The required problems are Exercises 7.1, 7.2, and 7.4. Please let me know if any of the problems are unclear or have typos.

For the next three problems we need the following definition. Suppose X is a topological space. We define CX to be the *cone* on X: that is,

$$CX = X \times I \Big/ (x,1) \sim (y,1) \text{ for all } x, y \in X.$$

The point $a = [(x,1)]$ is called the *apex* of the cone.

Exercise 7.1. Equip the integers $\mathbb{Z}$ with the discrete topology. Show that $C\mathbb{Z}$ is homeomorphic to the wedge sum of a countable collection of unit intervals.

Exercise 7.2. Let $I_n \subset \mathbb{R}^2$ to be the line segment connecting $(0,1)$ to $(n,0)$, for $n \in \mathbb{Z}$. Set $D = \cup_{n\in\mathbb{Z}} I_n$ and equip D with the subspace topology. Show that $C\mathbb{Z}$ is not homeomorphic to D.

Exercise 7.3. [Hard] For any space X, show that CX is contractible. Deduce that $\pi_1(CX, a)$ is trivial.

Exercise 7.4. Suppose G and H are nontrivial groups. Show that the free product $G * H$ is not isomorphic to $\mathbb{Z}^2$.

Exercise 7.5. Suppose that $\{G_\alpha\}$ is a countable collection of countable groups. Show that $*_\alpha G_\alpha$ is countable.

For the next two problems we need the following definition. Let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0) \in \mathbb{R}^2$. We define $H \subset \mathbb{R}^2$, the *Hawaiian earring*, to be the union $H = \cup_{n=1}^{\infty} C_n$, equipped with the subspace topology. We take H to be a pointed space, with basepoint at $h = (0,0)$. Let $\Gamma = \pi_1(H, h)$.

Exercise 7.6.

- For all $n > 0$ give a retraction $r_n \colon H \to C_n$. Explain why r_n is continuous.
- Show that $\Gamma = \pi_1(H, h)$ is uncountable. Briefly explain why Γ is not isomorphic to

$$\pi_1\left(\bigvee_{n\in\mathbb{N}} S^1\right) \cong \mathop{*}_{n\in\mathbb{N}} \mathbb{Z}.$$

Exercise 7.7.

- Show that $H \cong H \vee H$. (Recall that we use $h = (0,0)$ as the basepoint.)
- [Medium.] Show that the homeomorphism above does not induce an isomorphism between Γ and $\Gamma * \Gamma$.