Uniqueness and error analysis for Hamilton–Jacobi equations with discontinuities

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We consider the Hamilton–Jacobi equation of eikonal type
$$H(\nabla u) = f(x), \quad x \in \Omega,$$
where H is convex and f is allowed to be discontinuous. Under a suitable assumption on f we prove a comparison principle for viscosity sub- and supersolutions in the sense of Ishii. Furthermore, we develop an error analysis for a class of finite difference schemes, which are monotone, consistent and satisfy a suitable stability condition.

Keywords: Hamilton–Jacobi equation; viscosity solution; comparison principle; finite difference method; error bounds.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We consider the Hamilton–Jacobi equation

$$H(\nabla u) = f(x), \quad x \in \Omega, \tag{1.1}$$

$$u(x) = \phi(x), \quad x \in \partial\Omega, \tag{1.2}$$

where f and ϕ are given functions. The equation (1.1) occurs in a variety of applications including geometrical optics, computer vision and etching. In order to motivate the link to propagating fronts, let us suppose for a moment that such a front at time t can be described as the t-level set of an auxiliary function $u : \bar{\Omega} \to \mathbb{R}$, i.e. $\Gamma(t) = \{x \in \Omega \mid u(x) = t\}$. Then, formally, a unit normal ν to $\Gamma(t)$ and the corresponding normal velocity V are given by

$$\nu = \frac{\nabla u(x)}{|\nabla u(x)|}, \quad V = \frac{1}{|\nabla u(x)|}, \quad x \in \Gamma(t).$$

If in addition, u solves (1.1), f is positive and H is homogeneous of degree one, then

$$V = \frac{1}{|\nabla u(x)|} = \frac{1}{|\nabla u(x)|} \frac{H(\nabla u(x))}{f(x)} = \frac{1}{f(x)} H(\nu(x)), \quad x \in \Gamma(t),$$

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i.e. the front moves with a normal velocity which depends on the properties of the underlying space and the orientation of the front. Furthermore, we can interpret $u(x)$ as the first arrival time at the point x of a front which was the zero level set of u at the beginning of the evolution.

It is often desirable to consider situations in which the function f is allowed to be discontinuous, e.g. in geometrical optics, when light propagates through a layered medium. Thus, we shall be concerned both with the well-posedness of (1.1), (1.2) in the case that f is discontinuous and with the convergence of numerical algorithms which approximate the corresponding solution.

Various ways of defining a solution of (1.1), (1.2) for discontinuous f have been suggested. First, Ishii [8] (see §2 below) extended the concept of viscosity solution introduced in [5] to the case of discontinuous Hamiltonians. He obtained existence and uniqueness results for an equation of the form $H(x, u(x), \nabla u(x)) = 0$, in which the Hamiltonian $H(x, u, p)$ is allowed to be discontinuous in u. Using this notion of solution, Soravia [15] studies a class of Hamilton–Jacobi equations in a control-theoretic framework and gives necessary and sufficient conditions for uniqueness of solutions of boundary value problems. In [10], Newcomb & Su introduce a concept of solution for (1.1), (1.2) which is based on the optical length function $L(x, y)$ (see §2 below) and which they term Monge solution. For lower semicontinuous f they prove a comparison principle as well as existence and uniqueness for the Dirichlet problem. Recently, a further definition of solution was suggested in [4] by Camilli & Siconolfi for Hamilton–Jacobi equations of the form $H(x, Du) = 0$. They introduce a generalized notion of viscosity solution, which allows measurable dependence of H on x. In the case of the eikonal equation, i.e. $H(p) = |p|$, this definition involves the measure-theoretic notion of an approximate limit for a subsolution and an essential limit for a supersolution and hence is not symmetric. Comparison and uniqueness results are provided.

Problem (1.1), (1.2) also occurs in shape-from-shading. In this case, the right hand side f is related to the light intensity which can be discontinuous. In [16], Tourin establishes a comparison result for equations of the form $H(x, Du) = 0$, in which H is allowed to be discontinuous along a smooth surface. For the shape-from-shading problem, Rouy & Tourin [13] present a consistent and monotone scheme along with numerical calculations. A related problem is studied in [12], where the unique solution is obtained as the limit of sequences which arise from a suitable regularization of the intensity function.

Our work is based on Ishii's definition of solution, but let us emphasize that even though we allow discontinuities of f the solutions themselves will be Lipschitz continuous. In order to obtain uniqueness for the solution of (1.1), (1.2) an assumption on f is needed: this condition (see (F2) below) can be seen as a generalization of a condition which appears in [16], and it amounts to a one-sided continuity constraint along a fixed direction at each point in Ω. Under this assumption we are able to prove a comparison result in Theorem 2.3. In §3 we analyze a class of numerical schemes which approximate the solution of (1.1), (1.2). Denoting by h the grid size we obtain an order $\mathcal{O}(\sqrt{h})$ for finite difference schemes which are monotone, consistent and satisfy a suitable stability condition. In §4 we present examples of schemes which satisfy the above requirements, while §5 contains numerical tests.

2. Existence and uniqueness

Let us start by defining a viscosity solution of (1.1), (1.2). As already mentioned above, we use a concept which was introduced by Ishii in [8] and which is based on upper and lower semicontinuous envelopes. For a given function $v : \Omega \to \mathbb{R}$ let

$$v^*(x) := \limsup_{r \to 0} \{ v(y) \mid y \in B_r(x) \cap \Omega \},$$

$$v_*(x) := \liminf_{r \to 0} \{ v(y) \mid y \in B_r(x) \cap \Omega \}.$$

DEFINITION 2.1 A function $u \in C^0(\bar{\Omega})$ is called a *viscosity subsolution* (resp. *supersolution*) of (1.1) if for each $\zeta \in C^\infty(\Omega)$: if $u - \zeta$ has a local maximum (resp. minimum) at a point $x_0 \in \Omega$, then

$$H(\nabla \zeta(x_0)) \leqslant f^*(x_0) \quad (\text{resp. } \geqslant f_*(x_0)).$$

A *viscosity solution* of (1.1), (1.2) is then a function $u \in C^0(\bar{\Omega})$ which is both a viscosity sub- and supersolution and which satisfies $u(x) = \phi(x)$ for all $x \in \partial \Omega$.

Assumptions on the data. We shall assume that $H : \mathbb{R}^n \to \mathbb{R}$ satisfies:

(H1) $H(0) = 0$ and $H(p) > 0$ for all $p \in \mathbb{R}^n \setminus \{0\}$.

(H2) H is convex.

(H3) $H(p) \to \infty$ as $|p| \to \infty$.

Concerning the right hand side we make the following assumptions:

(F1) $f : \Omega \to \mathbb{R}$ is Borel measurable and there exist $0 < m \leqslant M < \infty$ such that

$$m \leqslant f(x) \leqslant M \quad \forall x \in \Omega. \tag{2.1}$$

(F2) For every $x \in \Omega$ there exist $\epsilon_x > 0$ and $n_x \in S^{n-1}$ so that for all $y \in \Omega, r > 0, d \in S^{n-1}$ with $|d - n_x| < \epsilon_x$ and $y + rd \in \Omega$ we have

$$f(y + rd) - f(y) \leqslant \omega(|y - x| + r), \tag{2.2}$$

where $\omega : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function with $\omega(0) = 0$.

For later purposes we shall also formulate a stronger version of this assumption, namely

(F2)* There exist $\epsilon > 0$ and $K \geqslant 0$ such that for all $x \in \Omega$ there is a direction $n = n_x \in S^{n-1}$ with

$$f(y + rd) - f(y) \leqslant Kr \tag{2.3}$$

for all $y \in \Omega, d \in S^{n-1}, r > 0$ with $|y - x| < \epsilon, |d - n| < \epsilon$ and $y + rd \in \Omega$.

Clearly, (F2) holds at all points x at which f is continuous, but it also allows for certain types of discontinuous behavior as shown by the following

EXAMPLE. Suppose that a surface Γ splits Ω into two subdomains Ω_1 and Ω_2 such that $f_{|\Omega_1} \in C^0(\bar{\Omega}_1), f_{|\Omega_2} \in C^0(\bar{\Omega}_2)$ and

$$\lim_{y \to x, \, y \in \Omega_1} f(y) < \lim_{y \to x, \, y \in \Omega_2} f(y) \quad \text{for all } x \in \Gamma.$$

In addition, assume that the following uniform cone property holds: for every $x \in \Gamma$ there exists a neighborhood U_x and a cone C_x (which is congruent to a fixed given cone C_0) such that $y \in U_x \cap \bar{\Omega}_1$ implies that $y + C_x \subset \Omega_1$. Then (F2) holds with $n = n_x$ given by the direction of the cone C_x.

To see this, observe that the cone condition prevents a situation where $y \in \bar{\Omega}_1$, $y + rd \in \Omega_2$, which would lead to a violation of (F2) (cf. [16], where Γ is assumed to be smooth).

One can also consider e.g. a two-dimensional domain Ω, where three curves of discontinuity meet at a triple junction. A suitable modification of the above also yields an example for $(F2)^*$.

In order to describe our assumptions on ϕ let us define $L : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$L(x, y) := \inf \left\{ \int_0^1 N(f^*(\gamma(t)), \gamma'(t)) \, dt \;\middle|\; \gamma \in W^{1,\infty}((0, 1); \bar{\Omega}) \text{ with } \gamma(0) = x, \ \gamma(1) = y \right\},$$

where

$$N(r, \zeta) := \sup\{-(\zeta, p) \mid H(p) = r\}.$$

We then suppose the following compatibility condition for the boundary data,

(C) $\phi(x) - \phi(y) \leqslant L(x, y)$ for all $x, y \in \partial\Omega$.

