On the Cahn–Hilliard equation with degenerate mobility

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Abstract

An existence result for the Cahn–Hilliard equation with a concentration dependent diffusional mobility is presented. In particular the mobility is allowed to vanish when the scaled concentration takes the values ± 1 and it is shown that the solution is bounded by 1 in magnitude. Finally applications of our method to other degenerate fourth order parabolic equations are discussed.

Keywords: Cahn–Hilliard equations, degenerate parabolic equations, nonlinear diffusion, phase transitions.

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1 Introduction

The Cahn–Hilliard equation

$$\begin{aligned} u_t &= -\nabla \cdot \mathbf{J}\,, \\ \mathbf{J} &= -B(u)\nabla w\,, \\ w &= -\gamma\Delta u + \Psi'(u)\,, \qquad \gamma \in \mathbb{R}^+ \end{aligned} \tag{1.1}$$

was introduced to study phase separation in binary alloys (see Cahn and Hilliard [8, 9]). Although the Cahn–Hilliard equation has been intensively studied, little mathematical analysis has been done for a diffusional mobility B which depends on u (where u is the difference of the mass density of the two components of the

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alloy). A concentration dependent mobility appeared in the original derivation of the Cahn–Hilliard equation (see [9]) and a thermodynamically reasonable choice is $B(u) = 1 - u^2$ (see [10, 11, 18]). The mathematical difficulty in studying the Cahn–Hilliard equation with a mobility like this lies in the degeneracy of B. On the other hand there is hope that solutions which initially take values in the interval $[-1, 1]$ will do so for all positive time (which is not true for fourth order parabolic equation without degeneracy). We remark that only values in the interval $[-1, 1]$ are physically meaningful.

The function Ψ represents the homogeneous free energy in the energy functional

$$\mathcal{E}(u) = \int_\Omega \left(\frac{\gamma}{2} \mid \nabla u \mid^2 + \Psi(u) \right) dx$$

where $\Omega \subset \mathbb{R}^n \quad (n \in \mathbb{N})$ is a bounded domain with sufficiently smooth boundary. Possible choices for Ψ are

$$\begin{aligned} \Psi(u) &= (1-u^2)^2 \quad \text{and} \\ \Psi(u) &= \frac{\theta}{2}\left((1+u)\ln(1+u) + (1-u)\ln(1-u)\right) + F_0(u) \end{aligned} \tag{1.2}$$

with a smooth function F_0. In the case $F_0(u) = 1 - u^2$ one gets in the limit as $\theta \searrow 0$ the double obstacle potential (see the papers of Blowey and Elliott [5, 6] and Elliott and Luckhaus [14])

$$\Psi(u) = \begin{cases} 1 - u^2 & \text{if } |u| \le 1\,, \\ \infty & \text{otherwise.} \end{cases}$$

In order to formulate an existence result for (1.1) in a general situation we make the following assumptions.
Let

$$\Psi(u) := \Psi_1(u) + \Psi_2(u)$$

with functions Ψ_1, Ψ_2 such that

$$\|\Psi_2\|_{C^2[-1,1]} \le C$$

and

$$\Psi_1 : (-1, 1) \to \mathbb{R}$$

is convex and is of the form

$$\Psi_1''(u) = (1-u^2)^{-m} F(u) \quad (m \ge 1)$$

with a C^1–function $F : [-1, 1] \to \mathbb{R}_0^+$. This means we allow Ψ to be singular in the convex part as $|u| \to 1$. In particular the logarithmic form (1.2) is a possible choice. Furthermore we assume that the mobility is of the form

$$B(u) = (1-u^2)^m \bar{B}(u)$$

with a C^1–function

$$\bar{B} : [-1,1] \to \mathbb{R}$$

which satisfies

$$b_0 \leq \bar{B}(u) \leq B_0 \qquad (B_0, b_0 > 0)$$

for $u \in [-1,1]$. We extend the defintion of B to all of $\mathbb{R}$ by $B(u) = 0$ for $|u| > 1$. Let

$$\Phi : (-1,1) \to \mathbb{R}_0^+$$

be defined by

$$\Phi''(u) = \frac{1}{B(u)} \;, \;\; \Phi'(0) = 0 \;\text{ and }\; \Phi(0) = 0\,.$$

The following theorem states the existence of a weak solution to the Cahn–Hilliard equation with a non–constant mobility as above on an arbitrary time interval $[0,T]$ $(T \in \mathbb{R}^+)$ which fulfills the boundary conditions

$$\mathbf{n} \cdot \mathbf{J} = 0 \quad \text{and} \quad \mathbf{n} \cdot \nabla u = 0 \quad \text{on} \quad \partial\Omega \times (0,T)\,,$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$.

Theorem 1: *Let either $\partial\Omega \in C^{1,1}$ or Ω convex and suppose that $u_0 \in H^1(\Omega)$ with $|u_0| \leq 1$ a.e. and*

$$\int_\Omega (\Psi(u_0) + \Phi(u_0)) \leq C\,, \qquad C \in \mathbb{R}^+$$

then there exists a pair $(u, \mathbf{J})$ such that

a) $u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$,

b) $u_t \in L^2(0,T;(H^1(\Omega))')$,

c) $u(0) = u_0$ *and* $\nabla u \cdot \mathbf{n} = 0$ *on* $\partial\Omega \times (0,T)$,

d) $|u| \leq 1$ *a.e in* $\Omega_T := \Omega \times (0,T)$,

e) $\mathbf{J} \in L^2(\Omega_T, \mathbb{R}^n)$,

which satisfies $u_t = -\nabla \cdot \mathbf{J}$ in $L^2(0,T;(H^1(\Omega))')$, i.e.

$$\int_0^T \langle \zeta(t), u_t(t) \rangle_{H^1,(H^1)'} = \int_{\Omega_T} \mathbf{J} \cdot \nabla\zeta$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ and

$$\mathbf{J} = -B(u)\nabla \cdot (-\gamma\Delta u + \Psi'(u))$$

in the following weak sense

$$\int_{\Omega_T} \mathbf{J} \cdot \boldsymbol{\eta} = -\int_{\Omega_T} [\gamma\Delta u \nabla \cdot (B(u)\boldsymbol{\eta}) + (B\Psi'')(u)\nabla u \cdot \boldsymbol{\eta}]$$

for all $\boldsymbol{\eta} \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \cap L^\infty(\Omega_T,\mathbb{R}^n)$ *which fulfill* $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ *on* $\partial\Omega\times(0,T)$.

We point out that the nonlinearity $(B\Psi'')(u) = \bar{B}(u)F(u) + B(u)\Psi_2''(u)$ is bounded and therefore the last integral in the formulation of the theorem is well defined.

An existence result for the Cahn–Hilliard equation with a degenerate mobility in a one–dimensional situation has been established by Yin Jingxue [23]. The existence result we present is for arbitrary space dimensions and uses a weak formulation which is different to the formulation of Yin Jingxue. Furthermore we allow the bulk energy Ψ to have singularities when B degenerates. We refer also to the work of Bernis and Friedman [3] for results on fourth order degenerate parabolic equations in one space dimension.

In section 4 we prove a similar existence result for a viscous Cahn–Hilliard type equation of the form

$$\begin{aligned} u_t &= -\nabla\cdot\mathbf{J}\,, \\ \mathbf{J} &= -B(u)\nabla w\,, \\ w &= -\gamma\Delta u + \Psi'(u) + \alpha u_t\,, \qquad \alpha\in\mathbb{R}^+ \end{aligned}$$

where we assume the mobility B and the homogeneous free energy Ψ to be as above.

We want to point out that our result includes the case $B(u) = 1-u^2$ and

$$\Psi(u) = \frac{\theta}{2}\left((1+u)\ln(1+u) + (1-u)\ln(1-u)\right) + \frac{1}{2}(1-u^2)\,. \tag{1.3}$$

In a recent work by Cahn, Elliott and Novick–Cohen [7] a formal asymptotic result for the deep quench limit ($\theta \searrow 0$) of the Cahn–Hilliard equation with $B(u) = 1-u^2$ and Ψ as in (1.3) has been established (they used the scaling $\gamma = \varepsilon^2, t\to\varepsilon^2 t$). They show that one gets in the limit $\varepsilon\searrow 0$ the following geometric motion for hypersurfaces

$$V = -D\Delta_S\kappa \qquad , \qquad D\in\mathbb{R}^+\,, \tag{1.4}$$

where V is the normal velocity, κ denotes the mean curvature and Δ_S is the surface Laplacian. Material scientists refer to this evolution law as motion by surface diffusion (see Cahn and Taylor [10], Davi and Gurtin [12] and Mullins [21]). The two components of the alloy are separated by a sharp free boundary which is evolving according to the law (1.4).

