \frac{1-b}{2} \\ 1 \end{bmatrix}$$

$\square$

Lemma 4.1.16. *Let R be a commutative ring with 2 invertible. Let (P, β) be a symmetric bilinear form module with P a finitely generated projective R-module. Let x and y be in P such that $\beta(x, x) = \beta(y, y) = 1$. Then there exists an element $A \in O(\beta \perp \beta)$ such that $A(x \perp 0) = y \perp 0$.*

Proof. Write β_x for $\beta|_{Rx}$ and β_y for $\beta|_{Ry}$. Since $\beta(x, x) = \beta(y, y) = 1$, we can apply the orthogonal decomposition lemma ([14], I.3.1) to write the commutative diagram of isomorphisms:

Now, orthogonal sum everywhere with a copy of $\beta|_{Rx} \perp (\beta|_{Rx})^\perp \cong \beta$ to obtain a new diagram

where the map
$$\phi : \beta_x \perp (\beta_x)^\perp \perp \beta_x \to \beta_y \perp (\beta_y)^\perp \perp \beta_x$$

is defined as follows; first observe that since β_x is isometric to β_y, we have a third orthogonal decomposition $d_{xy} : \beta \xrightarrow{\cong} (\beta_y)^\perp \perp \beta_x$ and therefore an isometry $d_{xy}d_x^{-1} : \beta_x \perp (\beta_x)^\perp \to (\beta_y)^\perp \perp \beta_x$. Then set $\phi = \psi \perp d_{xy}d_x^{-1}$ where ψ is the isometry from β_x to β_y which sends x to y.

Now consider the composition $(d_y \perp 1)^{-1} \circ (\phi \perp 1) \circ (d_x \perp 1)$. This is an isometry

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from $\beta \perp \beta_x \perp (\beta_x)^\perp$ to itself which sends $x \perp 0$ to $y \perp 0$. Composition with the isometry $\beta_x \perp (\beta_x)^\perp \cong \beta$ gives the required result. □

Proposition 4.1.17. *Let R be a commutative ring with 2 invertible. Let (P, β) be a symmetric bilinear form module with P a projective R-module. Let x and y be in P such that $\beta(x, x) = \beta(y, y)$. Then there exists a vector v in a finitely generated projective symmetric bilinear form module (P', β') and $A \in O(\beta \perp \beta')$ such that $A(x \perp v) = y \perp v$.*

Proof. Apply Lemma 4.1.15, so that we have $v' \in (R^2, H)$ with $(\beta \perp H)(x, \perp v', x \perp v') = (\beta \perp H)(y, \perp v', y \perp v') = 1$. Then apply Lemma 4.1.16 to the vectors $x \perp v'$ and $y \perp v'$, we have that $v = v' \perp 0, P' = (P \oplus H) \oplus (P \oplus H)$, and A is defined as in the proof of Lemma 4.1.16 , so that we have the required result. □

Proposition 4.1.18. *Under the same assumptions and notations as Proposition 4.1.13, the group homomorphism*

$$GW_0(\underline{R}) \to R \oplus GW_0(R)$$
$$[B, \beta] \mapsto (x_B^T \beta^{-1} x_B, [\beta])$$

as defined in Proposition 4.1.13 is injective.

Proof. Suppose $[B, \beta]$ and $[B', \beta']$ are elements of $GW_0(\underline{R})$ such that

$$(x_B^T \beta^{-1} x_B, [\beta]) = (x_{B'}^T \beta'^{-1} x_{B'}, [\beta']).$$

We have that β and β' are in the same class in $GW_0(R)$; this implies that β and β' have the same rank, n say. Moreover, there exists an $m \times m$ symmetric matrix σ and an element $A \in GL_{n+m}(R)$ such that $A^T(\beta \perp \sigma)A = \beta' \perp \sigma$. Let B_σ be a lift of σ under the restriction map R given in Definition 4.1.2: the surjectivity of $\rho : R[C_2] \to R$ implies one always exists. Consider the two $(R, \mathbb{A}(R))$-quadratic forms

$$(B \perp B_\sigma, \beta \perp \sigma), \quad (B' \perp B_\sigma, \beta' \perp \sigma),$$

which are, respectively, sent to

$$(x_{B \perp B_\sigma}^T (\beta^{-1} \perp \sigma^{-1}) x_{B \perp B_\sigma}, [\beta \perp \sigma]),$$
$$(x_{B' \perp B_\sigma}^T (\beta' \perp \sigma)^{-1} x_{B' \perp B_\sigma}, [\beta' \perp \sigma]),$$

both of which are elements of $R \oplus GW_0(R)$. In fact, we have

$$(x_{B \perp B_\sigma}^T (\beta^{-1} \perp \sigma^{-1}) x_{B \perp B_\sigma}, [\beta \perp \sigma]) = (x_{B' \perp B_\sigma}^T (\beta' \perp \sigma)^{-1} x_{B' \perp B_\sigma}, [\beta' \perp \sigma]). \quad (4.4)$$

Indeed, the existence of A such that $A^T(\beta \perp \sigma)A = \beta' \perp \sigma$ implies that $[\beta \perp \sigma] = [\beta' \perp \sigma]$. Moreover, we have

$$x_{B \perp B_\sigma}^T (\beta^{-1} \perp \sigma^{-1}) x_{B \perp B_\sigma} = x_B^T \beta^{-1} x_B + x_{B_\sigma}^T \sigma^{-1} x_{B_\sigma},$$

and

$$x_{B' \perp B_\sigma}^T (\beta'^{-1} \perp \sigma^{-1}) x_{B' \perp B_\sigma} = x_{B'}^T \beta'^{-1} x_{B'} + x_{B_\sigma}^T \sigma^{-1} x_{B_\sigma},$$

which, since we assume $x_B^T \beta^{-1} x_B = x_{B'}^T \beta'^{-1} x_{B'}$, implies the required equality. Applying again the fact that there exists A such that $A^T(\beta \perp \sigma)A = \beta' \perp \sigma$, we have

$$\begin{aligned}
x_{B' \perp B_\sigma}^T (\beta' \perp \sigma)^{-1} x_{B' \perp B_\sigma} &= x_{B' \perp B_\sigma}^T (A^T(\beta \perp \sigma)A)^{-1} x_{B' \perp B_\sigma} \\
&= (A^{T^{-1}} x_{B' \perp B_\sigma})^T (\beta^{-1} \perp \sigma^{-1}) (A^{T^{-1}} x_{B' \perp B_\sigma}) \\
&= x_{B \perp B_\sigma}^T (\beta^{-1} \perp \sigma^{-1}) x_{B \perp B_\sigma},
\end{aligned}$$

where the last equality is due to (4.4), so that Proposition 4.1.17 applies to the symmetric bilinear form $\beta^{-1} \perp \sigma^{-1}$ and the vectors $x_{B' \perp B_\sigma}$ and $A^{T^{-1}} x_{B' \perp B_\sigma}$, and there exists a vector v in a symmetric bilinear form module β'' and $A' \in O(\beta^{-1} \perp \sigma^{-1} \perp \beta'')$ such that

$$A'(x_{B \perp B_\sigma} \perp v) = A^{T^{-1}}(x_{B' \perp B_\sigma}) \perp v$$

and multiplying on the left by the matrix $A^T \perp id$, we have

$$(A^T \perp id) A'(x_{B \perp B_\sigma} \perp v) = x_{B' \perp B_\sigma} \perp v$$

which can be rewritten, using the fact that assignment $B \mapsto x_B$ respects block sum,

$$(A^T \perp id) A'(x_B \perp x_{B_\sigma} \perp v) = x_{B'} \perp x_{B_\sigma} \perp v,$$

so that, in view of Remark 4.1.14, (B, β) and (B', β') are stably equivalent and therefore have the same class in $GW_0(\underline{R})$ as required. $\qquad \square$

Proposition 4.1.19. *Under the same assumptions and notations as Proposition*

4.1.13, the group homomorphism

$$GW_0(\underline{R}) \to R \oplus GW_0(R)$$
$$[B, \beta] \mapsto (x_B^T \beta^{-1} x_B, [\beta])$$

as defined in Proposition 4.1.13 is surjective.

Proof. Recall the hyperbolic plane symmetric bilinear form module, defined by the symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Similarly to the proof of Lemma 4.1.15, since 2 is invertible, every element $a \in R$ can be written $x^T H x$ for some vector x, by choosing the entries of x to be $a/2$ and 1. In view of this, as well as the fact that $H^{-1} = H$, the element $(a, [H]) \in R \oplus GW_0(R)$ is in the image of the map for all $a \in R$.

Extending by linearity, the element $(b, -[H])$ is in the image of the map for all $b \in R$. In particular, taking $b = 0$ and summing the two elements, we have that $(a, 0)$ is in the image of the map for all $a \in R$. Moreover, the class $[T_{\frac{\beta}{2}}, \beta]$ is mapped to $(0, [\beta]) \in R \oplus GW_0(R)$. Combining all of this, we have that $(a, [\beta])$ is in the image of the map for all $a \in R$ and $[\beta] \in GW_0(R)$, so that the map is surjective as required. □

Theorem 4.1.20. *Let R be a commutative ring with $1/2 \in R$ and trivial involution. Assume also that finitely generated projective R-modules are free. Denote by $\underline{R}$ the Burnside form ring over R. Then the map*

$$GW_0(\underline{R}) \to R \oplus GW_0(R)$$
$$[B, \beta] \mapsto (x_B^T \beta^{-1} x_B, [\beta])$$

is a well-defined isomorphism of abelian groups.

Proof. Recalling Lemma 4.1.10 lets us view $\underline{R}$ as the form ring (4.1), Proposition 4.1.13 gives well-definedness and the fact that the map is a homomorphism. Proposition 4.1.18 is injectivity. Proposition 4.1.19 is surjectivity. □

4.2 The integer case

4.2.1 An affine action

We now turn our attention to the case of the Burnside form ring $\underline{\mathbb{Z}}$. The proof, very broadly speaking, has the same structure as in the 2 invertible case, but, as we will see, the fact 2 is not invertible in $\mathbb{Z}$ causes the need for some significant modifications. We begin with a lemma which proves every symmetric bilinear form over the integers is stably equivalent to a diagonal form, precisely, an orthogonal sum of $1's$ and $-1's$. This fact was essentially given, although without proof, in Example 2.3.8.

This will eventually allow us, as a simplification, to compute $GW_0(\underline{\mathbb{Z}})$ by considering only forms which restrict to diagonal symmetric bilinear forms.

Lemma 4.2.1 ([23], Théorème 1). *Let β be a symmetric bilinear form over $\mathbb{Z}$. Then there exist positive integers r and s and a symmetric bilinear form σ such that $\beta \perp \sigma$ is isometric to $r\langle 1 \rangle \perp s\langle -1 \rangle$. Therefore $GW_0(\underline{\mathbb{Z}})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. We say that a finitely generated projective (therefore free) symmetric bilinear form module (P, β) over $\mathbb{Z}$ is of type 1 if it contains a vector x such that $\beta(x, x)$ is an odd integer. Also, we say β is indefinite if $\beta(x, x)$ takes both positive and negative values. Per Theorem II.4.3 in [14], every type 1 indefinite symmetric bilinear form $\mathbb{Z}$ is isometric to one of the form $r\langle 1 \rangle \perp s\langle -1 \rangle$ for positive integers r and s. Furthermore, note that every symmetric bilinear form β over $\mathbb{Z}$ is stably equivalent to an indefinite type 1 form; to see this, consider the orthogonal sum $\beta \perp \langle 1 \rangle \perp \langle -1 \rangle$. This form is indefinite and of type 1, and is therefore isometric to $r\langle 1 \rangle \perp s\langle -1 \rangle$. Now consider $\beta' = (r-1)\langle 1 \rangle \perp (s-1)\langle -1 \rangle$. Then, we have that $\beta \perp \langle 1 \rangle \perp \langle -1 \rangle$ is isometric to $\beta' \perp \langle 1 \rangle \perp \langle -1 \rangle$; so take $\sigma = \langle 1 \rangle \perp \langle -1 \rangle$ to obtain the required result. The isomorphism to $\mathbb{Z} \oplus \mathbb{Z}$ is obtained by mapping β to (r, s) if it is stably isometric to $r\langle 1 \rangle \perp s\langle -1 \rangle$ and extending by linearity. $\square$

Corollary 4.2.2. *As in Proposition 4.1.7, denote by $\mathcal{P}(\underline{\mathbb{Z}})$ the abelian monoid of isometry classes of $\underline{\mathbb{Z}}$-quadratic forms. Also denote by $\mathcal{P}(\underline{\mathbb{Z}})_D$ the abelian monoid of isometry classes of $\underline{\mathbb{Z}}$-quadratic forms with restriction equal to $r\langle 1 \rangle \perp s\langle -1 \rangle$ for some non-negative integers r and s. Then the inclusion map*

$$\mathcal{P}(\underline{\mathbb{Z}})_D \to \mathcal{P}(\underline{\mathbb{Z}})$$

is cofinal; that is to say, for each $[B] \in \mathcal{P}(\underline{\mathbb{Z}})$, there exist $[D] \in \mathcal{P}(\underline{\mathbb{Z}})_D$ and $[B'] \in$

$\mathcal{P}(\mathbb{Z})$ with $B \perp B' \cong D$.

Proof. In view of the proof of Lemma 4.2.1, this can be done by taking B' to be any $\underline{\mathbb{Z}}$-quadratic form such that $R(B') = \langle 1 \rangle \perp \langle -1 \rangle$; such a form always exists because the map $\rho : \mathbb{Z}[C_2] \to \mathbb{Z}$ is surjective. If A is an invertible integer matrix such that $A^T(R(B) \perp \langle 1 \rangle \perp \langle -1 \rangle)A$ is a diagonal symmetric bilinear form, then we have $Q(A)(B \perp B') = D$ as required. $\square$

Proposition 4.2.3. *The inclusion map of abelian monoids*

$$\mathcal{P}(\underline{\mathbb{Z}})_D \to \mathcal{P}(\underline{\mathbb{Z}})$$

of Corollary 4.2.2 induces an isomorphism on Grothendieck groups.

Proof. Corollary II.1.3 in [29], together with Corollary 4.2.2, implies:

- $GW_0(\mathcal{P}(\underline{\mathbb{Z}})_D)$ is a subgroup of $GW_0(\mathcal{P}(\underline{\mathbb{Z}})) = GW_0(\underline{\mathbb{Z}})$,

- Every element in $GW_0(\underline{\mathbb{Z}})$ is of the form $[B] - [D]$, with $[D] \in \mathcal{P}(\underline{\mathbb{Z}})_D$.

This means that $[B] = [B'] \in GW_0(\underline{\mathbb{Z}})$ if and only if there exists a $\underline{\mathbb{Z}}$-quadratic form D with diagonal restriction such that $B \perp D \cong B' \perp D$. Therefore, we have proven the proposition if for any $\underline{\mathbb{Z}}$-quadratic form denoted B, we can find forms D and D' with diagonal restriction such that $B \perp D \cong D' \perp D$.

Taking again D to be a $\underline{\mathbb{Z}}$-quadratic form such that $R(D) = \langle 1 \rangle \perp \langle -1 \rangle$, we have that there exists A such that $A^T(R(B) \perp \langle 1 \rangle \perp \langle -1 \rangle)A = r\langle 1 \rangle \perp s\langle -1 \rangle$. Then we have $Q(A)(B \perp D) = D' \perp D$, where $R(D') = (r-1)\langle 1 \rangle \perp (s-1)\langle -1 \rangle$ as required. $\square$

Notation. In view of Proposition 4.2.3, from here onward, we write $GW_0(\underline{\mathbb{Z}})$ for the Grothendieck group of the abelian monoid of isometry classes of forms over $\underline{\mathbb{Z}}$ with diagonal restriction.

The natural step at this point is to look for an integer analogue of Proposition 4.1.11. However, in the case of $\underline{\mathbb{Z}}$, the choice $T^{\frac{\beta}{2}}$ is not available. One fairly natural alternative choice is β itself; the next lemma shows what happens when one does this. First, we reiterate some notation from the previous section.