We shall assume (H1)–(H3) throughout the paper, while the corresponding assumptions on f and ϕ will be stated explicitly.

THEOREM 2.2 Assume that (F1) and (C) hold. Then there exists a viscosity solution $u \in C^{0,1}(\bar{\Omega})$ of (1.1), (1.2).

Proof.   We regularize f using the sup-convolution, i.e.

$$f_\epsilon(x) := \sup_{y \in \Omega} \left\{ f(y) - \frac{1}{\epsilon} |x - y|^2 \right\}, \quad \epsilon > 0.$$

Clearly, f_ϵ is continuous and satisfies $f^*(x) \leqslant f_\epsilon(x)$ for all $x \in \Omega$. In view of assumption (C) and the monotonicity of N in the first variable we deduce that

$$\phi(x) - \phi(y) \leqslant L_\epsilon(x, y) \quad \forall x, y \in \partial\Omega,$$

where

$$L_\epsilon(x, y) := \inf \left\{ \int_0^1 N(f_\epsilon(\gamma(t)), \gamma'(t)) \, dt \;\middle|\; \gamma \in W^{1,\infty}((0, 1); \bar{\Omega}) \text{ with } \gamma(0) = x, \ \gamma(1) = y \right\}.$$

Therefore, the problem

$$H(\nabla u^\epsilon) = f_\epsilon(x), \quad x \in \Omega,$$
$$u^\epsilon(x) = \phi(x), \quad x \in \partial\Omega,$$

has a unique viscosity solution u^ϵ, which is given by the formula

$$u^\epsilon(x) = \inf_{y \in \partial\Omega} \{L_\epsilon(x, y) + \phi(y)\}.$$

It is not difficult to verify that

$$\|u^\epsilon\|_{C^{0,1}(\bar{\Omega})} \leqslant C(M, \Omega) \quad \text{uniformly in } \epsilon > 0.$$

Thus, there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ with $\epsilon_k \searrow 0$ as $k \to \infty$ and $u \in C^{0,1}(\bar{\Omega})$ such that

$$u^{\epsilon_k} \to u \text{ as } k \to \infty \quad \text{uniformly in } \bar{\Omega}. \tag{2.4}$$

Clearly, $u = \phi$ on $\partial\Omega$. We claim that u is a viscosity solution in the sense of Definition 2.1. Let $\zeta \in C^\infty(\Omega)$ and suppose that $u - \zeta$ has a local maximum at $x_0 \in \Omega$. In view of (2.4) there exist $x_k \in \Omega$ such that $x_k \to x_0$ as $k \to \infty$ and $u^{\epsilon_k} - \zeta$ has a local maximum at x_k. Then

$$H(\nabla\zeta(x_k)) \leqslant f_{\epsilon_k}(x_k), \tag{2.5}$$

and taking into account (2.1) we obtain

$$f_{\epsilon_k}(x_k) = \sup_{|x_k - y| \leqslant \sqrt{M\epsilon_k}} \left\{ f(y) - \frac{1}{\epsilon_k}|x_k - y|^2 \right\} \leqslant \sup\{f(y) \mid |y - x_0| \leqslant |x_k - x_0| + \sqrt{M\epsilon_k}\},$$

which implies by passing to the limit in (2.5)

$$H(\nabla\zeta(x_0)) \leqslant \limsup_{k \to \infty} f_{\epsilon_k}(x_k) \leqslant f^*(x_0).$$

On the other hand, if $u - \zeta$ has a local minimum at x_0, there exist $\tilde{x}_k \in \Omega$ with $\tilde{x}_k \to x_0$ as $k \to \infty$ such that $u^{\epsilon_k} - \zeta$ has a local minimum at $\tilde{x}_k$. Thus,

$$H(\nabla\zeta(\tilde{x}_k)) \geqslant f_{\epsilon_k}(\tilde{x}_k) \geqslant f_*(\tilde{x}_k).$$

Since f_* is lower semicontinuous, we deduce that

$$H(\nabla\zeta(x_0)) \geqslant f_*(x_0).$$

In conclusion, u is a viscosity solution of (1.1), (1.2). $\square$

Uniqueness of the viscosity solution is a consequence of the following comparison result.

THEOREM 2.3   Assume that (F1), (F2) hold, that $u \in C^0(\bar{\Omega})$ is a subsolution of (1.1), $v \in C^0(\bar{\Omega})$ is a supersolution of (1.1) and that at least one of the functions belongs to $C^{0,1}(\bar{\Omega})$. If $u \leqslant v$ on $\partial\Omega$ then $u \leqslant v$ in $\bar{\Omega}$.

Proof.   Let us assume that $v \in C^{0,1}(\bar{\Omega})$. We shall use the approach presented in [9] (see also [16]). Fix $\theta \in (0, 1)$ and define $u_\theta(x) := \theta u(x)$. Next, choose $x_0 \in \bar{\Omega}$ such that

$$u_\theta(x_0) - v(x_0) = \max_{x \in \bar{\Omega}}(u_\theta(x) - v(x)) =: \mu, \tag{2.6}$$

and suppose that $\mu > 0$. Upon replacing u, v by $u + k, v + k$, we may assume that $u \geqslant 0$ in $\bar{\Omega}$, so that $u_\theta \leqslant u$ in $\bar{\Omega}$. In particular, $u_\theta \leqslant v$ on $\partial\Omega$, which implies that $x_0 \in \Omega$. Let $\epsilon = \epsilon_{x_0}$ and $n = n_{x_0} \in S^{n-1}$ be the quantities which appear in (F2). It is not difficult to verify that (F2) implies

$$f^*(y + rd) - f_*(y) \leqslant \omega(|y - x_0| + r) \tag{2.7}$$

for all $y \in \Omega, r > 0$ and $d \in S^{n-1}, |d - n| < \epsilon$.

Let us define for $\lambda > 0$, $L \geqslant 1$ the function $\Phi : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$\Phi(x, y) := u_\theta(x) - v(y) - L\lambda\left|x - y - \frac{1}{\lambda}n\right|^2 - |x - x_0|^2.$$

Let $(x_\lambda, y_\lambda) \in \bar{\Omega} \times \bar{\Omega}$ be such that

$$\Phi(x_\lambda, y_\lambda) = \max_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} \Phi(x, y). \tag{2.8}$$

Since $x_0 - \frac{1}{\lambda}n \in \Omega$ for large λ, the relation $\Phi(x_\lambda, y_\lambda) \geqslant \Phi(x_0, x_0 - \frac{1}{\lambda}n)$ implies together with (2.6) that

$$L\lambda\left|x_\lambda - y_\lambda - \frac{1}{\lambda}n\right|^2 + |x_\lambda - x_0|^2 \leqslant u_\theta(x_\lambda) - v(y_\lambda) - u_\theta(x_0) + v\left(x_0 - \frac{1}{\lambda}n\right)$$

$$= (u_\theta(x_\lambda) - v(x_\lambda)) - (u_\theta(x_0) - v(x_0)) + v(x_\lambda) - v(y_\lambda) - v(x_0) + v\left(x_0 - \frac{1}{\lambda}n\right)$$

$$\leqslant \mathrm{lip}(v)|x_\lambda - y_\lambda| + \mathrm{lip}(v)\frac{1}{\lambda}$$

$$\leqslant \mathrm{lip}(v)\left|x_\lambda - y_\lambda - \frac{1}{\lambda}n\right| + 2\,\mathrm{lip}(v)\frac{1}{\lambda},$$

where $\mathrm{lip}(v)$ denotes the Lipschitz constant of v. Therefore,

$$L\lambda\left|x_\lambda - y_\lambda - \frac{1}{\lambda}n\right|^2 + |x_\lambda - x_0|^2 \leqslant C(\mathrm{lip}(v))\frac{1}{\lambda}, \tag{2.9}$$

so that

$$x_\lambda, y_\lambda \to x_0 \quad \text{as } \lambda \to \infty, \tag{2.10}$$

$$\lambda\left|x_\lambda - y_\lambda - \frac{1}{\lambda}n\right| \leqslant \frac{C}{\sqrt{L}} < \frac{\epsilon}{2+\epsilon} \tag{2.11}$$

provided that L is sufficiently large. Next, (2.8) implies that $u - \frac{1}{\theta}\zeta$ has a local maximum at x_λ where $\zeta(x) = \tilde{v}(y_\lambda) + L\lambda|x - y_\lambda - \frac{1}{\lambda}n|^2 + |x - x_0|^2$. Therefore,

$$H\left(\frac{1}{\theta}\left(2L\lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n\right) + 2(x_\lambda - x_0)\right)\right) \leqslant f^*(x_\lambda), \tag{2.12}$$

and similarly,

$$H\left(2L\lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n\right)\right) \geqslant f_*(y_\lambda). \tag{2.13}$$

Combining (2.12) and (2.13) and using assumptions (H1), (H2) we obtain

$$f_*(y_\lambda) \leqslant \theta H\left(\frac{1}{\theta}2L\lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n\right)\right) \tag{2.14}$$

$$\leqslant \theta\left(H\left(\frac{1}{\theta}2L\lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n\right)\right) - H\left(\frac{1}{\theta}\left(2L\lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n\right) + 2(x_\lambda - x_0)\right)\right)\right) + \theta f^*(x_\lambda).$$