Cahn and Taylor [10, 11] also propose the motions

$$V = \Delta_S\left(\frac{1}{M}\Delta_S - \frac{1}{D}\right)^{-1}\kappa\,, \qquad (M,D\in\mathbb{R}^+) \tag{1.5}$$

which formally give in the limit as $M\to\infty$ ($D\to\infty$ respectively) the laws

$$V = -D\Delta_S\kappa \qquad \text{if} \quad M\to\infty \tag{1.6}$$

and

$$V = M(\kappa - \kappa_\alpha) \qquad \text{if} \quad D \to \infty \tag{1.7}$$

where κ_α is the average mean curvature on the surface.

Formal asymptotic results suggest that the intermediate motion (1.5) is the asymptotic limit of the viscous Cahn–Hilliard equation with a mobility $B(u) = 1 - u^2$ (as before with a logarithmic free energy and in the deep quench limit with a scaling $\gamma = \varepsilon^2$ and $t \to \varepsilon^2 t$).

For the geometric motions (1.5)–(1.7) just a few results exist so far. We can prove local existence for the two dimensional case, i.e. for the evolution of curves in the plane and results for the global behaviour if the initial data are close to a circle (see [13]).

This paper is organized as follows. In section 2 we prove the existence of a solution to the Cahn–Hilliard equation with a mobility which is bounded away from zero. This result is used in section 3 to establish the existence of approximate solutions to the degenerate problem. We derive energy estimates for the approximate solutions which enable us to pass to the limit in the approximate equation to get the existence of a weak solution as stated in Theorem 1. Section 4 is devoted to other applications of our method. In particular the viscous Cahn–Hilliard equation and the deep quench limit are studied. Furthermore our method can be used to establish an existence result for degenerate parabolic equations of fourth order in arbitrary space dimensions. Finally we discuss some open questions and give suggestions for further research.

2 Existence theorems for positive mobilities

In this section we study the Cahn–Hilliard equation with a mobility which is bounded away from zero. We prove existence of solutions under various conditions on the bulk energy ψ. In section 3 we will use these solutions as approximate solutions for the degenerate case.

We consider the Cahn–Hilliard equation in the form

$$u_t = \nabla \cdot b(u) \nabla w \,, \tag{2.1}$$

$$w = -\gamma \Delta u + \psi'(u) \tag{2.2}$$

with Neumann and no–flux boundary conditions

$$\nabla u \cdot \mathbf{n} = 0 \quad \text{and} \quad \nabla w \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0,T)\,.$$

Here $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a bounded domain with Lipschitz boundary. We assume b and ψ to be such that

i) $b \in C(\mathbb{R}, \mathbb{R}^+)$ and there exist $b_1, B_1 > 0$ such that $b_1 \le |b(u)| \le B_1$ for all $u \in \mathbb{R}$,

ii) $\psi \in C^1(\mathbb{R}, \mathbb{R})$ and there exist constants $C_1, C_2, C_3 > 0$ such that

$$|\psi'(u)| \le C_1 |u|^q + C_2 \quad \text{and} \quad \psi(u) \ge -C_3$$

where $q = \frac{n}{n-2}$ if $n > 3$ and $q \in \mathbb{R}^+$ arbitrary if $n = 1, 2$.

Under these assumptions we can state the following theorem.

Theorem 2: *Suppose $u_0 \in H^1(\Omega)$. Then there exist a pair of functions (u, w) such that*

1) $u \in L^\infty(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega))$,

2) $u_t \in L^2(0, T; (H^1(\Omega))')$,

3) $u(0) = u_0$,

4) $w \in L^2(0, T; H^1(\Omega))$.

which satisfies equations (2.1) and (2.2) in the following weak sense

$$\int_0^T \langle \zeta(t), u_t(t) \rangle_{H^1,(H^1)'} = -\int_{\Omega_T} b(u) \nabla w \nabla \zeta \tag{2.3}$$

for all $\zeta \in L^2(0, T; H^1(\Omega))$ and

$$\int_\Omega w\phi = \gamma \int_\Omega \nabla u \nabla \phi + \int_\Omega \psi'(u)\phi \tag{2.4}$$

for all $\phi \in H^1(\Omega)$ and almost all $t \in [0, T]$.

Proof: To prove the theorem we apply a Galerkin approximation. Let $\{\phi_i\}_{i \in \mathbb{N}}$ be the eigenfunctions of the Laplace operator with Neumann boundary conditions, i.e.

$$-\Delta \phi_i = \lambda_i \phi_i \quad \text{in } \Omega \quad \text{and} \quad \nabla \phi_i \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega\,.$$

The eigenfunctions ϕ_i are orthogonal in the $H^1(\Omega)$ and the $L^2(\Omega)$ scalar product. We normalize the ϕ_i such that $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$. Furthermore we assume without loss of generality that $\lambda_1 = 0$.

Now we consider the following Galerkin ansatz for (2.1),(2.2)

$$\begin{aligned}
u^N(t,x) &= \sum_{i=1}^N c_i^N(t)\phi_i(x)\,,\ w^N(t,x) = \sum_{i=1}^N d_i^N(t)\phi_i(x)\,, && (2.5)\\
\int_\Omega \partial_t u^N \phi_j &= -\int_\Omega b(u^N)\nabla w^N \nabla \phi_j \quad \text{for} \quad j = 1,..,N\,, && (2.6)\\
\int_\Omega w^N \phi_j &= \gamma \int_\Omega \nabla u^N \nabla \phi_j + \int_\Omega \psi'(u^N)\phi_j \quad \text{for} \quad j = 1,..,N \quad \text{and} && (2.7)\\
u^N(0) &= \sum_{i=1}^N (u_0, \phi_i)_{L^2(\Omega)}\phi_i\,. && (2.8)
\end{aligned}$$

This gives an initial value problem for a system of ordinary differential equations for $(c_1, .., c_N)$:

$$\begin{aligned}
\partial_t c_j^N &= -\sum_{k=1}^N d_k^N \int_\Omega b\left(\sum_{i=1}^N c_i^N \phi_i\right) \nabla\phi_k \nabla\phi_j\,, && (2.9)\\
d_j^N &= \gamma\lambda_j c_j^N + \int_\Omega \psi'\left(\sum_{k=1}^N c_k^N \phi_k\right)\phi_j \quad \text{and} && (2.10)\\
c_j^N(0) &= (u_0, \phi_j)_{L^2(\Omega)} && (2.11)
\end{aligned}$$

which has to hold for $j = 1, .., N$. Since the right hand side in (2.9) depends continuously on $c_1, .., c_N$ the initial value problem has a local solution.

In order to derive a priori estimates we differentiate the energy $\mathcal{E}$ and get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\int_\Omega \left(\frac{\gamma}{2}|\nabla u^N|^2 + \psi(u^N)\right) , \\
&= \int_\Omega \left(\gamma \nabla u^N \nabla u_t^N + \psi'(u^N) u_t^N\right) , \\
&= \int_\Omega w^N u_t^N = - \int_\Omega b(u^N)|\nabla w^N|^2 .
\end{aligned}$$

This implies

$$\int_\Omega \frac{\gamma}{2}|\nabla u^N(t)|^2 + \int_\Omega \psi(u^N(t)) + \int_{\Omega_T} b(u^N)|\nabla w^N|^2 = \int_\Omega \frac{\gamma}{2}|\nabla u^N(0)|^2 + \int_\Omega \psi(u^N(0)) \le C . \tag{2.12}$$

