Please let me know if any of the problems are unclear or have typos.

In Exercises 11.1 and 11.3 we use a piece of notation from [Perspectives on Projective Geometry, Chapter 6]. If $P = (a : b)$ and $Q = (c : d)$ are points of $\mathbb{P}^1$ we define their *bracket* to be $[P, Q] = ad - bc$. Note that $[P, Q]$ is not well-defined, as $(a : b) = (\lambda a : \lambda b)$ for any non-zero λ.

Exercise 11.1. Suppose that $P, Q, R, S \in \mathbb{P}^1$ are distinct points. We define

$$(P, Q : R, S)' = \frac{[P, S]}{[P, R]} \cdot \frac{[Q, R]}{[Q, S]}$$

Give a direct proof that $(P, Q : R, S)'$ is a well-defined function of the (distinct) points $P, Q, R, S \in \mathbb{P}^1$.

Exercise 11.2. Suppose that $P, Q, R, S \in \mathbb{P}^1$ are distinct points. In lecture we defined $z = (P, Q : R, S)$ to be the *cross-ratio* of those four points, in that order. (That is, if $A \in \mathrm{PGL}(2, \mathbb{R})$ is the unique matrix with $A(P) = (1 : 0)$, $A(Q) = (0 : 1)$, and $A(R) = (1 : 1)$ then $A(S) = (1 : z)$.)

- Prove that $(P, Q : R, S) = (P, Q : R, S)'$.
- There are twenty-four possible orderings of the points P, Q, R, S. Compute the cross-ratio for each, in terms of $z = (P, Q : R, S)$. [Hint: use group theory.]
- Using the above give a map of the permutation group Sym_4 into $\mathrm{PGL}(2, \mathbb{R})$. Deduce that $\mathrm{Sym}_4 \cong K_4 \rtimes \mathrm{Sym}_3$. Here $K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the *Klein four group*. What three things is Sym_3 acting on?

Exercise 11.3. Suppose that $P, Q, R, S \in \mathbb{P}^1$. Verify the *Plücker relation*

$$[P, Q] \cdot [R, S] - [P, R] \cdot [Q, S] + [P, S] \cdot [Q, R] = 0.$$

Exercise 11.4. Here is a simplfied version of the *Ptolemy relation*. Suppose that $p, q, r, s \in \mathbb{R}^1$ are four distinct points, in that order. Prove that

$$|p - q||r - s| - |p - r||q - s| + |p - s||q - r| = 0.$$

For a discussion of this in the complex projective plane, and the connection to Plücker's relation, please see [Perspectives on Projective Geometry, Chapter 17].