Note that H is locally Lipschitz continuous (since it is convex) so that we may deduce from (2.11)

$$\left| H\left(\frac{1}{\theta} 2L\lambda \left(x_\lambda - y_\lambda - \frac{1}{\lambda} n \right) \right) - H\left(\frac{1}{\theta}\left(2L\lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n \right) + 2(x_\lambda - x_0) \right) \right) \right| \leqslant C|x_\lambda - x_0|.$$

Inserting this inequality into (2.14) we arrive at

$$(1 - \theta)f_*(y_\lambda) \leqslant C|x_\lambda - x_0| + \theta(f^*(x_\lambda) - f_*(y_\lambda)). \tag{2.15}$$

In order to treat the second term we write $x_\lambda = y_\lambda + r_\lambda d_\lambda$ with

$$d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda}|n + w_\lambda|, \quad w_\lambda = \lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n \right).$$

Now, (2.11) implies

$$|d_\lambda - n| \leqslant \frac{2|w_\lambda|}{1 - |w_\lambda|} \leqslant \frac{\frac{2\epsilon}{2+\epsilon}}{1 - \frac{\epsilon}{2+\epsilon}} = \epsilon \tag{2.16}$$

so that (2.7) yields

$$f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \leqslant \omega(|y_\lambda - x_0| + r_\lambda).$$

If we insert this estimate into (2.15) and recall (2.1) the result is

$$m(1 - \theta) \leqslant C|x_\lambda - x_0| + \omega(|y_\lambda - x_0| + r_\lambda).$$

Sending $\lambda \nearrow \infty$ yields $m(1 - \theta) \leqslant 0$ in view of (2.10), which is a contradiction. Thus, $u_\theta \leqslant v$ for all $\theta < 1$ and sending $\theta \nearrow 1$ finally yields the result. $\square$

3. Numerical scheme and error analysis

Numerical schemes on uniform grids for Hamilton–Jacobi equations have been developed on the basis of an upwind discretisation of the gradient; see [11], [14] for an outline of the underlying ideas. It is possible to derive these schemes as well as methods on unstructured grids by interpreting the corresponding viscosity solution as the value function of an optimal control problem and by using the dynamic programming principle. We refer to Appendix A, written by M. Falcone, in [1] for a description of basic results together with a comprehensive list of references.

There is a close connection between the static Hamilton–Jacobi equation (1.1) and time-dependent problems of the form

$$u_t + H(\nabla u) = 0. \tag{3.1}$$

Error estimates for finite difference approximations of (3.1) have been proved in [6], while [7] introduces a class of semi-Lagrangian schemes which are strongly related to Godunov methods in one space dimension.

We shall start from a class of finite difference schemes which are monotone and consistent and which satisfy a suitable stability condition. In order to keep the presentation simple we shall from now on assume that $\Omega = \prod_{i=1}^{n}(0, b_i)$. Let $h > 0$ be such that there exist $N_i \in \mathbb{N}$ with $b_i = N_i h, i = 1, \dots, n$, and define

$$\Omega_h := \mathbb{Z}_h^n \cap \Omega, \quad \partial\Omega_h := \mathbb{Z}_h^n \cap \partial\Omega, \quad \bar{\Omega}_h := \Omega_h \cup \partial\Omega_h,$$

where $\mathbb{Z}_h^n = \{x_\alpha = (h\alpha_1, \ldots, h\alpha_n) \mid \alpha_i \in \mathbb{Z}, i = 1, \ldots, n\}$. We shall approximate the viscosity solution u by a grid function $U : \bar{\Omega}_h \to \mathbb{R}, U_\alpha = U(x_\alpha), \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. For $x_\alpha \in \Omega_h$ and $k \in \{1, \ldots, n\}$ let

$$D_k^- U_\alpha := \frac{U_\alpha - U_{\alpha - e_k}}{h}, \quad D_k^+ U_\alpha := \frac{U_{\alpha + e_k} - U_\alpha}{h}$$

be the usual backward and forward difference quotients. The numerical scheme now reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$H_N(D_1^- U_\alpha, D_1^+ U_\alpha, \ldots, D_n^- U_\alpha, D_n^+ U_\alpha) = f(x_\alpha), \quad x_\alpha \in \Omega_h, \tag{3.2}$$

$$U_\alpha = \phi(x_\alpha), \quad x_\alpha \in \partial\Omega_h, \tag{3.3}$$

where $H_N : \mathbb{R}^{2n} \to \mathbb{R}, (p_1, q_1, \ldots, p_n, q_n) \mapsto H_N(p_1, q_1, \ldots, p_n, q_n)$ is the numerical Hamiltonian. It is convenient to also introduce $F_N : \mathbb{R}^{2n+1} \to \mathbb{R}, a = (a_0, a_1, a_2, \ldots, a_{2n-1}, a_{2n}) \mapsto F_N(a)$, as

$$F_N(a) := H_N(a_0 - a_1, a_2 - a_0, \ldots, a_0 - a_{2n-1}, a_{2n} - a_0). \tag{3.4}$$

In what follows we shall assume that H_N is locally Lipschitz continuous and has the following properties:

(HN1) *Consistency:*

$$H_N(p_1, p_1, \ldots, p_n, p_n) = H(p_1, \ldots, p_n) \quad \text{for all } p = (p_1, \ldots, p_n) \in \mathbb{R}^n. \tag{3.5}$$

(HN2) *Monotonicity:*

$$a_0 \mapsto F_N(a) \text{ is increasing}, \tag{3.6}$$

$$a_k \mapsto F_N(a) \text{ is decreasing for } k = 1, \ldots, 2n. \tag{3.7}$$

(HN3) *Stability:* there exists a function $Z : \bar{\Omega}_h \to \mathbb{R}$ which satisfies

$$H_N(D_1^- Z_\alpha, D_1^+ Z_\alpha, \ldots, D_n^- Z_\alpha, D_n^+ Z_\alpha) \geqslant f(x_\alpha), \quad x_\alpha \in \Omega_h, \tag{3.8}$$

$$Z_\alpha = \phi(x_\alpha), \quad x_\alpha \in \partial\Omega_h, \tag{3.9}$$

$$|D_k^- Z_\alpha|, |D_k^+ Z_\alpha| \leqslant R, \quad x_\alpha \in \Omega_h, \tag{3.10}$$

where R is independent of h.

The function Z which appears in (HN3) will act as a discrete supersolution for solutions of (3.2), (3.3). We shall examine some examples of choices of H_N in §4.

REMARK 3.1 Note that the function Z in (HN3) above satisfies

$$Z_\alpha \geqslant \phi_{\min} := \min_{x \in \partial\Omega} \phi(x), \quad x_\alpha \in \bar{\Omega}_h. \tag{3.11}$$

To see this, let $Z_\beta = \min_{x_\alpha \in \bar{\Omega}_h} Z_\alpha$ and assume that $x_\beta \in \Omega_h$. Then, (2.1), (3.8) and (3.7) would imply

$$m \leqslant f(x_\beta) \leqslant H_N(D_1^- Z_\beta, D_1^+ Z_\beta, \ldots, D_n^- Z_\beta, D_n^+ Z_\beta)$$
$$= F_N\left(\frac{Z_\beta}{h}, \frac{Z_{\beta-e_1}}{h}, \frac{Z_{\beta+e_1}}{h}, \ldots, \frac{Z_{\beta-e_n}}{h}, \frac{Z_{\beta+e_n}}{h}\right) \leqslant F_N\left(\frac{Z_\beta}{h}, \frac{Z_\beta}{h}, \frac{Z_\beta}{h}, \ldots, \frac{Z_\beta}{h}, \frac{Z_\beta}{h}\right)$$
$$= H_N(0, \ldots, 0) = H(0) = 0$$

in view of (H1), a contradiction. Thus, $x_\beta \in \partial\Omega_h$ and (3.11) follows.

Next, let us prove an auxiliary result which yields a kind of diagonal coercivity for the numerical Hamiltonian.

LEMMA 3.2 For $(a_1, a_2, \ldots, a_{2n-1}, a_{2n}) \in \mathbb{R}^{2n}$,

$$\lim_{t \to \infty} F_N(t, a_1, a_2, \ldots, a_{2n-1}, a_{2n}) = \infty.$$

Proof. Using the defintion of F_N along with (3.5)–(3.7) we deduce for $t \geqslant r := \max(a_1, a_2, \ldots, a_{2n-1}, a_{2n})$ that

$$
\begin{aligned}
F_N(t, a_1, a_2, \ldots, a_{2n-1}, a_{2n}) &\geqslant F_N(t, r, r, \ldots, r, r) = H_N(t - r, r - t, \ldots, t - r, r - t) \\
&= F_N(t - r, 0, 0, \ldots, 0, 0) \\
&\geqslant F_N(t - r, 0, 2(t - r), \ldots, 0, 2(t - r)) \\
&= H_N(t - r, t - r, \ldots, t - r, t - r) = H(t - r, \ldots, t - r) \\
&\to \infty \quad \text{as } t \to \infty,
\end{aligned}
$$

which proves the lemma. □

LEMMA 3.3 Assume that (HN1)–(HN3) and (F1) hold. Then there exists a solution U of (3.2), (3.3), which satisfies $\phi_{\min} \leqslant U_\alpha \leqslant Z_\alpha$ for all $x_\alpha \in \bar{\Omega}_h$.