The last inequality follows from (2.8), assumption ii) and the fact that $u_0 \in H^1(\Omega)$. Since $\frac{d}{dt}\int_\Omega u^N = 0$ (which follows from (2.6) with $j = 1$) Poincaré's inequality yields

$$\operatorname{ess\,sup}_{0<t<T} \|u(t)\|_{H^1(\Omega)} \le C .$$

This estimate implies that the $(c_1^N, .., c_N^N)$ are bounded and therefore a global solution to the initial value problem (2.9)–(2.11) exists.
If we denote by Π_N the projection of $L^2(\Omega)$ onto $span\{\phi_1, .., \phi_N\}$ we get

$$\begin{aligned}
|\int_{\Omega_T} \partial_t u^N \phi| &= |\int_{\Omega_T} \partial_t u^N \Pi_N \phi| \\
&= |\int_{\Omega_T} b(u^N)\nabla w^N \nabla \Pi_N \phi| \\
&\le \left(\int_{\Omega_T} |b(u^N)\nabla w^N|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_T} |\nabla \Pi_N \phi|^2\right)^{\frac{1}{2}} \\
&\le B_1 \left(\int_{\Omega_T} b(u^N)|\nabla w^N|^2\right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega_T)} \\
&\le C\|\nabla \phi\|_{L^2(\Omega_T)}
\end{aligned}$$

for all $\phi \in L^2(0, T; H^1(\Omega))$. This implies

$$\|\partial_t u^N\|_{L^2(0,T;(H^1(\Omega))')} \le C .$$

Using compactness results (see Lions [19] and Remark 1) we obtain for a subsequence (which we still denote by u^N)

$$\begin{array}{rcllll}
u^N & \longrightarrow & u & \text{weak} - * & \text{in} & L^\infty(0, T; H^1(\Omega)) , \\
u^N & \longrightarrow & u & \text{strongly} & \text{in} & C([0, T]; L^2(\Omega)) , \\
\partial_t u^N & \longrightarrow & \partial_t u & \text{weakly} & \text{in} & L^2(0, T; (H^1(\Omega))') \quad \text{and} \\
u^N & \longrightarrow & u & \text{strongly} & \text{in} & L^2(0, T; L^p(\Omega)) \text{ and a.e. in } \Omega_T
\end{array}$$

where $p < \frac{2n}{n-2}$. It remains to show the convegence of w^N. Choosing $j = 1$ in (2.7) gives $\int_\Omega w^N(t) = \int_\Omega \psi'(u^N(t))$ which together with (2.12), assumption ii) and Poincaré's inequality gives

$$\|w^N\|_{L^2(0,T;H^1(\Omega))} \leq C$$

This implies (again for a subsequence)

$$w^N \longrightarrow w \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\Omega)).$$

With the convergence properties proved so far and using the assumptions on b and ψ we can pass to the limit in (2.6) and (2.7) in a standard fashion (see Lions [19] for details) to get that (2.3) and (2.4) hold for (u, w).

The strong convergence of u^N in $C([0,T];L^2(\Omega))$ and the fact that $u^N(0) \to u_0$ in $L^2(\Omega)$ gives $u(0) = u_0$. This proves the theorem.

□

Remark 1: a) Let X, Y and Z be Banach spaces with a compact embedding $X \hookrightarrow Y$ and a continuous embedding $Y \hookrightarrow Z$. Then the embeddings

$$\{u \in L^2(0,T;X) \,|\, \partial_t u \in L^2(0,T;Z)\} \hookrightarrow L^2(0,T;Y) \tag{2.13}$$

and

$$\{u \in L^\infty(0,T;X) \,|\, \partial_t u \in L^2(0,T;Z)\} \hookrightarrow C([0,T];Y) \tag{2.14}$$

are compact (for a proof see Simon [22]).

b) In the proof of Theorem 2 we applied the above result for the case $X = H^1(\Omega), Y = L^2(\Omega)$ ($Y = L^p(\Omega)$ with $p < \frac{2n}{n-2}$ respectively) and $Z = (H^1(\Omega))'$.

c) The solution in Theorem 2 lies in $C([0,T];H^\beta(\Omega))$ (where $\beta < 1$). We get this by choosing $X = H^1(\Omega), Y = H^\beta(\Omega)$ and $Z = (H^1(\Omega))'$ in a).

□

The existence result in Theorem 2 requires a bulk energy which is bounded from below. It is possible to generalize this result if we assume further assumptions on $\partial\Omega$ and the growth of ψ.

We assume now either $\partial\Omega \in C^{1,1}$ or Ω is convex. Furthermore we replace assumption ii) by

iii) $\psi \in C^2(\mathbb{R}, \mathbb{R})$ and there exists a constant $D > 0$ such that $|\psi''(u)| \leq D$ for all $u \in \mathbb{R}$.

Theorem 3: *Assume i), iii) and $u_0 \in H^1(\Omega)$. Then there exists a function u such that*

1) $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$,

2) $u_t \in L^2(0,T;(H^1(\Omega))')$,

3) $u(0) = u_0$ *and* $\nabla u \cdot \mathbf{n} = 0$ *on* $\partial\Omega \times (0,T)$,

4) $\nabla\Delta u \in L^2(\Omega_T)$.

which satisfies the Cahn–Hilliard equation in the following sense

$$\int_0^T \langle \zeta(t), u_t(t) \rangle_{H^1,(H^1)'} = \int_{\Omega_T} b(u) \nabla(-\gamma \Delta u + \psi'(u)) \nabla \zeta$$

for all $\zeta \in L^2(0,T; H^1(\Omega))$.

Proof: As in the proof of Theorem 2 we apply a Galerkin approximation, but now we make an ansatz just in u:

$$\begin{aligned} u^N(t,x) &= \sum_{i=1}^N c_i^N(t) \phi_i(x) \,, \\ \int_\Omega \partial_t u^N \phi_j &= -\int_\Omega b(u_N)(-\gamma \nabla \Delta u^N + \psi''(u^N) \nabla u^N) \nabla \phi_j \text{ for } j = 1,..,N \,, \quad (2.15) \\ u^N(0) &= \sum_{i=1}^N (u_0, \phi_i)_{L^2(\Omega)} \phi_i \,. \end{aligned}$$

Instead of differentiating the energy we use Δu^N as a test function to get

$$\frac{1}{2} \partial_t \int_\Omega |\nabla u^N|^2 + \int_\Omega b(u^N) \gamma |\nabla \Delta u^N|^2 = \int_\Omega b(u^N) \psi''(u^N) \nabla u^N \nabla \Delta u^N \,.$$

Using Young's inequality and assumptions i) and iii) we derive

$$\partial_t \int_\Omega |\nabla u^N|^2 + b_1 \gamma \int_\Omega |\nabla \Delta u^N|^2 \leq C \int_\Omega |\nabla u^N|^2 \,.$$

A Gronwall argument now gives

$$\int_\Omega |\nabla u^N(t)|^2 + \int_{\Omega_T} |\nabla \Delta u^N|^2 \leq C(T) \,.$$

With this estimate the rest of the proof is straightforward using compactness results (see Lions [19] and Remark 1) and passing to the limit in equation (2.15).

□

3 Existence proof for the degenerate case

In this section we prove Theorem 1. Our approach is to approximate the degenerate problem by non–degenerate equations, i.e. by equations with a positive mobility. Furthermore we modify the bulk energy Ψ so that it is defined on all $\mathbb{R}$.

We introduce a positive mobility B_ε as

$$B_\varepsilon(u) := \begin{cases} B(-1+\varepsilon) & \text{for} \quad u \leq -1 + \varepsilon \,, \\ B(u) & \text{for} \quad |u| < 1 - \varepsilon \,, \\ B(1-\varepsilon) & \text{for} \quad u \geq 1 - \varepsilon \end{cases}$$

and we define Φ_ε such that $\Phi_\varepsilon''(u) = \frac{1}{B_\varepsilon(u)}$ and $\Phi_\varepsilon'(0) = \Phi_\varepsilon(0) = 0$. We point out that $\Phi_\varepsilon(u) = \Phi(u)$ when $|u| \le 1 - \varepsilon$.

The modified bulk energy $\Psi_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ is taken to be $\Psi_\varepsilon := \Psi_\varepsilon^1 + \Psi^2$ where

$$\left(\Psi_\varepsilon^1\right)''(u) := \begin{cases} \left(\Psi^1\right)''(-1+\varepsilon) & \text{for} \quad u \le -1+\varepsilon\,, \\ \left(\Psi^1\right)''(u) & \text{for} \quad |u| < 1-\varepsilon\,, \\ \left(\Psi^1\right)''(1-\varepsilon) & \text{for} \quad u \ge 1-\varepsilon \end{cases}$$

and $\Psi_\varepsilon^1(0) = \Psi^1(0)$, $\left(\Psi_\varepsilon^1\right)'(0) = \left(\Psi^1\right)'(0)$. As for Φ we get $\Psi_\varepsilon(u) = \Psi(u)$ if $|u| \le 1-\varepsilon$. Furthermore Ψ^2 is extended to be a function on all $\mathbb{R}$ such that $\|\Psi^2\|_{C^2(\mathbb{R})} \le C$.

