MA4J2 Three Manifolds

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Chapter 1

Introduction

1.1 Basic Definition and Examples

One goal of topology is to classify manifolds up to homeomorphism. In dimension $n \geq 4$, this problem is undecidable; no algorithm, given two manifolds as an input, can decide whether or not they are homeomorphic.* We will classify manifolds in dimensions 0, 1 and 2 in the next few pages. The general topic is to classify 3–manifolds.

Definition 1.1.1. An *n–manifold* M^n is a Hausdorff topological space with a countable basis and such that every point $p \in M$ has an open neighbourhood U which is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$.

Remark. $\mathbb{R}^n_+$ is called the upper half space, and $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n \leq 0\}$ is called the lower half space.

Definition 1.1.2. ∂M is the set of points p in M such that no neighbourhood of p is homeomorphic to $\mathbb{R}^n$.

Proposition 1.1.1. *∂M is an $(n-1)$–manifold, and $\partial\partial M = \varnothing$.*

Definition 1.1.3. $\mathrm{int}(M) = M - \partial M$.

Definition 1.1.4. We use $I = [0,1] \subseteq \mathbb{R}$, $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and $\mathbb{D}^2 = \mathbb{B}^2$.

Definition 1.1.5. We give several equivalent definitions of the *sphere*:

(i) A submanifold definition: $S^n = \partial\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

(ii) A *one-point compactification* definition: S^n is the one-point compactification of $\mathbb{R}^n$, that is $S^n = \mathbb{R}^n \cup \{\infty\}$ topologized such that for any compact $K \subseteq \mathbb{R}^n$, the set $(\mathbb{R}^n - K) \cup \{\infty\}$ is a neighbourhood of ∞. Note here that $\mathbb{B}^n$ is the one-point compactification of $\mathbb{R}^n_+$.

(iii) A gluing definition: $S^n = \mathbb{B}^n_0 \sqcup \mathbb{B}^n_1 / \sim$ where $(x,0) \sim (x,1)$ if and only if $x \in \partial\mathbb{B}^n$. For example, S^1 can be obtained by joining two copies of $\mathbb{B}^1$ by their boundaries, and similarly for S^2 and $\mathbb{B}^2$.

Definition 1.1.6. We now give several equivalent definitions of *projective spaces*:

(i) A covering space definition: $\mathbb{P}^n = S^n / \sim$ where $x \sim -x$, taking S^n as in definition (i) above.

(ii) A gluing definiton: $\mathbb{P}^n = \mathbb{B}^n / \sim$ where $x \sim -x$ if and only if $x \in \partial\mathbb{B}^n$.

(iii) A moduli space definition:

$$\mathbb{P}^n = \{L \subseteq \mathbb{R}^{n+1} : L \text{ is a line through the origin}\} = (\mathbb{R}^{n+1} - \{0\}) / \sim$$

where $x \sim \lambda x$ for $\lambda \in \mathbb{R} - \{0\}$.†

*This result is due to A.A. Markov (1958).
†We will sometimes use $\mathbb{R}^*$ for $\mathbb{R} - \{0\}$.

Definition 1.1.7. We have three equivalent definitions of *tori*:

(i) A *Cartesian product* definition: $\mathbb{T}^n = (S^1)^n$, taking the Cartesian product.

(ii) A covering space definition: $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim$ with $x \sim y$ if and only if $x - y \in \mathbb{Z}^n$.

(iii) A gluing definition: $\mathbb{T}^n = I^n/\sim$ where $(x, 0, y) \sim (x, 1, y)$ if and only if $x \in I^k$ and $y \in I^{n-k-1}$ for any $k \in \{0, ..., n-1\}$.

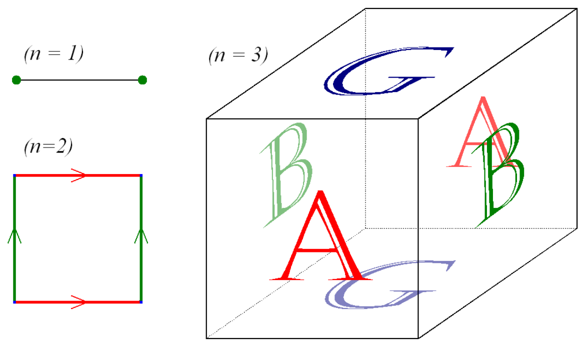

Figure 1.1: Construction of the first three n–tori $\mathbb{T}^n$. Identify opposite faces of I^n without twisting.

Note. $\mathbb{T}^1 \cong S^1$.

Exercise 1.1.1. For each set of three definitions above, prove that all three are equivalent.

1.2 The Classificaiton of Compact 2–Manifolds

In dimension zero, any compact manifold is a finite collection of points, so the classification is given by the number of points. All compact connected one-dimensional manifolds are homeomorphic to either S^1 or I.

Definition 1.2.1. Suppose M_i (for $i = 0, 1$) are orientable n–manifolds. Choose $\mathbb{B}_i^n \subseteq M_i$ and suppose $\varphi : \partial\mathbb{B}_0^n \to \partial\mathbb{B}_1^n$ is an orientation reversing homeomorphism. Define:

$$M_0 \# M_1 := ((M_0 - \text{int}(B_0^n)) \sqcup (M_1 - \text{int}(B_1^n)))/\sim$$

where $x \sim \varphi(x)$ whenever $x \in \partial\mathbb{B}_0^n$.

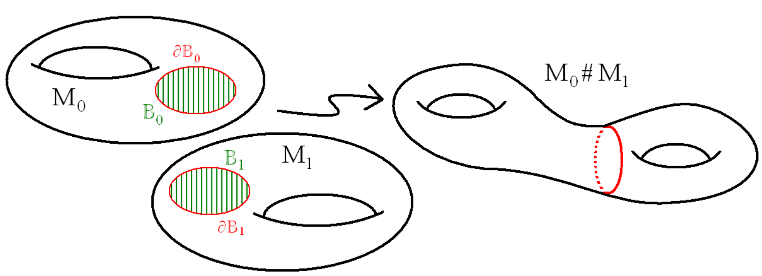

Figure 1.2: The connect sum. Remove the interiors of the disks $\mathbb{B}_i$ and glue along their boundaries.

Exercise 1.2.1. Show that $\#_3\mathbb{P}^2 \cong \mathbb{T} \# \mathbb{P}^2$.

Theorem 1.2.1. *Every compact connected two-dimensional manifold is homeomorphic to some $S_{g,n,c}$, where:*

$$S_{g,n,c} := (\#_g\mathbb{T}^2) \# (\#_n\mathbb{D}^2) \# (\#_c\mathbb{P}^2)$$

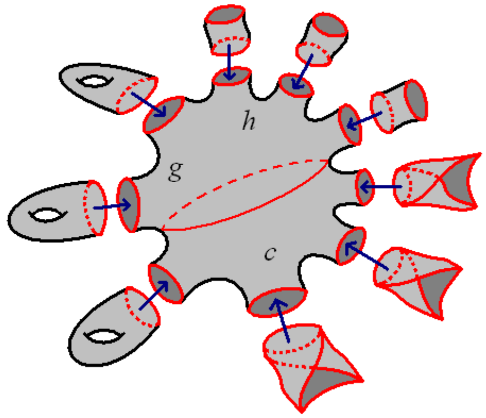

Figure 1.3: $S_{3,3,3}$ is the connect sum of the sphere with three tori, three Möbius strips and three 2–disks, glued along the boundary components (in red)

Example 1.2.1. Some spaces $S_{g,n,c}$ are homeomorphic, for example $S_{3,3,3} \cong S_{4,3,1}$.

Chapter 2

Alexander's Theorem and Incompressible Surfaces

2.1 Primes and Irreducibles

Example 2.1.1. We give some connect sums of three manifolds:

$$S^3 \# S^3 \cong S^3$$
$$\mathbb{T}^3 \# S^3 \cong \mathbb{T}^3$$

In general, S^n is a unit for the connect sum. $\mathbb{P}^3 \# \mathbb{P}^3$ is more interesting, as we will discuss later. On the other hand, $\mathbb{T}^3 \# \mathbb{T}^3$ invites splitting into two copies of $\mathbb{T}^3$ for a more interesting and fundamental geometry. In general, we shall find a decomposition theorem for 3–manifolds with respect to $\#$.

Definition 2.1.1. M^3 is *prime* if whenever $M = N \# L$ then either N or L is homeomorphic to S^3.

Remark. If $M = N \# L$ and $N \cong S^3$ then $L \cong M$, and vice versa.

Definition 2.1.2. M is *irreducible* if every smoothly embedded S^2 in M bounds a 3–ball.

Note. We have no examples yet of prime or irreducible 3–manifolds.

2.2 Alexander's Theorem

Definition 2.2.1. Suppose $X, Y \subseteq Z$. We say X is *ambient isotopic* (*diffeotopic*) to Y if there exists a continuous (smooth) map $F : Z \times I \to Z$ such that, defining $F_t(z) := F(t, z)$:

(i) For all $t \in I$, F_t is a homeomorphism (diffeomorphism).

(ii) $F_0 = \mathrm{Id}_Z$.

(iii) $F_1|X : X \to Y$ is a homeomorphism (diffeomorphism).

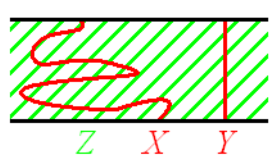

Figure 2.1: Here, X is ambient isotopic to Y in Z.

Theorem 2.2.1 (Alexander)**.** *Every smoothly embedded $S^2 \subset S^3$ is ambient isotopic to the equator.*

Compare this to:

Theorem 2.2.2 (Jordan-Schoenflies)**.** *Every smoothly embedded $S^1 \subset S^2$ is ambient isotopic to the equator.*

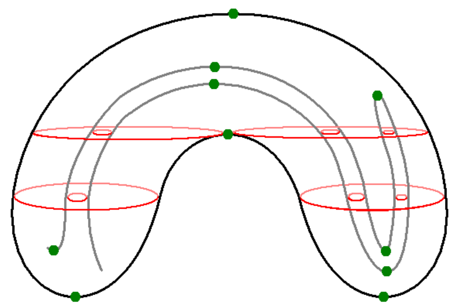

Figure 2.2: It is not always obvious which ball a sphere bounds

We will prove Alexander's theorem later, but for now give the following corollary.

Corollary 2.2.3. *S^3 is prime.*

Proof. Suppose $S^3 = M \#_S N$. By Alexander's theorem, S is ambient isotopic to a round embedding of S^2 in S^3 (say the equator). Thus $M - \operatorname{int}(\mathbb{B}^3) \cong N - \operatorname{int}(\mathbb{B}^3) \cong \mathbb{B}^3$, and hence $M \cong N \cong S^3 \cong \mathbb{B}^3 \cup_\partial \mathbb{B}^3$. □

It is important that the embedding is smooth, as the following result shows.

Theorem 2.2.4. *There exists a topological $S^2 \subset S^3$ which does not bound $\mathbb{B}^3$ on either side.*

Note. This is a generalization of the Alexander horned sphere.

Remark. The statement of Alexander's theorem with $S^2 \subset S^3$ replaced by $S^3 \subset S^4$ is an open problem, although it has been proved that a smoothly embedded $S^3 \subset S^4$ bounds a topological ball. Brown has proved the more general statement that a smoothly embedded $S^{n-1} \subset S^n$ bounds a topological ball.

Remark. It is worth making explicit the various categories involved:

(i) Topological (TOP).

(ii) Piecewise linear (PL).

(iii) Smooth (DIFF).

These categories are all equivalent in dimension at most 3, so we move between them freely.

Exercise 2.2.1.

(i) Prove that any irreducible manifold is prime.

(ii) Prove that if M is orientable and $S \subset M$ is a non-separating 2–sphere, then $M = N \# (S^2 \times S^1)$.

(iii) Suppose M is orientable. Then M is prime and reducible if and only if $M \cong S^2 \times S^1$. Prove the forward direction.

(iv) State and prove analogous statements to (ii) and (iii) for non-orientable manifolds.

We give one more corollary to Alexander's theorem:

Corollary 2.2.5. *If $M \subseteq S^3$ is compact and has $|\partial M| \leq 1$ (at most one boundary component) then M is irreducible.*

Example 2.2.1. We give further examples of irreducible manifolds. Suppose $K \subset S^3$ is a knot, that is a smooth embedding of S^1. Let $N(K) \subseteq S^3$ be a closed regular neighbourhood (i.e. a tubular neighbourhood) of the knot. Let $n(K) = \text{int}(N(K))$. Then the knot exterior $X_K := S^3 - n(K)$ is irreducible, by the previous corollary.

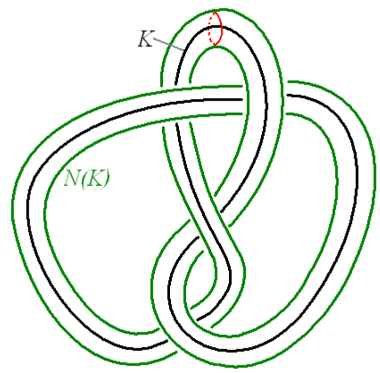

Figure 2.3: A tubular neighbourhood of the figure 8 knot

2.3 Proof

We now prove Alexander's theorem. More precisely, we will prove that any (smoothly) embedded $S^2 \subset \mathbb{R}^3$ bounds a 3-ball, from which the theorem can be deduced as a corollary.

Exercise 2.3.1. Show how Alexander's theorem follows from this statement.

We need the following lemma:

Lemma 2.3.1. *Suppose that a manifold M^n and $\mathbb{B}_1^{n-1} \subseteq \partial M^n$ are given, as is a diffeomorphism $\varphi : \mathbb{B}_0^{n-1} \to \mathbb{B}_1^{n-1}$, where $\mathbb{B}_0^{n-1} \subseteq \partial \mathbb{B}^n$. Then $M^n \cup_\varphi \mathbb{B}^n \cong M^n$, as per Figure 2.4.*

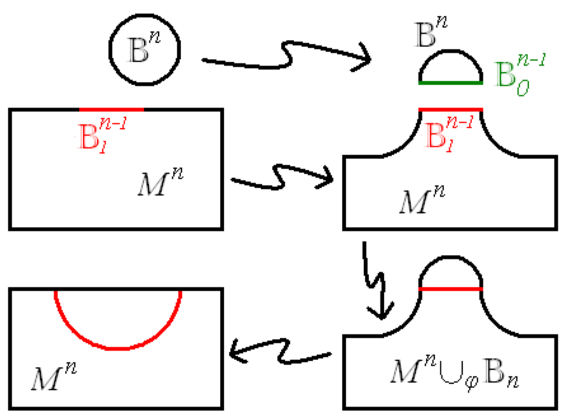

Figure 2.4: Glueing M^n to $\mathbb{B}^n$ along submanifolds of their boundaries is homeomorphic to M^n.

As a consequence, if B and B' are n–balls, then $B \cup_\partial B'$ is a ball (Figure 2.5(a)), as is $\overline{B - B'}$ if $B' \subset B$ and $\partial B' \cap \partial B \cong \mathbb{D}^{n-1}$ (Figure 2.5(b)).

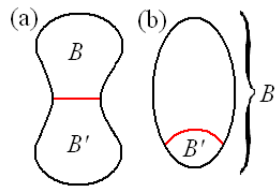

Figure 2.5: (a) B, B' balls $\Rightarrow B \cup_\partial B'$ a ball, and (b) $B' \subset B$ and $\partial B' \cap \partial B \cong \mathbb{D}^{n-1} \Rightarrow \overline{B - B'}$ a ball.

Theorem 2.3.2. *Any smoothly embedded $S^2 \subset \mathbb{R}^3$ bounds a 3–ball.*

Proof. Suppose $S^2 \cong S \subset \mathbb{R}^3$ is smooth. We can isotope S so that $z : S \to \mathbb{R}$ (the height function, giving the z co-ordinate) is a Morse function. Thus all critical points are of the standard three types; cups (minima), caps (maxima), and saddles, and all critical points occur at distinct heights (as illustrated in Figure 2.6).

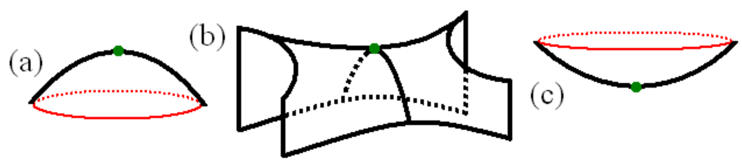

Figure 2.6: (a) A cap. (b) A saddle. (c) A cup.

Choose $a_i \in \mathbb{R}$ such that $(-\infty, a_1), (a_1, a_2), ..., (a_{n-1}, \infty)$ each contain exactly one critical value, as in Figure 2.7.

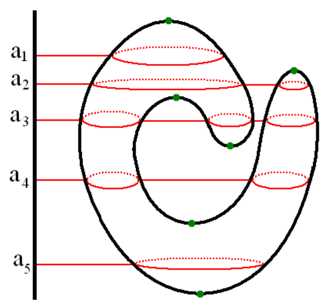

Figure 2.7: The red circles are regular values separating the critical points (green). Here we have $(n, w) = (6, 9)$.

Let:

$$L[a, b] := \{(x, y, z) : z \in [a, b]\}$$
$$L(a) := \{(x, y, z) : z = a\}$$
$$L_i := L(a_i)$$

Define $n(S)$ to be the number of critical points. Define the *width* by:

$$w(S) = \sum_{i=1}^{n-1} |S \cap L_i|$$

This is the number of red circles in Figure 2.7. We will induct on $(n(S), w(S))$ lexicographically. Note that the components of $L_i \cap S$ are all simple closed curves, because each a_i is a regular value. So by the Jordan-Schoenflies theorem, they all bound disks. Say that β, a component of $L_i \cap S$, is *innermost* if D_β, the disk bounded by β, has the property that $D_\beta \cap S = \beta$. Notice that β also bounds a pair of disks in S.

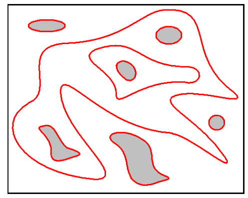

Figure 2.8: The intersection of the plane L_i with the sphere. Shaded components are innermost.

Label a_i with an A (resp. B) if there is some innermost curve $\beta \subseteq L_i \cap S$ such that one disk of $S - \beta$ contains exactly one critical point, a maximum (resp. minimum). Note that a_i could receive both labels. Note also that a_1 is labelled by B and a_{n-1} is labelled by A. We have cases:

Case 1: Some a_i is labelled both A and B.

Case 2: Some a_i is unlabelled.

Case 3: There exists i such that a_i is labelled B and a_{i+1} is labelled A.

Exercise 2.3.2. Check that we must always be in at least one of these cases.

We prove these in turn:

Case 1a: Some innermost $\beta \in L_i \cap S$ bounds a disk in S above and bounds a disk in S below, each with one critical point; this forms the base case of the induction, where $n(S) = 2$ and $w(S) = 1$. We claim that in this case S bounds a ball.

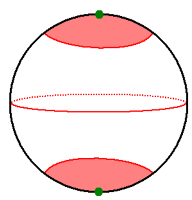

Figure 2.9: The base case.

To see this, cut off the two critical points with planes slightly above the minimum and below the maximum, removing two 3–balls from S, and giving a compact cylinder. We claim that for every $a \in \mathbb{R}$ such that the set $L(a)$ intersects this compact cylinder, there exists $\varepsilon > 0$ such that $S \cap L[a, a+\varepsilon]$ bounds a 3–ball in $L[a, a+\varepsilon]$. This can be proved by the implicit function theorem and the isotopy extension theorem. See Hatcher's *Notes on basic* 3*–manifold topology* for more details.

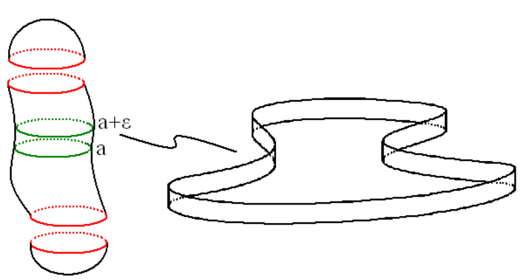

Figure 2.10: The slab bounded by $L_[a, a+\varepsilon]$.

Note that the intersection $L(a) \cap S$ is a curve, so bounds a disk. Note that finitely many of the $L[a, a+\varepsilon]$ cover the compact cylinder. Glue these slabs together, and re-attach the cap and cup. By Lemma 2.3.1, this gives a 3–ball.

Case 1b There are innermost $\alpha, \beta \subset L_i \cap S$ so that α bounds D above, β bounds E below. Let D', E' be the disks bounded by α, β, inside of L_i. So, by the base case $D \cup D'$ $(E \cup E')$ bounds a 3–ball. Use this 3–ball to define an ambient isotopy that flattens D (E), pushing the critical point just below (above) the plane L_i.

Exercise 2.3.1. Show that this reduces $w(S)$.

Case 2 The regular value a_i is not labelled. For this case, we first have to introduce surgery.

Definition 2.3.1. Suppose $F^2 \subset M^2$ is properly embedded (i.e. a submanifold, i.e. embedded and $F \cap \partial M = \partial F$). We say $(D^2, \partial D) \subset (M, F)$ is a *surgery disk* for F if $D \cap F = \partial D$. Let $n^\circ(\partial D)$ be an open annular neighbourhood of ∂D, in F. Let D_+, D_- be parallel copies of D in M. Define F *surgered along* D by $F_D := (F - n^\circ(\partial D)) \cup D_+ \cup D_-$, as in Figure 2.11.

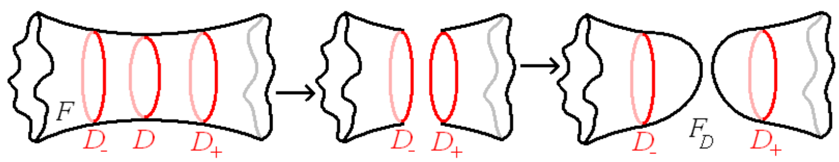

Figure 2.11: Surgery. $F_D := (F - n^\circ(\partial D)) \cup D_+ \cup D_-$.

We now return to case 2. Suppose $\beta \subset S \cap L_i$ is innermost. So, β bounds D above, E below, $D \cup_\beta E = S$ and D, E each contain at least 3 critical points. Say β bounds a disk $B \subset L_i$. So:

$$S_B = S_+ \cup S_-, \quad S_+ \cong D \cup B_+, \quad S_- \cong E \cup B_-.$$

Thus $n(S_+), n(S_-) < n(S)$ since $n(S_+) + n(S_-) = n(S) + 2$. By induction, S_+, S_- each bound a 3–ball X_+, X_- thus so did S, applying Lemma 1.3 in Hatcher's notes. In the first case $X_+ \cap X_- = B$ and so we take the union. In the second case $X_+ \subset X_-$, we take the difference. See Figure 2.12.