Notation. Given a vector λ with entries $\lambda_1, \ldots, \lambda_n \in R^n$, denote the diagonal matrix with non-zero entries $\lambda_1(1-t), \ldots, \lambda_n(1-t)$ by M_λ.

Lemma 4.2.4. *Recalling we assume $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$ on $P = \mathbb{Z}^{r+s}$, for any element $A \in O(\beta)$, we have the following commutative diagram:*

$$
\begin{array}{ccc}
Q_\beta & \xrightarrow{\;Q(A)\;} & Q_\beta \\
\downarrow{\scriptstyle -\beta} & & \downarrow{\scriptstyle -\beta} \\
Q_0 & \xrightarrow{\;\varphi_A\;} & Q_0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
\mathbb{Z}^{r+s} & \xrightarrow{\;A^T\cdot(-)+w_A\;} & \mathbb{Z}^{r+s}
\end{array}
$$

where the notation is as follows. In the top vertical maps, β means β considered as the matrix form over $\underline{\mathbb{Z}}$ by using the map $\mathbb{Z} \to \mathbb{Z}[C_2]$ which sends n to n. First, noting that every element of Q_0 is of the form M_λ for some $\lambda \in \mathbb{Z}^{r+s}$, f is the bijection which sends M_λ to λ.

Second, $\varphi_A(M_\lambda)$ is the diagonal matrix with ith diagonal entry given by

$$
Q(A)(M_\lambda)_{ii} + (1-t)\left(-\frac{\beta_{ii}}{2} + \sum_{j=1}^{r} \frac{A_{ji}}{2} - \sum_{j=r+1}^{r+s} \frac{A_{ji}}{2} \right).
$$

Finally, w_A is the vector with i-th entry given by the integer

$$
-\frac{\beta_{ii}}{2} + \sum_{j=1}^{r} \frac{A_{ji}}{2} - \sum_{j=r+1}^{r+s} \frac{A_{ji}}{2}
$$

Proof. We first check that

$$
-\frac{\beta_{ii}}{2} + \sum_{j=1}^{r} \frac{A_{ji}}{2} - \sum_{j=r+1}^{r+s} \frac{A_{ji}}{2}
$$

is indeed an integer for all i. This follows if we can show that

$$
-\beta_{ii} + \sum_{j=1}^{r} A_{ji} - \sum_{j=r+1}^{r+s} A_{ji}
$$

is even for all i.

To see this, note that the fact that $A \in O(\beta)$ means that

$$\beta_{ii} = \sum_{j=1}^{r} A_{ji}^2 - \sum_{j=r+1}^{r+s} A_{ji}^2.$$

Since for any integer x, $x^2 \equiv x \bmod 2$, reducing both sides of this equation mod 2 then adding β_{ii} to both sides gives

$$0 = \beta_{ii} + \sum_{j=1}^{r} A_{ji} - \sum_{j=r+1}^{r+s} A_{ji} \in \mathbb{Z}/2\mathbb{Z},$$

which implies the required result.

We now check that the diagram commutes. Gtiven M_λ in Q_0, $Q(A)(M_\lambda + \beta) - \beta = \varphi_A(M_\lambda)$. We will check this on the first diagonal entry. The rest are completely analogous.

First, note that $Q(A)(\beta)_{11}$ is equal to

$$\sum_{j=1}^{r} \eta(A_{j1}) - \sum_{j=r+1}^{r+s} \eta(A_{j1}).$$

We also have that, for any integer n,

$$\eta(n) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = (1+t)\frac{n^2}{2} + (1-t)\frac{n}{2}$$

Therefore, $Q(A)(\beta)_{11}$ is equal to

$$\sum_{j=1}^{r}(1+t)\frac{A_{j1}^2}{2} + (1-t)\frac{A_{j1}}{2} - \sum_{j=r+1}^{r+s}(1+t)\frac{A_{j1}^2}{2} + (1-t)\frac{A_{j1}}{2},$$

which is in turn equal to

$$\frac{1+t}{2} + (1-t)\left(\sum_{j=1}^{r}\frac{A_{j1}}{2} - \sum_{j=r+1}^{r+s}\frac{A_{j1}}{2}\right),$$

since the fact that $A \in O(\beta)$ implies $\sum_{j=1}^{r} A_{j1}^2 - \sum_{j=r+1}^{r+s} A_{j1}^2 = \beta_{11} = 1$. Therefore,

$Q(A)(\beta)_{11} - \beta_{11}$ is equal to

$$\frac{1+t}{2} - 1 + (1-t)\left(\sum_{j=1}^{r}\frac{A_{j1}}{2} - \sum_{j=r+1}^{r+s}\frac{A_{j1}}{2}\right)$$

$$= (1-t)\left(\frac{-1}{2} + \sum_{j=1}^{r}\frac{A_{j1}}{2} - \sum_{j=r+1}^{r+s}\frac{A_{j1}}{2}\right),$$

which, since $\beta_{11} = 1$, gives the required result on the first diagonal entry. Applying the same calculation to the other diagonal entries proves that the top square commutes. Then commutativity of the bottom square follows from the facts that f is additive and that $f(Q(A)(M_\lambda)) = A^T \cdot \lambda$. $\qquad\square$

Definition 4.2.5. Let P be a finitely generated projective (therefore free) $\mathbb{Z}$-module and let (P, β) be a symmetric bilinear form module. Then, for all $A \in O(\beta)$ we set

$$\tilde{A}(x) := Ax + w_A,$$

where w_A is the vector defined in Lemma 4.2.4.

At this point in the 2 invertible case, the way forward was clear; the isometry classes of forms over $\underline{R}$ are identified with orbits of the linear action of an orthogonal group, so we used the bilinear form associated to this orthogonal group for our proof. Here, we still have an action of the orthogonal group of β, but it acts by a linear map then translation by the vector w_A; that is to say, the action is *affine* rather than linear. All is not lost, however; by analogy with the previous case, one looks for another map which encodes information about the orbits of this affine action. This is provided by the *trace map*, which we define in the next subsection.

4.2.2 The trace map

We'll define the trace in general way for the moment, since doing so makes it easier to check well-definedness on isometry classes. Later, we will write down and work with a more explicit formula which takes our assumptions on β into account. We make no claim of originality for this section, since the trace map as given here is the π_0-level of a trace map which is given on the level of spectra, in, for example, Section 3.2 of [5].

We require some preliminary definitions.

Definition 4.2.6. Let R be a commutative ring, let M be a free R-module with basis $e_1, \ldots, e_n$, and denote the dual basis of $M^* = \text{Hom}(M, R)$ by $e_1^*, \ldots, e_n^*$. Then the *coevaluation map* is the R-linear map

$$\nabla_M : R \to M \otimes_R M^*$$

defined by setting

$$\nabla_M(1) = \sum_{i=1}^n e_i \otimes e_i^*$$

A priori, ∇_M seems to depend on the choice of basis. In fact, this is not the case, which will be important for our purposes:

Lemma 4.2.7. *The coevaluation map is basis-independent.*

Proof. Let $e_1', \ldots, e_n'$ be another choice of basis and call the change of basis linear map f. We have $f(e_i) = e_i'$ which implies $f^{-1^*}(e_i^*) = e_i^{*\prime}$. Then

$$\sum_{i=1}^n e_i' \otimes e_i^{*\prime} = \sum_{i=1}^n f(e_i) \otimes f^{-1^*}(e_i^*) = \sum_{i=1}^n f(e_i) \otimes e_i^* \circ f^{-1} = \sum_{i=1}^n e_i \otimes e_i^*$$

as required. $\qquad\square$

Given $\underline{\mathbb{Z}}$-quadratic forms $q : P \to \mathbb{Z}[C_2]$ and $q' : P' \to \mathbb{Z}[C_2]$, we will also require a way to combine them, giving a $\underline{\mathbb{Z}}$-quadratic form on $P \otimes P'$:

Definition 4.2.8 (cf. page 28 of [22]). Given $\underline{\mathbb{Z}}$-quadratic forms $q : P \to \mathbb{Z}[C_2]$ and $q' : P' \to \mathbb{Z}[C_2]$, with associated bilinear forms β_q and $\beta_{q'}$, define

$$q \hat{\otimes} q' : P \otimes P' \to \mathbb{Z}[C_2]$$

by saying

$$(q \hat{\otimes} q')\left(\sum_{i=1}^n x_i \otimes x_i' \right) = \sum_{i=1}^n q(x_i) q'(x_i') + \sum_{1 \leq i < j \leq n} \tau(\beta_q(x_i, x_j) \beta_{q'}(x_i', x_j'))$$

Remark 4.2.9. The construction given in Definition 4.2.8 is a special case of a more general construction given on page 28 of [22]. Given two form rings (R, Λ) and (S, Γ), denote the categories of (R, Λ)-quadratic forms and (S, Γ)-quadratic forms respectively by $R\text{Mod}_\Lambda$ and $S\text{Mod}_\Gamma$. Given an R-module M and an S-module N,

the general version of the construction is a map

$$Q_\Lambda(M)\hat{\otimes}Q_\Gamma(N) \to Q_{\Lambda\hat{\otimes}\Gamma}(M \otimes N). \tag{4.5}$$

The source of the map (4.5) comes from viewing $Q_\Lambda(M)$ and $Q_\Gamma(N)$ as the C_2-groups of certain C_2-Mackey functors and taking their tensor product in the manner of Definition 3.3.3. Recalling Remark 3.1.20, (4.5) and the functor defined by the tensor product of form rings (Definition 3.3.3 applied to the case of form rings) are both part of the data of a "form functor" between the "form categories"

$$R\mathrm{Mod}_\Lambda \otimes S\mathrm{Mod}_\Gamma \to (R \otimes S)\mathrm{Mod}_{\Lambda\hat{\otimes}\Gamma},$$

where the left hand side is the tensor product of form categories given in Definition 2.34 of [22].

Definition 4.2.10. Given (P, q, β), a $\underline{\mathbb{Z}}$-quadratic form with P finitely-generated and projective, the *trace* $\mathrm{tr}_\beta(q)$ is given by the following value:

$$tr_\beta(q) = q\hat{\otimes}q((1 \otimes \beta_q^{-1})(\nabla_P(1))) \in \mathbb{Z}[C_2]$$

where we view β_q as a map from P to P^*. That is to say, choosing a basis $e_1, \ldots, e_n$ for P:

$$tr_\beta(q) = \sum_{i=1}^n q(e_i)q(\beta_q^{-1}(e_i^*)) + \sum_{1 \le i < j \le n} \tau(\beta_q(e_i)(e_j) \cdot e_i^*(\beta_q^{-1}(e_j^*)))$$

Lemma 4.2.11. *The trace map as given in Definition 4.2.10 is well-defined on isometry classes of non-degenerate $\underline{\mathbb{Z}}$-quadratic forms.*

Proof. Let $f : P' \xrightarrow{\cong} P$ be an isomorphism of finitely generated projective $\mathbb{Z}$-modules. Then consider the $\mathbb{Z}$-quadratic forms (P, q, β) and $(P', q \circ f, f^*\beta f)$, where we write $f^*\beta f$ rather than $\beta \circ (f \otimes f)$ because we are still viewing β as a map $P \to P^*$. We aim to show that these give the same trace. Lemma 4.2.7 implies that the trace is independent of the choice of basis. With that in mind, choose a basis $e_1, \ldots, e_n$ of P' and take $f(e_1), \ldots, f(e_n)$ as the basis of P. Then (P, q, β_q) is mapped to

$$\sum_{i=1}^n q(f(e_i)) \cdot q(\beta^{-1}(f(e_i)^*))) + \sum_{1 \le i < j \le n} \tau(\beta(f(e_i))(f(e_j)) \cdot f(e_i)^*(\beta^{-1}(f(e_j)^*))$$

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and $(P', q \circ f, f^*\beta f)$ is sent to

$$\sum_{i=1}^{n}(q \circ f)(e_i) \cdot (q \circ f)([f^*\beta f]^{-1}(e_i^*)) + \sum_{1 \le i < j \le n} \tau((f^*\beta f))(e_i)(e_j) \cdot e_i^*([f^*\beta f]^{-1}(e_j^*))$$

which are seen to be equal by the following calculations:

$$(q \circ f)([f^*\beta f]^{-1}(e_i^*)) = (q \circ f)(f^{-1}\beta^{-1}f^{*^{-1}}(e_i^*))$$
$$= q(\beta^{-1}(f(e_i)^*)),$$

$$f^*\beta f(e_i)(e_j) = (\beta f(e_i) \circ f)(e_j)$$
$$= \beta f(e_i)(f(e_j)),$$

$$e_i^*([f^*\beta f]^{-1}(e_j^*)) = (e_i^* \circ f^{-1})(\beta^{-1}f^{*^{-1}}(e_j^*))$$
$$= f(e_i)^*(\beta^{-1}f(e_j)^*).$$

which use the identities $f^*(\varphi) = \varphi \circ f$ and $f^{*^{-1}}(e_i^*) = f(e_i)^*$. □

Lemma 4.2.12. *Let $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$ and let B_q in Q_β as defined in Proposition 4.1.6 be the matrix of a $\underline{\mathbb{Z}}$-quadratic form q. Then in the language of matrices the trace map tr from Q_β to $\mathbb{Z}[C_2]$ is given by:*

$$B_q \mapsto \sum_{i=1}^{r} B_{q_{ii}}^2 + t \sum_{i=r+1}^{r+s} B_{q_{ii}}^2$$

Proof. Since the matrix of β is diagonal, the τ terms in the formula for the trace map vanish. Also, since $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$, β and β^{-1} are equal as matrices, so, when viewed as maps from $\mathbb{Z}^{r+s}$ to its dual, they are defined by the same formula. Precisely, choosing an orthogonal basis $e_1, \ldots, e_{r+s}$ for $(\mathbb{Z}^{r+s}, \beta)$:

$$\beta(e_i) = \begin{cases} e_i^T & \text{if } i \le r \\ -e_i^T & \text{if } i > r \end{cases}$$

and analogously for β^{-1}. Then the result follows from the facts that $B_{q_{ii}} = q(e_i)$ and $q(-x) = \eta(-1)q(x) = tq(x)$. □

Since the trace is well-defined on isometry classes, for any finitely generated projec-

tive (therefore free) $\underline{\mathbb{Z}}$-quadratic form $(\mathbb{Z}^{r+s}, q, \beta = r\langle 1 \rangle \perp s\langle -1 \rangle)$ and $A \in O(\beta)$, by Lemma 4.2.4, the following diagram commutes

$$
\begin{array}{ccccccc}
\mathbb{Z}^{r+s} & \xrightarrow{f^{-1}} & Q_0 & \xrightarrow{+\beta} & Q_\beta & \xrightarrow{tr} & \mathbb{Z}[C_2] \\
{\scriptstyle A^T(-)+w_A}\downarrow & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle Q(A)(-)} & & \downarrow{\scriptstyle 1} \\
\mathbb{Z}^{r+s} & \xrightarrow{f^{-1}} & Q_0 & \xrightarrow{+\beta} & Q_\beta & \xrightarrow{tr} & \mathbb{Z}[C_2]
\end{array}
\qquad (4.6)
$$

where the notations are the same as those given in Lemma 4.2.4; that is to say, the map $x \mapsto tr(f^{-1}(x) + \beta)$ is well-defined on the space of orbits of the affine action (Definition 4.2.5) of $O(\beta)$ on $\mathbb{Z}^{r+s}$.