With this choice of B_ε and Ψ_ε Theorem 2 give the existence of a weak solution to the equation

$$\begin{aligned} u_t &= \nabla \cdot B_\varepsilon(u) \nabla w && \text{in} \quad \Omega_T\,, \\ w &= -\gamma \Delta u + \left(\Psi_\varepsilon\right)'(u) && \text{in} \quad \Omega_T\,, \\ \nabla u \cdot \mathbf{n} &= 0 \quad \text{and} \quad \nabla w \cdot \mathbf{n} = 0 && \text{on} \quad \partial\Omega \times (0,T)\,. \end{aligned}$$

We denote the solution by $(u_\varepsilon, w_\varepsilon)$. From now on we assume either $\partial\Omega \in C^{1,1}$ or Ω is convex. With this assumption we can state:

Lemma 1: *The solution u_ε belongs to the space $L^2(0,T;H^2(\Omega))$ and $\nabla\Delta u_\varepsilon \in L^2(\Omega_T)$.*

Proof: Since

$$\int_\Omega w_\varepsilon \phi = \int_\Omega \gamma \nabla u_\varepsilon \nabla \phi + \int_\Omega \Psi_\varepsilon'(u_\varepsilon)\phi$$

for all $\phi \in H^1(\Omega)$ and almost all $t \in (0,T)$ the first assertion follows from elliptic regularity theory. Because $\nabla w_\varepsilon \in L^2(\Omega_T)$ and $\nabla \Psi_\varepsilon'(u_\varepsilon) = \Psi_\varepsilon''(u_\varepsilon)\nabla u_\varepsilon \in L^2(\Omega_T)$ the identity $w_\varepsilon = -\gamma\Delta u_\varepsilon + \Psi_\varepsilon'(u_\varepsilon)$ gives $\nabla\Delta u_\varepsilon \in L^2(\Omega_T)$.

$\square$

Therefore we get

$$\int_0^T \langle \zeta, \partial_t u_\varepsilon \rangle_{H^1,(H^1)'} = -\int_{\Omega_T} B_\varepsilon(u_\varepsilon) \nabla(-\gamma\Delta u_\varepsilon + \Psi_\varepsilon'(u_\varepsilon)) \nabla\zeta \tag{3.1}$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$.

In the next step we prove the following energy estimates.

Lemma 2: *There exists an ε_0 such that for all $0 < \varepsilon \le \varepsilon_0$ the following estimates hold with a constant C independent of ε.*

$$
\begin{array}{ll}
a) & \operatorname{ess\,sup}_{0\le t\le T}\int_\Omega \left(\frac{\gamma}{2}|\nabla u_\varepsilon(t)|^2+\Psi_\varepsilon(u_\varepsilon(t))\right)+\int_{\Omega_T} B_\varepsilon(u_\varepsilon)|\nabla w_\varepsilon|^2\le C\,,\\
b) & \operatorname{ess\,sup}_{0\le t\le T}\int_\Omega \Phi_\varepsilon(u_\varepsilon(t))+\int_{\Omega_T}\left(\gamma|\Delta u_\varepsilon|^2+(\Psi^1_\varepsilon)''(u_\varepsilon)|\nabla u_\varepsilon|^2\right)\le C\,,\\
c) & \operatorname{ess\,sup}_{0\le t\le T}\int_\Omega (|u_\varepsilon|-1)_+^2\le C\varepsilon^m\,,\\
d) & \int_{\Omega_T}|\mathbf{J}_\varepsilon|^2\le C \quad \textit{where} \quad \mathbf{J}_\varepsilon:=B_\varepsilon(u_\varepsilon)\nabla w_\varepsilon\,.
\end{array}
$$

Proof: The function $w_\varepsilon = -\gamma\Delta u_\varepsilon + \Psi'_\varepsilon(u_\varepsilon) \in L^2(0,T;H^1(\Omega))$ is a valid test function in (3.1). Therefore we obtain

$$
\int_0^t \langle -\gamma\Delta u_\varepsilon + \Psi'_\varepsilon(u_\varepsilon), \partial_t u_\varepsilon\rangle_{H^1,(H^1)'} = -\int_{\Omega_t} B_\varepsilon(u_\varepsilon)|\nabla w_\varepsilon|^2 \tag{3.2}
$$

for all $t\in[0,T]$.

We claim: for almost all $t\in[0,T]$

$$
\int_0^t \langle -\gamma\Delta u_\varepsilon + \Psi'_\varepsilon(u_\varepsilon), \partial_t u_\varepsilon\rangle_{H^1,(H^1)'} = \int_\Omega |\nabla u_\varepsilon(t)|^2 + \int \Psi_\varepsilon(u_\varepsilon(t)) - \int_\Omega |\nabla u_0|^2 - \int_\Omega \Psi_\varepsilon(u_0) \tag{3.3}
$$

holds.

To prove this we define functions

$$
u_{\varepsilon h}(t,x) := \frac{1}{h}\int_{t-h}^t u_\varepsilon(\tau,x)d\tau \tag{3.4}
$$

where we set $u_\varepsilon(t,x)=u_0(x)$ when $t\le 0$. It is easily proved that

$$
\begin{array}{rcll}
\Delta u_{\varepsilon h} & \longrightarrow & \Delta u_\varepsilon & \text{strongly in} \quad L^2(0,T;H^1(\Omega))\,,\\
\text{and} \quad \Psi'(u_{\varepsilon h}) & \longrightarrow & \Psi'(u_\varepsilon) & \text{strongly in} \quad L^2(0,T;H^1(\Omega))\,,
\end{array}
$$

for at least a subsequence (as $h\searrow 0$). Furthermore we can show $\partial_t u_{\varepsilon h}\longrightarrow \partial_t u_\varepsilon$ strongly in $L^2(0,T;(H^1(\Omega))')$. For any $\zeta\in L^2(0,T;H^1(\Omega))$ we have

$$
\begin{aligned}
|\langle \zeta, \partial_t u_{\varepsilon h}-\partial_t u_\varepsilon\rangle_{L^2(H^1),L^2((H^1)')}| &= \frac{1}{h}\left|\int_0^T \langle \zeta, \int_{t-h}^t (\partial_t u_\varepsilon(\tau)-\partial_t u_\varepsilon(t))d\tau\rangle_{H^1,(H^1)'}dt\right|\\
&= \frac{1}{h}\left|\int_0^T \langle \zeta, \int_{-h}^0 (\partial_t u_\varepsilon(t+s)-\partial_t u_\varepsilon(t)ds\rangle dt\right|\\
&\le \frac{1}{h}\int_{-h}^0 \left|\int_0^T\int_\Omega \nabla\zeta\cdot(\mathbf{J}_\varepsilon(t+s)-\mathbf{J}_\varepsilon(t))dx\,dt\,\right|ds\\
&\le \|\nabla\zeta\|_{L^2(\Omega_T)} \sup_{-h\le s\le 0}\|\mathbf{J}_\varepsilon(.+s)-\mathbf{J}_\varepsilon(.)\|_{L^2(\Omega_T)}
\end{aligned}
$$

Since

$$\sup_{-h\le s\le 0} \|\mathbf{J}_\varepsilon(.+s) - \mathbf{J}_\varepsilon(.)\|_{L^2(\Omega_T)} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0\,.$$

it follows that

$$\partial_t u_{\varepsilon h} \longrightarrow \partial_t u_\varepsilon \quad \text{strongly in} \quad L^2(0,T;(H^1(\Omega))')\,.$$

Using $\partial_t u_{\varepsilon h} \in L^2(\Omega_T)$ we have for almost all $t \in [0,T]$:

$$\begin{aligned}
&\int_0^t \langle -\gamma\Delta u_{\varepsilon h} + \Psi_\varepsilon'(u_{\varepsilon h}), \partial_t u_{\varepsilon h}\rangle_{H^1,(H^1)'} = \\
&= \int_{\Omega_t} \left(-\gamma\Delta u_{\varepsilon h} + \Psi_\varepsilon'(u_{\varepsilon h})\right)\partial_t u_{\varepsilon h} \\
&= \partial_t \int_0^t\int_\Omega \left(\frac{\gamma}{2}|\nabla u_{\varepsilon h}|^2 + \Psi(u_{\varepsilon h})\right) \\
&= \int_\Omega \left(\frac{\gamma}{2}|\nabla u_{\varepsilon h}(t)|^2 + \Psi(u_{\varepsilon h}(t))\right) - \int_\Omega \left(\frac{\gamma}{2}|\nabla u_0|^2 + \Psi_\varepsilon(u_0)\right)
\end{aligned}$$

Passing to the limit ($h \searrow 0$) in this equation, where we apply the convergence properties of $u_{\varepsilon h}$ proved above and using (3.2) gives for almost all t

$$\int_\Omega \left(\frac{\gamma}{2}|\nabla u_\varepsilon(t)|^2 + \Psi_\varepsilon(u_\varepsilon(t))\right) + \int_{\Omega_t} B_\varepsilon(u_\varepsilon)|\nabla w_\varepsilon|^2 = \int_\Omega \left(\frac{\gamma}{2}|\nabla u_0|^2 + \Psi_\varepsilon(u_0)\right)\,.$$

Noting that $\Psi_\varepsilon(u) \le \Psi(u)$ for ε sufficiently small proves a).