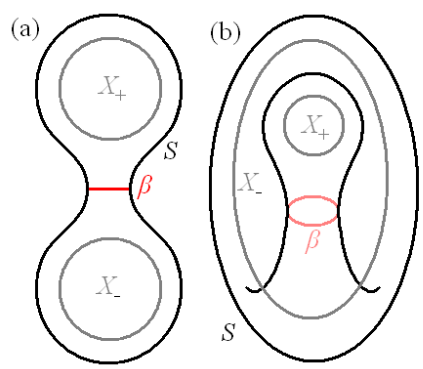

Figure 2.12: The case when (a) $x_+ \cap X_- = \varnothing$ or (b) $X_+ \subseteq X_-$.

Case 3 The regular value a_i is labelled only B and the regular value a_{i+1} is labelled only A. Between L_i and L_{i+1} we have $S \cap L[a_i, a_{i+1}]$ is a union of cylinders, caps, cups, pairs of pants, upside down pairs of pants and pants with inverted legs, as illustrated in Figure 2.13.

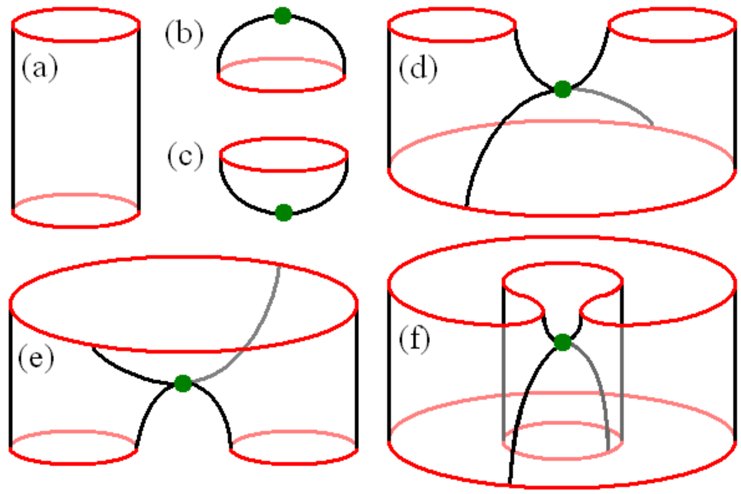

Figure 2.13: $S \cap L[a_i, a_{i+1}]$ is a union of (a) cylinders, (b) caps, (c) cups, (d) pairs of pants, (e) upside down pairs of pants, (f) pants with inverted legs and (g) an upside down version of (f) (not shown).

Note that there is at most one critical point in $S \cap L[a_i, a_{i+1}]$, so it is a saddle (check this using the labelling). Using the labelling deduce that either α or β is a cuff of the pants.

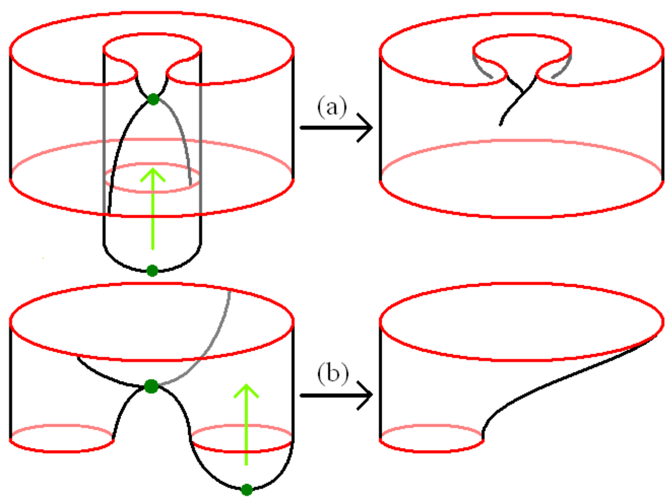

Figure 2.14: Two examples of how may isotope E to be in L_i and then upwards, canceling two critical points.

We have that β is innermost in L_i and β bounds (in S) a disk below, E, with a single critical point (minimum). Hence, by the base case, we may isotope E to be in L_i and then upwards to cancel two critical points, as in Figure 2.14. Thus, we have isotoped S to a sphere S' such that $n(S') = n(S) - 2$. This completes the induction step and so, the proof. □

2.4 Incompressible Surfaces

Definition 2.4.1. Say a 2–sphere $S \subset M^3$ is *essential* if no component of $M - n(S)$ is a 3–ball.

Definition 2.4.2. Suppose $F^2 \subset M^3$ is properly embedded. Suppose $(D, \partial D^2) \subset (M, F)$ is a surgery disk. Say that D is a *trivial surgery* disk if $\partial D \subset F$ is equal to $\mathbf{1} \in \pi_1(F)$ where $\pi_1(F)$ is the fundamental group of F. We say that D is a *compressing disk* if $\partial D \subset F$ is not equal to $\mathbf{1} \in \pi_1(F)$.

An alternative definition is: D is a trivial surgery disk if ∂D bounds a disk in F. See Figure 2.15.

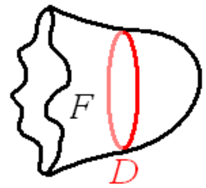

Figure 2.15: Here, D is a trivial surgery disc for F.

Exercise 2.4.1. Check that a simple closed curve $\alpha \subset F$ bounds a disk $E \subset F$ if and only if $[\alpha] = \mathbf{1} \in \pi_1(F)$.

Definition 2.4.3. Suppose $F \subset M$ is either properly embedded or $F \subset \partial M$ is a subsurface. Then we say that F is *compressible* if and only if there exists a compressing disk for F. Otherwise we call F *incompressible*.

Example 2.4.1. Let $T \subset S^3$ be the standard embedding, i.e. $\partial N(U)$ where U is the unknot. Then T is compressible since there are two compressing disks. We call them the *meridian disk* and the *longitude disk* respectively, as illustrated in Figure 2.16.

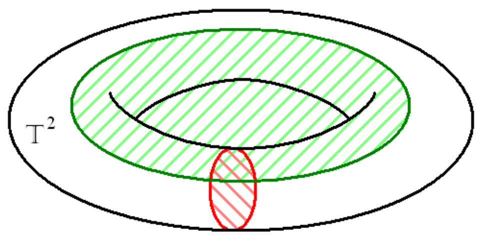

Figure 2.16: The meridian disk is in green while the longitude disk is red. The boundary of the meridian disk is a circle in T but its interior is in S^3.

Example 2.4.2. If $M = D \times S^1$ is a solid torus then $\partial M \subset M$ is compressible.

Exercise 2.4.2. Show that $T = \mathbb{T}^2 \times \{\frac{1}{2}\} \subset \mathbb{T}^2 \times I = M$ is incompressible.

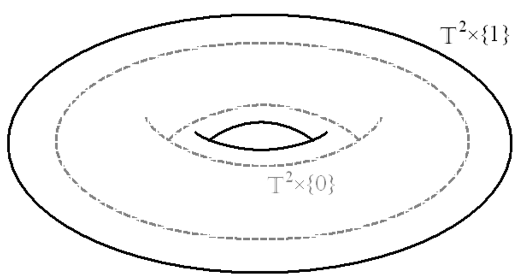

Figure 2.17: $T = \mathbb{T}^2 \times \{\frac{1}{2}\} \subset \mathbb{T}^2 \times I = M$ is incompressible.

Exercise 2.4.3. Suppose that M is an irreducible three-manifold and $F, G \subset \partial M$ are disjoint, incompressible subsurfaces. Suppose that $\varphi\colon F \longrightarrow G$ is a homeomorphism. Show that M/φ is irreducible.

Note. One can check $M = \mathbb{D}^2 \times S^1$ is irreducible but $D(M)$, the double of M, is not. Here $D(M) = M_0 \sqcup M_1/\sim$, where $(x, 0) \sim (x, 1)$ if and only if $x \in \partial M$ where $M_i = M \times \{i\}$.

Exercise 2.4.4.

1. If $F \subset S^3$ is closed, $F \neq S^2$, then F is compressible.
2. (Alexander) Any $\mathbb{T}^2 \subset S^3$ bounds a solid torus ($\mathbb{D}^2 \times S^1$) on at least one side.

Definition 2.4.4. Let V_g be the *handlebody* of genus g, i.e.

$$V_g = \underbrace{\mathbb{D}^2 \times S^1 \ \cup_{\mathbb{D}^2} \ \mathbb{D}^2 \times S^1 \ \cup_{\mathbb{D}^2} \ \ldots \ \cup_{\mathbb{D}^2} \ \mathbb{D}^2 \times S^1}_{g \text{ times}}$$

By convention, $V_0 = \mathbb{B}^3$. See Figure2.18.

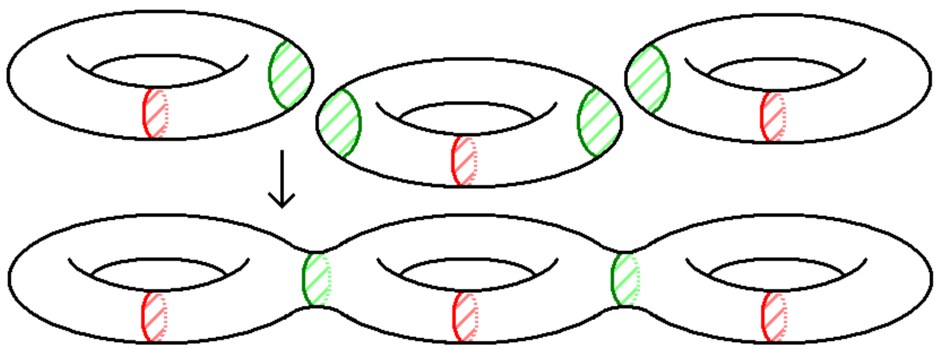

Figure 2.18: The handlebody V_3. Note that V_g is "solid", and not a surface.

Example 2.4.3. Find $S_2 \hookrightarrow S^3$ which does not bound a handlebody on either side. Here S_2 denotes a surface of genus 2.

Remark. $\partial V_g = \#_g \mathbb{T}^2 = S_g$ because $\partial(\mathbb{D}^2 \times S^1) = S^1 \times S^1 = \mathbb{T}^2$.

Chapter 3

Products and Bundles

3.1 Bundles and Neighbourhoods

Definition 3.1.1. A map $\rho\colon Z \longrightarrow X$ is a Y*–bundle* (or a *fibre bundle*) if for all $x \in X$ there exists a neighbourhood $x \in U \subset X$ and a homeomorphism $h_U\colon Y \times U \longrightarrow \rho^{-1}(U)$ such that the composition $\rho \circ h_U$ is the projection onto the second coordinate. Here, Z is called the *total space*, X the *base space*, Y the *fibre* and h_U is called a *local trivialization*.

Example 3.1.1. Let $Z = \mathbb{D}^2 \times S^1$ and denote ρ_i be the projection onto the i–th coordinate. Then $\rho_1\colon Z \longrightarrow \mathbb{D}^2$ is a S^1–bundle map and $\rho_2\colon Z \longrightarrow S^1$ is a $\mathbb{D}^2$–bundle map.

See Lackenby §6.

Definition 3.1.1. We say $Z \xrightarrow{\rho} X$, $Z' \xrightarrow{\rho'} X$ are *equivalent* Y–bundles if there is a homeomorphism $h\colon Z' \longrightarrow Z$ making the following diagram commute

$$\begin{tikzcd} Z' \arrow[r, "h"] \arrow[d, "\rho'"'] & Z \arrow[d, "\rho"] \\ X \arrow[r, "\mathrm{Id}_X"] & X \end{tikzcd}$$

Corollary 3.1.1 (See Corollary 6.3 in Lackenby's notes)**.** *If X is contractible then any Y–bundle $Z \xrightarrow{\rho} X$ is equivalent to the product bundle $Y \times X \xrightarrow{\rho_2} X$.*

Exercise 3.1.1. Prove this directly for $X = \mathbb{B}^1, \mathbb{B}^2$.

Exercise 3.1.2. Find a S^1–bundle over S^2 that is not equivalent to the product bundle. It follows that the fundamental group $\pi_1(X, x) = \{\mathbf{1}\}$ is not sufficient hypothesis for Corollary 6.1.

Lemma 3.1.2 (See Lemma 6.4 in Lackenby's notes)**.** *For all $n \in \mathbb{N}$ there are exactly two $\mathbb{B}^n$–bundles over S^1 up to equivalence. These are*

- *the trivial bundle $\mathbb{B}^n \times S^1$*
- *the twisted bundle $\mathbb{B}^n \widetilde{\times} S^1 = \mathbb{B}^n \times I/(x,0) \sim (r(x),1)$, where $r(x_1, \dots, x_n) = (x_1, \dots, -x_n)$ is a reflection.*

For an example, see Figure 3.1.

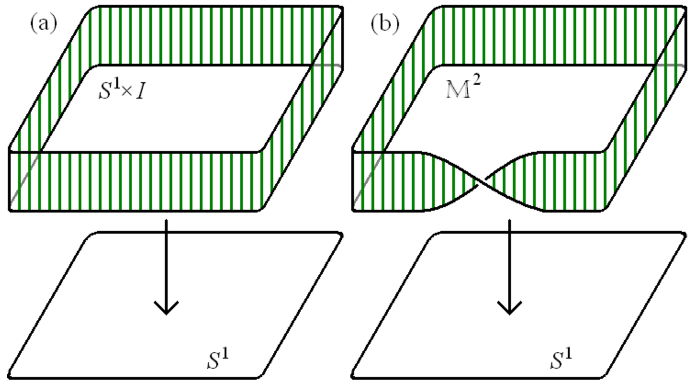

Figure 3.1: Two inequivalent bundles over S^1: (a) $\mathbb{B}^1 \times S^1$ and (b) $\mathbb{B}^1 \widetilde{\times} S^1$.

Version of the Tubular Neighbourhood Theorem

Definition 3.1.2. Suppose $\rho\colon Z \longrightarrow X$ is a bundle. Then a map $s\colon X \longrightarrow Z$ is a *section* of ρ if $\rho \circ s = \mathrm{Id}_X$.

Theorem 3.1.1. *Suppose $F^{n-k} \subset M^n$ is properly embedded. Then there is a closed neighbourhood $N = N(F) \subset M$ of F and a $\mathbb{B}^k$–bundle map such that*

1. *the inclusion $i\colon F \longrightarrow N(F)$ is the zero section, i.e. $i(x) = 0 \in \mathbb{B}^k = \rho^{-1}(x)$,*
2. *N is a codimension 0 submanifold of M (with corners) and*
3. *any $N'(F)$ satisfying the properties (1) and (2) is ambient isotopic to $N(F)$ fixing F pointwise.*

Notation. We denote by $n(F)$ the interior of $N(F)$. Furthermore, M *cut along* F, is the manifold (perhaps with corners) $M - n(F)$. When F is codimension 1 manifold there is a regluing map $M - n(F) \stackrel{\text{reglue}}{\longrightarrow} M$.

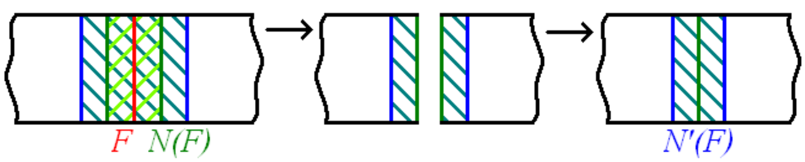

Figure 3.2: Cut open along $N(F)$ and glue back along $N'(F)$.

Exercise 3.1.3. All I–bundles over S^2 are trivial.

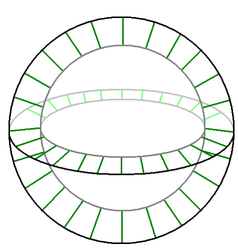

Figure 3.3: The trivial I–bundle over S^2.

3.2 Classification of I–Bundles over F^2

Suppose that $\rho\colon G^2 \to F^2$ is a double cover. Roughly, this corresponds to an index two subgroup of $\pi_1(F)$, and hence to a homomorphism $\pi_1(F) \to \mathbb{Z}/2\mathbb{Z}$. Then for all $x \in F$, $|\rho^{-1}(x)| = 2$, so there is a canonical involution $\tau\colon G \to G$, where $\tau(y)$ is defined to be the unique element of $\rho^{-1}(\rho(y)) - \{y\}$. For an example, see Figure 3.4.

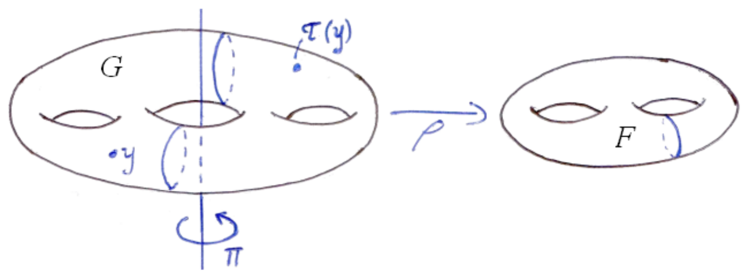

Figure 3.4: Here the involution τ is rotation by π about an axis.

Define $T = (G \times I)/\sim$, where $(y, 0) \sim (\tau(y), 0)$. Then $P\colon T \to F$ given by $(y, t) \mapsto \rho(y)$ is an I–bundle over F. Now suppose that $\rho\colon G \to F$ is the orientation double cover; so $G = F \times \{0, 1\}$ if F is orientable, and G is orientable if F is not; for example $\mathbb{T}^2 \overset{\times 2}{\to} \mathbb{K}^2$ (Figure 3.5).

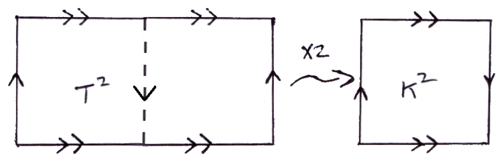

Figure 3.5: The torus is a double cover for the Klein bottle.

Then $P\colon T \to F$ as above is called the *orientation I–bundle* (Figure 3.6).

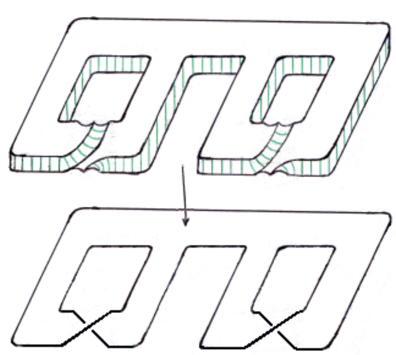

Figure 3.6: The orientation I–bundle over $\mathbb{K}^2 - \operatorname{int}\left(\mathbb{D}^2\right)$.

We have the following:

Theorem 3.2.1. *Suppose that $(F^2, \partial M) \subset (M^3, \partial M)$ is properly embedded. Then $N(F)$ is bundle equivalent to an I–bundle over F. If additionally M is orientable, then $N(F)$ is bundle equivalent to the orientation I–bundle over F.*

Example 3.2.1. Figure 3.7 shows the I–bundle for the punctured torus.

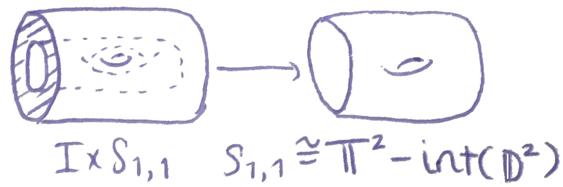

Figure 3.7: The orientation I–bundles are the only I–bundles one can draw in three-space.

Definition 3.2.1. Say $F \subset M$ is *one-sided* if F does not separate $N(F)$. Say F is *two-sided* if it does separate.

Example 3.2.2. The core curve α in the Möbius band $\mathbb{M}^2$ is one-sided. $\mathbb{D}^2 \times \{p\} \subset \mathbb{D}^2 \times S^1$ is two-sided for any $p \in S^1$. We can also find a Möbius band in $\mathbb{D}^2 \times S^1$ that is one-sided. $\mathbb{M}^2 \times \left\{\frac{1}{2}\right\}$ is two-sided in $\mathbb{M}^2 \times I$; see Figure 3.8.

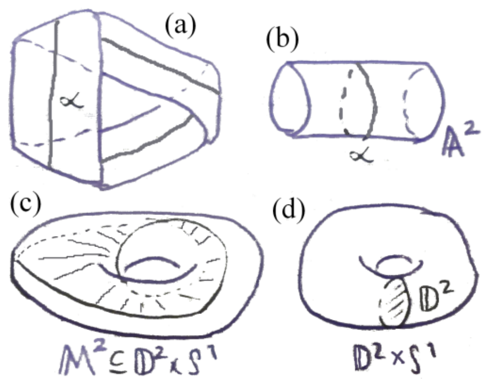

Figure 3.8: (a) α is one-sided in $\mathbb{M}^2$. (b) α is two-sided in $\mathbb{A}^2$ (c) $\mathbb{M}^2$ is one-sided in $\mathbb{D}^2 \times S^1$ (d) $\mathbb{D}^2$ is two-sided in $\mathbb{D}^2 \times S^1$.

Exercise 3.2.1. If $F \subset M$ is properly embedded, give a relationship between the orientability of M and F, and the number of sides of F.

Definition 3.2.2. If $\rho \colon T \to F$ is an I–bundle, then $X \subset T$ is *vertical* if X is a union of fibres.

Definition 3.2.3. The *vertical* boundary of an I–bundle $\rho \colon T \to F$ is $\partial_v T := \rho^{-1}(\partial F)$.

Definition 3.2.4. The *horizontal* boundary of an I–bundle $\rho \colon T \to V$ is $\partial_h T = \partial T - \mathrm{int}(\partial_v T)$.

Exercise 3.2.2. $\partial_v T$, $\partial_h T$ and the zero section are all incompressible in T, except for $\partial_v T$ when $T = I \times \mathbb{D}^2$.

Exercise 3.2.3. If $\partial F \neq \varnothing$, F is compact and connected, and $\rho \colon T \to F$ is the orientation I–bundle, then T is a handlebody.

Before moving on, we summarize examples of 3–manifolds discussed so far.

Example 3.2.3. We have seen:

(i) S^3, $\mathbb{P}^3$ and $\mathbb{T}^3$, which are closed.

(ii) V_g, the handlebodies.

(iii) I–bundles and S^1–bundles over surfaces.

Chapter 4

Triangulations and the Fundamental Group

4.1 Triangulations

Definition 4.1.1. Define the *k–simplex* by:

$$\Delta^k = \left\{ (x_0, \dots, x_k) \in \mathbb{R}^{k+1} : \sum x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}.$$

Definition 4.1.2. The *facet* $\delta_I \subset \Delta^k$ is the subsimplex of the form:

$$\delta_I = \{(x_0, ..., x_k) \in \Delta^k : x_i = 0 \text{ for all } i \in I\}.$$

Definition 4.1.3. If $\delta \subset \Delta$ and $\delta' \subset \Delta'$ are *faces* (codimension 1 facets), then a *face pairing* is an isometry $\varphi \colon \delta \to \delta'$.

Definition 4.1.4. We call a collection T of simplices and face pairings a *triangulation.*

Remark. We require that for every face pairing $\varphi \in T$ that if $\varphi \colon \delta \to \delta'$ then $\delta \neq \delta'$.

