

Please let me know if any of the problems are unclear or have typos. Please turn in a *single* worked exercise — write your name, the date, and the problem you are solving at the top of the page. If you collaborate with other students, please also include their names.

Exercise 4.1. [Hatcher page 147, the splitting lemma.] Suppose that \mathcal{C}_* is a chain complex. If $H_*(\mathcal{C}) = 0$ we call \mathcal{C}_* an *exact sequence*. (Equivalently, $Z_n = B_n$ for all n .) If, additionally, \mathcal{C}_* has at most three non-zero terms we call \mathcal{C}_* a *short exact sequence*. Show that every non-zero term is adjacent to another such. Now suppose

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

is a short exact sequence of Abelian groups. Show the following are equivalent.

- There is a homomorphism $r: B \rightarrow A$ so that $ri = \text{Id}_A$.
- There is a homomorphism $s: C \rightarrow B$ so that $ps = \text{Id}_C$.
- $B \cong A \oplus C$ and there is an isomorphism from the given sequence to the sequence $0 \rightarrow A \rightarrow A \oplus C \rightarrow C \rightarrow 0$ with the obvious inclusion and projection maps.

Such a sequence is called *split*; the maps r and s are called a *retraction* and a *section*, respectively.

Exercise 4.2. [Hatcher page 148.] With notation as in Exercise 4.1, show if C is free then the sequence is split.

Exercise 4.3.

1. For any $d \times n$ matrix M of integers the *Smith normal form* of M is a $d \times n$ matrix $D = [d_{i,j}]$ with $d_{i,j} \in \mathbb{N}$, with $d_{i,j} = 0$ (if $i \neq j$), with $d_{i,i}$ evenly dividing $d_{i+1,i+1}$, and having matrices $U \in \text{GL}(d, \mathbb{Z})$ and $V \in \text{GL}(n, \mathbb{Z})$ so that $D = UMV$. Learn how to compute Smith normal form in your favourite computer algebra system.
2. Find an algorithm that, given any chain complex $\mathcal{C}_* = \{C_n, \partial_n\}$ of finitely generated free Abelian groups, computes the homology groups $H_*(\mathcal{C})$. (Hints: Recall $B_n = \text{Im}(\partial_{n+1})$ and $Z_n = \text{Ker}(\partial_n)$. The inclusion of Z_n into C_n , and ∂_n , together give a short exact sequence $0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0$. Since B_{n-1} is free, this sequence splits. Choose a splitting $C_n \cong Z_n \oplus B_{n-1}$ extending the inclusion of Z_n into C_n . This induces an isomorphism $\text{Cok}(\partial_{n+1}) \cong H_n \oplus B_{n-1}$.)
3. Show that the following sequence of groups and homomorphisms \mathcal{C}_* is a chain complex.

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^5 \xrightarrow{N} \mathbb{Z} \longrightarrow 0$$

Here the homomorphisms M and N are given by the matrices

$$M = \begin{bmatrix} -6 & -26 & -82 \\ 0 & 4 & 3 \\ 1 & 0 & 7 \\ 0 & 2 & 5 \\ 2 & 10 & 30 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -2 & -2 & -4 & -2 & -4 \end{bmatrix}.$$

4. Take M , N , and \mathcal{C}_* as in part (3) above. Compute the Smith normal forms of M and N . Use the algorithm to compute $H_*(\mathcal{C})$.

Exercise 4.4. We say an exact sequence is *very short* if it has at most two non-zero terms.

- i. Show the central map of a very short exact sequence is an isomorphism.
- ii. Show a long exact sequence of free Abelian groups may be written as a direct sum of very short exact sequences.

Exercise 4.5. [Roberts.] For each exact sequence of Abelian groups below, say as much as possible about the group G and the homomorphism α .

- i. $0 \rightarrow \mathbb{Z}_2 \rightarrow G \rightarrow \mathbb{Z} \rightarrow 0$
- ii. $0 \rightarrow \mathbb{Z} \rightarrow G \rightarrow \mathbb{Z}_3 \rightarrow 0$
- iii. $0 \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow 0$
- iv. $0 \rightarrow G \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$
- v. $0 \rightarrow \mathbb{Z}_3 \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow 0$

Here we write \mathbb{Z}_n for the group $\mathbb{Z}/n\mathbb{Z}$.

Exercise 4.6. [Medium.] Suppose B and D are finitely generated free Abelian groups, $A < B$ and $C < D$ are subgroups, and $B/A \cong D/C$. Show the chain complexes $0 \rightarrow A \rightarrow B \rightarrow 0$ and $0 \rightarrow C \rightarrow D \rightarrow 0$ are chain homotopy equivalent. (This is a generalization of Exercise 3.10. Smith normal form may be useful. See also problem 43(b) on page 159 of Hatcher.)

Exercise 4.7. [Hard.] Show chain complexes \mathcal{C}_* and \mathcal{D}_* of finitely generated free Abelian groups are chain homotopy equivalent if and only if they have isomorphic homology groups: $H_*(\mathcal{C}) \cong H_*(\mathcal{D})$. (Hints are available at MathOverflow, question number 10974. Exercise 4.6 may be useful. See also problem 43(a) on page 159 of Hatcher.)

Exercise 4.8. [Medium.] Find two topological spaces X and Y , with isomorphic homology groups, that are not homotopy equivalent. (Thus Exercise 4.7 does not generalize to topological spaces.) State any theorems from Hatcher that you use; you will need to read ahead a bit.