To prove b) we want to use $\Phi_\varepsilon'(u_\varepsilon)$ as a test function in (3.1). Since Φ_ε'' is bounded we have $\Phi_\varepsilon'(u_\varepsilon) \in L^2(0,T;H^1(\Omega))$ and therefore is an admissible test function. With a similar argument as in the proof of a) we can prove that

$$\int_0^t \langle \Phi_\varepsilon'(u_\varepsilon), \partial_t u_\varepsilon\rangle_{H^1,(H^1)'} = \int_\Omega \Phi_\varepsilon(u_\varepsilon(t)) - \int_\Omega \Phi_\varepsilon(u_0)$$

is true for almost all $t \in [0,T]$. On the other hand we derive

$$\begin{aligned}
&\int_{\Omega_t} B_\varepsilon(u_\varepsilon)\nabla(-\gamma\Delta u_\varepsilon + \Psi_\varepsilon'(u_\varepsilon))\nabla\Phi_\varepsilon'(u_\varepsilon) = \\
&= \int_{\Omega_t} \left(-\gamma\nabla\Delta u_\varepsilon + \Psi_\varepsilon''(u_\varepsilon)\nabla u_\varepsilon\right) B_\varepsilon(u_\varepsilon)\Phi_\varepsilon''(u_\varepsilon)\nabla u_\varepsilon \\
&= \int_{\Omega_t} \left(\gamma|\Delta u_\varepsilon|^2 + \Psi_\varepsilon''(u_\varepsilon)|\nabla u_\varepsilon|^2\right)
\end{aligned}$$

It follows that

$$\int_\Omega \Phi_\varepsilon(u_\varepsilon(t)) + \int_{\Omega_t} \gamma|\Delta u_\varepsilon|^2 + \int_{\Omega_t} \left(\Psi_\varepsilon^1\right)''(u_\varepsilon)|\nabla u_\varepsilon|^2 \le \int_\Omega \Phi_\varepsilon(u_0) + \int_{\Omega_t} |\left(\Psi^2\right)''(u_\varepsilon)|\,|\nabla u_\varepsilon|^2\,.$$

Since $\Phi_\varepsilon(u) \le \Phi(u)$ for ε sufficiently small and $(\Psi^2)''$ is bounded we have proved b) (note that we have estimated $\int_\Omega |\nabla u_\varepsilon|^2$ in a)).

Now we can use the bound for $\int_\Omega \Phi_\varepsilon(u_\varepsilon)$ to derive a bound for $\int_\Omega (|u_\varepsilon|-1)_+^2$. If $z>1$ and $\varepsilon<1$ then we have

$$\begin{aligned}
\Phi_\varepsilon(z) &= \underbrace{\Phi(1-\varepsilon)}_{\geq 0} + \underbrace{\Phi'(1-\varepsilon)}_{\geq 0}\underbrace{(z-(1-\varepsilon))}_{\geq 0} + \frac{1}{2}\Phi''(1-\varepsilon)\,(z-(1-\varepsilon))^2 \\
&\geq \frac{1}{2}\Phi''(1-\varepsilon)(z-1)^2 = \frac{1}{2}\frac{1}{B(1-\varepsilon)}(z-1)^2 \\
&= \frac{1}{2}\frac{1}{(1-(1-\varepsilon)^2)^m\,\bar{B}(1-\varepsilon)}(z-1)^2 \geq C^{-1}\varepsilon^{-m}(z-1)^2\,.
\end{aligned}$$

It follows now $(z-1)^2 \leq C\varepsilon^m \Phi_\varepsilon(z)$. Similarly we obtain $(|z|-1)^2 \leq C\varepsilon^m\Phi_\varepsilon(z)$ for $z<-1$. This implies

$$\int_\Omega (|u_\varepsilon|-1)_+^2 \leq C\varepsilon^m \int_\Omega \Phi_\varepsilon(u_\varepsilon) \leq C\varepsilon^m$$

which proves c).

Assertion d) follows easily from a), and this finishes the proof of Lemma 2.

□

Since Δu_ε is uniformly bounded in $L^2(\Omega_T)$, $\nabla u_\varepsilon \cdot \mathbf{n} = 0$ and $\int_\Omega u_\varepsilon = \int_\Omega u_0$, elliptic regularity theory yields

$$\|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C\,.$$

Now we apply the compactness result mentioned in Remark 1 (2.13) with $X = H^2(\Omega)$, $Y = H^1(\Omega)$ and $Z = (H^1(\Omega))'$ to conclude the existence of a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ (which we still denote by $(u_\varepsilon)_{\varepsilon>0}$) such that

$$u_\varepsilon, \nabla u_\varepsilon \longrightarrow u, \nabla u \quad \text{strongly in} \quad L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T.$$

Furthermore using standard compactness properties we obtain the convergence

$$\begin{aligned}
\partial_t u_\varepsilon &\longrightarrow \partial_t u \quad \text{weakly in} \quad L^2(0,T;(H^1(\Omega))')\,, \\
\Delta u_\varepsilon &\longrightarrow \Delta u \quad \text{weakly in} \quad L^2(\Omega_T)\,, \\
\text{and} \quad \mathbf{J}_\varepsilon &\longrightarrow \mathbf{J} \quad \text{weakly in} \quad L^2(\Omega_T)\,.
\end{aligned}$$

Passing to the limit in

$$\int_\Omega (|u_\varepsilon|-1)_+^2 \leq C\varepsilon^m$$

yields $|u| \leq 1$ a.e. in Ω_T.

It remains to show that u fulfills the limit equation. The weak convergence of $\partial_t u_\varepsilon$ and $\mathbf{J}_\varepsilon$ gives in the limit

$$\int_0^T \langle \zeta, \partial_t u\rangle_{H^1,(H^1)'} = \int_{\Omega_T} \mathbf{J}\cdot\nabla\zeta$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$. Now we have to identify $\mathbf{J}$. Therefore we want to pass to the limit in the equation

$$\int_{\Omega_T} \mathbf{J}_\varepsilon \cdot \boldsymbol{\eta} = \int_{\Omega_T} B_\varepsilon(u_\varepsilon) \nabla \left(-\gamma \Delta u_\varepsilon + \Psi'_\varepsilon(u_\varepsilon)\right) \boldsymbol{\eta} \tag{3.5}$$

where $\boldsymbol{\eta} \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \cap L^\infty(\Omega_T,\mathbb{R}^n)$ with $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ on $\partial\Omega \times (0,T)$. The left hand side converges to $\int_{\Omega_T} \mathbf{J}\cdot\boldsymbol{\eta}$. Since $\nabla\Delta u_\varepsilon$ may not have a limit in $L^2(\Omega_T)$ we integrate the first term on the right hand side of (3.5) by parts to get

$$\int_{\Omega_T} B_\varepsilon(u_\varepsilon)\nabla(-\gamma\Delta u_\varepsilon)\boldsymbol{\eta} = \int_{\Omega_T} \gamma\Delta u_\varepsilon B_\varepsilon(u_\varepsilon)\nabla\cdot\boldsymbol{\eta} + \int \gamma\Delta u_\varepsilon B'_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\cdot\boldsymbol{\eta} =: I + II\,.$$

Using the fact that for all $z \in \mathbb{R}$

$$|B_\varepsilon(z) - B(z)| \le \sup_{1-\varepsilon\le|y|\le 1} |B(y)| \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0$$

it follows that $B_\varepsilon \longrightarrow B$ uniformly.
Hence we have

$$B_\varepsilon(u_\varepsilon) \longrightarrow B(u) \quad \text{a.e. in } \Omega_T\,.$$