Definition 4.1.5. The number of simplices is written $|T|$. The *underlying space* is written $||T||$, and is defined by

$$||T|| := \left(\bigsqcup \Delta_i \right) / \{\varphi_j\}.$$

Definition 4.1.6. The quotient map is given by $\pi \colon \bigsqcup \Delta_i \to ||T||$ and we define $\pi_i \colon \Delta_i \to ||T||$ by restriction: $\pi_i = \pi | \Delta_i$.

Example 4.1.1. If T is the pair of simplices in Figure 4.1 with face pairings given by the arrows, then $||T|| \cong \mathbb{T}^2$.

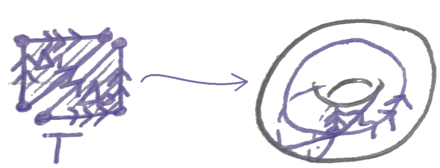

Figure 4.1: $||T|| \cong \mathbb{T}^2$.

Similarly, if we draw T as in Figure 4.2 then $||T|| = \mathbb{M}^2$.

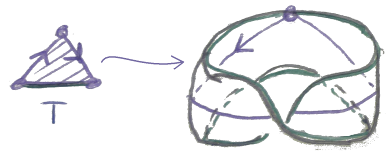

Figure 4.2: $||T|| \cong \mathbb{M}^2$.

Exercise 4.1.1. Find necessary and sufficient combinatorial conditions on T so that $||T||$ is a (PL) manifold of dimension 1, 2 or 3.

Hauptvermutung (Moise)**.** *Every topological 3–manifold admits a triangulation, unique up to subdivision. In particular, for any M^3, there exists a triangulation T such that $||T|| \cong M$.*

Remark. This is one important step in showing, in dimension three, that the categories TOP, PL and DIFF are all equivalent.

Definition 4.1.7. Suppose (M^3, T) is a triangulated manifold. An *orientation* of M is a choice of orientation for all $\Delta \in T$, such that all face pairings reverse the induced orientation on faces.

Example 4.1.2. The annulus is orientable, but the Möbius band is not. See Figure 4.3.

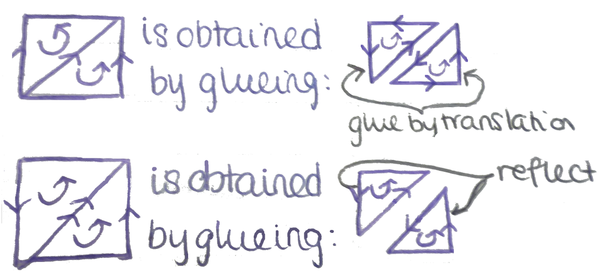

Figure 4.3: The annulus is orientable as all face pairings reverse the induced orientation on faces.

Proposition 4.1.1 (Proposition 6.5 in Lackenby)**.** *An n–manifold (M^n, T) is orientable if and only if for every simple closed curve $\alpha \in M$ we have $N(\alpha) \cong \mathbb{B}^{n-1} \times S^1$.*

Remark. We can also determine orientability in DIFF using $\mathrm{sign}(\det(Dh))$ where h ranges over the overlap maps, as in Figure 4.4. We can also define orientation in TOP using homology.

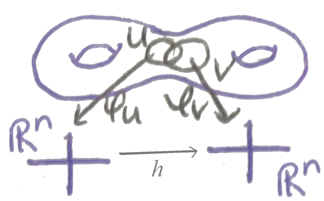

Figure 4.4: Orientation in DIFF arises from overlap maps of charts.

Definition 4.1.8. Define $\Delta^{(k)}$ to be the union of k–dimensional facets of Δ. If (M, T) is a triangulated 3–manifold, define $M^{(k)}$, the k–*skeleton* of M to be the manifold with triangulation $T = \bigcup_{i=1}^{|T|} \pi_i(\Delta^{(k)})$. Figure 4.5 shows the k-skeleta of Δ.

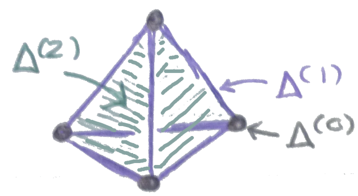

Figure 4.5: The k–skeleta of Δ.

Example 4.1.3. Figure 4.6 shows two examples of identifications.

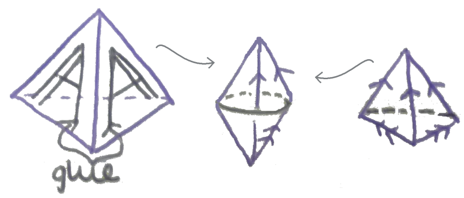

Figure 4.6: Two different views of the same triangulation for $\mathbb{B}^3$.

Exercise 4.1.2. Verify that the triangulation in Figure 4.7 is a three-manifold, and recognise it.

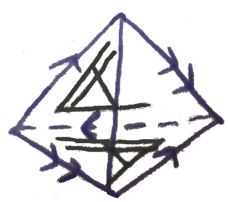

Figure 4.7: Which three-manifold is this?

4.2 Haken Kneser Finiteness

Definition 4.2.1. An isotopy $F\colon M \times I \to M$ is *normal* with respect to a triangulation T of M if for all $t \in I$, the homeomorphism F_t preserves $M^{(k)}$ for all k, and $F_0 = \mathrm{Id}_M$. See Figure 4.8 for an example.

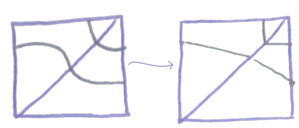

Figure 4.8: A normal isotopy.

Remark. Thus $M^{(0)}$ is fixed pointwise, and all other facets are fixed setwise.

Definition 4.2.2. Say an arc $(\alpha, \partial\alpha) \subset (\Delta^2, \partial\Delta)$ is *normal* if the points of $\partial\alpha$ are in distinct edges of Δ, and $\alpha \cap \Delta^{(0)} = \varnothing$. See Figure 4.9 for some examples and a non-example.

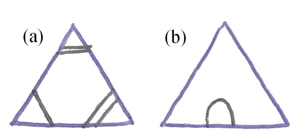

Figure 4.9: (a) Normal arcs. (b) This is not a normal arc.

Definition 4.2.3. A disk $(D, \partial D) \subset (\Delta^3, \partial\Delta)$ is a *normal disk* if ∂D is transverse to $\Delta^{(1)}$, ∂D meets each edge of $\Delta^{(1)}$ at most once, and $D \cap \Delta^{(0)} = \varnothing$. See Figures 4.10(a) and (b) for examples and 4.10(c) and (d) for non-examples.

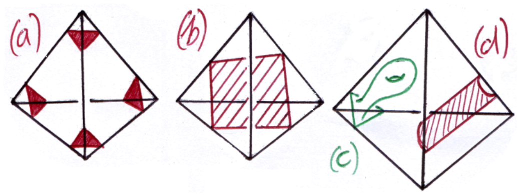

Figure 4.10: (a) There are four normal triangles. (b) There are three normal quadrilaterals. (c) This is not even a disk, let alone normal. (d) This is also not a normal disc.

Exercise 4.2.1. Prove that:

(i) There are only three normal arcs up to normal isotopy.

(ii) There are only seven normal disks up to normal isotopy.

Recall that $\pi_i \colon \Delta_i \to M$ is defined by $\pi_i = \pi|\Delta_i$, where π is the quotient map.

Definition 4.2.4. Suppose $S \subset M$ is a surface. Say S is normal if $\pi_i^{-1}(S)$ is a disjoint collection of *normal* disks for all i.

Example 4.2.1. The three normal disks in the tetrahedron shown in Figure 4.11 give a normal surface under the identification indicated by the arrows.

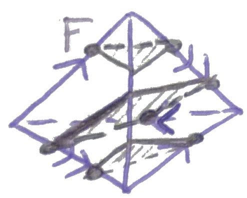

Figure 4.11: Recognise the normal surface F by computing $|\partial F|$, $\chi(F)$ and the orientability.

Exercise 4.2.2. Show that, with triangulations as in Figure 4.12, (a) and (b) are three manifolds, and recognise them.

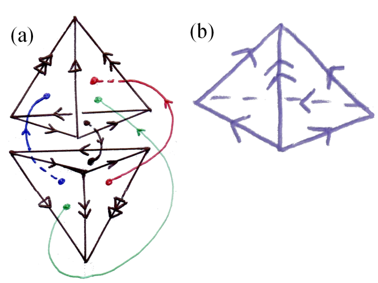

Figure 4.12: Show that (a) and (b) are three manifolds and recognise them.

Theorem 4.2.1 (Haken-Kneser Finiteness)**.** *Suppose (M, T) is a connected, compact triangulated 3–manifold. Suppose $S \subset (M, T)$ is an embedded normal surface. Then if $|S| \geq 20|T| + 1$ there are components $R, R' \subset S$ so that R, R' cobound a product component of $M - S$.*

Remark. Figures 4.13 and 4.14 show examples of parallel surfaces.

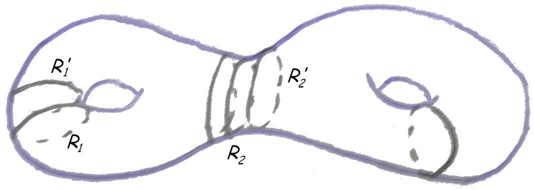

Figure 4.13: Here both R_1 & R_1' and R_2 & R_2' bound copies of $\mathrm{D}^2 \times I$.

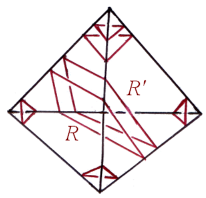

Figure 4.14: R and R' bound a product.

Proof of Theorem 4.2.1. Recall that $S \cap \Delta$ for $\Delta \in T$ is a finite collection of normal disks. Consider the subcollection of disks of a fixed *type*, that is a normal isotopy class. Call the outermost disks *ugly*, the second outermost disks *bad*, and all other disks *good*, as illustrated in Figure 4.15.

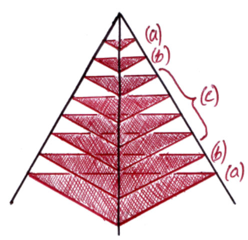

Figure 4.15: (a) Ugly disks. (b) Bad disks. (c) Good disks.

Thus there is a component $F \subset S$, such that F is a union of good disks. To see this, note that there are at most $20|T|$ ugly and bad disks in total. There are at most five types of disk in each $S \cap \Delta$, and at most four of each can be ugly or bad; see Figures 4.16(a) and (b).

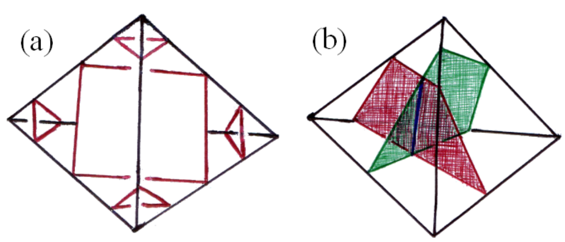

Figure 4.16: (a) There are at most five types of disk in each $S \cap \Delta$ because (b) two normal quadrilaterals of different types must intersect.

Now let N be the closure of the union, over all Δ_i, of all components of $\Delta_i - S$ that are adjacent to F, as in Figure 4.17.

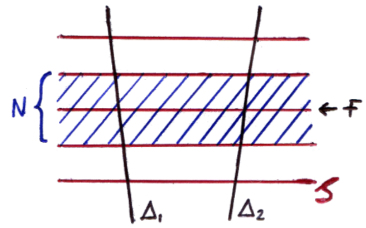

Figure 4.17: N is the closure of the union over all Δ_i of all components of $\Delta_i - S$ that are adjacent to F.

Exercise 4.2.3. Prove that N is an I–bundle and either N is ambient isotopic to $N(F)$ *or* F is two-sided and parallel to $\partial_h N$. □

4.3 The Fundamental Group

We recall properties of π_1:

Definition 4.3.1. Suppose A and B are groups. Then if $A = \langle a_i \mid r_k \rangle$ and $B = \langle b_j \mid s_l \rangle$, their free product $A * B$ is given by

$$A * B = \langle a_i, b_j \mid r_k, s_l \rangle .$$

Theorem 4.3.1 (van Kampen)**.** *If $W = X \cup_Z Y$ and Z is path connected (as in Figure 4.18), then, choosing a base point $p \in Z$, $\pi_1(W,p) \cong \pi_1(X,p) * \pi_1(Y,p)/N$, where N is the normal subgroup generated by:*

$$\{i_*(z)(j_*(z))^{-1} : z \in \pi_1(Z,p)\}$$

where $i : Z \hookrightarrow X$ and $j : Z \hookrightarrow Y$ are the inclusions.

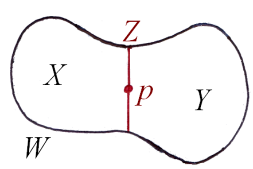

Figure 4.18: If $W = X \cup_Z Y$, then $\pi_1(W) = (\pi_1(X) * \pi_1(Y))/N$.

Corollary 4.3.1. *If $\pi_1(Y,p) = \{1\}$ then $\pi_1(W,p) = \pi_1(X,p)/N$ where N is the normal subgroup generated by:*

$$\{i_*(z) : z \in \pi_1(Z,p)\}.$$

Corollary 4.3.2. *If $\pi_1(Z,p) = \{1\}$ then $\pi_1(W,p) = \pi_1(X,p) * \pi_1(Y,p)$.*

Proposition 4.3.3. *If (M,T) is triangulated then $\pi_1(M) = \pi_1\left(T^{(2)}\right)$.*

Exercise 4.3.1. Prove Proposition 4.3.3. See Figure 4.19 for a hint.

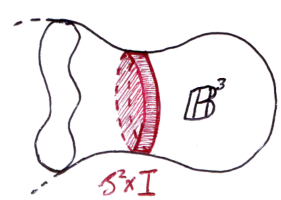

Figure 4.19: Hint: Attach 3-balls one by one.

Proposition 4.3.4. *$\pi_1(T^{(2)}) = \pi_1(T^{(1)})/N$ where N is the normal subgroup generated by boundaries of two-simplices in T. Note that $\pi_1(T^{(1)})$ is a free group, as $T^{(1)}$ is a connected graph.*

We now give several example computations.

Example 4.3.1. Consider Figure 4.20, where the faces are glued according to the arrows.

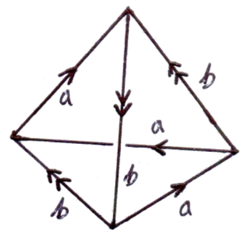

Figure 4.20: What is the fundamental group of this manifold?

Exercise 4.3.2. Check that this is a 3–manifold.

Step 1: Find a spanning tree for $T^{(1)}$. Here $T^{(1)}$ is the graph shown in Figure 4.21 and so the spanning tree is just the vertex.

Figure 4.21: $T^{(1)}$ in this case. The spanning tree is the single vertex circled in green.

Step 2: Give labels to the non-tree edges of $T^{(1)}$, as in Figure 4.21.

Step 3: Read off relations from faces of $T^{(2)}$. There is one relation per face in the quotient. Here we have $\langle a, b \mid a^2 = b,\ b^2 a = \mathbf{1} \rangle$.

Step 4: (optional) Use Tietze transformations to simplify:

$$\left\langle a, b \mid a^2 = b,\ b^2 a = \mathbf{1} \right\rangle \cong \left\langle a \mid (a^2)^2 a = \mathbf{1} \right\rangle \cong \mathbb{Z}/5\mathbb{Z}.$$

Example 4.3.2. (A non-Abelian example.) The *one-quarter turn space* Q is the quotient of the unit cube as shown in Figure 4.22:

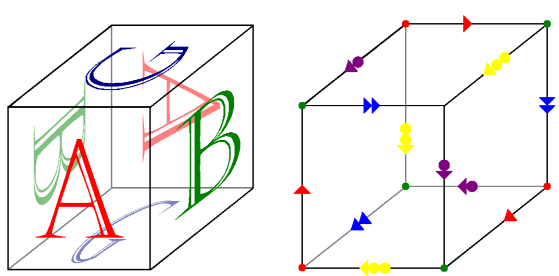

Figure 4.22: Two visualisations of how to glue faces to get Q.

Step 1: The 1–skeleton is the graph in Figure 4.23(a) with four edges and two vertices. We take the circled edge as the spanning tree.

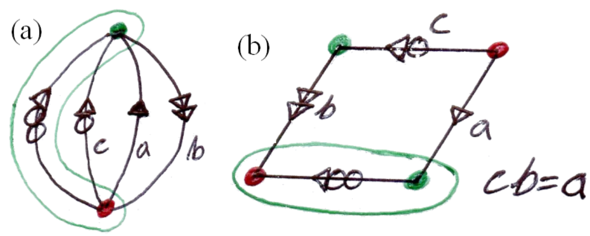

Figure 4.23: (a) The 1–skeleton and spanning tree. (b) After labelling the non-tree edges, read off relators from the faces. Edges of the spanning tree do not contribute to the relators.

Step 2: Label the non-tree edges with a, b, c.

Step 3: The three squares give relations and we have the following presentation

$$\pi_1(Q) = \langle a, b, c \mid a = cb,\ ba = c,\ abc = \mathbf{1} \rangle \,.$$

Exercise 4.3.3. Recognize $\pi_1(Q)$. In particular, it is not Abelian.

4.4 Abelian Groups

Definition 4.4.1. Suppose that Z is an Abelian group. Define $N := \{z \in Z : z \text{ is finite order}\}$. Then $N < Z$ is called the *torsion subgroup* of Z.

Recall that $A \oplus B$ is the *direct product* of A and B.

Proposition 4.4.1. *Suppose Z is a finitely generated Abelian group. Then there exist unique $k \in \mathbb{N}$ and N a finite group so that $Z \cong \mathbb{Z}^k \oplus N$.*

Proof. This follows from the classification of finitely generated Abelian groups. □

Definition 4.4.2. We call k the *rank* of Z, and use the notation $\mathrm{rk}(Z) = k$.

Definition 4.4.3. Let G be any (finitely generated) group. The *commutator subgroup* of G is $[G, G]$, the subgroup of G generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.

$$[G, G] = \left\langle xyx^{-1}y^{-1} \mid x, y \in G \right\rangle \triangleleft G.$$

Definition 4.4.4. We define the *Abelianization* of G to be $G^{\mathrm{Ab}} = G/[G, G]$.

Definition 4.4.5. We define the *first homology group* of M^3 to be $H_1(M, \mathbb{Z}) := [\pi_1(M)]^{\mathrm{Ab}}$.

Example 4.4.1. Let $M^3 = N^3 \# P^3$. Then it follows by van Kampen's theorem that $\pi_1(M) \cong \pi_1(N) * \pi_1(P)$. Therefore $H_1(M) = H_1(N) \oplus H_1(P)$.

Exercise 4.4.1. Show that $(A * B)^{\mathrm{Ab}} = A^{\mathrm{Ab}} \oplus B^{\mathrm{Ab}}$.

Example 4.4.2. As in the last example we have that

$$\pi_1(\#_g S^2 \times S^1) = F_g \cong *_g \mathbb{Z},$$

so $H_1(\#_g S^2 \times S^1) = \mathbb{Z}^g$ has rank g. We denote $\#_g S^2 \times S^1$ by M_g.

Chapter 5

The Connect Sum Decomposition

5.1 Existence

Proposition 5.1.1. *If M is connected, orientable, compact and $M \cong N \# M_g$, then $g \leq \mathrm{rk}(H_1(M))$.*

Note here that π_1 is finitely generated since M is compact.

Proof. We know that $H_1(M) = H_1(N) \oplus H_1(M_g)$, so:

$$\mathrm{rk}(H_1(M)) = \mathrm{rk}(H_1(N)) + g. \qquad \square$$

This is the first step in the existence proof for connect sum decompositions. For the next step, we need the following proposition:

Proposition 5.1.2. *Suppose M is connected, orientable and compact. Then there exists a decomposition*

$$M \cong \#_{i=1}^{k} N_i \# (\#_g S^2 \times S^1) \# (\#_n \mathbb{B}^3)$$

where each N_i is irreducible and not S^3, $\mathbb{B}^3$ or $S^2 \times S^1$.

Proof. The proof is split into three main steps.

Step 1: Let n be the number of components of ∂M that are 2–spheres. Let F be the frontier of a "tree-like" union of arcs and two-sphere boundary components, as shown in Figure 5.1. Form $M - n(F)$ and cap off $F^{\pm}$ by 3–balls. From now on we assume that $n = 0$.

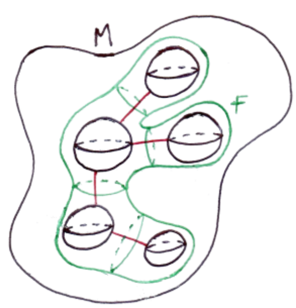

Figure 5.1: F is the frontier of a "tree-like" union of arcs and two-sphere boundary components.

Step 2: Proposition 5.1.1 gives us an upper bound on the number of summands of M homeomorphic to $S^2 \times S^1$. Thus from now on we may assume that $g = 0$. It follows that any 2–sphere embedded in M separates.

For Step 3, we require the following definitions.

Definition 5.1.1. We define $S^3_k := \#_{i=1}^k \mathbb{B}^3$ and we call this a *ball with holes* or a *punctured sphere*. See Figure 5.2.

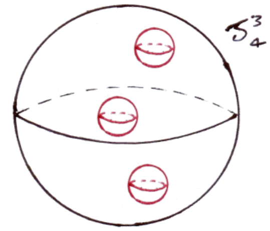

Figure 5.2: Here, $n = 4$.

Exercise 5.1.1. Show that $(\#_n\mathbb{B}^3) \cup_{S^2} (\#_m\mathbb{B}^3) \cong \#_{n+m-2}\mathbb{B}^3$.

Definition 5.1.2. We call $S \hookrightarrow M$ a *sphere system* if S is an embedding of a disjoint collection of 2–spheres; see Figure 5.3.

Definition 5.1.3. A system $S \hookrightarrow M$ is *reduced* if no component of $M - n(S)$ is homeomorphic to a punctured sphere. The sphere system in Figure 5.3 is reduced.

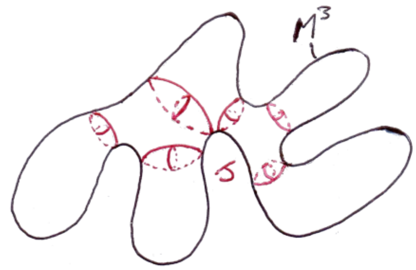

Figure 5.3: A reduced sphere system S in M.

Step 3: If M is irreducible we are done. If $M \cong S^3$ we are done by Alexander's theorem. So suppose that M contains an essential 2–sphere. For the reminder of the proof, we fix a finite triangulation T of M. So our assumptions give us a reduced sphere system $S \subset M$.

Normalization Lemma. For any reduced sphere system $S \subset M$ there is a normal, reduced sphere system S' such that $|S'| \geq |S|$.