We can modify the trace to obtain a map to $\mathbb{Z}$:

Lemma 4.2.13. *Let $(\mathbb{Z}^{r+s}, q, \beta = r\langle 1 \rangle \perp s\langle -1 \rangle)$ be a $\underline{\mathbb{Z}}$-quadratic form. Define the map $\tilde{tr}_\beta : Q_\beta \to ker(\rho) \subset \mathbb{Z}[C_2]$ by $\tilde{tr}_\beta(q) = tr_\beta(q) - tr_\beta(\beta)$, where we view β as an element of Q_β by the canonical lift which sends n to n for any $n \in \mathbb{Z}$. Then composition with the isomorphism $ker(\rho) \cong \mathbb{Z}$ which sends $n(1-t)$ to n gives a map $Q_\beta(P) \to \mathbb{Z}$.*

Proof. All that has to be checked is that $\tilde{tr}_\beta$ lands in $ker(\rho)$. Since ρ is a ring homomorphism, $\rho(\tilde{tr}_\beta(q)) = \rho(tr_\beta(q)) - \rho(tr_\beta(\beta))$. The trace of β is clearly $r + st$, so that $\rho(tr(\beta)) = r + s$.

Furthermore, again since ρ is a ring homomorphism:

$$
\begin{aligned}
\rho(tr_\beta(q)) &= \rho\Big(\sum_{i=1}^{r} B_{q_{ii}}^2 + t \sum_{i=r+1}^{r+s} B_{q_{ii}}^2 \Big) \\
&= \sum_{i=1}^{r} \rho(B_{q_{ii}})^2 + \sum_{i=r+1}^{r+s} \rho(B_{q_{ii}})^2,
\end{aligned}
$$

which is $r + s$ since $\rho(B_{q_{ii}}) = \pm 1$ for all i. So $\tilde{tr}(q)$ is in $ker(\rho)$ as required. $\qquad \square$

In the sequel, we will sometimes abuse notation by viewing $\tilde{tr}_\beta$ as a map to $\mathbb{Z}$. Now, analogously to the 2 invertible case, we use this to define a map $\mathbb{Z}^n \to \mathbb{Z}$:

Lemma 4.2.14. *Consider the symmetric bilinear form module $(\mathbb{Z}^{r+s}, \beta = r\langle 1 \rangle \perp s\langle -1 \rangle)$. Recalling the commutative diagram (4.6), the map $\mathbb{Z}^{r+s} \to \mathbb{Z}$ which sends x*

to $\tilde{tr}_\beta(f^{-1}(x) + \beta)$ *is given explicitly by*

$$x \mapsto 2\left(\beta(x,x) + \sum_{i=1}^{r+s} x_i\right)$$

Proof. The diagonal matrix $f^{-1}(x) + \beta$ has entries as follows:

$$(f^{-1}(x) + \beta)_{ii} = \begin{cases} x_i(1-t) + 1 & \text{if } 1 \leq i \leq r \\ x_i(1-t) - 1 & \text{if } r+1 \leq i \leq r+s \end{cases}$$

So that $tr_\beta(f^{-1}(x) + \beta)$ is equal to

$$\sum_{i=1}^{r}(x_i(1-t)+1)^2 + t\sum_{i=r+1}^{r+s}(x_i(1-t)-1)^2$$

which, multiplying out the brackets and using the fact $(1-t)^2 = 2(1-t)$, becomes

$$\sum_{i=1}^{r} 1 + (1-t)(2x_i + 2x_i^2) + t\sum_{i=r+1}^{r+s} 1 + (1-t)(-2x_i + 2x_i^2)$$

$$= \sum_{i=1}^{r} 1 + (1-t)(2x_i + 2x_i^2) + \sum_{i=r+1}^{r+s} t + (1-t)(2x_i - 2x_i^2)$$

which is equal to

$$r + st + 2\left(\beta(x,x) + \sum_{i=1}^{r+s} x_i\right)(1-t).$$

Subtracting $tr_\beta(\beta)$ gives $\tilde{tr}_\beta(f^{-1}(x) + \beta)$, but $tr_\beta(\beta) = r + st$, so that we have the required value for $\tilde{tr}_\beta(f^{-1}(x) + \beta)$. $\qquad\square$

Note that $\beta(x,x) + \sum_{i=1}^{r+s} x_i$ is always an even number, since it is the sum of terms of the form $x_i^2 + x_i = x_i(x_i + 1)$ and $-x_i^2 + x_i = x_i(1 - x_i)$ which are always even. This means that, from the point of view of classifying forms, there is no harm in dividing the value of $\tilde{tr}(f^{-1}(x) + \beta)$ by 4:

Definition 4.2.15. Given $x \in (\mathbb{Z}^{r+s}, \beta)$, where $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$ is a symmetric bilinear form, call the value

$$T_\beta(x) := \frac{\tilde{tr}(f^{-1}(x) + \beta)}{4} = \frac{1}{2}\left(\beta(x,x) + \sum_{i=1}^{r+s} x_i\right) \in \mathbb{Z}$$

the *trace signature.*

By well-definedness of the trace map, recalling that $f : Q_0 \to \mathbb{Z}^n$ sends a matrix $B \in Q_0$ to the factors of $(1 - t)$ in its diagonal entries, writing x_B for the vector $f(B - \beta) \in \mathbb{Z}^{r+s}$, and, in accordance with Proposition 4.2.3, viewing $GW_0(\underline{\mathbb{Z}})$ as the Grothendieck-Witt group of $\underline{\mathbb{Z}}$-quadratic forms with restriction equal to $r\langle 1 \rangle \perp s\langle -1 \rangle$, we have a well-defined function

$$GW_0(\underline{\mathbb{Z}}) \to \mathbb{Z} \oplus GW_0(\mathbb{Z})$$
$$[B, \beta] \mapsto (T_\beta(x_B), [\beta])$$

which we endeavour to show is a group homomorphism, injective, and surjective. The first of these is relatively simple to prove:

Lemma 4.2.16. *The function*

$$GW_0(\underline{\mathbb{Z}}) \to \mathbb{Z} \oplus GW_0(\mathbb{Z})$$
$$[B, \beta] \mapsto (T_\beta(x_B), [\beta])$$

is a group homomorphism, where T_β is the trace signature defined in Definition 4.2.15.

Proof. Per Proposition 4.2.3, let $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$ and $\beta' = r'\langle 1 \rangle \perp s'\langle -1 \rangle$. Since in the language of matrices the group operation in $GW_0(\underline{\mathbb{Z}})$ is the block sum, we have that $[B, \beta] + [B', \beta']$ is mapped to:

$$
\begin{aligned}
T_\beta(x_B) + T_{\beta'}(x_{B'}) &= \frac{1}{2}\left(\beta(x_B, x_B) + \sum_{i=1}^{r+s} x_{B_i} + \beta'(x_{B'}, x_{B'}) + \sum_{i=1}^{r'+s'} x_{B'_i}\right) \\
&= \frac{1}{2}(\beta \perp \beta'(x_B \perp x_{B'}, x_B \perp x_{B'}) + \sum_{i=1}^{r+s+r'+s'} (x_B \perp x_{B'})_i) \\
&= T_{\beta \perp \beta'}(x_B \perp x_{B'})
\end{aligned}
$$

where $x_B \perp x_{B'}$ is the concatenation of x_B and $x_{B'}$ defined in the proof of Proposition 4.1.13. The expression

$$T_{\beta \perp \beta'}(x_B \perp x_{B'})$$

is the image of $[B \perp B', \beta \perp \beta']$, so that we have a monoid homomorphism on isometry classes, and therefore a group homomorphism on the Grothendieck-Witt

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group.

$\square$

4.2.3 The proof of injectivity

For most of the results in this section, we assume that the symmetric bilinear form $\beta = r\langle 1\rangle \perp s\langle -1\rangle$ has signature 1. We will use the following lemma which shows that, for our purposes, it is safe to do so:

Lemma 4.2.17. *Let $(\mathbb{Z}^{r+s}, \beta = r\langle 1\rangle \perp s\langle -1\rangle)$ be a finitely generated projective (therefore free) symmetric bilinear form module over the integers. Then there exists a symmetric bilinear form module (P', β') such that $(P \oplus P', \beta \perp \beta')$ has signature 1 as a symmetric bilinear form module.*

Proof. Take $P' = \mathbb{Z}^{1+r+s}, \beta = (1+s)\langle 1\rangle \perp r\langle -1\rangle$. $\square$

In what follows, let v_β be the vector in P with entries given by the diagonal entries of the Gram matrix of the symmetric bilinear form $\beta = r\langle 1\rangle \perp s\langle -1\rangle$.

Lemma 4.2.18. *Let $(\mathbb{Z}^{r+s}, \beta)$ be a finitely generated projective (therefore free) symmetric bilinear form module over $\mathbb{Z}$. Assume $\beta = r\langle 1\rangle \perp s\langle -1\rangle$ has signature 1 as a symmetric bilinear form. Then we have an orthogonal decomposition*

$$(P, \beta) \cong (\mathbb{Z}v_\beta, \beta|_{\mathbb{Z}v_\beta}) \perp ((\mathbb{Z}v_\beta)^\perp, \beta|_{(\mathbb{Z}v_\beta)^\perp})$$

of symmetric bilinear forms over $\mathbb{Z}$.

Proof. Applying the symmetric bilinear form β to the vector v_β, we obtain

$$\beta(v_\beta, v_\beta) = r(1^2) - s(-1)^2 = r - s$$

which is equal to the signature of β, which is in turn equal to 1 by assumption. Thus, since 1 is a unit, we can apply the orthogonal decomposition lemma (I.3.1 in [14]) to obtain our result. $\square$

Lemma 4.2.19. *In the situation of Lemma 4.2.18, the orthogonal decomposition*

$$(\mathbb{Z}^{r+s}, \beta) \cong (\mathbb{Z}v_\beta, \beta|_{\mathbb{Z}v_\beta}) \perp ((\mathbb{Z}v_\beta)^\perp, \beta|_{(\mathbb{Z}v_\beta)^\perp})$$

gives rise to the following decomposition of the trace signature; given a vector $x \in \mathbb{Z}^{r+s}$, there exist vectors $r_x v_\beta \in \mathbb{Z} v_\beta$ and $u_x \in (\mathbb{Z} v_\beta)^\perp$, where $r_x \in \mathbb{Z}$, such that $T_\beta(x) = T_\beta|_{\mathbb{Z} v_\beta}(r_x v_\beta) + \frac{1}{2}\beta|_{(\mathbb{Z} v_\beta)^\perp}(u_x, u_x)$.

Proof. Using the orthogonal decomposition of Lemma 4.2.18:

$$
\begin{aligned}
T_\beta(x) &= \frac{1}{2}\Big(\beta(x,x) + \sum_{i=1}^{r+s} x_i\Big) \\
&= \frac{1}{2}(\beta(x + v_\beta, x)) = \frac{1}{2}(\beta(x,x) + \beta(v_\beta, x)) \\
&= \frac{1}{2}(\beta|_{\mathbb{Z} v_\beta}(r_x v_\beta, r_x v_\beta) + \beta|_{(\mathbb{Z} b_\beta)^\perp}(u_x, u_x)) + \beta(v_\beta, r_x v_\beta + u_x) \\
&= \frac{1}{2}(\beta|_{\mathbb{Z} v_\beta}(r_x v_\beta, r_x v_\beta) + \beta(v_\beta, r_x v_\beta) + \beta|_{(\mathbb{Z} b_\beta)^\perp}(u_x, u_x))
\end{aligned}
$$

for some $r_x v_\beta \in \mathbb{Z} v_\beta$ and $u_x \in (\mathbb{Z} v_\beta)^\perp$. The last equality is true because u_x is in the orthogonal complement of $\mathbb{Z} v_\beta$, so that $\beta(v_\beta, u_x) = 0$. We also have that $\beta(v_\beta, r_x v_\beta) = r_x$, so that we have the stated decomposition. $\square$

Lemma 4.2.19 implies we have the formula

$$
T_\beta(x) = \frac{r_x(r_x + 1)}{2} + \frac{\beta(u_x, u_x)}{2} \tag{4.7}
$$

where r_x is an integer determined by x and u_x is in the orthogonal complement of $\mathbb{Z} v_\beta$. The moral is that one can arrange things such that our theory differs from the classical theory of symmetric bilinear forms only in a rank 1 subspace. In fact, Lemma 4.2.21 shows that, after stabilisation and up to isometry, one can make things wholly classical. For the sake of concision we first define some notation, given rise to by (4.7):

Notation. Let $x \in (\mathbb{Z}^{r+s}, \beta)$, but make no assumptions about the signature of β. Then set $r_x = \beta(v_\beta, x) = \sum_{i=1}^{r+s} x_i$.

We have the following preliminary lemma:

Lemma 4.2.20. *Let $(\mathbb{Z}^{r+s}, \beta)$ be a free symmetric bilinear form module over $\mathbb{Z}$, and assume $\beta = r\langle 1 \rangle \perp s \langle -1 \rangle$ has signature 1. For $x \in \mathbb{Z}^{r+s}$, use Lemma 4.2.19 to write $x = r_x v_\beta + u_x$. Let $T \in O(\beta)$ be the linear map which acts by -1 on the subspace generated by v_β and the identity elsewhere. Then, recalling Definition*

4.2.5, the affine action $\tilde{T}(x)$ is given by the formula

$$\tilde{T}(x) := Tx + w_T = (-1 - r_x)v_\beta + u_x,$$

where w_T is the vector defined in the statement of Lemma 4.2.4.

Proof. In any basis consisting of v_β and a basis for $(\mathbb{Z}v_\beta)^\perp$, the matrix of T is diagonal, with -1 in the first diagonal entry and 1 in the rest. We have

$$Tx + w_T = T(r_x v_\beta + u_x) + w_T = -r_x v_\beta + u_x + w_T$$

so that, if we can show $w_T = -v_\beta$, we are done.

Applying the formulas from Lemma 4.2.4, we have that w_T has i-th entry given by the integer

$$-\frac{\beta_{ii}}{2} + \sum_{j=1}^{r} \frac{M_{ji}}{2} - \sum_{j=r+1}^{r+s} \frac{M_{ji}}{2},$$

where M is the matrix of T and where this expression is indeed an integer by the calculation at the beginning of the proof of Lemma 4.2.4.

We determine the entries of the vector w_T in our basis. If $i = 1$, we have $w_{T_1} = -\frac{1}{2} + -\frac{1}{2} = -1$. If $i \neq 1$ and $i \leq r$, we have $w_{T_i} = -\frac{1}{2} + \frac{1}{2} = 0$. If $i > r$, we have $w_{T_i} = \frac{1}{2} - \frac{1}{2} = 0$. Putting these together, in our basis w_T has -1 in its first entry and 0 elsewhere, so that $w_T = -v_\beta$ as required.

$\square$

Lemma 4.2.21. *Let (P, β) be a symmetric bilinear form module over $\mathbb{Z}$, with signature 1, and $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$. For any x and y in P, there exists $v \in (\mathbb{Z}^2, \beta \perp \langle 1 \rangle \perp \langle -1 \rangle)$, $A_0 \in O(\beta)$, and $A_1 \in O(\beta \perp \langle 1 \rangle \perp \langle -1 \rangle)$ such that*

$$r_{\tilde{A}_1(\tilde{A}_0(x) \perp v)} = r_{y \perp v}.$$

Proof. First note that, since the symmetric bilinear space $(\mathbb{Z}^2, \langle 1 \rangle \perp \langle -1 \rangle)$ has signature 0, the orthogonal sum $\beta \perp \langle 1 \rangle \perp \langle -1 \rangle$ still has signature 1 and therefore a decomposition in the form of Lemma 4.2.18. Next, note that for $v \in (\mathbb{Z}^2, \langle 1 \rangle \perp$

$\langle -1 \rangle),$

$$r_{x \perp v} = \beta \perp \langle 1 \rangle \perp \langle -1 \rangle (v_{\beta \perp \langle 1 \rangle \langle -1 \rangle}, x \perp v)$$
$$= \beta(v_\beta, x) + \langle 1 \rangle \perp \langle -1 \rangle (v_{\langle 1 \rangle \perp \langle -1 \rangle}, v)$$
$$= r_x + r_v$$

and similarly for $r_{y \perp v}$.

Now, consider the integer $-r_x - r_y - 1$. If this integer is even, we can make the following choices:

$$v = \begin{bmatrix} \frac{-r_x - r_y - 1}{2} \\ 0 \end{bmatrix},$$

$$A_0 = \mathrm{id},$$

$$A_1 = \mathrm{diag}(-1, 1, \ldots, 1).$$

Applying Lemma 4.2.20 to the linear map $A_1 = \mathrm{diag}(-1, 1, \ldots, 1)$ shows that these choices fulfil the required criteria.