Proof. We consider the following iteration: let $U^0 := Z$ and given $U^k : \bar{\Omega}_h \to \mathbb{R}$, let

$$U_\alpha^{k+1} = \inf \left\{ t \; \middle| \; F_N\left(\frac{t}{h}, \frac{U_{\alpha-e_1}^k}{h}, \frac{U_{\alpha+e_1}^k}{h}, \ldots, \frac{U_{\alpha-e_n}^k}{h}, \frac{U_{\alpha+e_n}^k}{h}\right) \geqslant f(x_\alpha) \right\}, \quad x_\alpha \in \Omega_h,$$

$$U_\alpha^{k+1} = \phi(x_\alpha), \quad x_\alpha \in \partial\Omega_h.$$

We claim that the sequence $(U^k)_{k \in \mathbb{N}}$ is well defined and that

$$\phi_{\min} \leqslant U^k \leqslant U^{k-1} \leqslant Z \quad \text{for all } k \in \mathbb{N}. \tag{3.12}$$

To see this, assume that (3.12) holds for all $1 \leqslant j \leqslant k$ and consider, for $x_\alpha \in \Omega_h$,

$$\eta(t) := F_N\left(\frac{t}{h}, \frac{U_{\alpha-e_1}^k}{h}, \frac{U_{\alpha+e_1}^k}{h}, \ldots, \frac{U_{\alpha-e_n}^k}{h}, \frac{U_{\alpha+e_n}^k}{h}\right), \quad t \in \mathbb{R}.$$

Clearly, η is continuous and increasing. Lemma 3.2 implies that $\eta(t) \to \infty$ as $t \to \infty$ so that U_α^{k+1} is well defined. Since $U^k \geqslant \phi_{\min}$ by our induction hypothesis, (3.7), (3.5) and (H1) yield

$$
\begin{aligned}
\eta(\phi_{\min}) &= F_N\left(\frac{\phi_{\min}}{h}, \frac{U_{\alpha-e_1}^k}{h}, \frac{U_{\alpha+e_1}^k}{h}, \ldots, \frac{U_{\alpha-e_n}^k}{h}, \frac{U_{\alpha+e_n}^k}{h}\right) \\
&\leqslant F_N\left(\frac{\phi_{\min}}{h}, \ldots, \frac{\phi_{\min}}{h}\right) = H(0) = 0 < f(x_\alpha), \tag{3.13}
\end{aligned}
$$

which implies that $U_\alpha^{k+1} \geqslant \phi_{\min}$. Also, as $U^k \leqslant U^{k-1}$, (3.7) yields

$$
\begin{aligned}
\phi(U_\alpha^k) &= F_N\left(\frac{U_\alpha^k}{h}, \frac{U_{\alpha-e_1}^k}{h}, \frac{U_{\alpha+e_1}^k}{h}, \ldots, \frac{U_{\alpha-e_n}^k}{h}, \frac{U_{\alpha+e_n}^k}{h}\right) \\
&\geqslant F_N\left(\frac{U_\alpha^k}{h}, \frac{U_{\alpha-e_1}^{k-1}}{h}, \frac{U_{\alpha+e_1}^{k-1}}{h}, \ldots, \frac{U_{\alpha-e_n}^{k-1}}{h}, \frac{U_{\alpha+e_n}^{k-1}}{h}\right) \geqslant f(x_\alpha)
\end{aligned}
$$

by the definition of U_α^k. Thus $U_\alpha^{k+1} \leqslant U_\alpha^k$. Using similar arguments and recalling (3.11) we infer that (3.12) holds for $k = 1$, so that we finally obtain (3.12) for all $k \in \mathbb{N}$. Note also that

$$F_N\left(\frac{U_\alpha^{k+1}}{h}, \frac{U_{\alpha-e_1}^k}{h}, \frac{U_{\alpha+e_1}^k}{h}, \ldots, \frac{U_{\alpha-e_n}^k}{h}, \frac{U_{\alpha+e_n}^k}{h}\right) = f(x_\alpha), \quad x_\alpha \in \Omega_h,\, k \in \mathbb{N}. \qquad (3.14)$$

From (3.12) we infer that $U_\alpha^k \to U_\alpha$ for all $x_\alpha \in \bar{\Omega}_h$ as $k \to \infty$. Clearly, $U_\alpha = \phi(x_\alpha)$ for $x_\alpha \in \partial\Omega_h$. Letting $k \to \infty$ in (3.14) and using the continuity of F_N finally implies that U satisfies (3.2). The bounds on U follow from (3.12). $\qquad\square$

Our main result of this paragraph is an error bound between a discrete solution U and the viscosity solution u. It is natural to model the error analysis on the uniqueness proof, but in our case a closer inspection of the proof of Theorem 2.3 shows that it is not obvious how to control the difference between the viscosity solution u and an approximation U. Therefore we recall a different approach to prove uniqueness, namely to apply the Kruzhkov transform. The advantage is that the equation satisfied by the transformed function contains an additional term of zero order (cf. (3.15)), which is subsequently exploited.

THEOREM 3.4 Assume that (F1), (F2)*, (C) as well as (HN1)–(HN3) hold. Let u be the viscosity solution of (1.1), (1.2) and U a solution of (3.2), (3.3) which satisfies $\phi_{\min} \leqslant U \leqslant Z$. Then there exists a constant C, which is independent of h, such that

$$\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)| \leqslant C\sqrt{h}.$$

Proof. As mentioned above we introduce the Kruzhkov transform of u and U, i.e. $\tilde{u} : \bar{\Omega} \to \mathbb{R}$, $\tilde{U} : \bar{\Omega}_h \to \mathbb{R}$ which are defined by

$$\tilde{u}(x) := -e^{-u(x)}, \ x \in \bar{\Omega}, \quad \tilde{U}_\alpha := -e^{-U_\alpha}, \ x_\alpha \in \bar{\Omega}_h.$$

Clearly, $\tilde{u}(x) = -e^{-\phi(x)}$ for $x \in \partial\Omega$ and one verifies (cf. [5]) that $\tilde{u}$ is a viscosity supersolution of

$$f(x)\tilde{u} - \tilde{u}H\left(-\frac{1}{\tilde{u}}\nabla\tilde{u}\right) = 0 \qquad (3.15)$$

in the sense that if $\zeta \in C^\infty(\Omega)$ and $\tilde{u} - \zeta$ has a local minimum at a point $x_0 \in \Omega$, then

$$f_*(x_0)\tilde{u}(x_0) - \tilde{u}(x_0)H\left(-\frac{1}{\tilde{u}(x_0)}\nabla\zeta(x_0)\right) \geqslant 0. \qquad (3.16)$$

Note also that

$$\tilde{H}(x, r, p) := f(x)r - rH\left(-\frac{1}{r}p\right), \quad (x, r, p) \in \Omega \times \mathbb{R} \setminus \{0\} \times \mathbb{R}^n,$$

satisfies in view of (2.1) and the convexity of H (cf. [5])

$$\frac{\partial\tilde{H}}{\partial r}(x, r, p) = f(x) - H\left(-\frac{1}{r}p\right) + \left(DH\left(-\frac{1}{r}p\right), -\frac{1}{r}p\right) \geqslant f(x) \geqslant m \qquad (3.17)$$

uniformly in (x, r, p).

Next, let $x_\beta \in \bar{\Omega}_h$ be such that

$$|\tilde{u}(x_\beta) - \tilde{U}_\beta| = \max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha|$$

and assume that $\tilde{U}_\beta \geqslant \tilde{u}(x_\beta)$, the case $\tilde{u}(x_\beta) > \tilde{U}_\beta$ being treated in a similar way.

Let us first consider the situation when

$$(x_\beta)_i \leqslant \sqrt{h} \quad \text{or} \quad (x_\beta)_i \geqslant b_i - \sqrt{h} \quad \text{for some } i \in \{1, \dots, n\}. \tag{3.18}$$

In the first case, let $x_{\beta_0} = (\beta_1 h, \dots, \beta_{i-1}h, 0, \beta_{i+1}h, \dots, \beta_n h) \in \partial\Omega$ and $\tilde{Z}_\alpha := -e^{-Z_\alpha}$. Since $\tilde{u}(x_{\beta_0}) = -e^{-\phi(x_{\beta_0})} = \tilde{Z}_{\beta_0}$ we deduce with the help of (3.10) that

$$\tilde{U}_\beta - \tilde{u}(x_\beta) = (\tilde{U}_\beta - \tilde{u}(x_{\beta_0})) + (\tilde{u}(x_{\beta_0}) - \tilde{u}(x_\beta)) \leqslant (\tilde{Z}_\beta - \tilde{Z}_{\beta_0}) + (\tilde{u}(x_{\beta_0}) - \tilde{u}(x_\beta))$$
$$\leqslant (C(R) + \mathrm{lip}(\tilde{u}))|x_\beta - x_{\beta_0}| \leqslant (C(R) + \mathrm{lip}(\tilde{u}))\sqrt{h}.$$

Arguing in a similar way if $(x_\beta)_i \geqslant b_i - \sqrt{h}$ we conclude that

$$\max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| = \tilde{U}_\beta - \tilde{u}(x_\beta) \leqslant C\sqrt{h} \tag{3.19}$$

if (3.18) holds. Now we consider the case

$$\sqrt{h} < (x_\beta)_i < b_i - \sqrt{h} \quad \text{for } i = 1, \dots, n. \tag{3.20}$$