If we assume this lemma, we get the following proposition:

Proposition 5.1.3. *(Existence) Let M be defined as above. Then $M \cong \#_{i=1}^n N_i$ such that all N_i are irreducible and $N_i \ncong S^3, \mathbb{B}^3$.*

Proof. Let S_1 denote an essential 2–sphere, so $M = N_1 \#_{S_1} N_2$. If N_1 is homeomorphic to $\#_k\mathbb{B}^3$ for $k \geq 1$, then we have a contradiction. So, $S = \{S_1\}$ is a reduced sphere system. Let $\overline{S}$ be a maximal sphere system (i.e. of maximal size). This exists because any normal reduced system has at most $20|T|$ components; this follows from the Haken-Kneser finiteness and the normalization lemma. Since $\overline{S}$ is maximal , if we cut M along $\overline{S}$ and cap off with 3–balls the resulting manifolds $\{N_i\}$ are all irreducible. □

This completes the proof of Proposition 5.1.2. □

5.2 Proof of the Normalisation Lemma

To prove the normalization lemma, we must *normalize* the given system S.

Proof of Normalization Lemma. Isotope S to be transverse to $T^{(k)}$ for $k = 0, 1, 2$, i.e. $S \cap T^{(0)} = \varnothing$, $|S \cap T^{(1)}| =: w(S)$ (the *weight* of S) is finite, $S \cap T^{(1)}$ is transverse and $S \cap \partial\Delta_i$ is a finite collection of simple closed curves; see Figure 5.4. We alternatingly apply *surgery* and the *baseball move*.

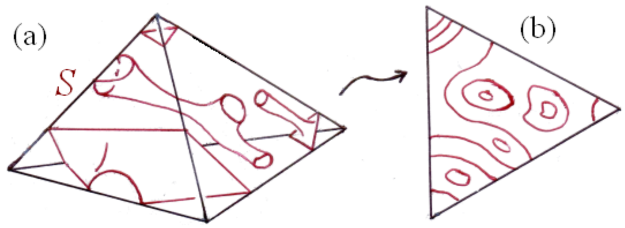

Figure 5.4: (a) The sphere system can look unpleasant in the triangulation. (b) A possible picture of $S \cap T^{(2)}$.

5.2.1 Surgery

Suppose $(D, \partial D) \subset (M, S)$ is a surgery disk, i.e. $D \cap S = \partial D$. Suppose $D \cap S \subset F$ is a component of S. As before, define $F_D = (F - n(D)) \cup D^+ \cup D^-$. Define $S_D = (S - F) \cup F_D$. Notice that ∂D separates F, so $F_D = F^+ \cup F^-$. See Figure 5.5.

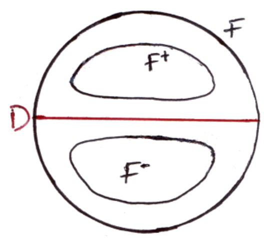

Figure 5.5: Notice that ∂D separates F, so $F_D = F^+ \cup F^-$.

Let $X, Y \subset M - n(S)$ be the components adjacent to F and suppose $D \cap X \neq \varnothing$. So let $X^+ \cup X_0 \cup X^- = X - n(F_D)$ where X_0 meets D and $X^\pm$ are adjacent to $F^\pm$, respectively. See Figure 5.6.

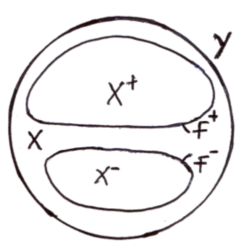

Figure 5.6: $X^+ \cup X_0 \cup X^- = X - n(F_D)$, where X_0 meets D, and $X^\pm$ are adjacent to $F^\pm$.

Note that $X_0 \cong \#_3\mathbb{B}^3$. Since we assumed S is a reduced sphere system, we find Y is not a punctured sphere.

Exercise 5.2.1. $Y \cup_F X_0$ is not a punctured sphere.

Claim. At most one of X^+, X^- is a punctured sphere.

Proof. If both are punctured spheres then so is $X = X^+ \cup_{F^+} X_0 \cup_{F^-} X^-$, a contradiction. This proves the claim. $\square$

Let $S' = S - F$. Thus either $S^+ = S' \cup F^+$ or $S^- = S' \cup F^-$ or $S_D = S' \cup F_D$ is a reduced system.

5.2.2 Using Surgery

For every tetrahedron $\Delta \in T^{(3)}$, the surface S meets $\partial\Delta$ is a collection of simple closed curves. See Figure 5.7 for a possible intersection pattern.

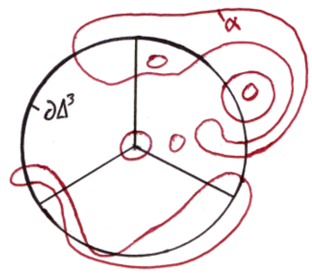

Figure 5.7: A possible intersection of S with the boundary of a tetrahedron.

For every simple closed curve $\alpha \subset \partial\Delta \cap S$ we do the following. Pick a disk $D \subset \partial\Delta$ bounded by α. Isotope D into Δ (∂D stays in S), as in Figure 5.8.

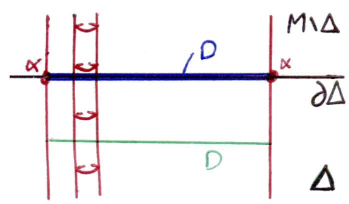

Figure 5.8: Isotope D into Δ.

Use D (in Δ) to surger all curves of $S \cap D$, innermost first. When this is done, $S \cap \Delta$ is a collection of disks (for all Δ).

Claim. After surgery, for all Δ and for all simple closed curves $\alpha \subset \partial\Delta \cap S$, α meets $\Delta^{(1)}$.

Proof. Suppose α has weight 0 and $\alpha \subset f \subset \Delta^{(2)}$ a face. We surgered along both $D^{\pm}$, so the component sphere containing α bounds a ball as in Figure 5.9.

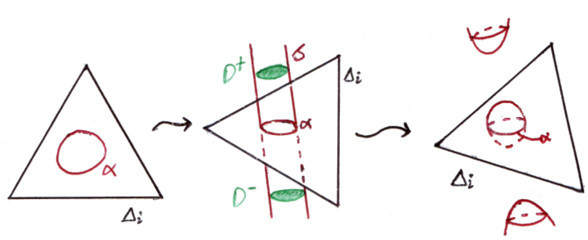

Figure 5.9: We surgered along both $D^{\pm}$, so the component sphere containing α bounds a ball.

But surgery deletes trivial spheres. This proves the claim. $\square$

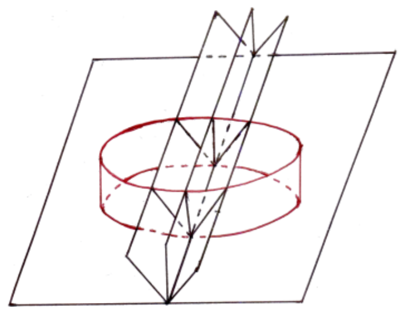

Figure 5.10: The intersection of S with the two-skeleton; outside of Δ it can be complicated.

5.2.3 The Baseball Move

We perform this move after surgery along all curves of $S \cap \partial\Delta$ for all $\Delta^3 \in T$. Suppose α is a simple closed curve of $S \cap \partial\Delta$, where $\Delta^3 \in T$. So α bounds disks D_0 and D_1 in $\partial\Delta$. Suppose that there is an edge $e \in \Delta^{(1)}$ with $|\alpha \cap e| \geq 2$, as illustrated in Figure 5.11.

Exercise 5.2.1. Without loss of generality, there is a component $d \subset D_0 \cap e$ such that $d \cap \Delta^{(0)} = \varnothing$, as in Figure 5.11.

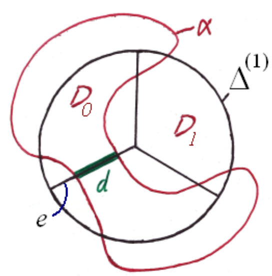

Figure 5.11: α bounds two disks D_0 and D_1, and there is an edge $e \in \Delta^{(1)}$ such that $|\alpha \cap e| = 2$.

Now let $D = D_0$. By an innermost arc argument we may assume that $d \cap S = \partial d$. Let $D' \subset S \cap \Delta$ be the disk bounded by α, as in Figure 5.12.

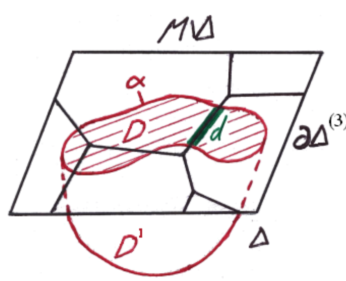

Figure 5.12: D' is the disk bounded by α.

Since $D \cup D' \cong S^2$, they cobound a three-ball, B, by Alexander's theorem, and so we may choose an embedded arc $d' \subset D'$ so that d and d' cobound a disk $E \subset B$, as in Figure 5.13.

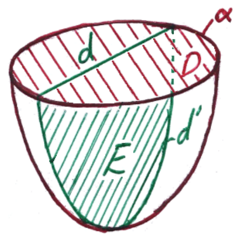

Figure 5.13: The arcs d and d' cobound a disk $E \subset B$.

Let C be the 3–ball obtained from $N(E)$ by cutting along S and retaining the component containing E; see figure 5.14.

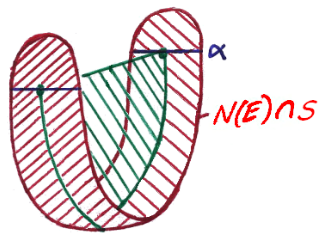

Figure 5.14: A picture of $N(E) \cap S$.

Write $\partial_- C = C \cap S$ and $\partial_+ C = \overline{\partial C - \partial_- C}$. The *baseball curve* is the common boundary $\partial\partial_+ C = \partial\partial_- C$, as in Figure 5.15.

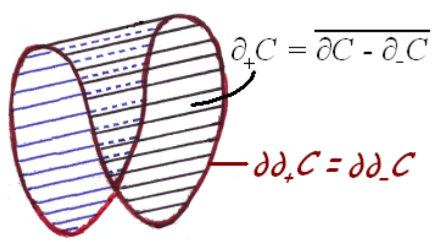

Figure 5.15: The baseball curve is the common boundary $\partial\partial_+ C = \partial\partial_- C$.

Since C is a 3–ball, there is an isotopy, called the *baseball move*, taking $\partial_- C$ to $\partial_+ C$; see Figures 5.16(a) or (b). This gives an isotopy of S to S'. Notice that $w(S') = w(S) - 2$.

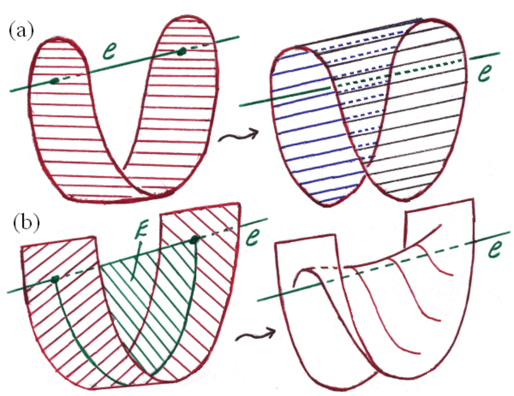

Figure 5.16: Two visualisations of the baseball move.

So alternate between surgery along all curves and single baseball moves. As $w(S)$ is decreasing, this process terminates with S in normal position. If $w(S) = 0$ then $S = \varnothing$ and this is a contradiction as surgery never decreases the initial number of essential spheres. So this completes the proof of existence. $\square$

5.3 Uniqueness

Following Hatcher, for uniqueness we use lemma 5.3.1.

Definition 5.3.1. If M is a 3–manifold, define $\widehat{M}$ to be M with all $S^2 \subset \partial M$ capped off by 3–balls, and discarding 3–sphere components.

Lemma 5.3.1. *Suppose that $S \subset M$ is a sphere system (not necessarily reduced) so that:*

$$\widehat{M - n(S)} = \bigsqcup_{i=1}^{k} N_i$$

is a disjoint union of irreducible manifolds. Suppose that $(D, \partial D) \subset (M, S)$ is a surgery disk. Then:

$$\widehat{M - n(S_D)} = \bigsqcup_{i=1}^{k} N_i.$$

Exercise 5.3.1. Prove this lemma. For a hint, see Figure 5.17.

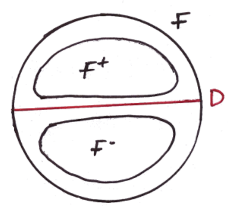

Figure 5.17: Hint for Exercise 5.3.1.

So we may now complete the proof of uniqueness of prime decomposition.

Proof of uniqueness. Suppose S and T are sphere systems so that:

$$M - n(S) = \bigsqcup_{i=1}^{k} P_i$$

and

$$N - n(T) = \bigsqcup_{j=1}^{l} Q_j$$

where the P_i and Q_j are irreducible. Now, if $S \cap T = \varnothing$ we have:

$$\begin{aligned} \bigsqcup P_i &= \widehat{\bigsqcup P_i - n(T)} \\ &= \widehat{M - n(S \cup T)} \\ &= \widehat{\bigsqcup Q_j - n(S)} = \bigsqcup Q_j \end{aligned}$$

On the other hand, if $S \cap T \neq \varnothing$ then surger S along an innermost disk of T and apply Lemma 5.3.1. Finally, if $M \cong N \# \left(\#_l S^2 \times S^1\right)$ and $M \cong N \# \left(\#_k S^2 \times S^1\right)$ then:

$$\operatorname{rank}(H_1(N)) + l = \operatorname{rank}(H_1(M)) = \operatorname{rank}(H_1(N)) + k$$

and so $l = k$. $\square$

Chapter 6

Torus Decompositions

6.1 Essential Tori

Exercise 6.1.1. Suppose that (M, T) is orientable, compact, connected, irreducible and triangulated. Suppose $F \subset M$ is embedded, closed ($\partial F = \varnothing$, compact) and orientable. Show that if G is incompressible, it is isotopic to a normal surface.

Definition 6.1.1. Say F properly embedded in M is *boundary parallel* if there is an isotopy (relative to ∂F) pusing F into ∂M. More precisely, there is an isotopy $H\colon F \times I \to M$ such that:

(i) H_t is an embedding of F into M for all $t < 1$.

(ii) H_1 is an embedding of F into ∂M.

(iii) $H_0 = \mathrm{Id}$.

(iv) $H_t|\partial F = \mathrm{Id}$.

Equivalently $M - n(F)$ has a component $X \cong F \times I$ with $F \times \{0\} = F^+ \subset N(F)$ and $F \times \{1\} \subseteq \partial M$. See Figure 6.1.

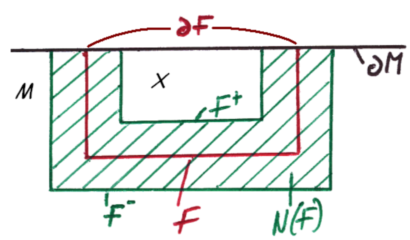

Figure 6.1: F is boundary parallel to M.

Example 6.1.1. (See Figure 6.2)

(i) The equatorial disk $\mathbb{B}^2 \subset \mathbb{B}^3$ is boundary parallel.

(ii) Take $K \subset T = \partial(\mathbb{D}^2 \times S^1)$. Let $N(K)$ be a closed neighbourhood in $\mathbb{D}^2 \times S^1$. Let $G = N(K) \cap T$. So $G \subset T = \partial(\mathbb{D}^2 \times S^1)$. Let $F = \overline{\partial N(K) - G}$, so F is boundary parallel; in fact parallel to G.

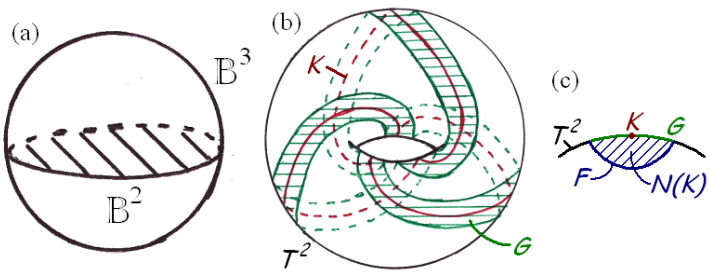

Figure 6.2: (a) Example (i). (b) Example (ii). (c) Cross section for Example (ii).

Note. F in example (ii) above is boundary parallel in essentially a unique way, unlike $\mathbb{B}^2 \subset \mathbb{B}^3$, or the following. Take $\mathbb{B}^1 \times S^1 \subseteq \mathbb{D}^2 \times S^1$. Then this is boundary parallel in two ways; see Figure 6.3.

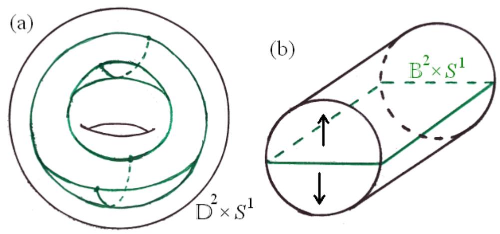

Figure 6.3: (b) is a cross section of (a), and $\mathbb{B}^1 \times S^1$ can be isotoped either up or down into $\mathbb{T}^2 = \partial\left(\mathbb{D}^2 \times S^1\right)$.

Example 6.1.2. $\mathbb{M}^2 \subseteq \mathbb{D}^2 \times S^1$ is not boundary parallel; see Figure 6.4.

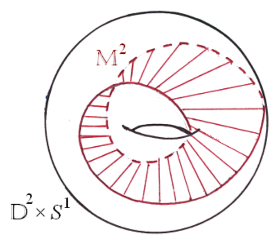

Figure 6.4: $\mathbb{M}^2$ is not boundary parallel in $\mathbb{D}^2 \times S^1$.

Definition 6.1.2. A torus $T \subset M$ is essential if it is incompressible and not boundary parallel.

Definition 6.1.3. Suppose M is irreducible, orientable, compact and connected. Then the manifold M is *toroidal* if there exists an essential torus $T \subset M$. M is *atoroidal* if there are no essential tori embedded in M.

Example 6.1.3. Suppose $K \subset S^3$ is a knot. Define the *knot exterior* $X_K := S^3 - n(K)$. If $K = L \# L'$ is a non-trivial connect sum of knots, then X_K is toroidal. See Figure 6.5 for example.

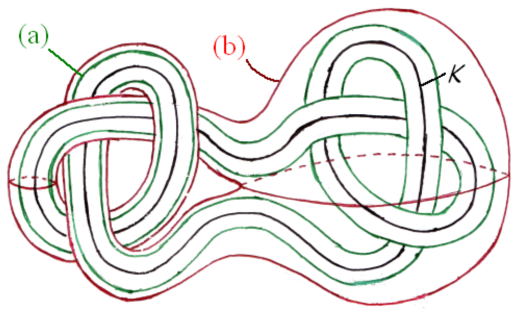

Figure 6.5: (a) $n(K)$. (b) An essential torus in X_K.

As shown earlier, when dealing with essential 2–spheres, we cut and cap off with 3–balls. However, there is no canonical way to cap off $\mathbb{T}^2 \subset \partial M$. So we must live with the possibility of incompressible tori, but at least we may eliminate essential tori.

Definition 6.1.4. Fix K, a knot in S^3, called the *companion knot*. Fix $L \subset \mathbb{D}^2 \times S^1$, the *pattern knot*. Fix a homeomorphism $\varphi\colon \mathbb{D}^2 \times S^1 \to N(K)$. Then $\varphi(L) \subset S^3$ is a *satellite knot* with pattern L and companion K. See Figure 6.6.

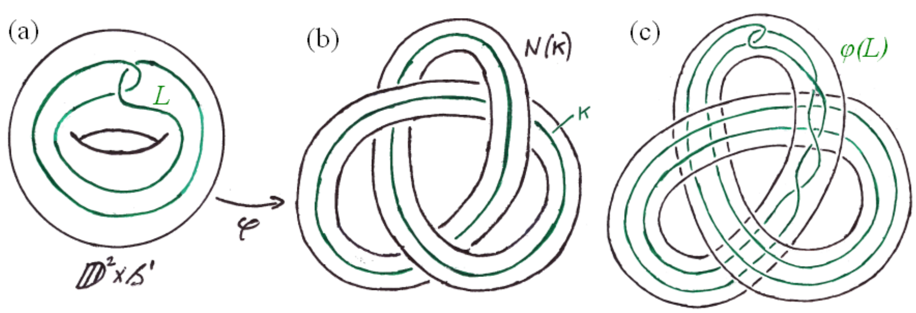

Figure 6.6: (a) L is the pattern knot, (b) K is the companion knot and (c) $\varphi(L)$ is the satelite knot.

Example 6.1.4. All non-trivial connect sums are satellite knots.

Remark. If K is not the unknot and $L \subset \mathbb{D}^2 \times S^1$ is *disk busting* (for all compressing disks $D \subset \mathbb{D}^2 \times S^1$, $|L \cap D| \geq 1$, and L is not isotopic to $\{0\} \times S^1$), then $X_{\varphi(L)}$ is toroidal.

Theorem 6.1.1 (Thurston)**.** *Every knot $K \subset S^3$ other than the unknot is either a satellite knot, a torus knot or a hyperbolic knot, as respectively X_K is toroidal, X_K is atoroidal but cylindrical, or X_K is atoroidal and acylindrical.*

Exercise 6.1.2. Show that X_K is irreducible.

Example 6.1.5. S^3 is atoroidal, but $\mathbb{T}^3$ is not; see Figure 6.7.

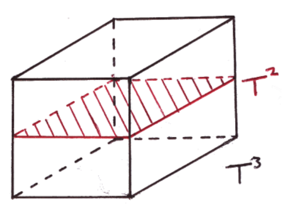

Figure 6.7: $\mathbb{T}^3$ contains $\mathbb{T}^2$ as an essential torus, and so is toroidal.

Exercise 6.1.3. Suppose $F \subset M$ is properly embedded and suppose that $i_*\colon \pi_1(F) \to \pi_1(M)$ is injective. Show that F is incompressible (i.e., all surgery disks are trivial).

The final part of the course will be devoted to proving a partial converse to Exercise 6.1.3, via the loop theorem, the disk theorem and Dehn's lemma. An application of this converse will give us the following example:

Example 6.1.6. A knot $K \subset S^3$ is isotopic to a round circle (that is K is unknotted) if and only if $\pi_1(X_K) \cong \mathbb{Z}$.

Definition 6.1.5. A *torus system* is a finite union of disjoint, non-parallel, essential tori.