On the other hand, if $-1 - r_x - r_y$ is odd, then $r_x - r_y$ is even. In this case, we can choose

$$v = \begin{bmatrix} \frac{r_x - r_y}{2} \\ 0 \end{bmatrix},$$

$$A_0 = \mathrm{diag}(-1, 1, \ldots, 1),$$
$$A_1 = \mathrm{diag}(-1, 1, \ldots, 1),$$

and, applying Lemma 4.2.20 to both A_0 and A_1 in this case, we have the required result. $\qquad \square$

We will make much use of the following lemma:

Lemma 4.2.22. *Let $(\mathbb{Z}^{r+s}, \beta = r\langle 1 \rangle \perp s\langle -1 \rangle)$ be a symmetric bilinear form module with signature 1. Then for any vector $u \in (\mathbb{Z}v_\beta)^\perp$, $\beta(u, u)$ is an even integer.*

Proof. Applying Lemma 4.2.19 and noting that $r_u = 0$, we obtain that the trace signature of Definition 4.2.15 is given by

$$T_\beta(u) = \frac{\beta(u, u)}{2},$$

which implies that $\beta(u, u)$ must be even because $T_\beta(u)$ is integer-valued. $\square$

We now, similarly to the proof of injectivity for the 2 invertible case, have a series of more general lemmas which, due to Lemma 4.2.22, apply in particular to our situation. We formulate them generally to avoid introducing too much new notation.

Lemma 4.2.23. *Let (P, β) be a symmetric bilinear form module over $\mathbb{Z}$. Let x and y in P be such that $\beta(x, x) = \beta(y, y) \in 2\mathbb{Z}$. Then there exists a vector v_{xy} in the hyperbolic plane $(\mathbb{Z}^2, H)$ such that $0 = (\beta \perp H)(x \perp v_{xy}, x \perp v_{xy}) = \beta \perp H(y \perp v_{xy}, y \perp v_{xy}) = 0$.*

Proof. The hyperbolic plane H, has Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that the vector $\begin{bmatrix} u \\ v \end{bmatrix}$ is mapped to

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 2uv$$

By Lemma 4.2.22, we can write $\beta(x, x) = \beta(y, y) = 2n$ for some integer n. Then setting $v_{xy} = \begin{bmatrix} -n \\ 1 \end{bmatrix} \in (\mathbb{Z}^2, H)$ gives the required result. $\square$

Definition 4.2.24. Let v be a vector in $\mathbb{Z}^n$. We say v is *indivisible* if there exists no vector v' and $n \in \mathbb{Z}$ such that $v = nv'$.

Lemma 4.2.25. *If a vector $v \in \mathbb{Z}^n$ is indivisible, it forms part of a basis for $\mathbb{Z}^n$.*

Proof. The indivisibility of v implies that the quotient group $\mathbb{Z}^n/\langle v \rangle$ has no non-zero element of finite order, so that it must be free abelian. Then the short exact sequence

$$0 \to \langle v \rangle \to \mathbb{Z}^n \to \mathbb{Z}^n/\langle v \rangle \to 0$$

is split, so that combining v and a basis for $\mathbb{Z}^n/\langle v \rangle$ gives a basis for $\mathbb{Z}^n$ containing v as required. $\square$

Lemma 4.2.26. *Let (P, β) be a symmetric bilinear form module over $\mathbb{Z}$. Let x and y be in P such that $\beta(x, x) = \beta(y, y) = 0$. Then there exists $v_I \in (\mathbb{Z}^2, H)$ such that $x \perp v_I$ and $y \perp v_I$ are indivisible, and such that $(\beta \perp H)(x \perp v_I, x \perp v_I) = (\beta \perp H)(y \perp v_I, y \perp v_I) = 0$.*

Proof. Set $v_I = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $x \perp v_I$ and $y \perp v_I$ are indivisible since a vector in $\mathbb{Z}^n$ for any n is indivisble if and only if the greatest common divisor of all its entries is 1. Also we have $(\beta \perp H)(x \perp v_I, x \perp v_I) = (\beta \perp H)(y \perp v_I, y \perp v_I) = 0$ because $H(v_I, v_I) = 0$. $\qquad\square$

We require the following definition:

Definition 4.2.27. Given a basis $e_1, \ldots, e_n$ for a free symmetric bilinear form module (P, β), the *dual basis* $e_1^\sharp, \ldots, e_n^\sharp$ of P is defined by the condition $\beta(e_i, e_j^\sharp) = \delta_{ij}$.

Lemma 4.2.28. *Given a basis $e_1, \ldots, e_n$ for a free symmetric bilinear form module (P, β), the dual basis $e_1^\sharp, \ldots, e_n^\sharp$ of P exists and is unique.*

Proof. The non-degeneracy of β implies the map $\varphi_\beta(v) = \beta(v, -)$ is an isomorphism from P to P^*. Now consider $e_1^*, \ldots, e_n^*$, the basis of P^* defined by setting

$$e_i^*(e_j) = \delta_{ij}$$

Now, set $e_i^\sharp = \varphi_\beta^{-1}(e_i^*)$ of P. Then we have

$$\beta(e_i, e_j^\sharp) = \beta(e_i, \varphi_\beta^{-1}(e_j^*)) = e_j^*(e_i) = \delta_{ij},$$

which proves existence.

For uniqueness, let $f_1, \ldots, f_n$ be another basis of P with $\beta(e_i, f_j) = \delta_{ij}$. Since $e_1^\sharp, \ldots, e_n^\sharp$ is a basis, one can write

$$f_j = \sum_{k=1}^n c_{jk} e_k^\sharp$$

for all $1 \le j \le n$. Applying $\beta(e_i, -)$ to both sides of this equation, we obtain

$$\delta_{ij} = \beta(e_i, f_j) = \sum_{k=1}^n c_{jk} \beta(e_i, e_k^\sharp) = c_{ji}$$

so that $f_j = e_j^\sharp$ for all $1 \le j \le n$. $\qquad\square$

We will require Definition 4.2.27 and Lemma 4.2.28 to prove the following lemma.

Lemma 4.2.29. *Let (P, β) be a projective (therefore free) symmetric bilinear form module over $\mathbb{Z}$ such that $\beta(x, x) \in 2\mathbb{Z}$ for all $x \in P$. Assume an element $y \in P$ is part of a basis for P, and that $\beta(y, y) = 0$. Then y is part of a basis for a hyperbolic plane which is an orthogonal summand of (P, β).*

Proof. Consider the element $y^\sharp$ in the dual basis. Then $\beta(y, y^\sharp) = 1$ by definition, so that the subspace spanned by y and $y^\sharp$ has Gram matrix of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}$$

where $a = \beta(y^\sharp, y^\sharp)$ is an even integer by assumption. This means we can take as a new basis y and $y^\sharp - \frac{a}{2} y$ which has Gram matrix
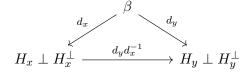
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that y is part of a basis for a hyperbolic plane H_y, and the orthogonal decomposition lemma ([14], I.3.1) implies $H_y \perp (H_y)^\perp \cong (P, \beta)$ as required. $\square$
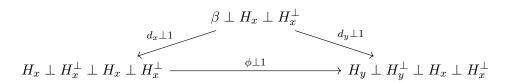
The last of our general lemmas about symmetric bilinear forms is the following.

Lemma 4.2.30. *Let (P, β) be a finitely generated projective (therefore free) symmetric bilinear form module over $\mathbb{Z}$, and let x and y both be part of bases for hyperbolic planes H_x and H_y which are orthogonal summands of (P, β). Then there exists $A \in O(\beta \perp \beta)$ such that $A(x \perp 0) = y \perp 0$.*

Proof. We have the following commutative diagram of isomorphisms:

$$
\begin{array}{ccc}
 & \beta & \\
 d_x \swarrow & & \searrow d_y \\
H_x \perp H_x^\perp & \xrightarrow{\; d_y d_x^{-1} \;} & H_y \perp H_y^\perp
\end{array}
$$

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Now add a copy of $H_x \perp H_x^\perp \cong \beta$ everywhere to get a new diagram

$$\beta \perp H_x \perp H_x^\perp$$

$$H_x \perp H_x^\perp \perp H_x \perp H_x^\perp \xrightarrow{\quad \phi \perp 1 \quad} H_y \perp H_y^\perp \perp H_x \perp H_x^\perp$$

with arrows $d_x \perp 1$ (left) and $d_y \perp 1$ (right) from the top.

where the map ϕ from $H_x \perp H_x^\perp \perp H_x$ to $H_y \perp H_y^\perp \perp H_x$ is defined as follows; first observe that since H_x is isometric to H_y, we have a third decomposition $d_{xy} : \beta \xrightarrow{\cong} H_y^\perp \perp H_x$ and therefore an isometry $d_{xy}d_x^{-1} : H_x \perp H_x^\perp \to H_y^\perp \perp H_x$. Then set $\phi = \psi \perp d_{xy}d_x^{-1}$ where ψ is the isometry from H_x to H_y that sends x to y and x' to y', where x' and y' are the other elements of the bases of H_x and H_y, respectively. Now consider the composition $(d_y \perp 1)^{-1} \circ (\phi \perp 1) \circ (d_x \perp 1)$. This is an isometry from $\beta \perp H_x \perp H_x^\perp$ to itself which sends $x \perp 0$ to $y \perp 0$. Composition with the isometry $H_x \perp H_x^\perp \cong \beta$ gives the required result. $\square$

We now combine Lemmas 4.2.23, 4.2.25, 4.2.26, 4.2.29, 4.2.30:

Proposition 4.2.31. *Let (P, β) be a finitely generated projective (therefore free) symmetric bilinear form module over $\mathbb{Z}$ with $\beta(u, u) \in 2\mathbb{Z}$ for all $u \in P$. Let x and y in P be such that $\beta(x, x) = \beta(y, y)$. Then there exists a vector v in a symmetric bilinear form module (P', β') and $A \in O(P \oplus P', \beta \perp \beta')$ such that $A(x \perp v) = y \perp v$.*

Proof. Applying Lemma 4.2.23 to x and y, we have that there exist $x \perp v_{xy}, y \perp v_{xy} \in (P \oplus \mathbb{Z}^2, \beta \perp H)$ with $(\beta \perp H)(x \perp v_{xy}, x \perp v_{xy}) = (\beta \perp H)(y \perp v_{xy}, y \perp v_{xy}) = 0$. Now, apply Lemma 4.2.26 to $x \perp v_{xy}$ and $y \perp v_{xy}$ to obtain $x \perp v_{xy} \perp v_I$ and $y \perp v_{xy} \perp v_I$, indivisible vectors in $P \oplus \mathbb{Z}^4$ with $(\beta \perp H \perp H)(x \perp v_{xy} \perp v_I, x \perp v_{xy} \perp v_I) = (\beta \perp H \perp H)(y \perp v_{xy} \perp v_I, y \perp v_{xy} \perp v_I) = 0$. Applying Lemma 4.2.25 and 4.2.29 to these indivisible vectors, we have that $x \perp v_{xy} \perp v_I$ is part of a basis for a hyperbolic plane H_x which is an orthogonal summand of $(P \oplus \mathbb{Z}^4, \beta \perp H \perp H)$, and similarly for $x \perp v_{xy} \perp v_I$ and a hyperbolic summand H_y. Then Lemma 4.2.30 implies that there exists $A \in O(\beta \perp H \perp H \perp \beta \perp H \perp H)$ such that $A(x \perp v_{xy} \perp v_I \perp 0) = y \perp v_{xy} \perp v_I \perp 0$. Therefore, in the statement of the proposition, taking $P' = \mathbb{Z}^4 \oplus P \oplus \mathbb{Z}^4, \beta' = H \perp H \perp \beta \perp H \perp H$, and $v = v_{xy} \perp v_I \perp 0$, we have the required result. $\square$

We now apply Proposition 4.2.31 to our specific situation:

Corollary 4.2.32. *Let $(P, \beta = r\langle 1 \rangle \perp s\langle -1 \rangle)$ have signature 1 as a projective (therefore free) symmetric bilinear form module. Recalling the orthogonal decomposition*

$$(P, \beta) \cong (\mathbb{Z}v_\beta, \beta|_{\mathbb{Z}v_\beta}) \perp ((\mathbb{Z}v_\beta)^\perp, \beta|_{(\mathbb{Z}v_\beta)^\perp})$$

of Lemma 4.2.18, if $x, y \in P$ are such that $\beta(u_x, u_x) = \beta(u_y, u_y)$, then there exists a vector v in a symmetric bilinear form module (P', β') and $A \in O(\beta|_{(\mathbb{Z}v_\beta)^\perp} \perp \beta')$ such that $A(u_x \perp v) = u_y \perp v$.

Proof. Lemma 4.2.22 implies that we can apply Proposition 4.2.31 to obtain the required result. $\qquad\square$

Proposition 4.2.33. *Let $(P, \beta = r\langle 1 \rangle \perp s\langle -1 \rangle)$ be a finitely generated projective (therefore free) symmetric bilinear form module over $\mathbb{Z}$ with signature 1. Write $\beta_1 = \beta \perp \langle 1 \rangle \perp \langle -1 \rangle$. Then, recalling the trace signature T_β from Definition 4.2.15, for any x and y such that $T_\beta(x) = T_\beta(y)$, there exists:*

- $v_0 \in (\mathbb{Z}^2, \langle 1 \rangle \perp \langle -1 \rangle)$,

- $A_0 \in O(\beta)$,

- $A_1 \in O(\beta \perp \beta_1)$,

- *a vector v_2 in a symmetric bilinear form module (P_2, β_2),*

- $A' \in O(\beta_1 \perp \beta_2)$,

such that

$$\tilde{A}'(\tilde{A}_1(\tilde{A}_0(x) \perp v_0) \perp v_2)) = y \perp v_0 \perp v_2. \tag{4.8}$$

Proof. Lemma 4.2.21 applied to x and y gives $v_0 \in (\mathbb{Z}^2, \langle 1 \rangle \perp \langle -1 \rangle)$, $A_0 \in O(\beta)$ and $A_1 \in O(\beta \perp \langle 1 \rangle \perp \langle -1 \rangle)$ such that

$$r_{\tilde{A}_1(\tilde{A}_0(x) \perp v_0)} = r_{y \perp v_0}, \tag{4.9}$$

recalling the orthogonal decomposition of Lemma 4.2.18. The assumption that $T_\beta(x) = T_\beta(y)$ implies that

$$T_{\beta_1}(\tilde{A}_1(\tilde{A}_0(x) \perp v_0)) = T_{\beta_1}(y \perp v_0),$$

which, taken together with (4.9) and (4.7), implies

$$\beta_1(u_{\tilde{A}_1(\tilde{A}_0(x)\perp v_0)}, u_{\tilde{A}_1(\tilde{A}_0(x)\perp v_0)}) = \beta_1(u_{y\perp v_0}, u_{y\perp v_0}).$$

Now, applying Corollary 4.2.32, there exists a vector v_2 in a symmetric bilinear form module (P_2, β_2) and $A_2 \in O(\beta_1|_{(\mathbb{Z}v_{\beta_1})^\perp} \perp \beta_2)$ such that

$$A_2(u_{\tilde{A}_1(\tilde{A}_0(x)\perp v_0)} \perp v_2) = u_{y\perp v_0} \perp v_2.$$

Which means that, setting $A' = \mathrm{id} \perp A_2 \in O(\beta_1 \perp \beta_2)$, A' takes $\tilde{A}_1(\tilde{A}_0(x) \perp v_0) \perp v_2$ to $y \perp v_0 \perp v_2$, as required. $\qquad\square$

We are now in position to state and prove the results concerning injectivity. We recall some notation. Let a $\underline{\mathbb{Z}}$-quadratic form be represented by a matrix B, and with associated symmetric bilinear form $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$. Recalling Lemma 4.2.4, we have the commutative diagram

$$
\begin{array}{ccc}
Q_\beta & \xrightarrow{\;Q(A)\;} & Q_\beta \\
\downarrow{\scriptstyle -\beta} & & \downarrow{\scriptstyle -\beta} \\
Q_0 & \xrightarrow{\;\varphi_A\;} & Q_0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
\mathbb{Z}^{r+s} & \xrightarrow{A^T\cdot(-)+w_A} & \mathbb{Z}^{r+s}
\end{array}
\qquad (4.10)
$$

and we write x_B for the vector $f(B - \beta)$.