Let $\epsilon > 0$, $K \geqslant 0$, $n = n_{x_\beta}$ be the quantities appearing in (2.3) and define $\Phi : \bar{\Omega} \times \bar{\Omega}_h \to \mathbb{R}$ by

$$\Phi(x, x_\alpha) := \tilde{U}_\alpha - \tilde{u}(x) - \frac{L_1}{\sqrt{h}}|x_\alpha - x - \sqrt{h}\,n|^2 - L_2\sqrt{h}\,|x_\alpha - x_\beta|^2,$$

where $L_1, L_2 \geqslant 0$ are constants that do not depend on h and which will be chosen later. There exists $(x_h, x_{\alpha_h}) \in \bar{\Omega} \times \bar{\Omega}_h$ such that

$$\Phi(x_h, x_{\alpha_h}) = \max_{(x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h} \Phi(x, x_\alpha).$$

In view of (3.20) we have $x_\beta - \sqrt{h}\,n \in \bar{\Omega}$ and therefore

$$\Phi(x_h, x_{\alpha_h}) \geqslant \Phi(x_\beta - \sqrt{h}\,n, x_\beta)$$

or equivalently

$$\tilde{U}_{\alpha_h} - \tilde{u}(x_h) - \frac{L_1}{\sqrt{h}}|x_{\alpha_h} - x_h - \sqrt{h}\,n|^2 - L_2\sqrt{h}\,|x_{\alpha_h} - x_\beta|^2 \geqslant \tilde{U}_\beta - \tilde{u}(x_\beta - \sqrt{h}\,n). \tag{3.21}$$

This implies

$$\frac{L_1}{\sqrt{h}} |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 + L_2 \sqrt{h}\, |x_{\alpha_h} - x_\beta|^2$$

$$\leqslant \tilde{u}(x_\beta - \sqrt{h}\, n) - \tilde{u}(x_h) + \tilde{U}_{\alpha_h} - \tilde{U}_\beta$$

$$\leqslant \tilde{u}(x_{\alpha_h}) - \tilde{u}(x_h) + ((\tilde{U}_{\alpha_h} - \tilde{u}(x_{\alpha_h})) - (\tilde{U}_\beta - \tilde{u}(x_\beta))) + \tilde{u}(x_\beta - \sqrt{h}\, n) - \tilde{u}(x_\beta)$$

$$\leqslant \mathrm{lip}(\tilde{u})|x_{\alpha_h} - x_h| + \sqrt{h}\, \mathrm{lip}(\tilde{u})$$

$$\leqslant \mathrm{lip}(\tilde{u})|x_{\alpha_h} - x_h - \sqrt{h}\, n| + 2\sqrt{h}\, \mathrm{lip}(\tilde{u})$$

$$\leqslant \frac{L_1}{2\sqrt{h}} |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 + \frac{\sqrt{h}}{2L_1} \mathrm{lip}(\tilde{u})^2 + 2\sqrt{h}\, \mathrm{lip}(\tilde{u}),$$

and therefore,

$$\frac{1}{h}|x_{\alpha_h} - x_h - \sqrt{h}\, n|^2 \leqslant \frac{1}{L_1^2} \mathrm{lip}(\tilde{u})^2 + \frac{4}{L_1} \mathrm{lip}(\tilde{u}) < \left(\frac{\epsilon}{2+\epsilon}\right)^2 \tag{3.22}$$

$$|x_{\alpha_h} - x_\beta|^2 \leqslant \frac{1}{2L_1 L_2} \mathrm{lip}(\tilde{u})^2 + \frac{2}{L_2} \mathrm{lip}(\tilde{u}) < \epsilon^2, \tag{3.23}$$

provided that L_1, L_2 are sufficiently large.

Let us first consider the case that $(x_h, x_{\alpha_h}) \in \Omega \times \Omega_h$. We infer from (3.16) that

$$f_*(x_h)\tilde{u}(x_h) - \tilde{u}(x_h)H\left(-\frac{1}{\tilde{u}(x_h)}\frac{2L_1}{\sqrt{h}}(x_{\alpha_h} - x_h - \sqrt{h}\, n)\right) \geqslant 0. \tag{3.24}$$

In order to derive a corresponding relation for the discrete solution, we consider the inequality $\Phi(x_h, x_{\alpha_h}) \geqslant \Phi(x_h, x_\alpha)$ for all $x_\alpha \in \bar{\Omega}$, which translates into

$$\tilde{U}_\alpha \leqslant \tilde{U}_{\alpha_h} + \frac{L_1}{\sqrt{h}}(|x_\alpha - x_h - \sqrt{h}\, n|^2 - |x_{\alpha_h} - x_h - \sqrt{h}\, n|^2)$$

$$+ L_2\sqrt{h}\,(|x_\alpha - x_\beta|^2 - |x_{\alpha_h} - x_\beta|^2)$$

$$=: \tilde{V}_\alpha. \tag{3.25}$$

Note first that

$$D_k^- \tilde{V}_\alpha = \frac{2L_1}{\sqrt{h}}(x_\alpha - x_h - \sqrt{h}\, n, e_k) + 2L_2\sqrt{h}(x_\alpha - x_\beta, e_k) - L_1\sqrt{h} - L_2 h^{3/2},$$

$$D_k^+ \tilde{V}_\alpha = \frac{2L_1}{\sqrt{h}}(x_\alpha - x_h - \sqrt{h}\, n, e_k) + 2L_2\sqrt{h}(x_\alpha - x_\beta, e_k) + L_1\sqrt{h} + L_2 h^{3/2},$$

and therefore by (3.22),

$$|D_k^- \tilde{V}_{\alpha_h}|, |D_k^- \tilde{V}_{\alpha_h}| \leqslant C,$$

$$\left| D_k^\pm \tilde{V}_{\alpha_h} - \frac{2L_1}{\sqrt{h}}(x_{\alpha_h} - x_h - \sqrt{h}\, n, e_k)\right| \leqslant C\sqrt{h}, \quad k = 1, \dots, n, \tag{3.26}$$

uniformly in h. Recalling that $U \leqslant Z$ we also deduce from (3.9) and (3.10) that

$$\tilde{V}_{\alpha_h} = \tilde{U}_{\alpha_h} = -e^{-U_{\alpha_h}} \leqslant -e^{-Z_{\alpha_h}} \leqslant -\bar{c}, \tag{3.27}$$

where $\bar{c} > 0$ depends on ϕ and R. Furthermore, (3.26) implies

$$\tilde{V}_{\alpha_h \pm e_k} \leqslant -\frac{1}{2}\bar{c} \quad \text{for } h \text{ sufficiently small.} \tag{3.28}$$

Thus we can define $V_\alpha := -\log(-\tilde{V}_\alpha)$ for $\alpha = \alpha_h, \alpha_h \pm e_k$ and the mean value theorem yields

$$D_k^- \tilde{V}_{\alpha_h} = e^{-\xi_k^-} D_k^- V_{\alpha_h}, \quad D_k^+ \tilde{V}_\alpha = e^{-\xi_k^+} D_k^+ V_\alpha, \quad k = 1, \ldots, n, \tag{3.29}$$

where ξ_k^- lies between $V_\alpha, V_{\alpha - e_k}$ and ξ_k^+ lies between $V_\alpha, V_{\alpha + e_k}$. In particular,

$$e^{\xi_k^\pm} \leqslant \max(e^{V_{\alpha_h \pm e_k}}, e^{V_{\alpha_h}}) = \max\left(\frac{-1}{\tilde{V}_{\alpha_h \pm e_k}}, \frac{-1}{\tilde{V}_{\alpha_h}}\right) \leqslant \frac{2}{\bar{c}} \tag{3.30}$$

by (3.28). Thus, (3.29), (3.26), (3.22), (3.27) together with the fact that $\phi_{\min} \leqslant U \leqslant Z$ imply

$$|D_k^\pm V_{\alpha_h}| \leqslant C, \quad k = 1, \ldots, n, \quad \left| -\frac{1}{\tilde{U}_{\alpha_h}} \frac{2L_1}{\sqrt{h}} (x_{\alpha_h} - x_h - \sqrt{h}\, n) \right| \leqslant C. \tag{3.31}$$

Next, we deduce from (3.25) that

$$U_{\alpha_h} = V_{\alpha_h}, \quad U_\alpha \leqslant V_\alpha, \ \alpha = \alpha_h \pm e_k, \ k = 1, \ldots, n,$$

so that the monotonicity property (3.7) and (3.2) imply

$$\begin{aligned}
f(x_{\alpha_h}) &= H_N(D_1^- U_{\alpha_h}, D_1^+ U_{\alpha_h}, \ldots, D_n^- U_{\alpha_h}, D_n^+ U_{\alpha_h}) \\
&\geqslant H_N(D_1^- V_{\alpha_h}, D_1^+ V_{\alpha_h}, \ldots, D_n^- V_{\alpha_h}, D_n^+ V_{\alpha_h}).
\end{aligned}$$