Since $\Delta u_\varepsilon \longrightarrow \Delta u$ weakly in $L^2(\Omega_T)$ and B_ε is uniformly bounded we conclude

$$\int_{\Omega_T} \gamma\Delta u_\varepsilon B_\varepsilon(u_\varepsilon)\nabla\cdot\boldsymbol{\eta} \longrightarrow \int_{\Omega_T} \gamma\Delta u B(u)\nabla\cdot\boldsymbol{\eta} \quad \text{as} \quad \varepsilon \longrightarrow 0\,.$$

Now we pass to the limit in II. First of all we consider the case $m > 1$. As for B we have $B'_\varepsilon \longrightarrow B'$ uniformly which gives

$$B'_\varepsilon(u_\varepsilon) \longrightarrow B'(u) \quad \text{a.e in } \Omega_T\,.$$

by using

$$\nabla u_\varepsilon \longrightarrow \nabla u \quad \text{in} \quad L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T\,,$$

and the fact that B'_ε is uniformly bounded a generalized version of the Lebesgue convergence theorem yields

$$B'_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \longrightarrow B'(u)\nabla u \quad \text{in} \quad L^2(\Omega_T)\,.$$

Hence

$$\int_{\Omega_T} \gamma\Delta u_\varepsilon B'_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\cdot\boldsymbol{\eta} \longrightarrow \int_{\Omega_T} \gamma\Delta u B'(u)\nabla u\cdot\boldsymbol{\eta}$$

where we use the fact that $\boldsymbol{\eta} \in L^\infty(\Omega_T)$.

In the case $m = 1$ the function B' is discontinuous and we have to use a more subtle argument.

We claim: $\quad B'_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \longrightarrow B'(u)\nabla u \quad \text{in} \quad L^2(\Omega_T)\,.$

We analyze the following integrals

$$\int_{\Omega_T} |B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon - B'(u)\nabla u|^2 =$$
$$\int_{\Omega_T\cap\{|u|<1\}} |B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon - B'(u)\nabla u|^2 + \int_{\Omega_T\cap\{|u|=1\}} |B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon - B'(u)\nabla u|^2 .$$

Since $\nabla u = 0$ on the set $\{|u|=1\}$ (see [16] Lemma 7.7) we obtain

$$\begin{aligned}\int_{\Omega_T\cap\{|u|=1\}} |B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon - B'(u)\nabla u|^2 &= \int_{\Omega_T\cap\{|u|=1\}} |B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon|^2 \\ &\le C\int_{\Omega_T\cap\{|u|=1\}} |\nabla u_\varepsilon|^2 \longrightarrow C\int_{\Omega_T\cap\{|u|=1\}} |\nabla u|^2 = 0\end{aligned}$$

On the set $\{|u|<1\}$ we know $B_\varepsilon'(u_\varepsilon) \longrightarrow B'(u)$ a.e.
Hence we have

$$B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon \longrightarrow B'(u)\nabla u \quad \text{a.e. in } \Omega_T .$$

The generalized Lebesgue convergence theorem now gives

$$\int_{\Omega_T\cap\{|u|<1\}} |B_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon - B'(u)\nabla u|^2 \longrightarrow 0$$

which proves our claim. Furthermore this proves that we can pass to the limit in II.

To complete the proof of Theorem 1 we have to show

$$\int_{\Omega_T} B_\varepsilon(u_\varepsilon)\Psi_\varepsilon''(u_\varepsilon)\nabla u_\varepsilon\cdot\boldsymbol{\eta} \longrightarrow \int_{\Omega_T} (B\Psi'')(u)\nabla u\cdot\boldsymbol{\eta} . \tag{3.6}$$

First of all we want to point out that $B_\varepsilon\Psi_\varepsilon''$ is uniformly bounded. Therefore it is sufficient to show

$$B_\varepsilon(u_\varepsilon)\Psi_\varepsilon''(u_\varepsilon) \longrightarrow (B\Psi'')(u) = \bar{B}(u)F(u) + B(u)\Psi_2''(u) \quad \text{a.e. in } \Omega_T . \tag{3.7}$$

If $|u(t,x))| < 1$ the convergence in (3.7) follows from the definition of B_ε and Ψ_ε ($B_\varepsilon(z) = B(z)$ and $\Psi_\varepsilon(z) = \Psi(z)$ if $|z| < 1-\varepsilon$). Now let us consider points (t,x) where $|u(t,x)| = 1$. Without loss of generality we assume $u_\varepsilon(t,x) \longrightarrow 1 = u(t,x)$.

For ε with $u_\varepsilon(t,x) \ge 1-\varepsilon$ we have

$$\begin{aligned} B_\varepsilon(u_\varepsilon(t,x))\Psi_\varepsilon''(u_\varepsilon(t,x)) &= \bar{B}(1-\varepsilon)F(1-\varepsilon) + B(1-\varepsilon)\Psi_2''(u_\varepsilon(t,x)) \\ &\longrightarrow \bar{B}(1)F(1) + B(1)\Psi_2''(1) = (B\Psi'')(u(t,x)) . \end{aligned}$$

On the other side if $u_\varepsilon(t,x) \le 1-\varepsilon$ and $u_\varepsilon(t,x) \longrightarrow 1$ we have

$$\begin{aligned} B_\varepsilon(u_\varepsilon(t,x))\Psi_\varepsilon''(u_\varepsilon(t,x)) &= B(u_\varepsilon(t,x))\Psi''(u_\varepsilon(t,x)) \\ &= \bar{B}(u_\varepsilon(t,x))F(u_\varepsilon(t,x)) + B(u_\varepsilon(t,x))\Psi_2''(u_\varepsilon(t,x)) \\ &\longrightarrow (B\Psi'')(u(t,x)) . \end{aligned}$$

We proved $B_\varepsilon(u_\varepsilon)\Psi''_\varepsilon(u_\varepsilon) \longrightarrow (B\Psi'')(u)$ a.e. in Ω_T, which together with the strong convergence of ∇u_ε in $L^2(\Omega_T)$ gives (3.6). This shows that u solves the Cahn–Hilliard equation in the sense of Theorem 1. The facts that $u \in C([0,T];L^2(\Omega))$ and $u_\varepsilon(0) = u_0$ follow as in the proof of Theorem 2 from an application of the compactness result mentioned in Remark 1. In fact it holds that $u \in C([0,T];H^\beta(\Omega))$ with $\beta < 1$.

$\square$

Remark 2 (generalized Lebesgue convergence theorem): Assume $E \subset \mathbb{R}^n$ is measurable, $g_n \longrightarrow g$ in $L^q(E)$ with $1 \leq q < \infty$ and $f_n, f : E \longrightarrow \mathbb{R}^n$ are measurable functions such that

$$\begin{aligned} f_n \longrightarrow f \qquad & \text{a.e. in } E\,, \\ |f_n|^p \leq |g_n|^q \qquad & \text{a.e. in } E \end{aligned}$$

with $1 \leq p < \infty$. Then $f_n \longrightarrow f$ in $L^p(E)$.

$\square$

For a proof see [1, A 1.26].

Remark 3: For $m \in [1,2)$ the functions Φ and Ψ are bounded on the interval $[-1,1]$ and therefore the assumption

$$\int_\Omega (\Phi(u_0) + \Psi(u_0)) \leq C$$

imposes no restrictions on the initial data. This is in particular true for the case $B(u) = 1 - u^2$ and Ψ of the logarithmic form (1.2).

$\square$

The following corollary gives an additional result in the case $m \geq 2$.

Corollary: *Assume $m \geq 2$ and u is the solution constructed in Theorem 1. Then*

a) $$\operatorname{ess\,sup}_{0\leq t\leq T} \int_\Omega (\Phi(u(t)) + \Psi(u(t))) \leq C$$

b) the set $\{x \mid |u(t,x)| = 1\}$ *has zero measure for almost all* $t \in [0,T]$.