We have the following result, which gives injectivity under the assumption that β has signature 1.

Proposition 4.2.34. *Let (B, β) and (B', β) be $\underline{\mathbb{Z}}$-quadratic forms, where $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$ has signature 1. If $T_\beta(x_B) = T_\beta(x'_B)$, then $[B] = [B']$ in $GW_0(\mathbb{Z})$.*

Proof. Applying Proposition 4.2.33 to the vectors x_B and $x_{B'}$, we can write the equation (4.8), replacing x with x_B and y with $x_{B'}$ In the orbit space of the action of $O(\beta \perp \langle 1 \rangle \perp \langle -1 \rangle)$, $[\tilde{A}_1(\tilde{A}_0(x_B) \perp v_0)] = [x_B \perp v_0]$. The equation (4.8) then implies that $x_B \perp v_0 \perp v_2$ and $x_{B'} \perp v_0 \perp v_2$ are in the same orbit. In view of the commutative diagram (4.10), this implies that the $\underline{\mathbb{Z}}$-quadratic forms $B \perp (f^{-1}(v_0) + \beta) \perp (f^{-1}(v_2) + \beta)$ and $B' \perp (f^{-1}(v_0) + \beta) \perp (f^{-1}(v_2) + \beta)$ are isometric; that is, B and B' are stably isometric, so $[B] = [B']$ in $GW_0(\mathbb{Z})$. $\qquad\square$

Finally, we invoke Lemma 4.2.17 to remove the assumption that the signature of β

is 1:

Corollary 4.2.35. *Let (B, β) and (B', β) be $\underline{\mathbb{Z}}$-quadratic forms, where $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$. If $T_\beta(x_B) = T_\beta(x'_B)$, then $[B] = [B']$ in $GW_0(\mathbb{Z})$.*

Proof. By Lemma 4.2.17, we have vectors $x_B \perp 0$ and $x_{B'} \perp 0$ in the symmetric bilinear form module $(\mathbb{Z}^{r+s} \oplus \mathbb{Z}^{1+r+s}, \beta \perp (1+s)\langle 1 \rangle \perp r\langle -1 \rangle)$ which has signature 1. Writing $\beta' = \beta \perp (1+s)\langle 1 \rangle \perp r\langle -1 \rangle$ for brevity, we have

$$T_{\beta'}(x_B \perp 0) = T_\beta(x_B) = T_\beta(x_{B'}) = T_{\beta'}(x_{B'} \perp 0),$$

so that we can apply Proposition 4.2.34 to the vectors $x_B \perp 0$ and $x_{B'} \perp 0$. This implies that $[B] + [\beta] = [B'] + [\beta]$ in $GW_0(\mathbb{Z})$, so that $[B] = [B']$ in $GW_0(\mathbb{Z})$, as required. $\qquad\square$

Proposition 4.2.36. *The map*

$$GW_0(\underline{\mathbb{Z}}) \to \mathbb{Z} \oplus GW_0(\mathbb{Z})$$
$$[B, \beta] \mapsto (T_\beta(x_B), [\beta])$$

is injective, where T_β is the trace signature given in Definition 4.2.15.

Proof. Let $[B_1, \beta_1]$ and $[B_2, \beta_2]$ be elements of $GW_0(\underline{\mathbb{Z}})$, and assume $(T_{\beta_1}(x_{B_1}), [\beta_1]) = (T_{\beta_2}(x_{B_2}), [\beta_2])$. Since we assume β_1 is stably equivalent to β_2, and recalling Proposition 4.2.3, we can take $\beta_1 = \beta_2 = r\langle 1 \rangle \perp s\langle -1 \rangle$. Then the result follows by applying Corollary 4.2.35 to B_1 and B_2. $\qquad\square$

As we have seen, injectivity takes a significant amount of work. Surjectivity is comparatively simpler:

Lemma 4.2.37. *The map*

$$GW_0(\underline{\mathbb{Z}}) \to \mathbb{Z} \oplus GW_0(\mathbb{Z})$$
$$[B, \beta] \mapsto (T_\beta(x_B), [\beta])$$

is surjective, where T_β is the trace signature given in Definition 4.2.15.

Proof. For clarity, write $\mathbb{Z} \oplus GW_0(\mathbb{Z})$ as $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the last two summands are r and s, recalling that $\beta = r\langle 1 \rangle \perp s\langle -1 \rangle$. We'll show that the elements $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ lie in the image of our map.

Take the class $[B, \beta] = [\langle 1 \rangle, \langle 1 \rangle]$. Then $x_B = f(B - \beta) = f(0) = 0$, so that $T_\beta(B_q) = 0$ and

$$[\langle 1 \rangle, \langle 1 \rangle] \mapsto (0, 1, 0).$$

Analogously, taking $[B, \beta] = [\langle -1 \rangle, \langle -1 \rangle]$ gives $(0, 0, 1)$.

Finally, taking $[B, \beta] = [\langle 2 - t \rangle, \langle 1 \rangle]$, $x_B = f(B - \beta) = f(1 - t) = 1$, so that $T_\beta(x_B) = \frac{1^2 + 1}{2} = 1$, which gives $(1, 1, 0)$. Therefore,

$$([\langle 2 - t \rangle, \langle 1 \rangle] - [\langle 1 \rangle, \langle 1 \rangle]) \mapsto (1, 0, 0).$$

$\square$

Proposition 4.2.36 and Lemma 4.2.37 combine to give:

Theorem 4.2.38. *The map*

$$GW_0(\mathbb{Z}) \to \mathbb{Z} \oplus GW_0(\mathbb{Z})$$

$$[B, \beta] \mapsto (T_\beta(x_B), [\beta])$$

is an isomorphism of abelian groups, where T_β is the trace signature given in Definition 4.2.15.

Proof. Combine Lemma 4.2.16, Proposition 4.2.36 and Lemma 4.2.37. $\square$

We now turn our attention to the ring structure of $GW_0(\mathbb{Z})$. We begin by defining the tensor product of $\underline{\mathbb{Z}}$-quadratic forms:

Definition 4.2.39. Let (P_1, q_1, β_1) and (P_2, q_2, β_2) be two $\underline{\mathbb{Z}}$-quadratic forms. Then their *tensor product*, written $(P_1 \otimes P_2, q_1 \otimes q_2, \beta_1 \otimes \beta_2)$, is defined thus:

- $P_1 \otimes P_2$ is the usual tensor product of the finitely-generated projective $\mathbb{Z}$-modules P_1 and P_2,

- $q_1 \otimes q_2 : P_1 \otimes P_2 \to \mathbb{Z}[C_2]$ is defined by setting $q_1 \otimes q_2(x_1 \otimes x_2) = q(x_1)q(x_2)$,

- $\beta_1 \otimes \beta_2 : (P_1 \otimes P_1) \otimes (P_2 \otimes P_2) \to \mathbb{Z}$ is the usual tensor product of symmetric bilinear forms; that is to say, $\beta_1 \otimes \beta_2((x_1, y_1) \otimes (x_2, y_2)) = \beta_1(x_1, y_1)\beta_2(x_2, y_2)$.

Remark 4.2.40. It is simple to verify that, given (P_1, q_1, β_1) and (P_2, q_2, β_2) as in Definition 4.2.39, their tensor product $(P_1 \otimes P_2, q_1 \otimes q_2, \beta_1 \otimes \beta_2)$ is also a $\underline{\mathbb{Z}}$-quadratic

form. For example, we have that, for $a \in \mathbb{Z}$:

$$q_1 \otimes q_2(a(x_1 \otimes x_2)) = q_1 \otimes q_2(ax_1 \otimes x_2)$$
$$= q_1(ax_1)q_2(x_2)$$
$$= \eta(a)q_1(x_1)q_2(x_2)$$
$$= \eta(a)(q_1 \otimes q_2(x_1 \otimes x_2)),$$

as required.

Proposition 4.2.41. *The Grothendieck-Witt group $GW_0(\mathbb{Z})$ has the structure of a unital commutative ring, with multiplication induced by the tensor product of forms over $\mathbb{Z}$.*

Proof. First, note that the form $(\mathbb{Z}, \langle 1 \rangle, \langle 1 \rangle)$ gives the multiplicative identity. Next, note that the tensor product of finitely-generated projective $\mathbb{Z}$-modules is associative, commutative, and distributive with respect to direct sum, up to isomorphism. Combining these facts with the associativity, commutativity, and distributivity of the multiplication in $\mathbb{Z}[C_2]$ yields the required result. $\square$

We have the following simplifying lemma, which gives a way of writing tensor products of $\mathbb{Z}$-quadratic forms of rank 1:

Lemma 4.2.42. *Let $(\langle \lambda_1 \rangle, \langle b_1 \rangle)$ and $(\langle \lambda_2 \rangle, \langle b_2 \rangle)$ be representations of the $\mathbb{Z}$-quadratic forms denoted (q_1, β_1) and (q_2, β_2); that is to say $q_1(1) = \lambda_1$ and $q_2(1) = \lambda_2$, and $b_1 = \rho(\lambda_1)$ and $b_2 = \rho(\lambda_2)$ are both ± 1. Then the rank 1 form $(q_1 \otimes q_2, \beta_1 \otimes \beta_2)$ has representation $(\langle \lambda_1 \lambda_2 \rangle, \langle b_1 b_2 \rangle)$.*

Proof. By definition, $q_1 \otimes q_2(1 \otimes 1) = q_1(1)q_2(1) = \lambda_1 \lambda_2$. Then the fact that $\beta_1 \otimes \beta_2$ is represented by $b_1 b_2$ follows from the fact that ρ is multiplicative. $\square$

We use this lemma, together with the basis

$$[\langle 1 \rangle, \langle 1 \rangle], \quad [\langle -1 \rangle, \langle -1 \rangle], \quad [\langle 2 - t \rangle, \langle 1 \rangle] - [\langle 1 \rangle, \langle 1 \rangle]$$

to calculate the ring structure of $GW_0(\mathbb{Z})$:

Proposition 4.2.43. *As a commutative ring, $GW_0(\mathbb{Z})$ is isomorphic to the quotient ring*

$$\mathbb{Z}[X, Y]/\langle X^2 - 1 \rangle, \langle XY + Y \rangle, \langle Y^2 - 8Y \rangle$$

Proof. Defining the map

$$GW_0(\underline{\mathbb{Z}}) \to \mathbb{Z}[X, Y]/\langle X^2 - 1\rangle, \langle XY + Y\rangle, \langle Y^2 - 8Y\rangle$$

by sending $[\langle 1\rangle, \langle 1\rangle]$ to 1, $[\langle -1\rangle, \langle -1\rangle]$ to X, and $[\langle 2 - t\rangle, \langle 1\rangle] - [\langle 1\rangle, \langle 1\rangle]$ to Y, we check that the quotient relations are satisfied. Note that by Lemma 4.2.42, $[\langle 1\rangle, \langle 1\rangle]$ is clearly the unit in $GW_0(\underline{\mathbb{Z}})$. We have:

$$(\langle -1\rangle, \langle -1\rangle) \otimes (\langle -1\rangle, \langle -1\rangle) = (\langle 1\rangle, \langle 1\rangle)$$

which gives $X^2 = 1$. Further:

$$([\langle -1\rangle, \langle -1\rangle]) \cdot ([\langle 2 - t\rangle, \langle 1\rangle] - [\langle 1\rangle, \langle 1\rangle]) = [\langle t - 2\rangle, \langle -1\rangle] - [\langle -1\rangle, \langle -1\rangle].$$

which under the map from Theorem 4.2.38 goes to $(-1, 0, 1) - (0, 0, 1) = (-1, 0, 0)$ which is mapped to $-Y$, so that the relation $XY = -Y$ is satisfied. Finally:

$$([\langle 2 - t\rangle, \langle 1\rangle] - [\langle 1\rangle, \langle 1\rangle])^2 = [\langle 5 - 4t\rangle, \langle 1\rangle] - 2[\langle 2 - t\rangle, \langle 1\rangle] + [\langle 1\rangle, \langle 1\rangle]$$

which is mapped to $(10, 1, 0) - (2, 2, 0) + (0, 1, 0) = (8, 0, 0)$, so that $Y^2 = 8Y$ as required. $\square$

Chapter 5

Burnside form rings over finite fields

5.1 $\underline{\mathbb{Z}/2\mathbb{Z}}$ and the trace map

As we have seen in Definitions 3.2.8 and 3.2.11, one can use an extension of scalars construction to define a notion of Burnside form ring $\underline{R}$ for any ring R. This final chapter is concerned with calculating $GW_0(\underline{R})$ when R is a finite field. More specifically, we are mainly concerned with the characteristic 2 case, since the odd characteristic case is covered by Theorem 4.1.20.

Along the way, we define a version of the determinant which generalises the classical determinant map from symmetric bilinear forms, and show that, with certain hypotheses, this determinant is well-defined.

We will start with the simplest case in which the characteristic is 2; the case of $R = \mathbb{Z}/2\mathbb{Z}$. The following proposition gives a concrete way of thinking about the form ring $\underline{\mathbb{Z}/2\mathbb{Z}}$.

Proposition 5.1.1. *We have an isomorphism of form rings*

$$
\begin{array}{ccccc}
\mathbb{Z}/2\mathbb{Z} & \underset{\tau}{\overset{\eta}{\rightrightarrows}} & \mathbb{A}(\mathbb{Z}/2\mathbb{Z}) & \overset{\rho}{\longrightarrow} & \mathbb{Z}/2\mathbb{Z} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} \\
\mathbb{Z}/2\mathbb{Z} & \underset{\cdot 2}{\overset{e}{\rightrightarrows}} & \mathbb{Z}/4\mathbb{Z} & \overset{r}{\longrightarrow} & \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

where $e(0) = 0$ and $e(1) = 1$, r is reduction mod 2 ,and f is the ring homomorphism

defined by setting $f([1, 1]) = 1$.

Proof. Applying Definitions 3.2.8 and 3.2.11, the ring $\mathbb{A}(\mathbb{Z}/2\mathbb{Z})$ is additively generated by the symbols $[1]$ and $[1, \lambda]$ where λ ranges over all of $\mathbb{A}(\mathbb{Z}) = \mathbb{Z}[C_2]$. However, relation 6 from Definition 3.2.8 implies that $[1, 1] = [1, t]$. This fact combined with relation 4 from the same definition implies that $[1, \lambda] = [1, \rho(\lambda)]$, so that we can always assume $[1, \lambda] = [1, n]$ for some integer n. Also, relation 5 implies that $[1, 1] + [1, 1] = [1, 2] = [1]$ which implies $[1, 4] = 0$ since $[1] + [1] = 0$ by relation 2. Combining all of these facts, we have that $\mathbb{A}(\mathbb{Z}/2\mathbb{Z})$ is additively generated by $[1, 1]$, and that $[1, 1]$ has order 4.