Multiplying the above inequality by $\tilde{U}_{\alpha_h} < 0$, using (3.5) along with (3.31) and the local Lipschitz continuity of H_N we infer

$$\begin{aligned}
f(x_{\alpha_h}) &\tilde{U}_{\alpha_h} - \tilde{U}_{\alpha_h} H\left(-\frac{1}{\tilde{U}_{\alpha_h}} \frac{2L_1}{\sqrt{h}} (x_{\alpha_h} - x_h - \sqrt{h}\, n)\right) \\
&\leqslant \tilde{U}_{\alpha_h}\left(H_N(D_1^- V_{\alpha_h}, D_1^+ V_{\alpha_h}, \ldots, D_n^- V_{\alpha_h}, D_n^+ V_{\alpha_h}) - H\left(-\frac{1}{\tilde{U}_{\alpha_h}} \frac{2L_1}{\sqrt{h}} (x_{\alpha_h} - x_h - \sqrt{h}\, n)\right)\right) \\
&\leqslant C \max_{k=1,\ldots,n} \left| D_k^\pm V_{\alpha_h} + \frac{1}{\tilde{U}_{\alpha_h}} \frac{2L_1}{\sqrt{h}} (x_{\alpha_h} - x_h - \sqrt{h}\, n, e_k) \right| \\
&\leqslant C \max_{k=1,\ldots,n} \left(e^{\xi_k^\pm} \left| D_k^\pm \tilde{V}_{\alpha_h} - \frac{2L_1}{\sqrt{h}} (x_{\alpha_h} - x_h - \sqrt{h}\, n, e_k) \right| \right. \\
&\qquad \left. + |e^{\xi_k^\pm} - e^{V_{\alpha_h}}| \left| \frac{2L_1}{\sqrt{h}} (x_{\alpha_h} - x_h - \sqrt{h}\, n, e_k) \right| \right) \\
&\leqslant C\sqrt{h} + Ch. \tag{3.32}
\end{aligned}$$

Note that the last estimate is a consequence of (3.26) and (3.30). If we combine (3.24) with (3.32) and use the definition of $\tilde{H}$ we obtain

$$\tilde{H}\left(x_{\alpha_h}, \tilde{U}_{\alpha_h}, \frac{2L_1}{\sqrt{h}}(x_{\alpha_h} - x_h - \sqrt{h}\,n)\right) - \tilde{H}\left(x_{\alpha_h}, \tilde{u}(x_h), \frac{2L_1}{\sqrt{h}}(x_{\alpha_h} - x_h - \sqrt{h}\,n)\right)$$
$$\leqslant C\sqrt{h} + e^{-u(x_h)}(f(x_{\alpha_h}) - f_*(x_h)).$$

Note first that

$$\tilde{U}_{\alpha_h} - \tilde{u}(x_h) = \Phi(x_h, x_{\alpha_h}) + \frac{L_1}{\sqrt{h}}|x_{\alpha_h} - x_h - \sqrt{h}\,n|^2 + L_2\sqrt{h}\,|x_{\alpha_h} - x_\beta|^2$$
$$\geqslant \Phi(x_\beta, x_\beta) = \tilde{U}_\beta - \tilde{u}(x_\beta) - L_1\sqrt{h}.$$

Thus, in view of (3.17),

$$m(\tilde{U}_\beta - \tilde{u}(x_\beta)) \leqslant e^{-u(x_h)}(f(x_{\alpha_h}) - f_*(x_h)) + C\sqrt{h}. \tag{3.33}$$

Let us write $x_{\alpha_h} = x_h + r_h d_h$ with

$$d_h = \frac{n + w_h}{|n + w_h|}, \quad r_h = \sqrt{h}\,|n + w_h|, \quad w_h = \frac{1}{\sqrt{h}}(x_{\alpha_h} - x_h - \sqrt{h}\,n).$$

Note that (3.23) implies that $|x_{\alpha_h} - x_\beta| < \epsilon$; (3.22) and a similar argument as in (2.16) yield $|d_h - n| < \epsilon$, so that (2.3) gives

$$f(x_{\alpha_h}) - f_*(x_h) \leqslant f^*(x_h + r_h d_h) - f_*(x_h) \leqslant K r_h \leqslant C\sqrt{h}. \tag{3.34}$$

Combining (3.33) and (3.34) finally yields

$$\max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| = \tilde{U}_\beta - \tilde{u}(x_\beta) \leqslant C\sqrt{h}. \tag{3.35}$$

It remains to consider the case when $x_{\alpha_h} \in \partial\Omega_h$ or $x_h \in \partial\Omega$. If $x_{\alpha_h} \in \partial\Omega_h$, it follows from (3.21), the fact that $\tilde{u}(x_{\alpha_h}) = \tilde{U}_{\alpha_h}$ and (3.22) that

$$\tilde{U}_\beta - \tilde{u}(x_\beta) \leqslant \tilde{u}(x_\beta - \sqrt{h}\,n) - \tilde{u}(x_\beta) + \tilde{u}(x_{\alpha_h}) - \tilde{u}(x_h)$$
$$\leqslant \text{lip}(\tilde{u})(\sqrt{h} + |x_{\alpha_h} - x_h|)$$
$$\leqslant \text{lip}(\tilde{u})(2\sqrt{h} + |x_{\alpha_h} - x_h - \sqrt{h}\,n|) \leqslant C\sqrt{h}.$$

Let us finally assume that $x_h \in \partial\Omega$. Since $\tilde{u}(x_h) = \phi(x_h) = \tilde{Z}_{x_h}$ and $\tilde{U} \leqslant \tilde{Z}$ we obtain

$$\tilde{U}_\beta - \tilde{u}(x_\beta) \leqslant \tilde{U}_{\alpha_h} - \tilde{u}(x_h) + \tilde{u}(x_\beta - \sqrt{h}\,n) - \tilde{u}(x_\beta)$$
$$\leqslant \tilde{Z}(x_{\alpha_h}) - \tilde{Z}(x_h) + \tilde{u}(x_\beta - \sqrt{h}\,n) - \tilde{u}(x_\beta) \leqslant C\sqrt{h}$$

as above. If we combine (3.19), (3.35) with the above estimates we obtain $\max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| \leqslant C\sqrt{h}$. Transforming back to u and U implies the desired error bound.   □

4. Examples of numerical Hamiltonians

Let us consider some examples of numerical Hamiltonians H_N. In order to simplify matters we restrict ourselves to the case of two space dimensions and a domain Ω of the form $\Omega = (0, b_1) \times (0, b_2)$ with $b_2 \leqslant b_1$.

4.1 *Viscous regularization*

Suppose that H is globally Lipschitz continuous with Lipschitz constant L. We define $H_N : \mathbb{R}^4 \to \mathbb{R}$ by

$$H_N(p_1, q_1, p_2, q_2) := -\frac{L}{2} \sum_{k=1}^{2} (q_k - p_k) + H\left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}\right)$$

and verify (HN1)–(HN3).

(HN1) Clearly, $H_N(p_1, p_1, p_2, p_2) = H(p_1, p_2)$.

(HN2) The function F_N defined in (3.4) is given by

$$F_N(a_0, a_1, \ldots, a_4) = 2La_0 - \frac{L}{2} \sum_{k=1}^{2} (a_{2k-1} + a_{2k}) + H\left(\frac{a_2 - a_1}{2}, \frac{a_4 - a_3}{2}\right),$$

and (3.6) is evident. We check (3.7) for $k = 1$: from the Lipschitz continuity of H we deduce for $a_1 < \tilde{a}_1$ that

$$F_N(a_0, a_1, a_2, a_3, a_4) - F_N(a_0, \tilde{a}_1, a_2, a_3, a_4)$$
$$= \frac{L}{2}(\tilde{a}_1 - a_1) + H\left(\frac{a_2 - a_1}{2}, \frac{a_4 - a_3}{2}\right) - H\left(\frac{a_2 - \tilde{a}_1}{2}, \frac{a_4 - a_3}{2}\right)$$
$$\geqslant \frac{L}{2}(\tilde{a}_1 - a_1) - L\left|\frac{a_2 - a_1}{2} - \frac{a_2 - \tilde{a}_1}{2}\right| = 0,$$

i.e. $a_1 \mapsto F_N(a_0, a_1, a_2, a_3, a_4)$ is decreasing.

(HN3) Let us consider $d(x) := \text{dist}(x, \partial\Omega)$, $x = (x_1, x_2)$. It is not difficult to see that

$$d(x) = \begin{cases} x_1 & \text{in } \Omega_1 = \{x \in \Omega \mid 0 \leqslant x_2 \leqslant b_2, 0 \leqslant x_1 \leqslant \min(b_2 - x_2, x_2)\}, \\ b_1 - x_1 & \text{in } \Omega_2 = \{x \in \Omega \mid 0 \leqslant x_2 \leqslant b_2, b_1 \geqslant x_1 \geqslant \max(b_1 - x_2, b_1 - b_2 + x_2)\}, \\ x_2 & \text{in } \Omega_3 = \{x \in \Omega \mid 0 \leqslant x_1 \leqslant b_1, 0 \leqslant x_2 \leqslant \min(x_1, b_2/2, b_1 - x_1)\}, \\ b_2 - x_2 & \text{in } \Omega_4 = \{x \in \Omega \mid 0 \leqslant x_1 \leqslant b_1, b_2 \geqslant x_2 \geqslant \max(b_2 - x_1, b_2/2, x_1 + b_2 - b_1)\}. \end{cases}$$

We may assume that ϕ is defined as a Lipschitz continuous function on $\bar{\Omega}$ with Lipschitz constant L_ϕ (one such extension is e.g. given by the viscosity solution of (1.1), (1.2)).

Define $Z : \bar{\Omega}_h \to \mathbb{R}$ by $Z_\alpha := \phi_\alpha + \rho d(x_\alpha)$, where $\phi_\alpha = \phi(x_\alpha)$ and ρ is independent of h. Clearly, Z satisfies (3.9) as well as (3.10) with $R = L_\phi + \rho$. We claim that (3.8) holds provided that ρ is sufficiently large and verify this for a point $x_\alpha = (x_{1\alpha}, x_{2\alpha}) \in \Omega_1$.