Proof: We have proved

$$\int_\Omega \Phi_\varepsilon(u_\varepsilon(t)) \leq C$$

for almost all $t \in [0,T]$. Since $\Phi_\varepsilon(u_\varepsilon) \geq 0$ the Lemma of Fatou gives

$$\int_\Omega \liminf_{\varepsilon\searrow 0} \Phi_\varepsilon(u_\varepsilon(t)) \leq \liminf_{\varepsilon\searrow 0} \int_\Omega \Phi_\varepsilon(u_\varepsilon(t)) \leq C\,.$$

We claim:

$$\liminf_{\varepsilon\searrow 0} \Phi_\varepsilon(u_\varepsilon) = \begin{cases} \Phi(u) & \text{if } \; |u| < 1\,, \\ \infty & \text{elsewhere}\,. \end{cases}$$

If $|u| < 1$ it is clear that $\lim_{\varepsilon\searrow 0}\Phi_\varepsilon(u_\varepsilon) = \Phi(u)$. Now we consider points (t,x) where $\lim_{\varepsilon\searrow 0} u_\varepsilon(t,x) = 1$. In this case we have

$$\Phi_\varepsilon(u_\varepsilon(t,x)) \geq \min\left(\Phi(1-\varepsilon), \Phi(u_\varepsilon(t,x))\right) \longrightarrow \infty\,.$$

as $\varepsilon \longrightarrow 0$. The same argument can be applied for $\lim_{\varepsilon\searrow 0} u_\varepsilon(t,x) = -1$, which proves the claim. Therefore the set $\{x \mid |u(t,x)| = 1\}$ has zero measure and

$$\liminf_{\varepsilon\searrow 0} \Phi_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\searrow 0} \Phi_\varepsilon(u_\varepsilon) = \Phi(u) \quad \text{a.e. in } \ \Omega_T\,.$$

The estimate $\int_\Omega \Psi(u(t)) \leq C$ is proved similarly.

□

Remark 4: Since F can vanish at ± 1, Ψ'' can be less singular than of order m. In particular the smooth double well potential $\Psi(u) = (1-u^2)^2$ and the double obstacle potential are possible choices for all $m \geq 1$.

4 Some generalizations

4.1 The viscous Cahn–Hilliard equation

In this subsection we consider the viscous Cahn–Hilliard equation with a non–constant mobility

$$\begin{aligned} u_t &= -\nabla\cdot\mathbf{J}\,, \\ \mathbf{J} &= -B(u)\nabla w\,, \\ w &= -\gamma\Delta u + \Psi'(u) + \alpha u_t\,, \qquad \alpha \in \mathbb{R}^+ \end{aligned}$$

supplemented with the boundary conditions $\mathbf{J}\cdot\mathbf{n} = 0$ and $\nabla u\cdot\mathbf{n} = 0$ on $\partial\Omega\times(0,T)$. For a mobility $B \equiv 1$ this is the usual viscous Cahn–Hilliard equation as studied by Novick–Cohen, Elliott, Stuart and others [2, 15, 20].

In a first step we state a theorem for the non–degenerate case. Therefore we assume b and ψ to fulfill assumptions i) and ii) in section 2).

Theorem 4: *Suppose $u_0 \in H^1(\Omega)$ and $\partial\Omega$ Lipschitz. Then there exist a pair (u,w) such that*
1) $u \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))\,,$
2) $u_t \in L^2(\Omega_T)\,,$
3) $u(0) = u_0\,,$
4) $w \in L^2(0,T;H^1(\Omega))\,.$

which satisfies

$$\int_{\Omega_T} \zeta u_t = -\int_{\Omega_T} b(u)\nabla w\nabla\zeta$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ *and*

$$\int_\Omega w\phi = \gamma \int_\Omega \nabla u \nabla \phi + \int_\Omega \psi'(u)\phi$$

for all $\phi \in H^1(\Omega)$ *and almost all* $t \in [0,T]$.

Proof: As in the proof of Theorem 2 we apply a Galerkin approximation

$$\begin{aligned}
u^N(t,x) &= \sum_{i=1}^N c_i^N(t)\phi_i(x)\,,\; w^N(t,x) = \sum_{i=1}^N d_i^N(t)\phi_i(x)\,,\\
\int_\Omega \partial_t u^N \phi_j &= -\int_\Omega b(u_N)\nabla w^N \nabla \phi_j \quad \text{for} \quad j=1,..,N\,,\\
\int_\Omega w^N \phi_j &= \gamma \int_\Omega \nabla u^N \nabla \phi_j + \int_\Omega \psi'(u^N)\phi_j + \alpha \int_\Omega \partial_t u^N \phi_j \quad \text{for} \quad j=1,..,N\\
u^N(0) &= \sum_{i=1}^N (u_0,\phi_i)_{L^2(\Omega)}\phi_i\,.
\end{aligned}$$

which gives

$$\partial_t c_j^N = -\sum_{k=1}^N d_k^N \int_\Omega b\left(\sum_{i=1}^N c_i^N \phi_i\right)\nabla\phi_k\nabla\phi_j\,, \tag{4.1}$$

$$d_j^N = \gamma\lambda_j c_j^N + \int_\Omega \psi'\left(\sum_{l=1}^N c_l^N\phi_l\right)\phi_j + \alpha\partial_t c_j \quad \text{and} \tag{4.2}$$

$$c_j^N(0) = (u_0,\phi_j)_{L^2(\Omega)}\,. \tag{4.3}$$

These equations have to hold for $j = 1,..,N$. This yields to the following initial value problem for $(c_1^N,..,c_N^N)$:

$$\begin{aligned}
\partial_t c_j^N + \alpha\sum_{k=1}^N \partial_t c_k \int_\Omega b\left(\sum_{i=1}^N c_i^N\phi_i\right)\nabla\phi_k\nabla\phi_j = \quad (4.4)\\
-\sum_{k=1}^N\left(\gamma\lambda_k c_k^N + \int_\Omega \psi'\left(\sum_{l=1}^N c_l^N\phi_l\right)\phi_k\right)\int_\Omega b\left(\sum_{i=1}^N c_i^N\phi_i\right)\nabla\phi_k\nabla\phi_j\,,
\end{aligned}$$

$$c_j^N(0) = (u_0,\phi_j)_{L^2(\Omega)}\,. \tag{4.5}$$

Since the matrix $(g_{jk})_{j,k=1,N}$ with

$$g_{jk} = \int_\Omega b\left(\sum_i c_i^N\phi_i\right)\nabla\phi_k\nabla\phi_j$$

is positive definite, the initial value problem (4.4),(4.5) has a local solution.

Now we use

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_T}\left(|\nabla u^N|^2+\psi(u^N)\right) &= \int_{\Omega_T}\left(-\gamma\Delta u^N+\psi'(u^N)\right)u_t^N \\
&= \int_{\Omega} w^N u_t^N - \int_{\Omega}\alpha\left(u_t^N\right)^2 \\
&= -\int_{\Omega} b(u^N)|\nabla w^N|^2 - \int_{\Omega_T}\alpha\left(u_t^N\right)^2
\end{aligned}
$$

to establish a priori estimates. The rest is proved in a similar way as in the proof of Theorem 2.

□

Having proved an existence theorem for a positive mobility we are now in a position to prove existence for the degenerate case. We assume Ψ and B to be as in the introduction. Furthermore we assume either $\partial\Omega \in C^{1,1}$ or Ω convex.

Theorem 5: *Let $u_0 \in H^1(\Omega)$ satisfy $|u_0| \le 1$ a.e. in Ω_T and*

$$\int_{\Omega}(\Psi(u_0)+\Phi(u_0)) \le C\,, \quad C \in \mathbb{R}^+ .$$

Then there exists a pair $(u, \mathbf{J})$ such that
a) $u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$,
b) $u_t \in L^2(\Omega_T)$,
c) $u(0) = u_0$ *and* $\nabla u \cdot \mathbf{n} = 0$ *on* $\partial\Omega \times (0,T)$,
d) $|u| \le 1$ *a.e in* $\Omega_T := \Omega \times (0,T)$,
e) $\mathbf{J} \in L^2(\Omega_T, \mathbb{R}^n)$,

which satisfies $u_t = -\nabla\cdot\mathbf{J}$ in $L^2(0,T;(H^1(\Omega))')$, i.e.

$$\int_{\Omega_T}\zeta u_t = \int_{\Omega_T}\mathbf{J}\cdot\nabla\zeta$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ and

$$\mathbf{J} = -B(u)\nabla\cdot(-\gamma\Delta u + \Psi'(u) + \alpha u_t)$$

in the following weak sense

$$\int_{\Omega_T}\mathbf{J}\cdot\boldsymbol{\eta} = -\int_{\Omega_T}\left[(\gamma\Delta u - \alpha u_t)\,\nabla\cdot(B(u)\boldsymbol{\eta}) + (B\Psi'')(u)\nabla u\cdot\boldsymbol{\eta}\right]$$

for all $\boldsymbol{\eta} \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \cap L^\infty(\Omega_T,\mathbb{R}^n)$ which fulfill $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ on $\partial\Omega\times(0,T)$.