Per Lemma 3.2.10, we have that the ring structure is given by $[1, n] \cdot [1, m] = [1, nm]$, so that $\mathbb{A}(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ as a ring via the map f, as required. Commutativity of the required squares is clear for η and τ. For ρ, recall that $\rho([1, n]) = \rho(n) \bmod 2 = n \bmod 2$, as required. $\square$

Now, in an abuse of notation analogous to the one after Lemma 4.1.10, set

$$\underline{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \underset{\cdot 2}{\overset{e}{\rightrightarrows}} \mathbb{Z}/4\mathbb{Z} \overset{r}{\longrightarrow} \mathbb{Z}/2\mathbb{Z}$$

Remark 5.1.2. Observe that we have an isomorphism from $\ker(r)$, which consists of 0 and 2, to $\mathbb{Z}/2\mathbb{Z}$ given by division by 2. Also, since $\mathbb{Z}/2\mathbb{Z}$ is a field, we have that projective modules are free, so that we can view a quadratic form over $\underline{\mathbb{Z}/2\mathbb{Z}}$ as a symmetric matrix with diagonal entries in $\mathbb{Z}/4\mathbb{Z}$ and non-diagonal entries in $\mathbb{Z}/2\mathbb{Z}$. In view of Chapter 4, it may seem that we can use an argument analogous to that used in the integer case. Let (P, q, β) be a $\underline{\mathbb{Z}/2\mathbb{Z}}$-quadratic form module. Recall that, letting $e_1, \ldots, e_n$ be a basis for P, we have

$$tr_\beta(q) = \sum_{i=1}^n q(e_i) q(\beta^{-1}(e_i^*)) + \sum_{1 \leq i < j \leq n} \tau(\beta(e_i)(e_j) \cdot e_i^*(\beta^{-1}(e_j^*)))$$

Since β is a symmetric bilinear form over the quadratically closed field $\mathbb{Z}/2\mathbb{Z}$, it is isometric to one of the form $n\langle 1 \rangle$. We can therefore assume $\beta = n\langle 1 \rangle$, so that the trace reduces to

$$tr_\beta(q) = \sum_{i=1}^n q(e_i)^2,$$

but, since q is non-degenerate, $q(e_i) = 1 \bmod 2$ for all $1 \leq i \leq n$, so that $q(e_i) = \pm 1$ for all i, and $tr_\beta(q) = n \bmod 4$, the rank of P reduced mod 4. Therefore, the trace map encodes no more information about the class of (P, q, β) in $GW_0(\underline{\mathbb{Z}/2\mathbb{Z}})$ than

the rank of P. We will see in the sequel that rank is not sufficient to give a complete set of invariants for stable equivalence classes of forms over $\underline{\mathbb{Z}/2\mathbb{Z}}$.

We have a map of form rings

$$
\begin{array}{ccccc}
\mathbb{Z}/4\mathbb{Z} & \xrightarrow[\cdot 2]{(-)^2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{\ 1\ } & \mathbb{Z}/4\mathbb{Z} \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle r} \\
\mathbb{Z}/2\mathbb{Z} & \xrightarrow[\cdot 2]{e} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{\ r\ } & \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

where the top form ring is that which gives rise to classical symmetric bilinear forms over $\mathbb{Z}/4\mathbb{Z}$. We therefore have a map $GW_0(\mathbb{Z}/4\mathbb{Z}) \to GW_0(\underline{\mathbb{Z}/2\mathbb{Z}})$, and we can use the properties of $GW_0(\mathbb{Z}/4\mathbb{Z})$ to investigate those of $GW_0(\underline{\mathbb{Z}/2\mathbb{Z}})$.

5.2 A generalised determinant

Among other invariants, one may use the classical determinant map from $GW_0(\mathbb{Z}/4\mathbb{Z})$ to $(\mathbb{Z}/4\mathbb{Z})^*/(\mathbb{Z}/4\mathbb{Z})^{*2}$ to investigate the properties of $GW_0(\mathbb{Z}/4\mathbb{Z})$. For example, it shows immediately that $\langle 1 \rangle$ and $\langle -1 \rangle$ are not stably isometric to each other, so that $GW_0(\mathbb{Z}/4\mathbb{Z})$ is not a cyclic group. This raises the question: can a determinant map be defined for any form ring?

More precisely, for a commutative form ring (R, Λ), let (P, q, β) be a (R, Λ)-quadratic form where P is a free module. Recall that we denote the quadratic action of R on Λ by $Q(-) : R \to \mathrm{End}_{\mathbb{Z}}(\Lambda)$. Define an equivalence relation on Λ by setting $\lambda \sim \mu$ if there exists a unit $u \in R$ with $Q(u)(\lambda) = \mu$. Suppose q is represented by a matrix M. Is there a map

$$
Q(P) \xrightarrow{\tilde{det}} \rho^{-1}(R^*)/\sim
$$

which is well-defined on isometry classes, such that $\rho \circ \tilde{det}(M) = det(R(M)) \in R^*/R^{*2}$? This section gives conditions under which this is indeed the case.

Our starting point is the following proposition:

Proposition 5.2.1. *Let A be an $n \times n$ symmetric matrix with entries in a commutative ring R. Then the determinant of A is given by the formula*

$$
\sum_{\substack{\sigma \in S_n, \\ \sigma^2 = 1}} sgn(\sigma) \prod_{\sigma(i)=i} A_{ii} \prod_{\sigma(i) \neq i} A_{i\sigma(i)}^2 + \sum_{\substack{[\sigma] \in S_n/C_2, \\ \sigma^{-1} \neq \sigma}} sgn(\sigma) \prod_{\sigma(i)=i} A_{ii} \cdot 2 \prod_{\sigma(i) \neq i} A_{i\sigma(i)},
$$

where C_2 acts on S_n by $\sigma \mapsto \sigma^{-1}$.

Proof. First, suppose $\sigma^2 = 1$. This means that σ can be written as a product of disjoint 2-cycles. Recalling the usual determinant formula, the summand corresponding to σ is given by

$$sgn(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)},$$

which, since $A_{i\sigma(i)} = A_{\sigma(i)i}$, is equal to

$$sgn(\sigma) \prod_{\sigma(i)=i} A_{ii} \prod_{\sigma(i)\neq i} A_{i\sigma(i)}^2$$

as required.

If σ has order greater than 2, symmetry of A, together with the fact that σ and σ^{-1} have the same fixed points, implies that the summand corresponding to σ and the summand corresponding to σ^{-1} are both equal to

$$sgn(\sigma) \prod_{\sigma(i)=i} A_{ii} \prod_{\sigma(i)\neq i} A_{i\sigma(i)}.$$

Which implies that, if we multiply summands by 2, we can sum over representatives $[\sigma] \in S_n/C_2$, as required. □

Assume R has trivial involution, so that, recalling Remark 3.2.6, we can view a commutative form ring (R, Λ) as a Tambara functor

$$R \underset{\tau}{\overset{\eta}{\rightrightarrows}} \Lambda \xrightarrow{\rho} R.$$

Since, for all $a \in R$, $\rho\tau(a) = 2a$ and $\rho\eta(a) = a^2$, in view of Proposition 5.2.1, there is an obvious candidate for $\tilde{det}$. Recall that the matrix M of an (R, Λ)-quadratic form is a symmetric matrix which has diagonal entries in Λ and non-diagonal elements from R. Denote the diagonal elements by λ_i and the non-diagonal elements by A_{ij}. Then one possible definition of the determinant is given by

$$\sum_{\substack{\sigma \in S_n, \\ \sigma^2=1}} sgn(\sigma) \prod_{\sigma(i)=i} \lambda_i \prod_{\sigma(i)\neq i} \eta(A_{i\sigma(i)}) + \sum_{\substack{[\sigma] \in S_n/C_2, \\ \sigma^{-1}\neq\sigma}} sgn(\sigma) \prod_{\sigma(i)=i} \lambda_i \cdot \tau \prod_{\sigma(i)\neq i} A_{i\sigma(i)}. \quad (5.1)$$

As the next proposition shows, this requires a modification to be well-defined on isometry classes of (R, Λ)-quadratic forms.

Proposition 5.2.2. *Let R be a commutative ring with trivial involution, and let (R, Λ) be a commutative form ring with surjective restriction map ρ. Let $Q_n(R)$ be as in Definition 4.1.2, and define a map $\tilde{det} : Q_n(R) \to \Lambda/J$ by the formula (5.1), where $J \subset \Lambda$ is the ideal generated by the elements*

$$\tau(1) - 2, \quad \lambda^2 - \eta(\rho\lambda),$$

where λ ranges over all of Λ. For any $M \in Q_n(R)$ and $A \in GL_n(R)$,

$$\tilde{det}(Q(A)(M)) = \eta(det(A))\tilde{det}(M).$$

Proof. Our assumptions give a map of commutative form rings

$$
\begin{array}{ccc}
\Lambda/J \underset{\cdot 2}{\overset{(-)^2}{\rightrightarrows}} \Lambda/J \xrightarrow{\ 1\ } \Lambda/J \\
\Big\downarrow{\scriptstyle\rho} \qquad \Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle\rho} \\
R \underset{\tau}{\overset{\eta}{\rightrightarrows}} \Lambda/J \xrightarrow{\ \rho\ } R
\end{array}
$$

where the top form ring is the one which gives rise to classical symmetric bilinear forms over the commutative ring Λ/J. Note that $\rho : \Lambda/J \to R$ is well-defined since every element in the ideal J is in $\ker\rho$.

Let M be a symmetric matrix over Λ/J, and, analogously to Lemma 4.1.8, let $(\rho, 1)_*$ be the map on such matrices which is the identity on diagonal entries, and sends each M_{ij} with $i \neq j$ to $\rho(M_{ij})$. By definition of $\tilde{det}$ and the ideal J, we have

$$\tilde{det}(\rho, 1)_*(M)) = det(M), \tag{5.2}$$

where $det(M)$ is the usual determinant of matrices over Λ/J. In abuse of notation, let ρ be the map from matrices over Λ/J to matrices over R which applies ρ to every entry. Since ρ is surjective, for every matrix $A \in M_n(R)$, there exists $A' \in M_n(\Lambda/J)$ such that $\rho(A') = A$. Similarly, for every $M \in Q(R^n)$ there exists an $n \times n$ symmetric matrix M' with entries in Λ/J such that $(\rho, 1)_*(M') = M$.

The fact that $(\rho, 1)$ is a map of form rings means that the map $(\rho, 1)_*$ commutes

with the quadratic actions, so we have

$$\tilde{det}(Q(A)(M)) = \tilde{det}((\rho, 1)_*(A'^T M' A'))$$
$$= det(A'^T M' A')$$
$$= det(A')^2 det(M')$$
$$= \eta(det(A))\tilde{det}(M),$$

where the last equality is true because

$$\eta(det(A)) = \eta(det(\rho(A'))) = \eta\rho(det(A')) = det(A')^2,$$

where we use $(det(\rho(A'))) = \rho(det(A'))$, a fact which is implied by the fact that $\rho : \Lambda/J \to R$ is a ring homomorphism. $\qquad\square$

Remark 5.2.3. It really is necessary to quotient by the ideal J. For example, for a commutative form ring (R, Λ), consider the diagonal (R, Λ)-quadratic form given by the matrix

$$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and let A be a 2×2 matrix with entries in R such that

$$A^T \begin{bmatrix} \rho(\lambda_1) & 0 \\ 0 & \rho(\lambda_2) \end{bmatrix} A = \begin{bmatrix} \rho(\lambda_1) & 0 \\ 0 & \rho(\lambda_2) \end{bmatrix}. \qquad (5.3)$$

We have

$$Q(A)(M) = \begin{bmatrix} \eta(A_{11})\lambda_1 + \eta(A_{21})\lambda_2 & 0 \\ 0 & \eta(A_{12})\lambda_1 + \eta(A_{22})\lambda_2 \end{bmatrix}$$

so that

$$\tilde{det}(Q(A)(M)) = (\eta(A_{11})\lambda_1 + \eta(A_{21})\lambda_2)(\eta(A_{12})\lambda_1 + \eta(A_{22})\lambda_2)$$
$$= \lambda_1^2 \eta(A_{11}A_{12}) + \lambda_2^2 \eta(A_{21}A_{22}) + \lambda_1\lambda_2(\eta(A_{11}A_{22}) + \eta(A_{12}A_{21})).$$

In general, this is not equal to the desired result. Indeed, we have

$$\eta(det(A))\tilde{det}(M) = \eta(A_{11}A_{22} - A_{12}A_{21})\lambda_1\lambda_2$$
$$= (\eta(A_{11}A_{22}) + (\tau(1) - 1)(\eta(A_{12}A_{21}) + \tau(-A_{11}A_{22}A_{12}A_{21}))\lambda_1\lambda_2$$

where the second equality uses the easily-checked fact that $\eta(-1) = \tau(1) - 1$. However, after quotienting by J, we have

$$\eta(det(A))\tilde{det}(M) = (\eta(A_{11}A_{22}) + \eta(A_{12}A_{21}) + \tau(-A_{11}A_{22}A_{12}A_{21}))\lambda_1\lambda_2$$

because $\tau(1) = 2$, so $\tau(1) - 1 = 1$.
On the other hand, we have

$$\tilde{det}(Q(A)(M)) = \lambda_1^2\eta(A_{11}A_{12}) + \lambda_2^2\eta(A_{21}A_{22}) + \lambda_1\lambda_2(\eta(A_{11}A_{22}) + \eta(A_{12}A_{21}))\lambda_1\lambda_2$$
$$= \eta(\rho(\lambda_1)(A_{11}A_{12}) + \eta(\rho(\lambda_2)(A_{21}A_{22}) + \lambda_1\lambda_2(\eta(A_{11}A_{22}) + \eta(A_{12}A_{21})).$$

Also:

$$\eta(\rho(\lambda_1)(A_{11}A_{12}) + \eta(\rho(\lambda_2)(A_{21}A_{22})$$
$$=\eta(\rho(\lambda_1)A_{11}A_{12} + \rho(\lambda_2)A_{12}A_{22}) - \tau(\rho(\lambda_1)\rho(\lambda_2)A_{11}A_{12}A_{21}A_{22}),$$

which is equal to $-\tau(\rho(\lambda_1)\rho(\lambda_2)A_{11}A_{12}A_{21}A_{22})$ because (5.3) implies $\rho(\lambda_1)A_{11}A_{12} + \rho(\lambda_2)A_{12}A_{22} = 0$.

Remark 5.2.4. Returning to the case of $\underline{\mathbb{Z}/2\mathbb{Z}}$ viewed as the form ring

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow[\cdot 2]{e} \mathbb{Z}/4\mathbb{Z} \xrightarrow{r} \mathbb{Z}/2\mathbb{Z},$$

notice that the ideal $J = 0$ here. Indeed, the transfer of 1 is $2 \in \mathbb{Z}/4\mathbb{Z}$ by definition, and we have

$$e(r(0)) = e(0) = 0 = 0^2$$
$$e(r(1)) = e(1) = 1 = 1^2$$
$$e(r(2)) = e(0) = 0 = 2^2$$
$$e(r(3)) = e(1) = 1 = 3^2$$

This implies that the generalised determinant $\tilde{det}$ is a map to $\mathbb{Z}/2\mathbb{Z}$, in the guise of the group of units of $\mathbb{Z}/4\mathbb{Z}$. Moreover, for a form q over $\underline{\mathbb{Z}/2\mathbb{Z}}$, we have $\tilde{det}(q) = det(q')$, where q' is any lift of q under the surjective map $(\rho, 1)_*$, and where the right hand determinant is the classical one of symmetric bilinear forms over $\mathbb{Z}/4\mathbb{Z}$. One consequence of this is that in this case, the generalised determinant over $\underline{\mathbb{Z}/2\mathbb{Z}}$ agrees with the usual determinant over $\Lambda = \mathbb{Z}/4\mathbb{Z}$. If $J \neq 0$, this is not the case in

general.

Remark 5.2.5. It is unknown whether the condition that the restriction map $\rho : \Lambda \to R$ is surjective is necessary to have a well-defined determinant map. This assumption merely makes well-definedness easier to prove. It may be the case that the formula (5.1) is well-defined on isometry classes of forms over any form ring.

5.3 The nilpotency of kerρ

Let (R, Λ) be a commutative form ring with surjective restriction ρ and the ideal $J \subset \Lambda$ as defined in the statement of Proposition 5.2.2 equal to 0. Although Remark 5.2.4 focuses on the case of $\underline{\mathbb{Z}}/2\mathbb{Z}$, its arguments imply that, whenever we have these assumptions, the generalised determinant $\tilde{det}$ over (R, Λ) is essentially given by the usual determinant of symmetric bilinear forms over Λ. The goal of this section is to go further, and show that, given certain conditions, we have that $GW_0(R, \Lambda) \cong GW_0(\Lambda)$ where the group on the right is the Grothendieck-Witt group of symmetric bilinear forms over the commutative ring Λ.