Case 1: $x_{2\alpha} - h \leqslant x_{1\alpha} \leqslant x_{2\alpha}$. Since $x_{\alpha+e_1}, x_{\alpha-e_2} \in \Omega_3$ it follows that $d(x_{\alpha-e_1}) = x_{1\alpha} - h$, $d(x_{\alpha+e_1}) = x_{2\alpha}, d(x_{\alpha-e_2}) = x_{2\alpha} - h, d(x_{\alpha+e_2}) = x_{1\alpha}$ and therefore

$$D_1^- Z_\alpha = D_1^- \phi_\alpha + \rho, \quad D_1^+ Z_\alpha = D_1^+ \phi_\alpha + \rho \frac{x_{2\alpha} - x_{1\alpha}}{h} \leqslant D_1^+ \phi_\alpha + \rho,$$

$$D_2^- Z_\alpha = D_2^- \phi_\alpha + \rho \frac{x_{1\alpha} - x_{2\alpha} + h}{h} \geqslant D_2^- \phi_\alpha, \quad D_2^+ Z_\alpha = D_2^+ \phi_\alpha. \tag{4.1}$$

Recalling that $H \geqslant 0$ and using (2.1) we deduce

$$H_N(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha)$$

$$= -\frac{L}{2} \sum_{k=1}^{2} (D_k^+ Z_\alpha - D_k^- Z_\alpha) + H\left(\frac{D_1^+ Z_\alpha + D_1^- Z_\alpha}{2}, \frac{D_2^+ Z_\alpha + D_2^- Z_\alpha}{2} \right)$$

$$\geqslant -\frac{L}{2} \sum_{k=1}^{2} (D_k^+ \phi_\alpha - D_k^- \phi_\alpha) + L\rho \geqslant -2LL_\phi + L\rho \geqslant M \geqslant f(x_\alpha) \tag{4.2}$$

provided that ρ is sufficiently large.

Case 2: $b_2 - x_{2\alpha} - h \leqslant x_{1\alpha} \leqslant b_2 - x_{2\alpha}$. In this case we have $x_{\alpha+e_1}, x_{\alpha+e_2} \in \Omega_4$ and therefore $d(x_{\alpha-e_1}) = x_{1\alpha} - h, d(x_{\alpha+e_1}) = b_2 - x_{2\alpha}, d(x_{\alpha-e_2}) = x_{1\alpha}, d(x_{\alpha+e_2}) = b_2 - x_{2\alpha} - h$. Thus

$$D_1^- Z_\alpha = D_1^- \phi_\alpha + \rho, \quad D_1^+ Z_\alpha = D_1^+ \phi_\alpha + \rho \frac{b_2 - x_{2\alpha} - x_{1\alpha}}{h} \leqslant D_1^+ \phi_\alpha + \rho,$$

$$D_2^- Z_\alpha = D_2^- \phi_\alpha, \quad D_2^+ Z_\alpha = D_2^+ \phi_\alpha + \rho \frac{b_2 - x_{2\alpha} - h - x_{1\alpha}}{h} \leqslant D_2^+ \phi_\alpha. \tag{4.3}$$

The inequality (4.2) then follows in the same way as in Case 1.

Case 3: $x_{1\alpha} < x_{2\alpha} - h/2$ and $x_{1\alpha} < b_2 - x_{2\alpha} - h$. We now have $x_{\alpha\pm e_1}, x_{\alpha\pm e_2} \in \Omega_1$, which implies

$$D_1^\pm Z_\alpha = D_1^\pm \phi_\alpha + \rho, \quad D_2^\pm Z_\alpha = D_2^\pm \phi_\alpha, \tag{4.4}$$

so that

$$H_N(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha)$$

$$= -\frac{L}{2} \sum_{k=1}^{2} (D_k^+ \phi_\alpha - D_k^- \phi_\alpha) + H\left(\frac{D_1^+ \phi_\alpha + D_1^- \phi_\alpha}{2} + \rho, \frac{D_2^+ \phi_\alpha + D_2^- \phi_\alpha}{2} \right)$$

$$\geqslant -2LL_\phi + H\left(\frac{D_1^+ \phi_\alpha + D_1^- \phi_\alpha}{2} + \rho, \frac{D_2^+ \phi_\alpha + D_2^- \phi_\alpha}{2} \right) \geqslant M \geqslant f(x_\alpha) \tag{4.5}$$

in view of (H3) provided that ρ is large enough. Other points can be treated in a similar way.

4.2 *Godunov Hamiltonian*

In [2] the following formula was derived from the solution of the Riemann problem:

$$H_N(p_1, q_1, p_2, q_2) := \underset{\xi \in I[p_1, q_1]}{\text{ext}} \underset{\eta \in I[p_2, q_2]}{\text{ext}} H(\xi, \eta),$$

where

$$\underset{\xi \in I[p,q]}{\text{ext}} = \begin{cases} \underset{\xi \in [p,q]}{\min} & \text{if } p \leqslant q, \\ \underset{\xi \in [q,p]}{\max} & \text{if } p > q. \end{cases}$$

Let us again verify (HN1)–(HN3).

(HN1) Clearly, $H_N(p_1, p_1, p_2, p_2) = H(p_1, p_2)$.

(HN2) In order to verify (3.6) and (3.7) it is sufficient to check that

$$\begin{aligned} p_k &\mapsto H_N(p_1, q_1, p_2, q_2) \text{ is increasing,} \\ q_k &\mapsto H_N(p_1, q_1, p_2, q_2) \text{ is decreasing,} \end{aligned} \quad k = 1, 2. \tag{4.6}$$

As an example, assume that $p_1 > q_1$ and consider $q_2 \mapsto H_N(p_1, q_1, p_2, q_2)$.

Case 1: $q_2 < \tilde{q}_2 < p_2$:

$$H_N(p_1, q_1, p_2, q_2) = \max_{\xi \in [q_1, p_1]} \max_{\eta \in [q_2, p_2]} H(\xi, \eta) \geqslant \max_{\xi \in [q_1, p_1]} \max_{\eta \in [\tilde{q}_2, p_2]} H(\xi, \eta) = H_N(p_1, q_1, p_2, \tilde{q}_2).$$

Case 2: $q_2 < p_2 \leqslant \tilde{q}_2$:

$$\begin{aligned} H_N(p_1, q_1, p_2, q_2) &= \max_{\xi \in [q_1, p_1]} \max_{\eta \in [q_2, p_2]} H(\xi, \eta) \geqslant \max_{\xi \in [q_1, p_1]} H(\xi, p_2) \\ &\geqslant \max_{\xi \in [q_1, p_1]} \min_{\eta \in [p_2, \tilde{q}_2]} H(\xi, \eta) = H_N(p_1, q_1, p_2, \tilde{q}_2). \end{aligned}$$

Case 3: $p_2 \leqslant q_2 < \tilde{q}_2$:

$$H_N(p_1, q_1, p_2, q_2) = \max_{\xi \in [q_1, p_1]} \min_{\eta \in [p_2, q_2]} H(\xi, \eta) \geqslant \max_{\xi \in [q_1, p_1]} \min_{\eta \in [p_2, \tilde{q}_2]} H(\xi, \eta) = H_N(p_1, q_1, p_2, \tilde{q}_2).$$

Other situations are treated in a similar way.

(HN3) We use the same function Z as in the case of viscous regularization to verify (3.8)–(3.10). Again we examine the situation at a point $x_\alpha = (x_{1\alpha}, x_{2\alpha}) \in \Omega_1$.

Case 1: $x_{2\alpha} - h \leqslant x_{1\alpha} \leqslant x_{2\alpha}$. Combining (4.1) with (4.6) yields

$$H_N(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha) \geqslant H_N(D_1^- \phi_\alpha + \rho, D_1^+ \phi_\alpha + \rho, D_2^- \phi_\alpha, D_2^+ \phi_\alpha). \tag{4.7}$$

Case 2: $b_2 - x_{2\alpha} - h \leqslant x_{1\alpha} \leqslant b_2 - x_{2\alpha}$. Arguing as in Case 1, but using (4.3) instead of (4.1) we again derive (4.7).

Case 3: $x_{1\alpha} < x_{2\alpha} - h/2$ and $x_{1\alpha} < b_2 - x_{2\alpha} - h$. We infer from (4.4) that

$$H_N(D_1^- Z_\alpha, D_1^+ Z_\alpha, D_2^- Z_\alpha, D_2^+ Z_\alpha) = H_N(D_1^- \phi_\alpha + \rho, D_1^+ \phi_\alpha + \rho, D_2^- \phi_\alpha, D_2^+ \phi_\alpha).$$

Thus it remains to estimate the last expression from below. Choosing a priori $\rho \geqslant 2L_\phi$ we have $D_1^- \phi_\alpha + \rho \geqslant \rho/2$, $D_1^+ \phi_\alpha + \rho \geqslant \rho/2$ so that with the help of (H3) we obtain

$$H_N(D_1^- \phi_\alpha + \rho, D_1^+ \phi_\alpha + \rho, D_2^- \phi_\alpha, D_2^+ \phi_\alpha) \geqslant M \geqslant f(x_\alpha),$$

provided that ρ is large enough. Other points can be treated analogously.