Proof: We modify B and Ψ in the same manner as in the proof of Theorem 1 to get functions B_ε and Ψ_ε. For the modified equation we proved existence in Theorem

4. In a similar fashion as in the proof of Theorem 1 we can derive the following identities for the approximating solutions $(u_\varepsilon, w_\varepsilon)$.

$$\operatorname{ess\,sup}_{0\le t\le T}\int_\Omega \left(\tfrac{\gamma}{2}|\nabla u_\varepsilon(t)|^2 + \Psi(u_\varepsilon(t))\right) + \int_{\Omega_T} B_\varepsilon(u_\varepsilon)|\nabla w_\varepsilon|^2 + \\ +\alpha\int_{\Omega_T}(\partial_t u_\varepsilon)^2 = \int_\Omega \left(\tfrac{\gamma}{2}|\nabla u_0|^2 + \Psi_\varepsilon(u_0)\right)$$

and

$$\operatorname{ess\,sup}_{0\le t\le T}\int_\Omega \left(\Phi_\varepsilon(u_\varepsilon(t)) + \tfrac{\alpha}{2}|\nabla u_\varepsilon(t)|^2\right) + \int_{\Omega_T}\gamma|\Delta u_\varepsilon|^2 + \\ \int_{\Omega_T}\psi_\varepsilon''(u_\varepsilon)|\nabla u_\varepsilon|^2 = \int_\Omega \left(\Phi_\varepsilon(u_0) + \tfrac{\alpha}{2}|\nabla u_0|^2\right) .$$

With this estimates the remaining part of the proof follows the outline of the proof of Theorem 1. One uses compactness results to conclude the existence of a converging subsequence and passes to the limit in the approximating equation.

□

4.2 The deep quench limit

Now we consider the case $B(u) = 1 - u^2$ and

$$\Psi_\theta(u) = \frac{\theta}{2}\left((1+u)\ln(1+u) + (1-u)\ln(1-u)\right) + \frac{1}{2}(1-u^2) \tag{4.6}$$

where $\theta > 0$. Let us denote the solution we constructed in the proof of Theorem 1 by u_θ. Cahn, Elliott and Novick–Cohen [7] studied the deep quench limit ($\theta \searrow 0$) of these solutions. The purpose of this subsection is to show that the solutions u_θ converge to weak solutions of the Cahn–Hilliard equation with a mobility $B(u) = 1 - u^2$ and a bulk energy $\Psi(u) = 1 - u^2$, which is the case where we set $\theta = 0$ in (4.6).
For u_θ we have the following a priori estimates, which follow from the estimates derived in Lemma 2 and the weak lower semi–continuity of the L^2–norm.

$$\operatorname{ess\,sup}_{0\le t\le T}\int_\Omega |\nabla u_\theta(t)|^2 + \int_{\Omega_T}|\mathbf{J}_\theta|^2 \le \int_\Omega \left(\frac{\gamma}{2}|\nabla u_0|^2 + \Psi_\theta(u_0)\right) \le C$$

and

$$\int_{\Omega_T}\gamma|\Delta u_\theta|^2 \le \int_\Omega \Phi(u_0) + \int_{\Omega_T}|\nabla u_0|^2 \le C$$

with a constant C independently of θ. From these estimates we obtain

$$\|\partial_t u_\theta\|_{L^2(0,T;(H^1(\Omega))')} + \|u_\theta\|_{L^2(0,T;H^2(\Omega))} \le C .$$

Using the same compactness results as before we get (for a subsequence $\theta \searrow 0$)

$$\begin{array}{rclllll} u_\theta, \nabla u_\theta & \longrightarrow & u, \nabla u & \text{strongly} & \text{in} & L^2(\Omega_T) \text{ and a.e. in } \Omega_T\,, \\ \Delta u_\theta & \longrightarrow & \Delta u & \text{weakly} & \text{in} & L^2(\Omega_T)\,, \\ \partial_t u_\theta & \longrightarrow & \partial_t u & \text{weakly} & \text{in} & L^2(0,T;(H^1(\Omega))') \quad \text{and} \\ \mathbf{J}_\theta & \longrightarrow & \mathbf{J} & \text{weakly} & \text{in} & L^2(\Omega_T)\,. \end{array}$$

Since $|u_\varepsilon| \leq 1$ a.e. in Ω_T, the same is true for u. Furthermore we get $\partial_t u = \nabla \cdot \mathbf{J}$ in $L^2(0,T;(H^1(\Omega))')$. It remains to pass to the limit in

$$\int_{\Omega_T} \mathbf{J}_\theta \cdot \boldsymbol{\eta} = -\int_{\Omega_T} \left[\gamma \Delta u_\theta \nabla \cdot (B(u_\theta)\boldsymbol{\eta}) + \theta \nabla u_\theta \boldsymbol{\eta} - B(u_\theta)\nabla u_\theta \boldsymbol{\eta}\right]\,.$$

The fact that ∇u_θ is uniformly bounded in $L^2(\Omega_T)$ yields

$$\int_{\Omega_T} \theta \nabla u_\theta \boldsymbol{\eta} \longrightarrow 0\,.$$

All other terms can be handled as in the proof of Theorem 1 to get

$$\int_{\Omega_T} \mathbf{J} \cdot \boldsymbol{\eta} = -\int_{\Omega_T} \left[\gamma \Delta u \nabla \cdot (B(u)\boldsymbol{\eta}) - B(u)\nabla u \cdot \boldsymbol{\eta}\right]\,.$$

This proves that u is a weak solution in the case $B(u) = 1 - u^2$ and $\Psi(u) = \frac{1}{2}(1-u^2)$.

We note that we have not proved the convergence of the whole sequence. This is due to the fact that so far there is no uniqueness result for the Cahn–Hilliard equation with a degenerate mobility.

4.3 Other applications

In a paper by Bernis and Friedman [3] the equation

$$u_t = -(f(u)u_{xxx})_x \tag{4.7}$$

where

$$f(u) = |u|^m f_0(u)\,, \quad f_0 \in C^{1+\alpha}(\mathbb{R}), \quad f_0 > 0 \quad \text{and} \quad m \geq 1 \tag{4.8}$$

was studied. They proved the existence of a nonnegative continuous solution and properties of the support of the solution. For example they proved that the support increases when $m \geq 2$. We also refer to [3] for other applications of degenerate parabolic equations of higher order.

With straightforward modifications we can apply our techniques to the following generalization of (4.7) in several space dimensions

$$u_t = \nabla \cdot (f(u)\nabla(-\Delta u + \Psi'(u)))$$

supplemented with Neumann– and no–flux boundary conditions. Under appropriate conditions on the nonlinearities f and Ψ our method gives the existence of a nonnegative solution in the sense of Theorem 1. In particular we have to assume that f, f' and $\Psi'' f$ are bounded.

Degenerate parabolic equations of the form

$$u_t = -\nabla \cdot (f(u)(\nabla \Delta u + \nabla u)) + g(t, x, u) \tag{4.9}$$

arising in the theory of plasticity have been independently studied by Grün [17].

5 Conclusion

We proved the existence of a weak solution to the Cahn–Hilliard equation with a degenerate mobility. As was pointed out, our method is also applicable for other fourth order degenerate parabolic equations. So far an uniqueness result for fourth order degenerate parabolic equations has not been established. Methods for proving uniqueness in the case of second order degenerate parabolic equations seem not to be applicable directly.

Beside studying the question of uniqueness it is important to get a better understanding of the qualitative behaviour of solutions. Questions are for example: What kind of singularities occur when $|u| \longrightarrow 1$? What is the evolutionary behaviour of the set $\{|u| = 1\}$? In the case of the deep quench limit for example one would expect that the set $\{|u| = 1\}$ develops an interior. If this is the case one would get a free boundary problem for $\partial\{|u| = 1\}$.

Furthermore we are interested in the asymptotic behaviour of solutions as $t \longrightarrow \infty$. For second order degenrate parabolic equations similarity solutions were important for the understanding of the asymptotic behaviour of solutions. There are results by Bernis, Peletier and Williams [4] on similarity solutions in one space dimension. It would be interesting to study if similarity solutions in higher space dimensions exist.

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