Proposition 6.1.2 (Corollary 1.8 in Hatcher)**.** *Suppose that M is compact, connected, orientable and irreducible. Then there is a torus system $S \subset M$ (where we allow $S = \varnothing$), so that all components of $M - n(S)$ are atoroidal.*

Proof. If M is atoroidal then take $S = \varnothing$. Otherwise, fix a triangulation T of M and suppose that $F \subset M$ is an essential torus. So $S = \{F\}$ is a torus system. We now induct on $|S|$. By Exercise 6.1.1 we may normalize S. By Haken-Kneser finiteness we find that $|S| \leq 20|T|$, so if there exists a component $N \subseteq M - n(S)$ which is toroidal then we find $F' \subset N$ an essential torus. So F' is not parallel to any component of S. Let $S' = S \cup \{F'\}$. Then S' is again a torus system. $\square$

Remark. The final step uses Exercise 4.5 in Exercise Sheet 4.

Example 6.1.7. Suppose $\varphi\colon F \to F$ is a homeomorphism of a surface F. Define $M_\varphi = F \times I/(x,1) \sim (\varphi(x), 0)$. Then M_φ is a surface bundle over S^1 via $\rho\colon M_\varphi \to S^1$, where $\rho\colon (x,t) \mapsto t \in \mathbb{R}/\mathbb{Z}$; see Figure 6.8.

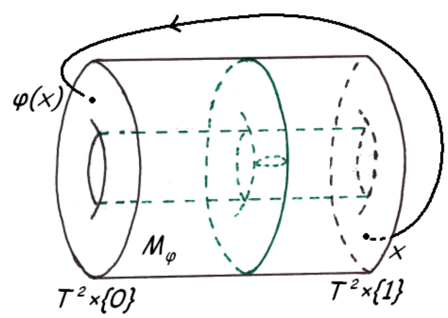

Figure 6.8: M_φ is a $\mathbb{T}^2$–bundle over S^1.

Exercise 6.1.4. Show that every fibre $T_t = \rho^{-1}(t)$ is incompressible (in fact π_1–injective) in M_φ.

Note. If $F = T \cong \mathbb{T}^2$, and $T \subset M_\varphi$ is a fibre, then $M_\varphi - n(T) \cong T \times I$. So we cannot avoid sometimes having a product component after cutting.

Remark. We have that $\mathbb{T}^3$ is the torus bundle M_{Id} in the above notation.

6.2 Lens Spaces

We now discuss lens spaces. Take $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 2\}$. Let y be the loop $\{|w| = 2\}$ and x be the loop $\{|z| = 2\}$, oriented as shown in Figure 6.9.

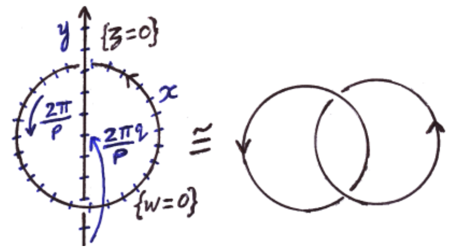

Figure 6.9: The great circles $\{z = 0\}$ and $\{w = 0\}$ in $S^3 \subset \mathbb{C}^2$ with this orientation are together homeomorphic to the right Hopf link.

Then define:

$$\begin{aligned} V &= \{(z,w) \in S^3 : |w| \le 1\}, \\ W &= \{(z,w) \in S^3 : |z| \le 1\}, \\ T &= V \cap W \\ &= \{(z,w) \in S^3 : |z| = |w| = 1\} \cong \mathbb{T}^2. \end{aligned}$$

Recall that $D \times S^1$ is a *solid torus*. We refer to any curve of the form $\partial D \times \{z\} \subset D \times S^1$ as a *meridian*. Now, as indicated in Figure 6.10 we take μ and λ to be generators of $\pi_1(T)$. Thus μ and λ are meridians of the solid tori V and W, respectively. We give μ and λ the orientations shown in Figure 6.10.

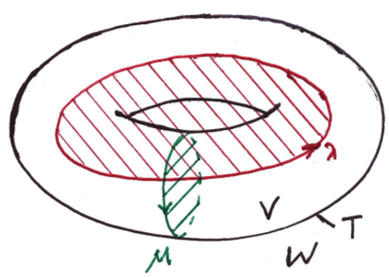

Figure 6.10: The curves μ and λ are oriented so that μ, λ and the outward normal for V form a right-handed frame.

Definition 6.2.1. Write $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{\alpha \in \mathbb{C} : \alpha^p = 1\}$ for $p \neq 0$, and fix $q \in \mathbb{Z}$ with $\gcd\{q,p\} = 1$. This acts on S^3 via:

$$\alpha \cdot (z,w) = (\alpha z, \alpha^q w).$$

Definition 6.2.2. Define $L(p,q) = \mathbb{Z}_p \backslash S^3$, the (p,q)–*lens space.*

Exercise 6.2.1. $L(p,q)$ is an orientable 3–manifold.

Example 6.2.1. We have $L(1,0) = S^3$.

Exercise 6.2.2. Show that $L(2,1) \cong P^3$.

Proposition 6.2.1. *Suppose $V, W \cong \mathbb{D}^2 \times S^1$ and $\varphi \colon \partial W \to \partial V$ is a homeomorphism. Show that $M = V \cup_\varphi W$ is either a lens space or is $S^1 \times S^2$.*

Note. We have $\pi_1(L(p,q)) \cong \mathbb{Z}_p$. Thus if $L(p',q') \cong L(p,q)$ then $p' = p$.

Exercise 6.2.3. Show that if $q' = \pm q^{\pm 1}$ modulo p, then $L(p,q') \cong L(p,q)$.

Remark. The converse holds, but is much harder to prove (see Brody 1960).

Remark. Whitehead (1941) showed that $L(p,q) \simeq L(p,q')$ (the spaces are homotopy equivalent) if and only if $qq' = \pm k^2$ modulo p for some k.

Example 6.2.2. We have $L(7,1) \simeq L(7,2)$, but these spaces are not homeomorphic.

For lens spaces, we have the following definitions:

- The quotient space $\mathbb{Z}_p \backslash S^3$.
- The gluing $V \cup_\varphi W$, the union of solid tori, which is either a lens space or $S^2 \times S^1$.
- The following construction: let $B = \{(z,t) \in \mathbb{C} \times \mathbb{R} : \ |z|^2 + t^2 \le 1\}$, a 3–ball. Let $D^\pm$ be the upper (respectively lower) hemisphere of ∂B, as in Figure 6.11.

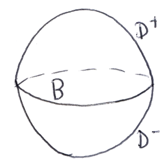

Figure 6.11: $D^\pm$ are the upper and lower hemispheres of ∂B.

Fix $\alpha = \exp(2\pi i/p)$ and glue D^- to D^+ by $\varphi \colon D^- \longrightarrow D^+$, where $\varphi(z,t) = (\alpha^q z, -t)$. See Figure 6.12.

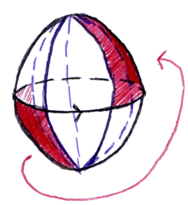

Figure 6.12: The lower hemisphere is glued to the upper by a $2\pi \cdot q/p$ twist.

Notice that, as Figure 6.12 indicates, there is a nice triangulation of B by a collection of p tetrahedra, all sharing the z–axis as an edge. Notice also that a neighborhood of the midpoint of any edge is a half-ball

$$B^3_+ \cong \{(x,y,z) : \ z \ge 0, \ x^2 + y^2 + z^2 \le 1\}$$

and p copies of these are glued, each to the next. So "geometrically", an edge has $p\pi$ dihedral angle which is $(p-2)\pi$ too much. So we consider a lens with dihedral angle $2\pi/p$ at the equator, as in Figure 6.13.

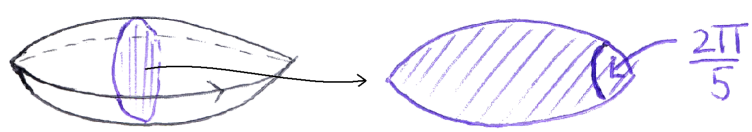

Figure 6.13: A lens with dihedral angle $2\pi/p$ at the equator (here, $p = 5$).

Now we can glue and get the right amount of dihedral angle. More precisely, the lens should live in S^3 and be cut out by great hemispheres, each meeting the next at angle $2\pi/p$. In Figure 6.14, you can see the lenses for $p = 10$. Glue pairs of these together to get lenses for $p = 5$.

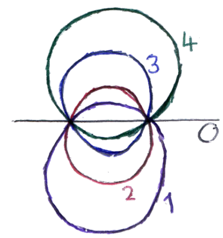

Figure 6.14: 10 copies of the lens tile S^3.

Exercise 6.2.1. Check that the three definitions agree.

6.3 Lens Spaces and Tori

Recall that we defined the meridian and longitude μ, λ for the torus $T = V \cap W \subset S^3$. See Figure 6.15.

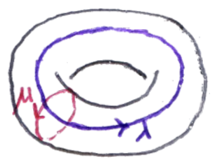

Figure 6.15: The torus T with meridian μ and longitude λ. Note that the orientation of μ, that of λ, and the outward normal to V, in that order, obey the right-hand rule.

Definition 6.3.1. If $K = s\mu + r\lambda$ then the *slope* of K is r/s.

Let $K = s\mu + r\lambda \in \pi_1(T)$, a simple closed curve. In Figure 6.16 for example, $K = 3\mu + 2\lambda$ has slope $2/3$ in T.

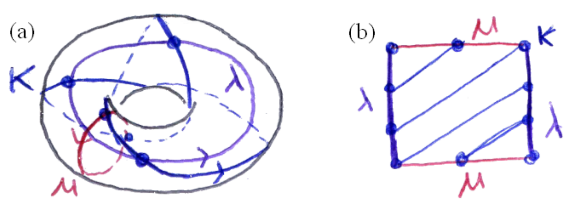

Figure 6.16: The right handed trefoil knot K has slope $2/3$. (a) K as seen in the torus T, and (b) K as seen in $\mathbb{R}^2/\mathbb{Z}^2 \cong T$.

Notation. For $\alpha, \beta \in \pi_1(T)$ we define $\alpha \cdot \beta$ to be the signed intersection number. So

$$\begin{array}{cc} \mu \cdot \mu = 0 & \mu \cdot \lambda = +1 \\ \lambda \cdot \mu = -1 & \lambda \cdot \lambda = 0 \end{array}$$

and thus $\mu \cdot K = r$ and $K \cdot \lambda = s$.

Definition 6.3.2. Suppose $r, s \in \mathbb{Z}$ are coprime, with $|r|, |s| > 1$. We call $K = r\lambda + s\mu \subset T \subset S^3$ the (r, s)*–torus knot*. Then we define $X_K := S^3 - n(K)$, the *knot exterior*. Moreover, we define $V_K := V - n(K)$, $W_K := W - n(K)$ and $A = T_K = T - n(K)$.

In Figure 6.17, z is the core curve of $A = V_K \cap W_K$.

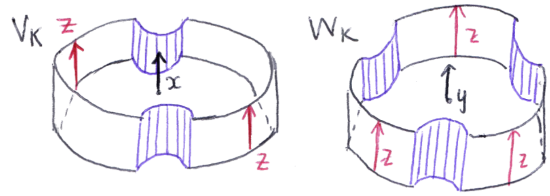

Figure 6.17: The cross-sections of V_K and W_K. The loop z is the core curve of $A = V_K \cap W_K$, and the loops x and y are the generators of $\pi_1(V_k)$ and $\pi_1(W_k)$ respectively.

Recall that the inclusions $i : A \hookrightarrow V_K$ and $j : A \hookrightarrow W_K$ induce maps i_* and j_* giving the following diagram:

$$\begin{tikzcd} & \pi_1(A) = \langle z \rangle \arrow[dl, hook, "i_*"'] \arrow[dr, hook, "j_*"] & \\ \pi_1(V_k) = \langle x \rangle & & \pi_1(W_k) = \langle y \rangle \end{tikzcd}$$

Exercise 6.3.1. Show that $i_*(z) = x^r$ and $j_*(z) = y^s$ hence i_* and j_* are injective, where x and y are the loops shown in Figure 6.17.

By Seifert-van Kampen, assuming that $r, s \neq 0$, we get the following push-out where the lower maps are again inclusions:

$$\begin{tikzcd} & \pi_1(A) = \langle z \rangle \arrow[dl, hook, "i_*"'] \arrow[dr, hook, "j_*"] & \\ \pi_1(V_k) = \langle x \rangle \arrow[dr, hook] & & \pi_1(W_k) = \langle y \rangle \arrow[dl, hook] \\ & \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} \cong \langle x, y \,|\, x^r = y^s \rangle =: \Gamma_{r,s} & \end{tikzcd}$$

Via group theory, one can show that $\Gamma_{r,s} \cong \Gamma_{p,q}$ if and only if $\{|p|, |q|\} = \{|r|, |s|\}$.

Aside. Note that

- $SO(2) \cong S^1$,
- $SO(3) \cong \mathbb{P}^3$ and
- $SL(2, \mathbb{R}) \cong \mathrm{int}(\mathbb{D} \times S^1) \cong \mathbb{R}^2 \times S^1$, the latter is not an isomorphism of groups.

Remark. We now have the following remarkable fact. Let $K \subset S^3$ be the trefoil knot and define $Y_K = S^3 - K$ be the *knot complement*, an open three-manifold. Then Y_K is homeomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$.

For the following we assume that K is not the unknot, i.e. $|p|, |q| \geq 2$.

Theorem 6.3.1. *Suppose $K = K_{p,q}$ is the (p, q)–torus knot, then the annulus $A = T - n(K)$ is the unique* essential *annulus in X_K, up to isotopy.*

We will prove this later in the course.

Corollary 6.3.1. *Define $X_{p,q} = X_K$, where $K = K_{p,q}$. Then $X_{p,q} \cong X_{r,s}$ if and only if $\{|p|, |q|\} = \{|r|, |s|\}$.*

6.4 Non-Uniqueness of Torus Decompositions

Now we closely follow Hatcher. Let $V_i \cong \mathbb{D} \times S^1$, $i = 1, 2, 3, 4$. Let $A_i \subset \partial V_i$ be an embedded annulus and suppose A_i winds q_i times about V_i with $q_i \geq 2$; for examples see Figure 6.18.

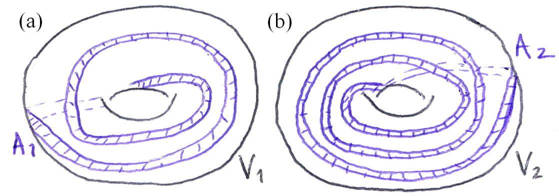

Figure 6.18: Two examples of a winding annulus; in (a) $q_1 = 2$ and in (b) $q_2 = 3$.

Another way to define q_i is the following: Let α_i be a core curve of A_i and define q_i via $q_i = |\alpha_i \cdot \partial D_i|$. Let $A'_i = \overline{\partial V_i - A_i}$ and pick $\varphi\colon A'_i \longrightarrow A_{i+1}$ where we take the indices modulo 4. Let $M = \sqcup V_i/\varphi_i$; see Figure 6.19(a). Let B_i denote the image of A_i in M. Now we define $M_i = V_i \cup_{\varphi_i} V_{i+1}$. Let $T_1 = B_1 \cup B_3$ and $T_2 = B_2 \cup B_4$. Thus $M = M_1 \cup_{T_1} M_3 = M_2 \cup_{T_2} M_4$.

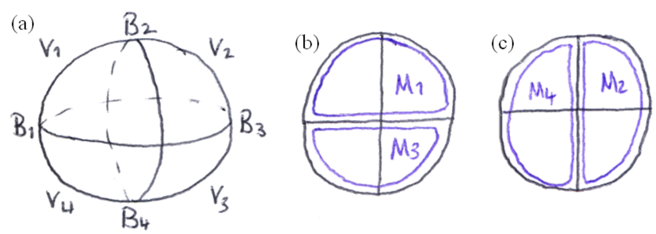

Figure 6.19: (a) A schematic of M. B_i is the image of A_i in M. (b) and (c) are schematics of two different torus decompositions.

Finally, we claim that $B_1 \cup B_3$ and $B_2 \cup B_4$ are incompressible tori in M. If we now choose the q_i to all be distinct and coprime then, for $i = 1, 2, 3, 4$, then manifold M_i is a torus knot exterior. So we have, for these choices of q_i, that M_1 is not homeomorphic to M_2 or M_4 and M_3 is not homeomorphic to M_2 or M_4. Thus the torus decompositions T_1 and T_2 are different; see Figures 6.19(b) and (c).

Remark (17.2)**.** This requires the following facts. If $X_{p,q} = S^3 - n(K_{p,q})$, then

- $\partial X_{p,q}$ is incompressible,
- $X_{p,q}$ is atoroidal and
- Theorem 6.3.1.

We will prove these facts later. To do so, and so to understand the non-uniqueness of torus decompositions, we must first understand *Seifert fibred spaces.*

6.5 Seifert Fibred Spaces

Fibre $D \times I$ by intervals of the form $\{x\} \times I$. We call $\{0\} \times I$ the *central fibre*. Let $\varphi\colon D \times \{1\} \longrightarrow D \times \{0\}$ be a $2\pi q/p$ rotation, $\varphi(z, 1) = (\alpha^q z, 0)$ where as usual p and q are coprime. Define $V_{p,q} = D \times I/\varphi$, the (p, q)*–fibred solid torus.* Notice that $\{0\} \times I$ now gives a circle as does the set of fibres $\{\alpha^k \cdot (z \times I) : \ \alpha^p = 1\}$. Note that $V_{p,q}$ is given a *fibring*, i.e. a decomposition into circles.

Definition 6.5.1. A *Seifert fibring* of a three-manifold M is a partition $\mathcal{F}$ of M into circles (the *fibres*) such that every fibre $\lambda \in \mathcal{F}$ has arbitrary small regular neighbourhoods $N(\lambda)$ all homeomorphic to $V_{p,q}$ for some fixed p, q. Here the homeomorphisms are all fibre-preserving.

Remark. The integers p, q only depend on λ.

Definition 6.5.2. We call p the *multiplicity* of λ.

Note that the space $V_{p,q}$ is Seifert fibred itself and the central fibre has multiplicity p while all other fibres have multiplicity equal to 1.

Definition 6.5.3. If λ has multiplicity greater than 1, then we call λ a *singular* fibre. All other fibres are called *generic*. See Figure 6.20.

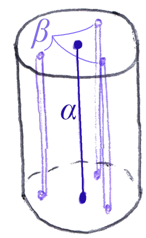

Figure 6.20: Inside of $V_{3,1}$ the central fibre α is singular (with multiplicity three) while all others, for example β, are generic.

Exercise 6.5.1. If M is compact then there are only finitely many singular fibres, all contained in the interior of M.

Exercise 6.5.2. Show that $L_{p,q}$ is a Seifert fibred space with at most two singular fibres. Compute their multiplicities.

Exercise 6.5.3. Let $K = K_{p,q}$ be the (p, q)–torus knot. Show that X_K is a Seifert fibered space. Find the singular fibres and their multiplicities.

Example 6.5.1. Let $M = V_1 \cup V_2 \cup V_3 \cup V_4$ as in the last lecture. Then M is a Seifert fibred space with 4 singular fibres.

Definition 6.5.4. Suppose $(M, \mathcal{F})$ is a Seifert fibred space. Let $B = M/S^1$ be the *base orbifold*; that is, the quotient of M sending fibres to points.

Example 6.5.2. Suppose $M = V_{p,q}$. The quotient M/S^1 is a disk D with a cone point at the centre. The angle at the cone point is $2\pi/p$; see Figure 6.21.

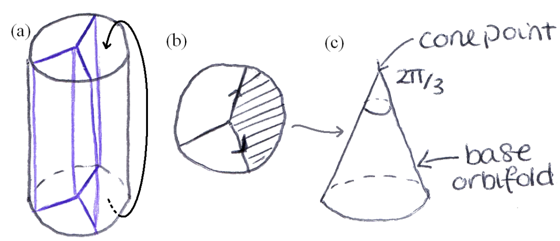

Figure 6.21: (a) The solid torus $V = V_{3,1}$. (b) A meridian disk for V. (c) The quotient V/S^1 is a cone with angle $2\pi/3$ at the cone point.

Exercise 6.5.4. In exercises 6.5.1 and 6.5.2, identify the base orbifolds.

Example 6.5.3. Notice that if $\rho\colon T \longrightarrow F$ is an S^1–bundle then $T/S^1 \cong F$.

Theorem 6.5.1 (1.9 in Hatcher)**.** *Let M be compact, irreducible and orientable. There exists a torus system $T \subset M$ such that all components of $M - n(T)$ are either atoroidal or Seifert fibred spaces. Furthermore any minimal such system is unique up to isotopy.*

Chapter 7

Essential Surfaces

7.1 Bigons

Suppose $F \subset M$ is properly embedded, and M is compact, irreducible and orientable. Recall that $(D, \partial D) \subset (M, F)$ is a *surgery disk* for F if $D \cap F = \partial D$. D is *trivial* if ∂D bounds a disk in F. If D is not trivial, then it is a *compressing disk* for F.

Definition 7.1.1. A disk D with $\partial D = \alpha \cup \beta$ such that α and β are connected and $\alpha \cap \beta = \partial\alpha = \partial\beta$ is a *bigon*; see Figure 7.1.

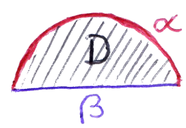

Figure 7.1: A bigon D.

Definition 7.1.2. Say $D \subset M$ is a *surgery bigon* for $F \subset M$ if D is a bigon, $D \cap F = \alpha$ and $D \cap \partial M = \beta$. Say that D is *trivial* if there is a bigon $D' \subset F$ so that $\partial D' = \alpha' \cup \beta'$, $\alpha = \alpha'$ and $D' \cap \partial M = \beta'$, as in Figure 7.2. If D is not trivial, call it a *boundary compressing bigon*, or simply a *boundary compression*.

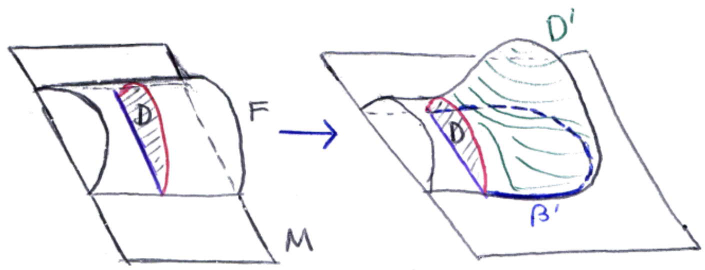

Figure 7.2: D is a trivial surgery bigon. Note that D' is not properly embedded in M but contained entirely in F.

Recall that a two-sided simple closed curve $\alpha \subset F^2$ is *essential* if α does not bound a disk on either side (Figure 7.3(a)). A sphere $S \subset M^3$ is *essential* if it does not bound a three-ball on either side (Figure 7.3(b)). If M is irreducible then a disk $(D, \partial D) \subset (M, \partial M)$ is *essential* if ∂D is essential in ∂M (Figure 7.3(c)).