The method of doing so is as follows. It is a well known fact that, for R a commutative ring and $I \subset R$ a nilpotent ideal, given a finitely generated projective R-module P, the assignment

$$P \mapsto P/IP$$

defines a functor from projective R-modules to finitely generated projective R/I-modules which is full, conservative, ($P/IP_1 \cong P_2/IP_2 \implies P_1 \cong P_2$) and essentially surjective (every finitely generated projective R/I module is isomorphic to one of the form P/IP); this implies it induces a bijection between isomorphism classes of finitely generated projective R-modules and isomorphism classes of projective R/I-modules. Fullness and conservativeness are proved in Proposition III.2.12 of [2], and essential surjectivity is proved in Lemma 10.77.5 of [24].

Now, if we assume ρ is surjective, we have $\Lambda/\ker\rho \cong R$ by the first isomorphism theorem. It turns out that, if the kernel of ρ squares to 0, and if finitely-generated projective R-modules are free, we can define a functor on categories of forms which extends the functor on projective modules, and which is also full, faithful, and essentially surjective. Thus, the bijection between isomorphism classes of projective modules can be expanded to the level of forms, which ultimately implies $GW_0(R, \Lambda) \cong GW_0(\Lambda)$, where $GW_0(\Lambda)$ means the Grothendieck-Witt group of symmetric bilinear forms over Λ.

We start by defining a functor on categories of forms:

Lemma 5.3.1. *Let (R, Λ) be a commutative form ring where R has trivial involution and where the ideal J as defined in Proposition 5.2.2 is equal to 0. Let $P(\Lambda, \Lambda)$ be the category of finitely generated projective Λ-modules equipped with a symmetric bilinear form, and let $P(\Lambda/ker\rho, \Lambda)$ be the category of finitely generated projective $\Lambda/ker\rho$-modules equipped with a $(\Lambda/ker\rho, \Lambda)$-quadratic form. Consider the assignment $F_{ker\rho} : P(\Lambda, \Lambda) \to P(\Lambda/ker\rho, \Lambda)$ which sends a finitely generated projective symmetric bilinear form module (P, β) to the $(\Lambda/ker\rho, \Lambda)$-quadratic form $(P/ker\rho P, q_\beta, \bar{\beta})$ where q_β and $\bar{\beta}$ are defined:*

- $q_\beta(\bar{x}) = \beta(x, x)$

- $\bar{\beta}(\bar{x}, \bar{y}) = \overline{\beta(x, y)}$.

This assignment defines a functor $P(\Lambda, \Lambda) \to P(\Lambda/ker\rho, \Lambda)$.

Proof. First, we check that q_β and $\bar{\beta}$ are well-defined. Let x, y and z be in P, such that $x - y = \lambda z$ for some $\lambda \in ker\rho$ so that we have $\bar{x} = \bar{y}$. Then we have

$$q_\beta(\bar{x}) = \beta(x, x) = \beta(y + (x - y), y + (x - y))$$
$$= \beta(y, y) + 2\lambda\beta(y, z) + \lambda^2\beta(z, z)$$

But, since $J = 0$, we have

$$2\lambda = \tau(1)\lambda = \tau(\rho\lambda) = \tau(0) = 0$$

and

$$\lambda^2 = \eta(\rho\lambda) = \eta(0) = 0,$$

so that $q_\beta(\bar{x}) = \beta(y, y) = q_\beta(\bar{y})$ as required. Well-definedness of $\bar{\beta}$ comes from bilinearity combined with the fact it sends $ker\rho$ to 0. The axioms of Definition 3.1.10 follow from the commutative diagram

$$
\begin{array}{ccccc}
\Lambda & \underset{\cdot 2}{\overset{(-)^2}{\rightrightarrows}} & \Lambda & \overset{1}{\longrightarrow} & \Lambda \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\
\Lambda/ker\rho & \underset{\tau\circ\rho}{\overset{\eta\circ\rho}{\rightrightarrows}} & \Lambda & \longrightarrow\!\!\!\!\!\!\to & \Lambda/ker\rho
\end{array}
$$

For example, $q(\overline{a}\overline{x}) = \beta(ax, ax) = a^2\beta(x, x) = \eta(\bar{a})q(\bar{x})$.

It remains to be shown that $F_{ker\rho}$ respects composition of maps of forms. To see this, take two symmetric bilinear form modules over Λ, denoted (P_1, β_1) and (P_2, β_2).

Suppose $f : P_1 \to P_2$ is a map such that $\beta_1 = \beta_2 \circ (f \otimes f)$. To prove the lemma, it suffices to show that

$$q_{\beta_1} = q_{\beta_2} \circ \bar{f}, \quad \bar{\beta}_1 = \bar{\beta}_2 \circ (\bar{f} \otimes \bar{f})$$

where $\bar{f}$ is the map $P_1/IP_1 \to P_2/IP_2$ defined by setting $\bar{f}(\bar{x}) = \overline{f(x)}$. But this is true, since we have

$$
\begin{aligned}
q_{\beta_1}(\bar{x}) &= \beta_1(x, x) \\
&= \beta_2(f(x), f(x)) \\
&= q_{\beta_2}(\overline{f(x)}) \\
&= q_{\beta_2}(\bar{f}(\bar{x}))
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\beta}_1(\bar{x}, \bar{y}) &= \overline{\beta_1(x, y)} \\
&= \overline{\beta_2(f(x), f(y))} \\
&= \bar{\beta}_2(\overline{f(x)}, \overline{f(y)}) \\
&= \bar{\beta}_2(\bar{f}(\bar{x}), \bar{f}(\bar{y}))
\end{aligned}
$$

as required. $\qquad\square$

Lemma 5.3.1 gives a functor from the category of finitely generated projective symmetric bilinear form modules over Λ to the category of finitely generated projective $\Lambda/\ker\rho$-modules equipped with a $(\Lambda/\ker\rho, \Lambda)$-quadratic form. The next two lemmas show that, if we assume $\ker\rho$ is a nilpotent ideal, then this functor is essentially surjective and conservative.

Lemma 5.3.2. *Under the assumptions of Lemmas 5.3.1, assume further that $\ker\rho$ is a nilpotent ideal. Then the functor $F_{\ker\rho}$ defined in Lemma 5.3.1 is essentially surjective; that is, for every $(\Lambda/\ker\rho, \Lambda)$-quadratic form (P_1, q_1, β_1) with P_1 a finitely generated projective $\Lambda/\ker\rho$-module, there exists a finitely generated projective symmetric bilinear Λ-module (P_2, β_2) such that $(P_2/\ker\rho P_2, q_{\beta_2}, \bar{\beta}_2)$ is isometric to (P_1, q_1, β_1) as a module over $(\Lambda/\ker\rho, \Lambda)$.*

Proof. Nilpotency of $\ker\rho$, per Lemma 10.77.5 of [24], implies that there exists a finitely generated projective Λ-module P_2 and an isomorphism $f : P_2/\ker\rho P_2 \to P_1$

of $\Lambda/\ker\rho$-modules. Therefore, we have an isometry

$$(P_2/\ker\rho P_2, q_1 \circ f, \beta_1 \circ (f \otimes f)) \cong (P_1, q_1, \beta_1).$$

so that we are done if we can show that $(q_1 \circ f, \beta_1 \circ (f \otimes f))$ is the image of a symmetric bilinear form on the Λ-module P_2 under the assignment given in the statement of Lemma 5.3.1.

We will show, more generally, that every $(\Lambda/\ker\rho, \Lambda)$-quadratic form $(P_2/\ker\rho P_2, q, \beta)$ is the image of some symmetric bilinear form β' on P_2. We can obtain a map β' on $P_2 \times P_2$ by setting $\beta'(x, y)$ to be a chosen lift of $\beta(\bar{x}, \bar{y})$ under the quotient map $\Lambda \to \Lambda/\ker\rho$ for $x \neq y \in P_2$. Further, set $\beta'(x, x) = q(\bar{x})$.

The lifts can be chosen such that β' is a symmetric bilinear form. Indeed, the symmetry of β implies we have, for all x and y in P_2,

$$\beta(\bar{x}, \bar{y}) = \beta(\bar{y}, \bar{x}),$$

so that one can obtain symmetry of β' by choosing the same lift for $\beta(\bar{x}, \bar{y})$ and $\beta(\bar{y}, \bar{x})$. A completely analogous argument shows that lifts can be chosen to make β' bilinear.

Finally, we have

$$q_{\beta'}(\bar{x}) = \beta'(x, x) = q(\bar{x})$$

and

$$\bar{\beta}'(\bar{x}, \bar{y}) = \overline{\beta'(x, y)}$$
$$= \beta(\bar{x}, \bar{y})$$

for all $x, y \in P_2$. $\qquad\square$

Remark 5.3.3. Note that, in general, it really is necessary in Lemma 5.3.2 to add the assumption that $\ker\rho$ is nilpotent. The ideal J being equal to 0 only implies that $\ker\rho$ is nil, and the two notions are not equivalent in general. This does not affect our aims, since in the case of $\underline{\mathbb{F}_q}$, $(\ker\rho)^2 = 0$, but it is worthwhile to keep in mind.

Lemma 5.3.4. *Under the same assumptions as Lemma 5.3.2, the functor defined in Lemma 5.3.1 is conservative; that is, if the image of a map f of symmetric bilinear form Λ-modules is an isometry of $(\Lambda/\ker\rho, \Lambda)$-quadratic forms, then f is an*

isometry.

Proof. Proposition III.2.12 in [2], combined with Proposition III.2.2 in the same text, says that, if an ideal I in a commutative ring R is such that $1 + x \in R^*$ for all $x \in I$, then the functor $P \mapsto P/IP$ on finitely generated projective modules is conservative.

This applies in our situation, since, if I is nilpotent, $1 + x \in R^*$ for all $x \in I$. To see this, note that every $x \in I$ is nilpotent, so that there exists a natural number n such that $x^n = 0$. Then we have

$$(1 + x)(1 - x + x^2 - \cdots + (-1)^{n-1}x_{n-1}) = 1 + x^n = 1,$$

so that $1 + x$ is a unit as required.

Applying this to our situation, suppose we have a map $f : P_1 \to P_2$ of projective Λ-modules, where P_2 is equipped with a symmetric bilinear form β and P_1 is equipped with the symmetric bilinear form $\beta \circ (f \otimes f)$. Suppose further that the map $\bar{f} : P_1/\mathrm{ker}\rho P_1 \to P_2/\mathrm{ker}\rho P_2$ is an isomorphism, so that $(P_1/\mathrm{ker}\rho P_1, q_\beta \circ \bar{f}, \bar{\beta} \circ (\bar{f} \otimes \bar{f}))$ is isometric to $(P_2/\mathrm{ker}\rho P_2, q_\beta, \bar{\beta})$. Then Propositions III.2.2 and III.2.12 in [2] imply f is an isomorphism, and $(P_1, \beta \circ (f \otimes f))$ is isometric to (P_2, β) as required. $\square$

So far, we have proven that the functor $F_{\mathrm{ker}\rho}$ defined in Lemma 5.3.1 is essentially surjective and conservative. To prove that it is full, it will be convenient to make some additional assumptions. We will show that these assumptions hold for the case $\underline{\mathbb{F}_q}$ where $\mathbb{F}_q$ is a finite field of characteristic 2. We begin with the following lemma, which shows that, if R is a commutative ring such that finitely generated projective modules are free, the same is true of Λ, so that we can write our proof of fullness in the language of matrices.

Lemma 5.3.5. *Let (R, Λ) be a commutative form ring under the same assumptions as Lemma 5.3.2. Assume further that finitely generated projective R-modules are free. Then projective Λ-modules are free.*

Proof. The first isomorphism theorem tells us that $R \cong \Lambda/\mathrm{ker}\rho$, so that projective $\Lambda/\mathrm{ker}\rho$-modules are free. Moreover, since $\mathrm{ker}\rho$ is nilpotent, combining Proposition III.2.12 in [2] and Lemma 10.77.5 in [24] says that every projective $\Lambda/\mathrm{ker}\rho$-module is isomorphic to one of the form $P/\mathrm{ker}\rho P$, and that two such modules $P_1/\mathrm{ker}\rho P_1$ and $P_2/\mathrm{ker}\rho P_2$ are isomorphic if and only if P_1 and P_2 are isomorphic projective Λ-modules. The result follows. $\square$

We will also require the following observation: we can replace the functor $F_{\ker\rho}$: $P(\Lambda, \Lambda) \to P(\Lambda/\ker\rho, \Lambda)$ with its restriction $\bar{F}_{\ker\rho}$: $\iota P(\Lambda, \Lambda) \to \iota P(\Lambda/\ker\rho, \Lambda)$, where, for a category $\mathcal{C}$, $\iota\mathcal{C}$ denotes the subcategory with the same objects as $\mathcal{C}$ but only invertible morphisms. This is since the abelian monoids of isometry classes of $\iota P(\Lambda, \Lambda)$ and $\iota P(\Lambda/\ker\rho, \Lambda)$ are equal to those for $P(\Lambda, \Lambda)$ and $P(\Lambda/\ker\rho, \Lambda)$. The functor $\bar{F}_{\ker\rho}$ is essentially surjective and conservative for the same reasons $F_{\ker\rho}$ is. The next lemma shows that, subject to the assumption $(\ker\rho)^2 = 0$, it is also full.

Lemma 5.3.6. *Under the same assumptions as Lemma 5.3.5, assume further that $(ker\rho)^2 = 0$. Then the functor $\bar{F}_{ker\rho} : \iota P(\Lambda, \Lambda) \to \iota P(\Lambda/ker\rho, \Lambda)$ is full.*

Proof. In view of Lemma 5.3.5 and Section 4.1.1, we can view an object of $\iota P(\Lambda, \Lambda)$ as a symmetric matrix with entries in Λ. Likewise, we can view an object of $\iota P(\Lambda/\ker\rho, \Lambda)$ as a symmetric matrix with diagonal entries in Λ and non-diagonal entries in $\Lambda/\ker\rho$. Then the functor $\bar{F}_{\ker\rho}$ is the identity on the diagonal and sends non-diagonal entries to their class in $\Lambda/\ker\rho$. A morphism from B_1 to B_2 in $\iota P(\Lambda, \Lambda)$ is an invertible matrix A with entries in Λ such that $A^T B_2 A = B_1$. Likewise, a morphism from C_1 to C_2 in $\iota P(\Lambda/\ker\rho, \Lambda)$ is an invertible matrix A with entries in $(\Lambda/\ker\rho)$ such that $Q(A)(C_2) = C_1$. On matrices, the functor $\bar{F}_{\ker\rho}$ sends each entry of a matrix to its class in $\Lambda/\ker\rho$.

In view of all this, we have proved the lemma if we can prove that, for each pair of objects B_1 and B_2 of $\iota P(\Lambda, \Lambda)$, if there is an invertible matrix $\bar{A}$ with entries in $\Lambda/\ker\rho$ with $Q(A)\bar{F}_{\ker\rho}(B_1) = \bar{F}_{\ker\rho}(B_2)$, then there exists an invertible lift of $\bar{A}$, denoted A, such that $A^T B_1 A = B_2$.