As a special case one obtains for the eikonal equation $H(p) = |p|$ the scheme

$$H_N(p_1, q_1, p_2, q_2) = \sqrt{(\max(p_1^+, -q_1^-))^2 + (\max(p_2^+, -q_2^-))^2}, \qquad (4.8)$$

where $p^+ = \max(p, 0)$ and $p^- = \min(p, 0)$. This scheme was examined in [13] in the context of shape-from-shading and convergence of approximations was proved with the help of a result of Barles & Souganidis [3]. In a recent paper, Zhao [18] shows $O(|h \ln(h)|)$ convergence for this scheme when $f \equiv 1$ in the case of approximating the distance function to a point.

5. Numerical results

In this section we present some results of numerical calculations for (1.1), (1.2) with $H(p) = |p|$. As a first test example, let $\Omega := (-1, 1) \times (0, 2)$ and $f : \Omega \to \mathbb{R}$ be defined by $f(x_1, x_2) := 1$, $x_1 < 0$, $f(0, x_2) := 3/4$, $f(x_1, x_2) := 1/2$, $x_1 > 0$. It is not difficult to see that f satisfies (2.3) and one verifies that

$$u(x_1, x_2) := \begin{cases} \frac{1}{2}x_2, & x_1 \geqslant 0, \\ -\frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, & -\frac{1}{\sqrt{3}}x_2 \leqslant x_1 \leqslant 0, \\ x_2, & x_1 < -\frac{1}{\sqrt{3}}x_2, \end{cases}$$

is a viscosity solution of $|\nabla u| = f$ in the sense of Definition 2.1. Furthermore, let $\phi := u_{|\partial \Omega}$. Since H is globally Lipschitz continuous with constant 1, the numerical scheme induced by viscous regularization reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$-\frac{1}{2}(U_{\alpha+e_1} + U_{\alpha-e_1} + U_{\alpha+e_2} + U_{\alpha-e_2} - 4U_\alpha)$$
$$+ \frac{1}{2}\sqrt{(U_{\alpha+e_1} - U_{\alpha-e_1})^2 + (U_{\alpha+e_2} - U_{\alpha-e_2})^2} = hf(x_\alpha), \quad x_\alpha \in \Omega_h,$$
$$U_\alpha = \phi(x_\alpha), \qquad x_\alpha \in \partial \Omega_h.$$

The system of equations was solved with the help of Newton's method and we calculated

$$E_{VR,h} := \max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U_\alpha|$$

together with the experimental order of convergence

$$\text{eoc} = \frac{\ln(E_{h_2}/E_{h_1})}{\ln(h_2/h_1)}$$

for various choices of h. We then used the numerical Hamiltonian (4.8) to approximate the viscosity solution. Observing that

$$\max((D_k^- U_\alpha)^+, -D_k^+ U_\alpha)^-) = \frac{1}{h}(U_\alpha - \min(U_{\alpha-e_k}, U_{\alpha+e_k}))^+$$

for $k = 1, \ldots, n$, the discrete problem reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$\left(\sum_{k=1}^2 ((U_\alpha - \min(U_{\alpha-e_k}, U_{\alpha+e_k}))^+)^2\right)^{1/2} = hf(x_\alpha), \quad x_\alpha \in \Omega_h, \qquad (5.1)$$

$$U_\alpha = \phi(x_\alpha), \qquad x_\alpha \in \partial \Omega_h. \qquad (5.2)$$

The discrete solution was calculated with the help of the *Fast Sweeping Method* (see e.g. [18] for a description) and the corresponding errors $E_{FS,h}$ and eoc's are shown in Table 1 together with the results from the method of viscous regularization. Figure 1 shows various level lines of the solution.

TABLE 1

Absolute error in maximum norm and experimental order of convergence for the first test problem

h	$E_{VR,h}$	eoc	$E_{FS,h}$	eoc
0.1	1.24348e−1	-	5.59016e−2	-
0.05	7.22984e−2	0.78	2.79508e−2	1.00
0.025	4.08509e−2	0.82	1.39754e−2	1.00
0.0125	2.26691e−2	0.85	6.98771e−3	1.00
0.0063	1.24385e−2	0.87	3.49386e−3	1.00
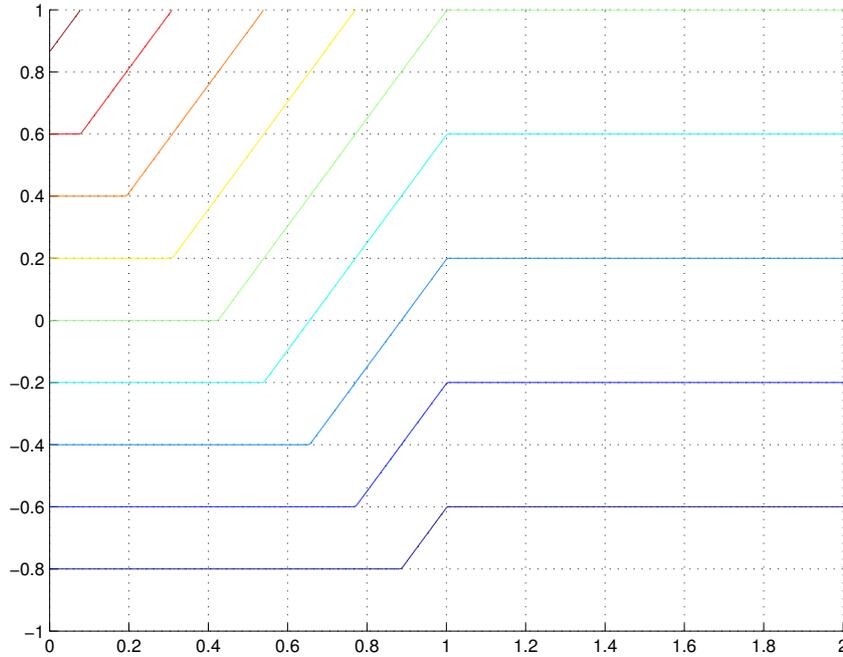
FIG. 1. Level lines of the solution from the first test problem.

We observe linear convergence in h for the method (5.1), (5.2) (cf. also [18], where a one-sided bound of the form $u_h - u \leqslant Ch \log(1/h)$ is proved for $f \equiv 1$).

In our second example we consider $\Omega = (-1, 1)^2$, $\phi \equiv 0$ and

$$f(x_1, x_2) := \begin{cases} 2, & (x_1 - 1/2)^2 + x_2^2 \leqslant 1/8 \text{ and } x_2 \geqslant x_1 - 1/2, \\ 3, & (x_1 - 1/2)^2 + x_2^2 \leqslant 1/8 \text{ and } x_2 < x_1 - 1/2, \\ 1, & \text{otherwise.} \end{cases}$$

Note that in this case discontinuities of f occur both along curved lines and along a straight line which is not aligned with the grid. Furthermore, the three regions, in which f takes different values, meet at the triple points $(3/4, 1/4)$, $(1/4, -1/4)$. It is not difficult to check that f satisfies (2.3). The numerical solutions were again calculated with the help of viscous regularization and (5.1), (5.2). In the absence of an exact solution we compared the discrete solutions for various grid sizes with an approximation U_f on a fine grid ($h = 1/640$). The results are displayed in Table 2, while Figure 2 shows some level curves of the solution.

TABLE 2

Absolute error in maximum norm and experimental order
of convergence for the second test problem

h	$E_{VR,h}$	eoc	$E_{FS,h}$	eoc
0.1	9.07922e−2	-	1.10429e−1	-
0.05	1.29179e−1	−0.51	1.28365e−1	−0.22
0.025	1.04327e−1	0.31	8.68148e−2	0.56
0.0125	7.35184e−2	0.50	5.00901e−2	0.79
0.0063	4.33351e−2	0.76	2.65873e−2	0.91
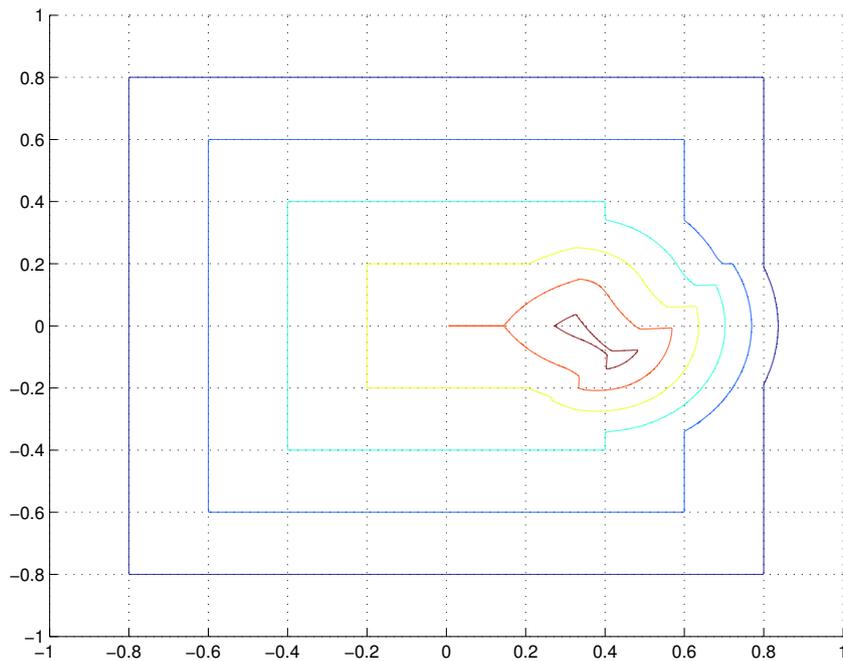
FIG. 2. Level lines of the solution from the second test problem.

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