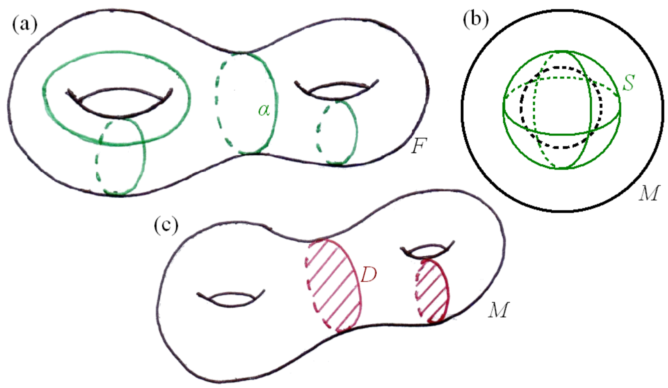

Figure 7.3: (a) All the green curves here are essential. (b) Here S is essential in M. (c) These disks are essential in M.

Definition 7.1.3. Suppose that $S \subset M$ is a properly embedded, connected, two-sided surface that is not a disk or a sphere. We say S is *essential* if it is incompressible and boundary incompressible.

Definition 7.1.4. If all surgery disks are trivial, we call F *incompressible*; similarly, if all surgery bigons are trivial, call F *boundary incompressible.*

Exercise 7.1.1. Suppose $S \subset M$ is an essential surface. Show that $\partial S \subset \partial M$ is essential.

Proposition 7.1.1. *If $S \subset D^2 \times S^1$ is essential then S is isotopic to $D^2 \times \{z\}$ for some $z \in S^1$.*

Proof. Let $\mu_z = \partial D^2 \times \{z\}$. We call μ_z the meridian curves. Abusing notation, let $D = D^2 \times \{1\}$. Then by Exercise 7.1.1, ∂S is essential so we may isotope components of ∂S so that all are either equal to meridian curves, or are transverse to all meridian curves, as in Figures 7.4(a) and (b).

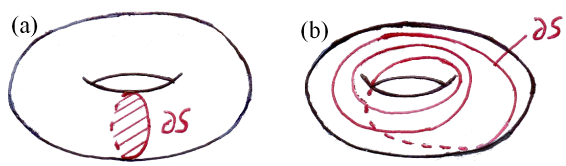

Figure 7.4: (a) Here the component of ∂S is meridian curve. (b) Here ∂S is transverse to all meridian curves.

Thus, we may assume that ∂S is transverse to μ_1, and via isotopy relative to ∂M, we may assume that S is transverse to D. Then $S \cap D$ is a collection of arcs and loops, as in Figure 7.5.

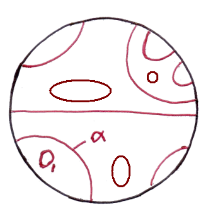

Figure 7.5: $S \cap D$ is a collection of arcs and loops.

We proceed as follows:

Step 1: First suppose $\alpha \subset D \cap S$ is an innermost loop, so α bounds a disk $D_1 \subset D$ such that $D_1 \cap S = \partial D_1$. So D_1 is a surgery disk for S and thus, as S is incompressible, there is a disk $E \subset S$ with $\partial E = \partial D_1 = \alpha$, as in Figure 7.6(a). So $D_1 \cup E$ is a 2–sphere. As $D \times S^1$ is irreducible, $D_1 \cup E$ bounds a 3–ball B, so there is an isotopy supported in $n(B)$ moving E past D_1; see Figure 7.6(b).

This gives an isotopy of S, reducing $|S \cap D|$. So without loss of generality, we may assume that $D \cap S$ consists only of arcs.

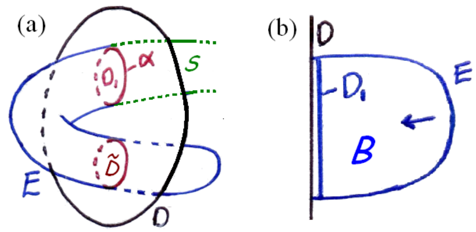

Figure 7.6: (a) $E \cup D_1$ bounds a 3–ball B, so (b) we may isotope E through $n(B)$ past D_1 to reduce $|S \cap D|$.

Step 2: Now suppose $\alpha \subset D \cap S$ is an outermost arc. So α cuts off from D a surgery bigon D_1. Since S is boundary incompressible, α cuts off a bigon E from S. Let $\gamma = E \cap \partial(D \times S^1)$ and $\beta = D_1 \cap \partial(D \times S^1)$. See Figure 7.7.

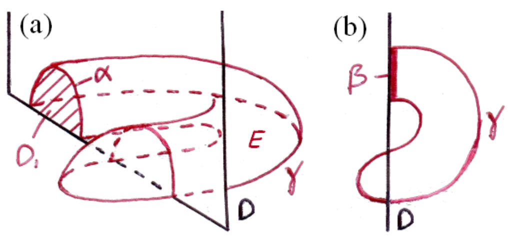

Figure 7.7: (a) α cuts a surgery bigon D_1 from D and E from S. (b) A plan view of (a).

Notice that $D_1 \cup E$ is a disk, with $D_1 \cap E = \alpha$. Thus $D_1 \cup E$ lifts to $\widetilde{D \times S^1} \cong D \times \mathbb{R}$, as in Figure 7.8.

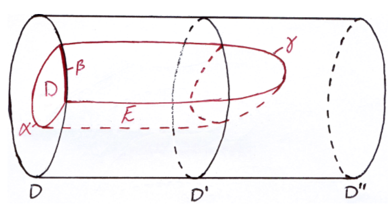

Figure 7.8: $D_1 \cup E$ lifts to $\widetilde{D \times S^1} \cong D \times \mathbb{R}$.

Let $h : D \times \mathbb{R} \to \mathbb{R}$ be projection to the second factor, and notice that:

$$h(\partial_+\gamma) = h(\partial_-\gamma)$$

as $\partial_\pm\gamma \in \partial D$. So by Rolle's theorem, $(h|\gamma)'$ has a zero, so γ is not transverse to μ_z for some $z \in S^1$, giving a contradiction. Thus without loss of generality, we may assume $S \cap D = \varnothing$.

Step 3: Next, define $B = (D \times S^1) - n(D)$. This is a 3–ball, and $S \subset B$. Pick any component $\delta \subset \partial S$. So δ divides ∂B into disks C and C'. So push C, say, into B, keeping ∂C inside of S. This gives a disk in the interior of B. See Figure 7.9. If $C \cap S \neq \partial C$, then we may isotope S, as in Step 1, to reduce $|S \cap C|$. So C gives a surgery disk for S. Thus S is a disk.

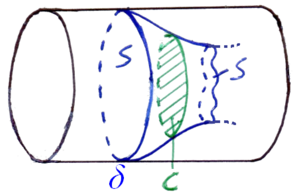

Figure 7.9: Push C into B (keeping ∂C inside of S) to get a disk in the interior of B.

Finally, Alexander's theorem implies that S is isotopic to $D \times \{z\}$ for some $z \in S^1$, fixing δ pointwise. □

Note. All surgery disks for S^2 are trivial, and all surgery disks and bigons for $\mathbb{D}^2$ are trivial, hence they are excluded from the statement of Proposition 7.1.1.

Definition 7.1.5. Suppose $(\alpha, \partial\alpha) \subset (A^2, \partial A^2)$ is an arc in an annulus. It is *trivial* if it cuts a bigon off of A, and *essential* otherwise. See Figure 7.10.

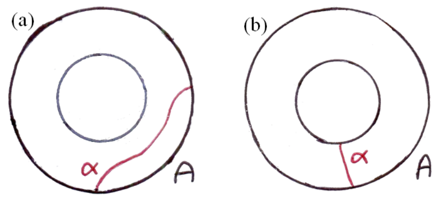

Figure 7.10: (a) A trivial arc. (b) An essential arc.

Exercise 7.1.2. Suppose $F \subset M$ is two-sided and incompressible. Suppose $D \subset M$ is a surgery bigon for F and suppose F_D is the result of surgery. Show that $F_D \subset M$ is incompressible.

Exercise 7.1.3. Deduce from the above that if $\rho : T \to F$ is an I–bundle then $\partial_h T$ is boundary incompressible.

Lemma 7.1.2 (1.10 in Hatcher)**.** *Suppose that $S \subset M$ is a connected, two-sided, incompressible surface, and M is irreducible. Suppose S admits a boundary compressing bigon D with $\partial D = \alpha \cup \beta$, $\alpha = D \cap S$, $\beta = D \cap \partial M$ and β is contained in a torus component $T \subset \partial M$. Then S is a boundary parallel annulus.*

Proof. By Exercise 7.1.1, $\partial S \cap T$ is essential in T. Let $A = T - n(\partial S)$, so A is a collection of annuli. So $\beta \subset A$ is either trivial or essential, as in Figure 7.11(a).

Case 1: Suppose that $\beta \subset A$ is trivial. So β cuts a bigon E off of A. Then $D \cup E$ is a disk. Isotope $D \cup E$, keeping $\partial(D \cup E)$ in S, to get a surgery disk for S; see Figure 7.11(b).

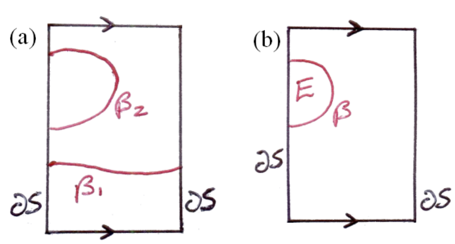

Figure 7.11: (a) β_1 is essential while β_2 is trivial. (b) Trivial arcs define a surgery bigon for S.

Since S is incompressible, $D \cup E$ cuts a disk D' out of S, and hence D was a trivial surgery bigon, as in Figure 7.12.

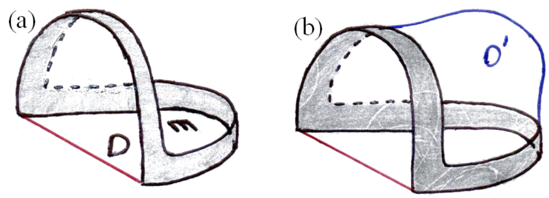

Figure 7.12: $D \cup E$ cuts a disk D' from S and so D is trivial.

Case 2: Suppose β is essential in A. If ∂B is contained in a single component of ∂S, then S is one-sided, giving a contradiction. To see this, we can orient β and ∂S so that both intersections have positive sign, as in Figure 7.13.

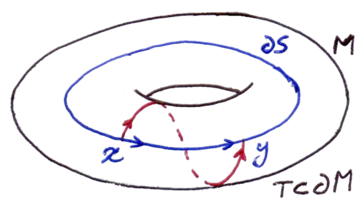

Figure 7.13: We can orient β and ∂S so that both intersections have positive sign.

Then following α we find that S is one-sided, as in Figure 7.14.

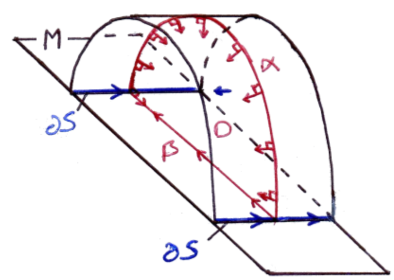

Figure 7.14: Carrying the orientation along α gives a different orientation to carrying along ∂S, a contradiction.

So we have that β connects distinct components of ∂S, as in Figure 7.15.

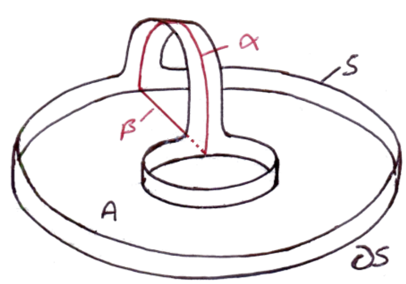

Figure 7.15: β connects distinct components of ∂S.

Boundary compress S along D to get S_D. Note that S_D is incompressible, by Exercise 7.1.2, and that S_D has a trivial boundary component, so S_D is a disk. To see this, say ∂S_D bounds E in T. So isotope E into E' in M, keeping ∂E in S_D, as in Figure 7.16.

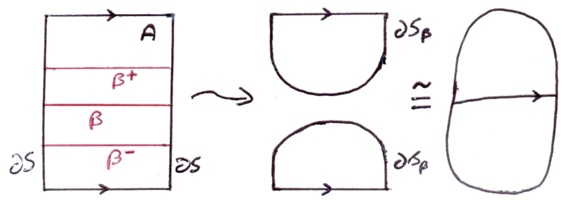

Figure 7.16: Cutting along β gives two components of ∂S_β, and the identification gives a trivial curve in ∂S_D.

Since S_D is incompressible, $\partial E'$ must cut a disk out of S_D, so S_D is a disk. Since M is irreducible, S_D is boundary parallel; in fact it is parallel to the original E.

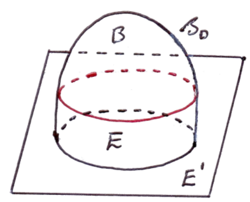

Figure 7.17: S_D is boundary parallel.

So S_D cuts a 3–ball B out of M. Letting $V = B \cup N(D)$, this is a solid torus, giving a parallelism of S with the annulus A, as in Figure 7.18.

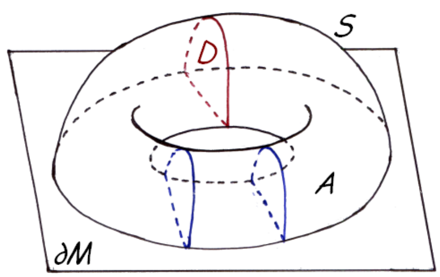

Figure 7.18: S is boundary parallel to the annulus A.

This completes the proof. □

7.2 Vertical and Horizontal Surfaces

Definition 7.2.1. Suppose that $(M, \mathcal{F})$ is Seifert fibred. Then we say that a properly embedded surface $S \subset M$ is *vertical* if S is a union of fibres, and it is *horizontal* if S is transverse to the fibres. We make the same definitions for $S \subset T$ for an I–bundle $\rho : T \to F$.

Exercise 7.2.1. All essential surfaces $S \subset T$, where $\rho : T \to F$ is an I–bundle, are isotopic to either vertical or horizontal surfaces.

Lemma 7.2.1. *[1.11 in Hatcher] Suppose that $(M, \mathcal{F})$ is compact, connected and irreducible. Supppose $S \subset M$ is essential. Then after a proper isotopy, S is either vertical or horizontal.*

Proof. Let $Z := \{\alpha_i\}_{i=1}^k$ be the set of singular fibres of $\mathcal{F}$; if M has no singular fibres, and $\partial M = \varnothing$, then let $\{\alpha_1\}$ be a single generic fibre. Let $M_0 = M - n(Z)$. Let $B = M/S^1$ and let $B_0 = M_0/S^1$. Note that $\partial B_0 \neq \varnothing$. In fact B_0 is B with neighbourhoods of cone points removed, as in Figure 7.19.

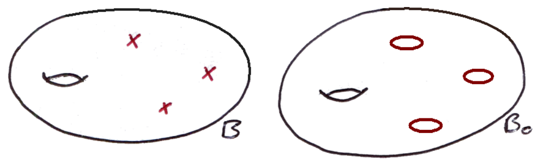

Figure 7.19: B_0 is B with neighbourhoods of cone points removed.

Example 7.2.1. If $M = V_{p,q}$ then Z is just the central fibre. Then $M_0 = \mathbb{A}^2 \times S^1$ and $B_0 = \mathbb{A}^2$; See Figure 7.20.

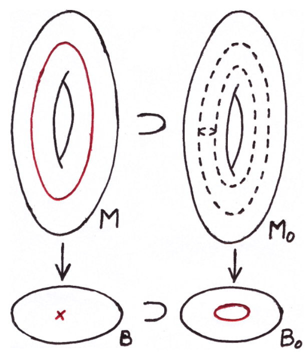

Figure 7.20: $M_0 = \mathbb{A}^2 \times S^1$ and $B_0 = \mathbb{A}^2$.

Choose a system of arcs in B_0 cutting B_0 into a disk, i.e. as in Figure 7.21.

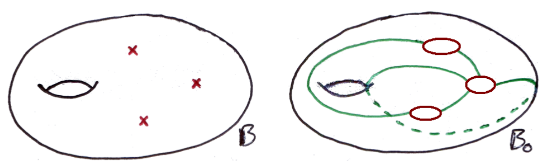

Figure 7.21: We may choose a system of arcs cutting B_0 into a disk.

Let $A \subset M_0$ be the vertical annuli above this system of arcs. So $M_0 - n(A) =: M_1$ is a solid torus, fibred by $\mathcal{F}|M_1$, with all fibres generic. Given an essential surface S, all components of ∂S are essential in ∂M.

(i) We may isotope them to all be vertical or horizontal with respect to the fibring $\mathcal{F}|\partial M$.

(ii) Isotope S (relative to ∂S) so that S meets Z transversely, and so meets $n(Z)$ in horizontal disks. Define $S_0 = S \cap M_0$, and make S_0 intersect A transversely. Consider the arcs and loops of $S_0 \cap A$, as in Figure 7.22.

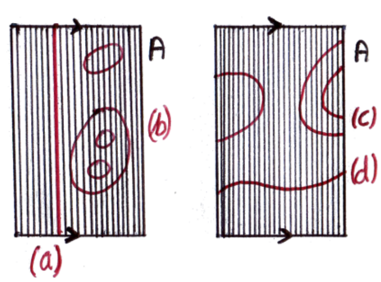

Figure 7.22: (a) An essential loop. (b) Trivial loops. (c) Trivial arcs. (d) An essential arc.

(iii) If there is a trivial loop, then there is an innermost such. Now, using incompressibility of S and irreducibility of M, there is an isotopy of S reducing $|S \cap A|$ as usual. So without loss of generality, there are no trivial loops.

(iv) Suppose $\beta \subset S \cap A$ is an outermost trivial arc and let D be the bigon cut our of A by β. If $\partial\beta \subset \partial M$ then D is a surgery bigon for S, but as in Proposition 7.1.1, ∂S is either contained in or transverse to $\mathcal{F}|\partial M$, giving a contradiction. To see this, since S is boundary incompressible, there is a bigon E contained in S, as in Figure 7.23.

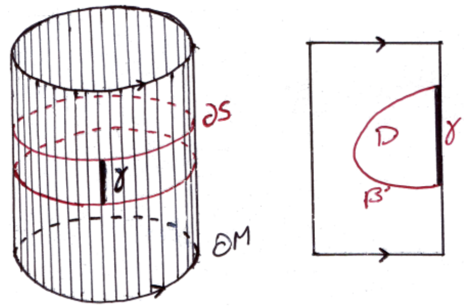

Figure 7.23: γ is parallel to the fibres.

So letting $\partial E = \beta \cup \gamma'$, we find that γ' is not transverse to $\mathcal{F}|\partial M$. On the other hand, if $\partial\beta \subset \partial M_0 - \partial M$, then a baseball move across D reduces $|S \cap (Z)|$ by two. Now without loss of generality, every component of $S \cap A$ is either horizontal or vertical.

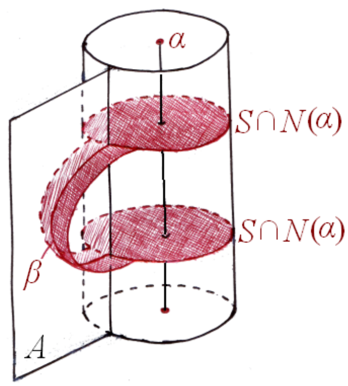

Figure 7.24: A baseball move across α reduces $|S \cap Z|$ by 2.

(v) Define $S_1 = S_0 \cap M = S_0 - n(A)$. So $\partial S_1 \subset M_1$ is completely horizontal or completely vertical. We may assume that S_1 is incompressible in M_1. Thus S_1 is either a collection of horizontal meridian disks, or a collection of boundary parallel annuli. If S_1 contains an annulus with slope that of the meridian, then S_1 is compressible. If S_1 contains an annulus $B \subset S_1$ with ∂B horizontal, then we see a surgery bigon with vertical boundary. So do a baseball move and return to case (iv).

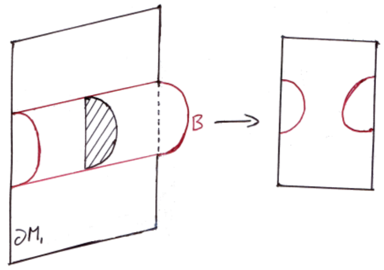

Figure 7.25: If S_1 contains an annulus B with ∂B horizontal, we may do a baseball move and reduce to case (iv).

So S_1 is now a collection of horizontal meridian disks, or a collection of boundary parallel vertical annuli. It follows that S_0, and so S, is either horizontal or vertical. $\square$

Remark. Vertical surfaces are easy to classify. They are orientable or not, and the base is I or S^1.

Base Orbifold	I	S^1	
	A^2	T^2	orientable
	M^2	K^2	non-orientable

Notation. Suppose F is not orientable. Let $F \tilde{\times} I$ denote the orientation I–bundle over F. Likewise define $F \tilde{\times} S^1$.

Exercise 7.2.1. Show that $P^2 \tilde{\times} I$ is homeomorphic to $P^3 - \text{int}(B^3)$.

7.3 Orbifolds and Covers

Definition 7.3.1. We say that $B = (S, Z)$ is an 2–*orbifold* if S is a surface and $Z \subset \text{int}(S)$ is a finite set such that for every $z \in Z$ we have an *order* $p_z \in \mathbb{Z}_+$. We call Z the *singular set*. A point $z \in Z$ is a *cone point* if $p_z > 1$.

Example 7.3.1. A surface is an orbifold with $Z = \varnothing$.

Example 7.3.2. The square pillow case, $S^2(2, 2, 2, 2)$, shown in Figure 7.26, is an orbifold.

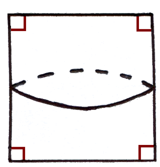

Figure 7.26: A picture of the square pillow case $S^2(2, 2, 2, 2)$.

Definition 7.3.2. If S is a surface with a triangulation T then we define the *Euler characteristic* of S to be $\chi(S) = V - E + F$ where V denotes the number of vertices, E the number of edges and F the number of triangles (faces).

Exercise 7.3.1. Show that χ stays unchanged under the Pachner moves. Figure 7.27 shows the Pachner moves. Since any two triangulations of a fixed closed surface are related by Pachner moves, the Euler characteristic is independent of the choice of T.

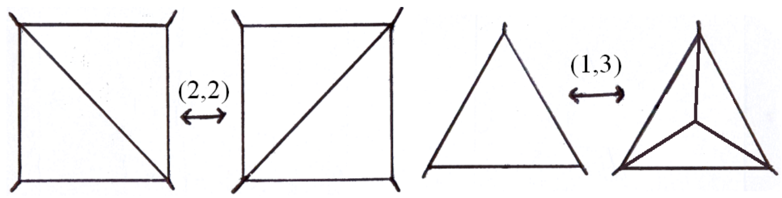

Figure 7.27: The Pacher moves.