Write $A = \bar{A} + V$, where we view $\bar{A}$ as a matrix with entries in Λ by sending $[A_{ij}]$ to A_{ij}, and where V has entries in $\ker\rho$. We aim to show that V can be chosen in such a way that $A^T B_1 A = B_2$. Since $\bar{F}_{\ker\rho}$ is the identity on diagonal elements, we have that $\bar{A}^T B_1 \bar{A}$ and B_2 have the same diagonal entries, and that their non-diagonal entries differ by elements of $\ker\rho$. Therefore write

$$B_2 = U^T + U + (\bar{A})^T B_1 (\bar{A})$$

where U is upper triangular with entries in $\ker\rho$. Further setting $(\bar{A} + V)^T B_1 (\bar{A} + V) = B_2$, we have

$$B_2 = (\bar{A} + V)^T B_1 (\bar{A} + V) = U^T + U + (\bar{A})^T B_1 (\bar{A})$$

Multiplying out the brackets gives

$$\bar{A}^T B_1 \bar{A} + V^T B_1 \bar{A} + \bar{A}^T B_1 V + V^T B_1 V = U^T + U + (\bar{A})^T B_1 (\bar{A})$$

Now, our assumption that $(\ker\rho)^2 = 0$ implies that $V^T B_1 V = 0$, and symmetry of B_1 implies that $(V^T B_1 \bar{A})^T = \bar{A}^T B_1 V$, so that we can solve the equation, and complete the proof, if we can show that V can be chosen such that

$$\bar{A}^T B_1 V = U$$

Since we assume $\bar{A}^T$ and B_1 are invertible, set $V = (\bar{A}^T)^{-1} B_1^{-1} U$. $\qquad\square$

Proposition 5.3.7. *Under the assumptions of Lemmas 5.3.2, 5.3.4, and 5.3.6 the functor defined in Lemmas 5.3.1 and in the discussion before 5.3.6 induces an isomorphism from the abelian monoid of isometry classes of symmetric bilinear forms over Λ to the abelian monoid of isometry classes of $(\Lambda/\ker\rho, \Lambda)$-quadratic forms.*

Proof. Lemma 5.3.2 implies the induced map is surjective, and Lemmas 5.3.4 and 5.3.6 implies it is injective. Therefore, all that has to be checked is that the map respects orthogonal sums. For any commutative ring R, ideal $I \subset R$, and R-modules M and N, we have

$$(M \oplus N)/(M \oplus N)I \cong M/IM \oplus N/IN.$$

Moreover, for (P_1, β_1) and (P_2, β_2), finitely generated projective symmetric bilinear form Λ-modules, we have

$$
\begin{aligned}
q_{\beta_1 \perp \beta_2} \overline{(x_1, x_2)} &= (\beta_1 \perp \beta_2)((x_1, x_2), (x_1, x_2)) \\
&= \beta_1(x_1, x_1) + \beta_2(x_2, x_2) \\
&= q_{\beta_1}(\bar{x}_1) + q_{\beta_2}(\bar{x}_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\beta_1 \perp \beta_2}(\overline{(x_1, x_2)}, \overline{(y_1, y_2)}) &= \overline{\beta_1 \perp \beta_2((x_1, x_2), (y_1, y_2))} \\
&= \overline{\beta_1(x_1, y_1)} + \overline{\beta_2(x_2, y_2)} \\
&= \bar{\beta}_1(\bar{x}_1, \bar{y}_1) + \bar{\beta}(\bar{x}_2, \bar{y}_2),
\end{aligned}
$$

so that the functor respects orthogonal sum and we therefore have an isomorphism of abelian monoids. $\qquad\square$

Corollary 5.3.8. *Let (R, Λ) be a commutative form ring where R has trivial involution and surjective restriction, finitely generated projective R-modules are free, the ideal J as defined in Proposition 5.2.2 is equal to 0, and the ideal $\ker\rho$ of Λ has square 0. Then $GW_0(\Lambda) \cong GW_0(R, \Lambda)$.*

Proof. Proposition 5.3.7 implies $GW_0(\Lambda) \cong GW_0(\Lambda/\ker\rho, \Lambda)$. The result follows from the fact that $\rho : \Lambda/\ker\rho \to R$ is an isomorphism by the first isomorphism theorem. $\square$

The last step is to show that Corollary 5.3.8 can be applied in the case where $(R, \Lambda) = (\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$, the Burnside form ring given in Definition 3.2.11 where $\mathbb{F}_q$ is a finite field of characteristic 2.

First, we show that $\underline{\mathbb{F}_q}$ has surjective restriction.

Lemma 5.3.9. *Let $\mathbb{F}_q$ be a finite field of characteristic 2. Then its Burnside form ring $(\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$ has surjective restriction.*

Proof. Recall that $\mathbb{A}(\mathbb{F}_q)$ is generated by symbols

$$[x], \quad [y, \lambda]$$

where $x, y \in \mathbb{F}_q$ and $\lambda \in \mathbb{Z}[C_2]$, subject to the relations 1-6 given in Definition 3.2.8. Denoting the unique ring homomorphism $\mathbb{Z} \to \mathbb{F}_q$ by f, the restriction is given by

$$[x] \mapsto 2x = 0$$
$$[y, \lambda] \mapsto y^2 f(\rho(\lambda)).$$

In particular, for any $x \in \mathbb{F}_q$, the element $[x, 1]$ is mapped to x^2. This means that, if we can show that finite fields of characteristic 2 are quadratically closed, we have the required result.

To see this, consider the squaring map $\mathbb{F}_q^* \xrightarrow{(-)^2} \mathbb{F}_q^*$. Since characteristic 2 implies $1 = -1$, this map has trivial kernel and is therefore injective. Since it is, in particular, a map of finite sets, injectivity implies surjectivity, so that $\mathbb{F}_q$ contains all square roots as required, and $\underline{\mathbb{F}_q}$ has surjective restriction. $\square$

Next, we show that $J = 0$ in $\underline{\mathbb{F}_q}$.

Lemma 5.3.10. *Let $\mathbb{F}_q$ be a finite field of characteristic 2, and let $\underline{\mathbb{F}_q} = (\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$ be its Burnside form ring as defined in 3.2.11. Then the ideal $J \subset \mathbb{A}(\mathbb{F}_q)$ as defined in Proposition 5.2.2 is equal to 0.*

Proof. Recall that $\mathbb{A}(\mathbb{F}_q)$ is generated by symbols

$$[x], \quad [y, \lambda]$$

where $x, y \in \mathbb{F}_q$ and $\lambda \in \mathbb{Z}[C_2]$, subject to the relations 1-6 given in Definition 3.2.8. Recall that, for a general commutative form ring (R, Λ), $J \subset \Lambda$ is the ideal generated by

$$\tau(1) - 2, \quad \eta(\rho(\lambda)) - \lambda^2$$

where λ ranges over all of Λ. Denote the unique ring homomorphism $\mathbb{Z} \to \mathbb{F}_q$ by f. In the specific situation of $\underline{\mathbb{F}_q}$, the restriction is given by

$$[x] \mapsto 2x = 0$$
$$[y, \lambda] \mapsto y^2 f(\rho(\lambda)),$$

the transfer is given by

$$x \mapsto [x]$$

and the map controlling the quadratic action is given by

$$x \mapsto [x, 1]$$

This means that we have proven that $J = 0$, at least when λ is one of the symbols that generate $\mathbb{A}(\mathbb{F}_q)$, if we can show that

$$2 = [1, 1] + [1, 1] = [1] \in \mathbb{A}(\mathbb{F}_q),$$

$$[y, \lambda]^2 = [y^2 f \rho(\lambda), 1] \in \mathbb{A}(\mathbb{F}_q),$$

and

$$[y]^2 = [2y, 1] = [0, 1] = 0 \in \mathbb{A}(\mathbb{F}_q),$$

where $x, y \in \mathbb{F}_q$, and $\lambda \in \mathbb{Z}[C_2]$.

For the first equation, note that relation 5 in Definition 3.2.8 implies

$$[1, 1] + [1, 1] = [2, 1] - [1]$$
$$= [1]$$

where the second equality is implied by the assumption of characteristic 2.

For the second equation, apply relation 6 to obtain

$$[y^2 f\rho(\lambda), 1] = [y^2, \eta(\rho(\lambda))]$$

Notice that relation 6 also implies $[x, t] = [-x, 1] = [x, 1]$ since characteristic 2 implies $x = -x$. Since $\rho(\lambda_1 + \lambda_2 t) = \lambda_1 + \lambda_2$, this fact together with relation 4 from Definition 3.2.8 implies, for all $x \in \mathbb{F}_q$ and $\lambda \in \mathbb{Z}[C_2]$, we have

$$[x, \lambda] = [x, \rho(\lambda)]$$

where we view $\rho(\lambda) \in \mathbb{Z}$ as an element of $\mathbb{Z}[C_2]$ by composing with the map $a \mapsto a + 0t \in \mathbb{Z}[C_2]$. Applying this, we have

$$[y^2, \eta(\rho(\lambda))] = [y^2, \rho(\eta(\rho(\lambda)))]$$
$$= [y^2, \rho(\lambda)^2]$$
$$= [y^2, \rho(\lambda^2)]$$
$$= [y^2, \lambda^2] = [y, \lambda]^2$$

as required.

For the final equation, recall that the multiplication on $\mathbb{A}(\mathbb{F}_q)$ as given in Lemma 3.2.10 says

$$[y]^2 = [y^2] + [y^2]$$

which, via relation 2 in Definition 3.2.8, is equal to

$$[y^2 + y^2]$$

which is equal to 0 by the assumption of characteristic 2. Since all three equations are true, $J = 0$ for $\underline{\mathbb{F}_q}$ on generators. It remains to see that this extends to the whole of $\underline{\mathbb{F}_q}$. It suffices to check that, for a general element

$$\mu = \sum_{i=1}^{n} [y_i, \lambda_i] + \sum_{j=1}^{m} [x_i]$$

we have $\mu^2 = \eta\rho(\mu)$. To see this, first note we have

$$\mu^2 = \left(\sum_{i=1}^{n} [y_i, \lambda_i] + \sum_{j=1}^{m} [x_j] \right)^2$$

$$= \left(\sum_{i=1}^{n} [y_i, \lambda_i] \right)^2 + \left(\sum_{j=1}^{m} [x_j] \right)^2 + 2 \left(\sum_{i=1}^{n} [y_i, \lambda_i] \cdot \sum_{j=1}^{m} [x_j] \right)$$

$$= \left(\sum_{i=1}^{n} [y_i, \lambda_i] \right)^2$$

$$= \sum_{i=1}^{n} [y_i^2, \lambda_i^2] + 2 \sum_{i<k}^{n} [y_i y_k, \lambda_i \lambda_k]$$

where the third equality is since characteristic 2 implies $[x]^2 = 2[x] = 0$ for any $x \in \mathbb{F}_q$.

Checking $\eta\rho$, we have

$$\eta\rho(\mu) = \eta \left(\sum_{i=1}^{n} y_i^2 f(\rho(\lambda_i)) \right)$$

$$= \left[\sum_{i=1}^{n} y_i^2 f(\rho(\lambda_i)), 1 \right]$$

$$= \sum_{i=1}^{n} [y_i^2 f(\rho(\lambda_i)), 1] + \sum_{i<k}^{n} [y_i^2 y_k^2 f(\rho(\lambda_i)) f(\rho(\lambda_k))]$$

which is equal to μ^2 by combining the facts

$$[y_i^2 f(\rho(\lambda_i))] = [y_i^2, \lambda_i^2]$$
$$[y_i^2 y_k^2 f(\rho(\lambda_i)) f(\rho(\lambda_k))] = [1] \cdot [y_i y_k, \lambda_i \lambda_k]$$
$$[1] = 2.$$

The second of these facts is from the definition of the multiplication given in Lemma 3.2.10. We have already proven the first and third of them. $\square$

Since all modules over $\mathbb{F}_q$ are free, the final step is to show that the kernel of the restriction in $\underline{\mathbb{F}_q}$ squares to 0. We first show that, if $J = 0$, then all of the elements in the kernel are nilpotent.

Lemma 5.3.11. *Let (R, Λ) be a commutative form ring such that the ideal J as*

*defined in the statement of Proposition 5.2.2 is equal to 0. Then the ideal $\ker\rho \subset \Lambda$
is nil; specifically every element $\lambda \in \ker\rho$ squares to 0.*

Proof. Let $\lambda \in \ker\rho$. Since $J = 0$, we have $\eta(\rho(\lambda)) = \lambda^2 = 0$, as required. □

Lemma 5.3.12. *In the Burnside form ring $\underline{\mathbb{F}_q}$ for a finite field of characteristic 2,
the kernel of the restriction map $\mathbb{A}(\mathbb{F}_q) \to \mathbb{F}_q$ has square 0.*

Proof. Combining Lemmas 5.3.10 and 5.3.11, we obtain that every element in the
kernel squares to 0. We aim to show that $\ker\rho$ is the principal ideal generated by $[1]$.
This gives the required result, Lemma 5.3.10 implies $[1]^2 = 0$. To see this, consider
the diagram
$$\mathbb{F}_q \xrightarrow{\eta} \mathbb{A}(\mathbb{F}_q)/[1]\mathbb{A}(\mathbb{F}_q) \xrightarrow{\rho} \mathbb{F}_q$$
For all $x \in \mathbb{F}_q$, $\rho(\eta(x)) = x^2$. The proof of Lemma 5.3.9 shows that the squar-
ing map is an isomorphism of rings, where additivity comes from the assumption
of characteristic 2. Moreover, η is a ring homomorphism here, since $\eta(x + y) =
\eta(x) + \eta(y) + \tau(\rho(xy))$ and $\tau(\rho(xy)) = xy\tau(1) = xy[1]$.
In fact, η is surjective: to see this, recall that $\mathbb{A}(\mathbb{F}_q)$ is generated by elements of the
form $[x]$ for $x \in \mathbb{F}_q$ and $[y, \lambda]$ for $y \in \mathbb{F}_q$ and $\lambda \in \mathbb{Z}[C_2]$. Recall that $\eta(x) = [x, 1]$.
Using the fact that the squaring map is surjective, for each $x \in \mathbb{F}_q$, we can write
$[x] = [y^2] = [\rho(\eta)(y)] = [1] \cdot \eta(y)$. So every element of the form $[x]$ is zero after
quotienting by $[1]\mathbb{A}(\mathbb{F}_q)$. Moreover, every element of the form $[y, \lambda]$ is in the image
of η, since, as in the proof of Lemma 5.3.10, $[y, \lambda] = [y, \rho(\lambda)]$, so that any element of
the form $[y, \lambda]$ can be obtained by adding copies of $[y, 1]$, which is in the image of η.
Now, the fact that $\rho\eta$ is equal to the ring isomorphism $(-)^2$ implies that η is in-
jective, so that η is an isomorphism, which, together with the fact that $(-)^2$ is an
isomorphism, implies that $\rho : \mathbb{A}(\mathbb{F}_q)/[1]\mathbb{A}(\mathbb{F}_q) \to \mathbb{F}_q$ is an isomorphism. Therefore
we have that $\ker\rho = [1]\mathbb{A}(\mathbb{F}_q)$, so that $(\ker\rho)^2 = 0$ as required. □

Theorem 5.3.13. *Let $\mathbb{F}_q$ be a finite field of characteristic 2, and let $\underline{\mathbb{F}_q} = (\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$
be its Burnside form ring as defined in Definition 3.2.11. Then we have*
$$GW_0(\underline{\mathbb{F}_q}) \cong GW_0(\mathbb{A}(\mathbb{F}_q)),$$
*where the right hand group is the Grothendieck-Witt group of symmetric bilinear
forms.*

Proof. Lemmas 5.3.9, 5.3.10, and 5.3.12 combine to imply that $\underline{\mathbb{F}_q}$ fulfils the hypotheses of Corollary 5.3.8. The result follows. $\qquad\square$

Corollary 5.3.14. *Let $\mathbb{F}_q$ be a finite field. Denoting its Burnside form ring $\underline{\mathbb{F}_q} = (\mathbb{F}_q, \mathbb{A}(\mathbb{F}_q))$, we have*

$$GW_0(\underline{\mathbb{F}_q}) \cong \begin{cases} \mathbb{F}_q \oplus GW_0(\mathbb{F}_q) & \text{if } \mathbb{F}_q \text{ has odd characteristic} \\ GW_0(\mathbb{A}(\mathbb{F}_q)) & \text{if } \mathbb{F}_q \text{ has characteristic 2} \end{cases}$$

Proof. Theorem 5.3.13 covers the characteristic 2 case. Theorem 4.1.20 covers the odd characteristic case. $\qquad\square$

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