Example 7.3.3. You can see by the triangulation shown in Figure 7.28(a) that $\chi(S^2) = 4-6+4 = 2$. Similarly, Figure 7.28(b) shows that $\chi(T^2) = 1-3+2 = 0$.

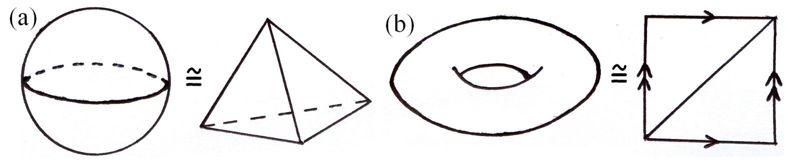

Figure 7.28: (a) A triangulation of a 2–sphere. (b) A triangulation of the 2–dimensional torus.

Definition 7.3.3. We define the *Euler characteristic* of an orbifold via

$$\chi_{\mathrm{orb}}(B) = \chi(S) + \sum_{z \in Z} \left(\frac{1}{p_z} - 1 \right).$$

Example 7.3.4. $\chi_{\mathrm{orb}}(S^2(2,2,2,2)) = 2 + 4(1/2 - 1) = 0$.

Exercise 7.3.2. List all 2–orbifolds B so that $\chi_{\mathrm{orb}}(B) = 0$.

Exercise 7.3.3. What can you say about B so that $\chi_{\mathrm{orb}}(B) > 0$?

Example 7.3.5. The map from $D \subset \mathbb{C} \to D$ which sends z to z^n is an orbifold map of order n. In Figure 7.29, $n = 3$.

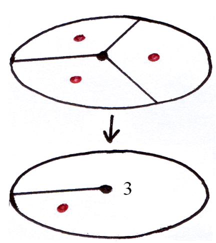

Figure 7.29: The map $z \mapsto z^3$ from $D \subset \mathbb{C}$ to itself is a three-fold cover.

Definition 7.3.4. If C, B are 2–orbifolds then $\varphi\colon C \to B$ is a *cover* if

1. $\varphi^{-1}(Z_B) = Z_C$,
2. $\varphi|(C - Z_C)\colon C - Z_C \to B - Z_B$ is a d–fold cover and
3. for every point $z \in Z_B$, we have $d/p_z = \sum\limits_{y \in \varphi^{-1}(z)} 1/p_y$.

Note that φ restricted to any regular neighbourhood of a point $z \in Z_C$ is modelled on the example $z \longmapsto z^n$.

Example 7.3.6. The quotient of T^2 via the $180°$ rotation shown in Figure 7.30 is a degree two orbifold cover.

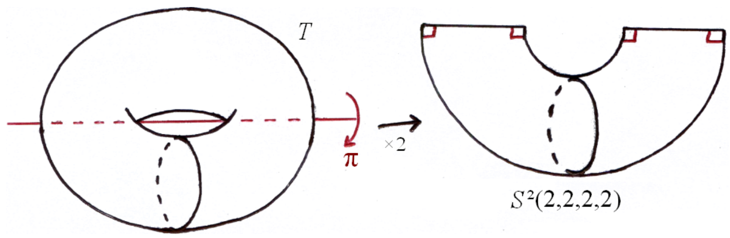

Figure 7.30: The quotient map of the 2–dimensional torus via the $180°$ rotation.

Exercise 7.3.4. Show that if $\varphi\colon C \to B$ is a d–fold orbifold cover then $\chi_{\text{orb}}(C) = d \cdot \chi_{\text{orb}}(B)$. As warm-up, show that if $\varphi\colon T \to S$ is a d–fold cover of surfaces then $\chi(T) = d \cdot \chi(S)$.

Exercise 7.3.5. List all 2–fold covers of $S^2(2, 2, 2, 2)$.

The following question is known as the *Hurwitz problem* and still open in general: Given B, C such that $\chi_{\text{orb}}(C)/\chi_{\text{orb}}(B) \in \{2, 3, 4, \ldots\}$ does there exists a d–fold cover?

Example 7.3.7. For $n \geq 2$, $S^2(n)$ is a *bad orbifold*, meaning it is not covered by a surface. Hence $S^2(n)$ is not covered by S^2. You can also see this because $2/(2 + (1/n - 1)) \notin \mathbb{N}$.

7.4 Back to Horizontal Surfaces

We now return to our original topic, horizontal surfaces. Suppose that $S \subset (M, \mathcal{F})$ is horizontal. As in the proof of Lemma 21.4, we may form $M \supset M_0 \supset M_1$ and $S \supset S_0 \supset S_1$. Let λ be any generic fibre and $d = |S \cap \lambda|$, so S_1 is a collection of d horizontal disks. Recall that Z is the set of all singular fibres. Thus $S \cap (N(Z))$ is also a collection of disks. Then S is formed by gluing horizontal disks along horizontal loops in $\partial N(Z)$ and horizontal arcs in A. Thus the quotient $\rho\colon M \to M/S^1 = B$ restricts to S to give a d–fold cover $\rho\colon S \to B$. So

$$\chi(S) = d \cdot \chi_{\text{orb}}(B) = d \cdot \left(\chi(B) + \sum_{z \in Z} \left(\frac{1}{p_z} - 1 \right) \right).$$

Proof. See Hatcher. □

To answer the question of a student, we will expand the definition of a *boundary compression*.

Definition 7.4.1. Suppose $S \subset \partial M$ is a subsurface. Then we say S is *boundary compressible* if there is a bigon D with $\partial D = \alpha \cup \beta$ so that $D \cap S = \alpha$, $D \cap \overline{\partial M - S} = \beta$ and α does not cut a bigon out of S. Say that S is *boundary incompressible* if no such bigon exists.

Now we continue our discussion of horizontal surfaces. Suppose that $S \subset (M, \mathcal{F})$ is two-sided, horizontal and connected. Then we get the following corollary of Proposition 7.2.1 (1.11 in Hatcher).

Corollary 7.4.1. *The manifold $M - n(S)$ is an I–bundle.*

Proof sketch. Recall that S_1 was a collection of horizontal disks in $M_1 \cong D \times S^1$. So $n(S_1)$ cuts M_1 into cylinders foliated by intervals. The vertical sides of these solid cylinders glue to give the desired I–bundle. □

Let $\rho\colon M - n(S) \longrightarrow F$ be the I–bundle map. Then there are two cases.

(Case 1) The manifold $M - n(S)$ is connected. So $M - n(S) \cong S \times I$ and thus $\partial_h(M - n(S)) = S \sqcup S$ and so $F \cong S$ and we find that M is an S–bundle over S^1. See Figure 7.31.

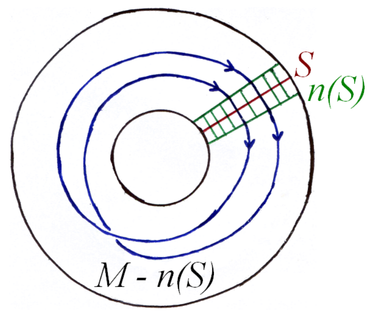

Figure 7.31: A picture of $M - n(S)$ as an S–bundle over S^1. The blue curve represents a generic fibre.

So the I–fibres in $N(S)$ and in $M - n(S)$ glue to give the Seifert fibring, $\mathcal{F}$. I.e., there is a *monodromy* (a homeomorphism $\varphi\colon S \longrightarrow S$) such that $M \cong S \times I/(x, 1) \sim (\varphi(x), 0) =: M_\varphi$ and finally $S/\varphi \cong B$. The monodromy is *periodic* of period $d = |S \cap \lambda|$, i.e. $\varphi^d = \mathrm{Id}_S$. See Figure 7.32.

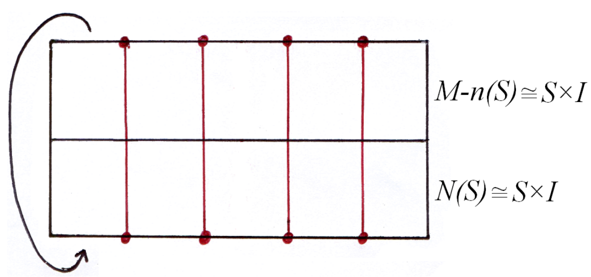

Figure 7.32: $M = (M - n(S)) \cup N(S) \cong M_\varphi$. Here φ has periodicity 4.

Example 7.4.1. Let φ be the hyperelliptic involution on the 2–torus shown in Figure 7.30. This is periodic.

Example 7.4.2. Glue the cube as shown in Figure 7.33 and note that planes parallel to the xy–plane glue to give tori.

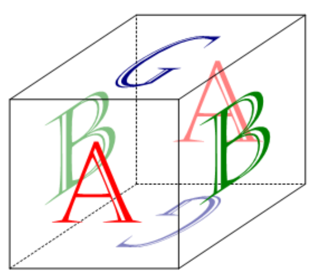

Figure 7.33: A cube with face pairings. The front and back are glued by the identity as are the left and right face. The bottom and top face are glued together by a 180° rotation.

Note that intervals parallel to the z–axis glue to give circles, 4 of length 1 and the rest of length 2.

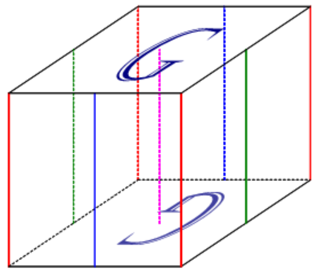

Figure 7.34: A picture of the different circles achieved by gluing intervals parallel to the z–axis. The gluings of the vertical faces are the same as in Figure 7.33 and are omitted.

All of the singular fibres in Figure 7.34 have length one, while all other vertical circles have length two. All other vertical circles have length 2. So $B \cong S^2(2,2,2,2)$ is the base orbifold, double covered by double covered by any horizontal surface, all of which are tori. See Figure 7.35.

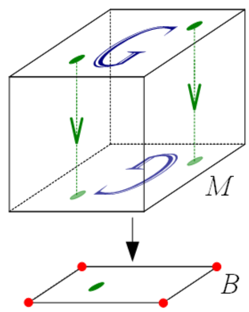

Figure 7.35: The base orbifold is a copy of the square pillow case: $B \cong M/S^1 \cong S^2(2,2,2,2)$, and double covered by T.

(Case 2) If $M - n(S)$ has two components then each is a twisted I–bundle over F and these glue to $N(S) \cong S \times I$ giving a *semibundle* (also called a *fibroid*). See Figure 7.36.

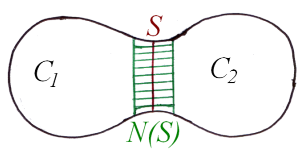

Figure 7.36: A picture of the two twisted I–bundles over F.

So letting T_1 and T_2 be the two I–bundles, we obtain M by gluing T_1 and T_2 to $N(S)$ and find involutions $\tau_i\colon S \longrightarrow S$ such that $T_i = S \times I/(x,0) \sim (\tau_i(x),0)$. Here the homeomorphism $\varphi = \tau_1 \circ \tau_2$ is again *periodic*.

Example 7.4.3. As an exercise, we showed that $P^3 - \mathrm{int}(B^3) = P^2 \tilde{\times} I$. Here $\partial_h T \cong S^2$ and the involution τ is the antipodal map. So if we consider $T_1 \cup_S T_2$ where $T_i \cong P^2 \tilde{\times} I$, we find that $P^3 \# P^3$ is Seifert fibred. Check that $\tau_1 \circ \tau_2 = \tau^2 = \varphi = \mathrm{Id}_S$ and so it is periodic.

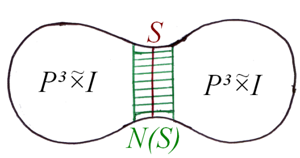

Figure 7.37: A picture of the gluing of $T_1 \cup_S T_2$.

Example 7.4.4. Consider the cube with face pairings given in Figure 7.28. Notice that the intervals parallel to the x–axis also define a Seifert fibring with $B = K^2$, the Klein bottle, and all fibres are generic, as in Figure 7.38(a). The planes $y = 1/4$ and $y = 3/4$ define a 2–torus $S \subset M$ and $M - n(S)$ has two components, both homeomorphic to $K \tilde{\times} I$.

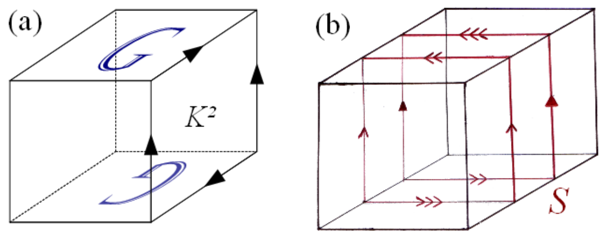

Figure 7.38: (a) Intervals parallel to the x–axis give a fibring with $B = K^2$. (b) Both components of $M - n(S)$ are homeomorphic to $K^2 \tilde{\times} I$.

Exercise 7.4.1. Check that these planes give a 2–torus with the claimed properties. Find the involutions τ_1, τ_2.

Recall that every essential arc in $A^2 \cong S^1 \times I$ is isotopic to $\{\text{pt}\} \times I$, as in Figure 7.39.

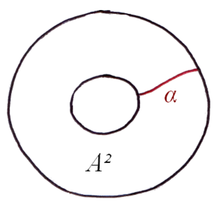

Figure 7.39: An essential arc α in the annulus A^2.

Exercise 7.4.2. Classify up to isotopy the essential arcs and loops in $\#_3 D^2$, the pair of pants.

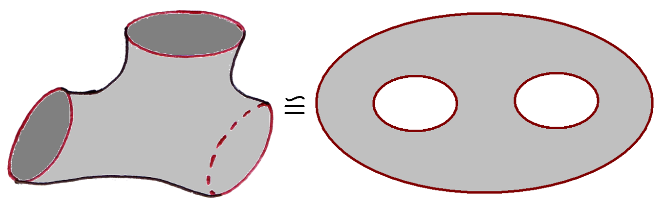

Figure 7.40: Two diagrams of the Pair of Pants.

Recall that if $X = X_K$ where $K = K_{p,q}$ is the (p, q)–torus knot then $B = X/S^1$ is the orbifold $D^2(p, q)$.

Exercise 7.4.3. Classify essential arcs and loops in $D(p, q)$. Deduce that the only essential vertical annulus in X is $A = V_K \cap W_K$. (Care is required if p or q is equal to 2, as then X contains a vertical Mobius band.)

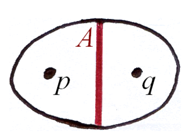

Figure 7.41: A diagram of $D^2(p, q)$. Note that the A here is the projection of the annulus into the orbifold.

Exercise 7.4.4. Use orbifold Euler characteristic to show that any horiontal surface $S \subset X$ has $\chi(X) \leq p+q-pq < 0$ as $p, q \geq 2$ and $p \neq q$. Deduce that X is atoroidal and A is the unique essential annulus in X, up to isotopy.

Exercise 7.4.5 (Harder)**.** Use Exercise 7.4.4 to prove that

$$g(K_{p,q}) = \frac{(p-1)(q-1)}{2}$$

where $g(K)$ is the minimal genus of a spanning surface for K.

Furthermore, X is a surface bundle over S^1 with monodromy of order pq. To show this, let S be the minimal spanning surface and consider $X - n(S)$.

Aside. To answer the question of a student, we define the Euler characteristic of an n–manifold.

Definition 7.4.2. We define $\chi(M^n)$ by taking a finite triangulation of M and setting $\chi(M) = \sum_{k=0}^{n}(-1)^k|T^{(k)}|$ where $|T^{(k)}|$ denotes the number of k-simplices in the image $\|T\|$.

7.5 Some Theorems From Hatcher

Proposition 7.5.1 (1.12 in Hatcher)**.** *Suppose $(M, \mathcal{F})$ is compact, connected and Seifert fibred. Then M is irreducible or M is homeomorphic to one of $S^2 \times S^1, S^2 \tilde{\times} S^1$ or $P^3 \# P^3$.*

Proof. Suppose $S \subset M$ is an essential 2–sphere. Following the proof of Proposition 7.2.1 (1.11 in Hatcher) with surgery of essential surfaces replacing isotopy of essential spheres, we find an essential 2–sphere S' such that S' is vertical or horizontal. Since S' is not A^2, T^2, M^2 or K^2, we find S' must be horizontal.

1. If S' is non-separating, then $M - n(S')$ is homeomorphic to $S^2 \times I$. So $M \cong S^2 \times S^1$ or $S^2 \tilde{\times} S^1$.
2. If S' separates, then it is an exercise to show that $M \cong P^3 \# P^3$. □

Proposition 7.5.2 (1.13 in Hatcher)**.** *Let $(M, \mathcal{F})$ be as above. Then*

1. *every horizontal 2–sided surface is essential and*
2. *every vertical 2–sided surface is essential except for tori bounding fibred solid tori and boundary parallel annuli cutting off fibred solid tori.*

Proof. Suppose that D is a surgery disk or bigon for $S \subset M$.

1. Suppose S is horizontal. By the previous discussion, $M - n(S)$ is an I–bundle and D gives a surgery for $\partial_h(M - n(S))$. But the horizontal boundary of an I–bundle is always essential.

   Exercise 7.5.1. The horizontal boundary of an I–bundle is always essential.

2. Suppose S is vertical. So D gives a surgery in $M' \subset M - n(S)$ where M' is the component of $M - n(S)$ containing D. Suppose D is essential. Since $D \subset M'$ is essential, D must be vertical or horizontal, hence horizontal. Let $B = M'/S^1$.

   Exercise 7.5.2. Show that B is a disk with at most one orbifold point. Hint: use that $d \cdot \chi_{\text{orb}}(B) = \chi(D) = 1$.

   Thus M' is a solid torus. If D was a bigon, then, as $D \cap \partial M = D \cap \partial M'$ is a single arc, the fibring of M' is the trivial fibring, so $M' \cong V_{1,0}$. □

Lemma 7.5.3 (1.14 in Hatcher)**.** *Let $A \subset (M, \mathcal{F})$ be an essential annulus. Then A can be properly isotoped to be vertical with respect to $\mathcal{F}$, possibly after changing $\mathcal{F}$ if M is $T \times I$, $T \tilde{\times} I$, $K \times I$ or $K \tilde{\times} I$.*

Proof. Since A is essential, it may be isotoped to be vertical or horizontal. Suppose A is horizontal. So $M - n(A)$ is an I–bundle with annuli as horizontal boundary components.

(i) If $M - n(A)$ is connected, then $M - n(A) \cong A \times I$. So

$$M = {}^{A \times I}\big/_{(x,1) \sim (\varphi(x), 0)} =: M_\varphi,$$

as in Figure 7.42.

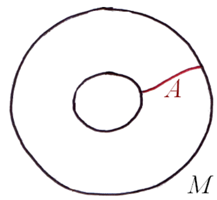

Figure 7.42: $M = (A \times I)/((x,1) \sim (\varphi(x), 0))$.

But there are only four possibilities for φ, up to isotopy: the identity, reflections switching or preserving the boundary components, and the rotation given by composing these reflections. See Figure 7.43.

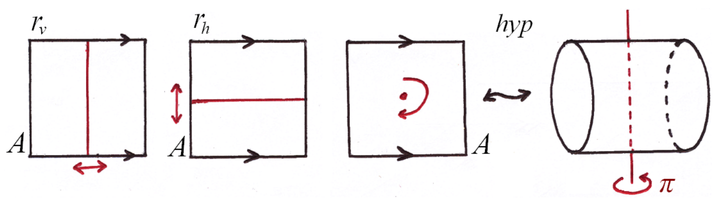

Figure 7.43: The three non-trivial possibilities for φ.

Exercise 7.5.1. Show that $\mathrm{MCG}(A) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here $\mathrm{MCG}(S)$ is the mapping class group of S, the group of homeomorphisms of S, up to isotopy.

These four maps give the four exceptions.

Exercise 7.5.2. Check this.

(ii) If $M - n(A)$ has two components, as in Figure 7.44, then $M - n(A) \cong \mathbb{M}^2 \mathbin{\tilde{\times}} I \sqcup \mathbb{M}^2 \mathbin{\tilde{\times}} I$.

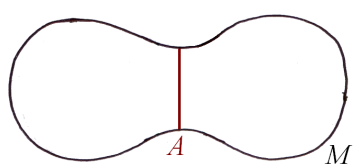

Figure 7.44: $M - n(A)$ may have two components.

Note that $\mathbb{M}^2 \mathbin{\tilde{\times}} I$ is a cube with a pair of opposite faces glued by a π twist, shown in Figure 7.45.

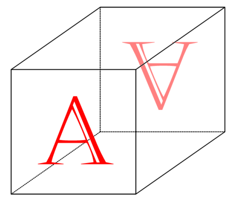

Figure 7.45: A picture of $\mathbb{M}^2 \tilde{\times} I$.

Exercise 7.5.3. Find the Möbius bands in this cube.

It is again an exercise to show that all four gluings give $K \tilde{\times} I$ with base orbifold $D^2(2,2)$. $\square$

Note. We have an exact sequence of groups: $S^1 \to K \tilde{\times} I \to D^2(2,2)$

$$\begin{array}{ccccccccc} \mathbf{1} & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(K^2) & \longrightarrow & D_\infty & \longrightarrow & \mathbf{1} \\ \mathbf{1} & \longrightarrow & \left\langle a^2 \right\rangle & \longrightarrow & \left\langle a,b \mid a^2 = b^2 \right\rangle & \longrightarrow & \left\langle a,b \mid a^2 = b^2 = 1 \right\rangle & \longrightarrow & \mathbf{1} \end{array}$$

coming from the long exact sequence for the Seifert fibering. See Theorem 4.41 page 276 of Hatcher's *Algebraic Topology* for more details.

Lemma 7.5.4 (1.15 in Hatcher)**.** *Let $(M, \mathcal{F})$ be as above. Then the slopes of $\mathcal{F}|\partial M$ are determined by M only, unless M is $V_{p,q}$ or one of the four exceptions above.*

Proof. If $\partial M = \varnothing$ then we have nothing to prove. If $B = M/S^1$ has no essential arcs, then $B = D^2(p)$.

Exercise 7.5.4. Check this.

Then $M \cong D \times S^1$ and we are done. So let $\alpha \subset B$ be an essential arc. See Figure 7.46.

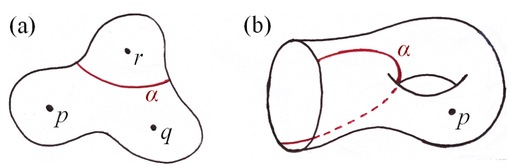

Figure 7.46: Two examples of essential arcs in (a) where $B = D^2(p,q,r)$ with $p,q,r > 1$, and (b) where $B = T^2 \# D^2(p)$.

Let $A \subset M$ be the vertical annulus above α. In this case:

(i) A is essential by Lemma 1.13 in Hatcher.

(ii) A is vertical in any fibering of M, with exceptions as above, by Lemma 1.14 in Hatcher.

So ∂A is determined by M alone, and we are done. $\square$

Remark. Note that in the above we used the fact that solid Klein bottles are not Seifert fibered spaces.

Exercise 7.5.5. Show that the solid Klein bottle can be partitioned as a disjoint union of circles. Show, nonetheless, that the solid Klein bottle cannot be Seifert fibered.

Exercise 7.5.6. Show that $K \times I$ contains a solid Klein bottle, yet is still a Seifert fibered space.

Lemma 7.5.5 (1.16 in Hatcher)**.** *Suppose M is connected, compact, orientable, irreducible and atoroidal. Suppose $A \subset M$ is an essential annulus with ∂A contained in torus components of ∂M. Then M admits a Seifert fibering.*

Proof. Let M, A be as above. Let T be the components of ∂M meeting A. Let $N = N(A \cup T)$. So there are three cases:

(i) A meets two boundary components, T_1 and T_2, as in Figure 7.47.

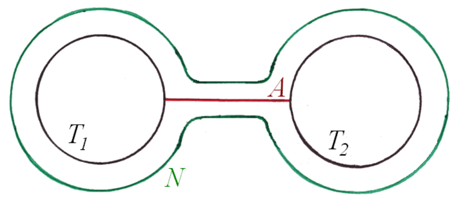

Figure 7.47: A meets two boundary components.

(ii) A meets a single boundary component without twisting, as shown in Figure 7.48.

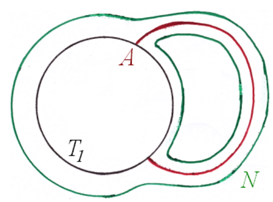

Figure 7.48: A meets a single boundary component without twisting.

(iii) A meets a single boundary component with a twist, as shown in Figure 7.49.

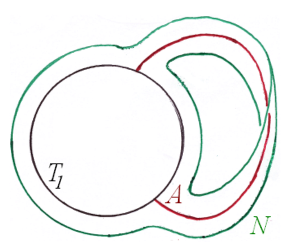

Figure 7.49: A meets a single boundary component with a twist.

Note. Note that Figures 7.47, 7.48 and 7.49 give a cross section of N. For example in Figure 7.47, the entirety of N is shown in Figure 7.50. Unfortunately the neighborhood N, in the third situation, does not embed in $\mathbb{R}^3$.

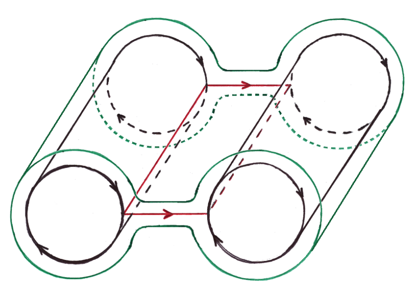

Figure 7.50: The whole of N in case (i), of which Figure 7.47 is a cross section. The front and back faces and edges are identifed.

Note that $N(A)$ and $N(T)$ are Seifert fibered, and we may glue these fibrings to get a fibering of N. Fix F, a component of $\partial N - \partial M$. In other words, a component of the *frontier* of N in M. Note that $F \cong \mathbb{T}^2$.

(i) Suppose that F compresses in M via a disk $(D, \partial D) \subset (M, F)$. Since A is essential we may arrange via an isotopy to have $A \cap D = \varnothing$. So we may assume that $D \cap N = \partial D$; thus F compresses to the "outside" of N. So F_D is a 2–sphere bounding a ball $B \subset M$. Note that $N \subset B$ is a contradiction as $\partial M \cap \partial N \neq \varnothing$. So $X = B \cup N(D)$ is a solid torus attached to F.

(ii) Suppose F is boundary parallel. Say $M - n(F)$ contains X, with $X \cong F \times I$ the parallelism. Since A is essential, we find that $X \cap N = F$, as $N \subset F$ leads to a contradiction.

So the fibering on N extends to a fibering on $N \cup X$. We do the same for all components of $\partial N - \partial M$. □

Exercise 7.5.7. Read the proof of Theorem 1.9 in Hatcher.

Chapter 8

Haken Manifolds and Hierarchies

8.1 Haken Manifolds and the Poincaré Conjecture

We now state the Poincaré conjecture, proved by Perelman, following a program of Hamilton.

Poincaré Conjecture. *Suppose M^3 is closed and simply connected. Then M is homeomorphic to S^3.*

Recall that closed means that M is compact and $\partial M = \varnothing$. Simply connected means that M is connected and $\pi_1(M) = \{\mathbf{1}\}$. Note that the equivalent statement in dimension two follows from the classification of surfaces and the Seifert-van Kampen theorem. In dimensions greater than three, the conjecture was solved previously by (among others) Smale, Stallings, and for dimension four, Freedman.

Remark. Poincaré originally conjectured that if $H_1(M, \mathbb{Z}) = 0$ then $M = S^3$. He then gave a counterexample to this, called the *Poincaré homology sphere*. Let D be the dodecahedron and let $P = D/\sim$, where we glue opposite faces with a 1/10 right-handed twist, as in Figure 8.1.

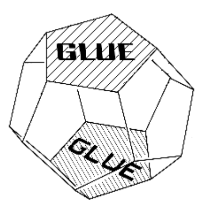

Figure 8.1: The Poincaré homology sphere. This diagram is adpated from one in The Shape of Space by J. Weeks.

Exercise 8.1.1. Let $\Gamma = \pi_1(P)$. Give a presentation of Γ and check that $\Gamma^{\text{ab}} = 0$.

Exercise 8.1.2. What if we use a 5/10 twist?

Remark. If we use a 3/10 twist we get the *Seifert-Weber dodecahedron space*. See Figure 8.2.

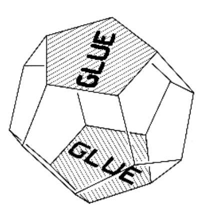

Figure 8.2: The Seifert-Weber dodecahedron space. This diagram is adapted from one in The Shape of Space by J. Weeks.

Definition 8.1.1. We say a knot $K \subset S^3$ is *spanned* by a surface $F \subset S^3$ if F is embedded and two-sided away from ∂F, and $\partial F = K$. In other words, the boundary of F wraps exactly once about K. See Figure 8.3a. Equivalently, $S \subset X_K$ is a *spanning surface* for K if it is two-sided, embedded, $|\partial S| = 1$ and the following holds. Let $N = N(K)$ and let $(D, \partial D) \subset (N, \partial N)$ be a meridian disk. Let $\mu = \partial D$. Then the transverse intersection $\mu \cap \partial S$ is a single point. See Figure 8.3b.

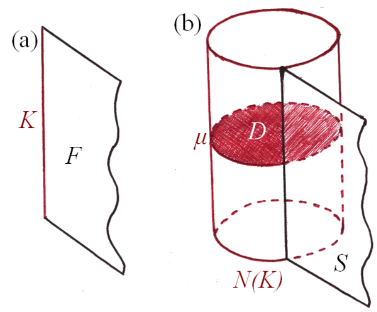

Figure 8.3: Diagrams of equivalent definitions of the spanning surface.

Recall that a knot K is the *unknot* if K is isotopic to a round circle.

Theorem 8.1.1. *Suppose $K \subset S^3$ is a knot. The following are equivalent:*

(i) *K is the unknot.*

(ii) *K is spanned by a disk E.*

(iii) *$X_K = S^3 - n(K)$ is a solid torus.*

(iv) *$\pi_1(X_K) \cong \mathbb{Z}$.*

See Figure 8.4.

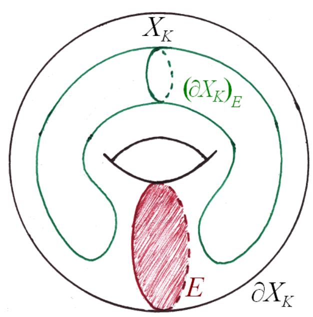

Figure 8.4: Illustration of Theorem 8.1.1.

Proof.

(i) $\Longrightarrow$ (ii) Use ambient isotopy.

(ii) $\Longrightarrow$ (iii) Use irreducibility of X_K and the fact that $(\partial X_K)_E \cong S^2$. Note that $E \subset X_K$ is essential as $\partial E \cap \mu$ is a point. So if $(\partial X_K)_E$ bounds a 3–ball B, we have $B \cup N(E) \cong E \times S^1$ is a solid torus.

(iii) $\Longrightarrow$ (i) This follows from Exercises 2.2 and 6.6.

(iii) $\Longrightarrow$ (iv) Since $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, we have $\pi_1(X_K) \cong \pi_1(S^1) = \mathbb{Z}$.

(iv) $\implies$ (iii) We must show that if M is irreducible, $\partial M = \mathbb{T}^2$ and $\pi_1(M) \cong \mathbb{Z}$, then $M \cong D \times S^1$. This requires Dehn's Lemma. □

Exercise 8.1.3. Deduce (iv) $\implies$ (iii) from the following lemma.

Dehn's Lemma (Papakyriakopoulos, 1957)**.** *Suppose $\alpha \subset \partial M$ is a simple closed curve, bounding a singular disk in M. Then α bounds an embedded disk in M.*

Loop Theorem. *Suppose F is a component of ∂M, and $i_*\colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential simple closed curve $\alpha \subset F$ such that $[\alpha] = \mathbf{1} \in \pi_1(M)$.*

This leads nicely to the following conjecture.

Simple Loop Conjecture. *If $i\colon F \looparrowright M$ is a two-sided map, and i_* is not injective, then there is an essential simple loop in the kernel.*

This has been proved by Gabai if M is a surface, and by Hass if M is Seifert fibered.

Exercise 8.1.4. Prove the simple loop conjecture when F is two-sided and properly embedded in M.

Disk Theorem. *Suppose that $F \subset \partial M$ is a component, and $i_*\colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential disk $(D, \partial D) \subset (M, F)$.*

Exercise 8.1.5. Show that the Disk Theorem is implied by the Loop Theorem and Dehn's Lemma.

The Disk Theorem is the first "promotion" theorem, among many others. For example we have the following:

Sphere Theorem. *Suppose M is an orientable 3–manifold with $\pi_2(M)$ non-trivial. Then there is an embedded 2–sphere $S \subset M$ such that $[S] \neq \mathbf{1} \in \pi_2(M)$.*

In general we assume that there is an essential map $(F, \partial F) \looparrowright (M, \partial M)$. The corresponding promotion theorem gives us an embedding. For example, F could be a disk or sphere (due to Papakyriakopoulos), a projective plane (due to Epstein), an annulus or torus, or indeed any F with $\chi(F) \geq 0$.

8.2 Heirarchies

We now discuss hierarchies. Suppose that $M_0 = M$, suppose that $S_i \subset M_i$ is a properly embedded two-sided surface, and define:

$$M_{i+1} := M_i - n(S_i).$$

So we have a sequence of manifolds:

$$M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} M_n.$$

Definition 8.2.1. Call a sequence $\{M_i, S_i\}$ a *partial hierarchy* if every S_i is essential in M_i.

Note. Some authors only require S_i to be incompressible.

The following example demonstrates why we require the S_i to be essential.

Example 8.2.1. Take annuli in V_2, the genus 2 handlebody, as in the right hand side of Figure 8.5, and glue them to give $M_0 \cong V_2$. Let S_0 be the single annulus given by the image of the two annuli under the gluing map. Then cutting along S_0 gives $M_1 \cong V_2$, so we could continue the process indefinitely.

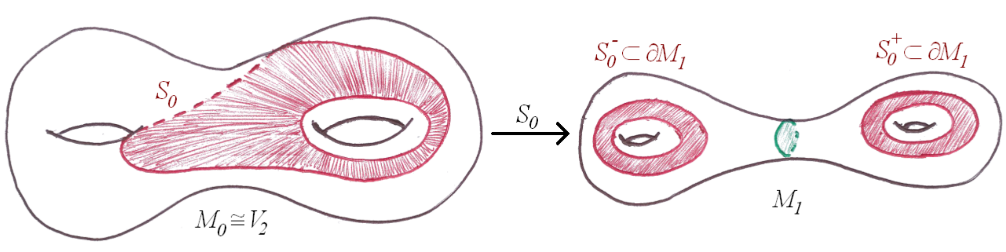

Figure 8.5: Note that S_0 is inside M_0, not on the boundary (although $\partial S_0 \subset \partial M_0$).

Equivalently, one can think of V_2 as $\big(T^2 - n\big((B^2)\big)\big) \times I$, as in Figure 8.6.

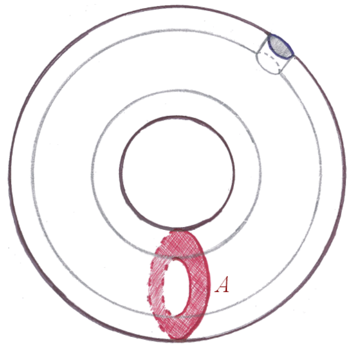

Figure 8.6: Another way to look at V_2.

Cut along A to get the pair of pants $\times I$, as in Figure 8.7.

Figure 8.7: Let $F = T - \mathrm{int}(D)$ be a once-holed torus. Let G be a pair of pants. Cutting $F \times I$ along a vertical annulus gives a copy of $G \times I$. As $F \times I \cong G \times I$ this could lead to an infinite hierarchy, were we to allow non-essential surfaces.

Definition 8.2.2. If M_n is a collection of 3–balls, then the partial hierarchy is simply called a *hierarchy*.

Example 8.2.2. Let $M_0 = \mathbb{T}^3$, thought of as the unit cube in $\mathbb{R}^3$ with face pairings. Let $S_0 \subset M_0$ be the image of the xy–plane, so $S_0 \cong T^2$. Then $M_1 \cong T \times I$. Let S_1 be the image of the yz–plane, so $S_1 \cong A^2$, and $M_2 \cong D \times S^1$. Let S_2 be the image of the zx–plane, a disk. Then $M_3 \cong B^3$. See Figure 8.8.

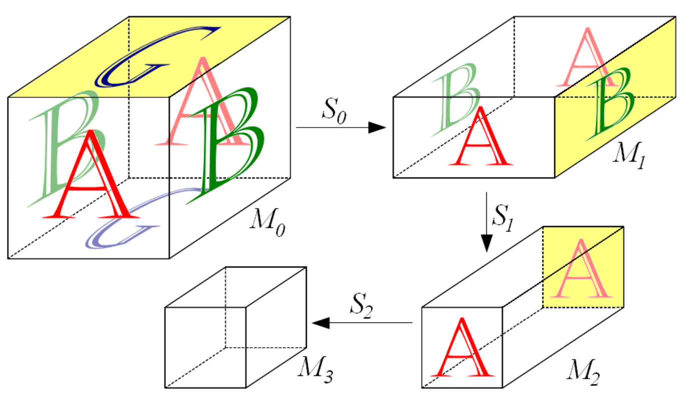

Figure 8.8: A hierarchy of length three for the three-torus.

Example 8.2.3. Let $M_0 = X_K$, where K is the (p, q)–torus knot, as shown in Figure 8.9, and let $S_0 = A$, the unique essential annulus. Then $X_K \xrightarrow{A} V_K \sqcup W_K = M_1$. Now letting S_1 be a pair of meridian disks, one in each of V_K and W_K, we find that $M_2 \cong B^3_1 \sqcup B^3_2$. See Figures 8.10 and 8.11.

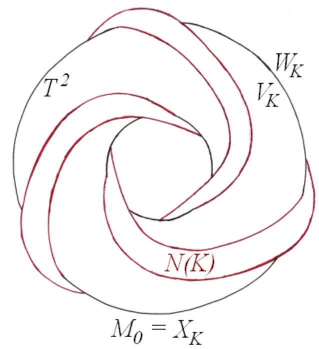

Figure 8.9: The (p, q)–torus knot complement, M_0.

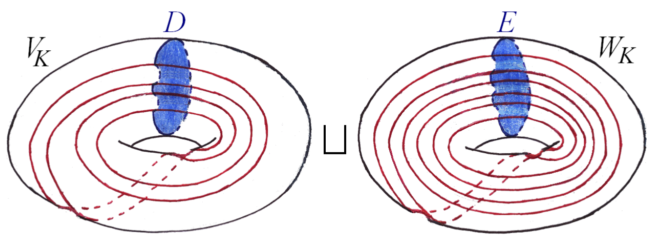

Figure 8.10: Compressing disks for V_K and W_K.

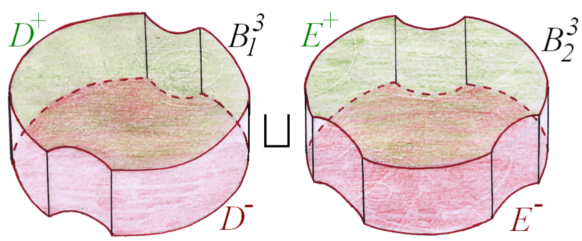

Figure 8.11: The final stage of the heirarchy.

Definition 8.2.3. If M is compact, orientable and irreducible, and $S \subset M$ is properly embedded, two-sided and essential, then M is called *Haken.*

Theorem 8.2.1. *If M is compact, orientable, irreducible and $\partial M \neq \varnothing$, then either M is a 3–ball or M is Haken.*

This theorem is implied by the following:

Theorem 8.2.2. *If M is compact, orientable and irreducible, and if*

$$\operatorname{rank}(H_1(M, \mathbb{Z})) \geq 1$$

then M is Haken.

Definition 8.2.1. Suppose M, N are 3–manifolds and $D \subset \partial M$ and $E \subset \partial N$ are disks. Let $\varphi\colon D \longrightarrow E$ be an orientation reversing homeomorphism. Then we define the *boundary connect sum* of M and N to be $M \#_\partial N := M \sqcup N/\varphi$. See Figure 8.12.

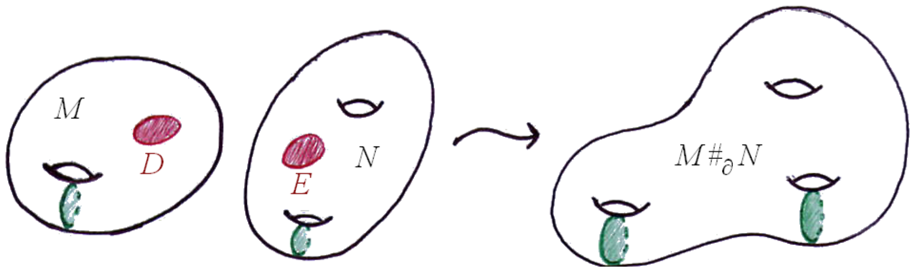

Figure 8.12: An example of the boundary connect sum.

Recall that φ only matters up to isotopy.

Definition 8.2.2. Suppose V is a handlebody and $F = \sqcup F_i$ is a collection of closed orientable surfaces, none of which is a two-sphere. Then $C := V \#_\partial (\#_\partial F_i \times I)$ is a *compression body.* We define the *inner boundary* $\partial_- C = \sqcup_{F_i \times \{0\}} C$ and the *outer boundary* $\partial_+ C = \partial C - \partial_- C$.

Example 8.2.1. See Figure 8.13.

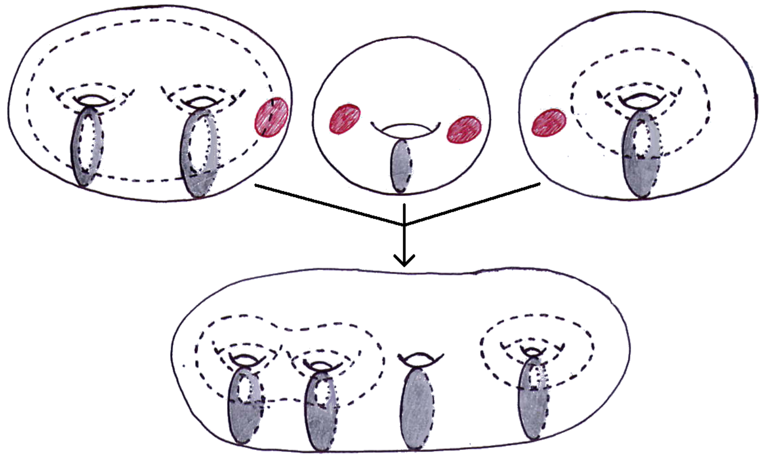

Figure 8.13: Another example of the boundary connect sum. Note that the third grey surface is a disk while the others are all annuli.

Exercise 8.2.1. Show that $\#_\partial$ is associative, commutative and B^3 is the unit.

Exercise 8.2.2. Show that the essential surfaces in C are

- essential disks compressing $\partial_+ C$,
- components of $\partial_- C$ and
- annuli meeting both $\partial_+ C$ and $\partial_- C$.

Example 8.2.2. See Figure 8.14.

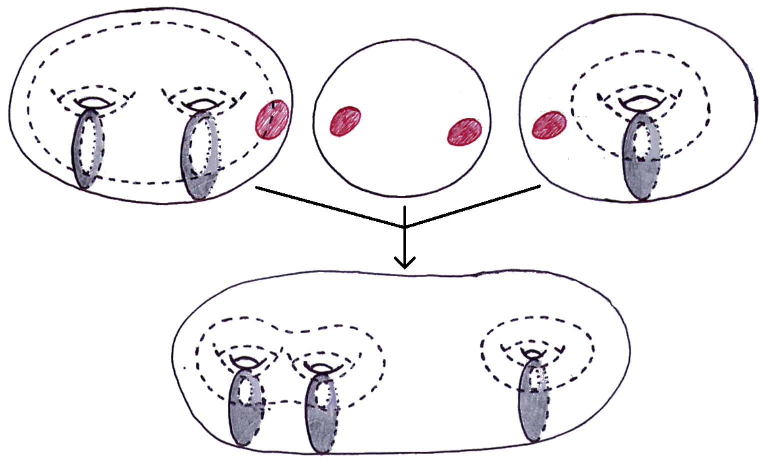

Figure 8.14: An example of the boundary connect sum.

8.3 The Existence of Short Hierarchies

Now we demonstrate the existence of short hierarchies, following Jaco. Suppose that M_0 is Haken and additionally that ∂M_0 is incompressible. Let $S_0 \subset M_0$ be a maximal collection of disjoint, non-parallel, closed, incompressible, two-sided surfaces in M_0 none of which are spheres. Since M_0 is Haken, S_0 is non-empty and it is finite by Haken-Kneser finiteness. See Figure 8.15.

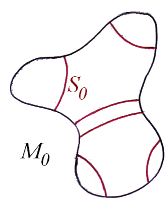

Figure 8.15: $S_0 \subset M_0$ is non-empty and finite. It is convenient to take $\partial M_0 = \varnothing$.

Aside. Note that closed incompressible surfaces, which are not spheres, are essential.

Note that every component $N \subset M_1 := M - n(S_0)$ has boundary with genus ≥ 1. So N contains some essential surface by Theorem 27.5. Let $S_1 \subset M_1$ be a maximal collection of disjoint, nonparallel, two-sided, essential surfaces in M_1: these are the green lines in Figure 8.16. Again, S_1 cuts every component of M_1 and S_1 is finite by Haken-Kneser finiteness in the bounded case. See the addedum to Exercise 5.5. Define $M_2 := M_1 - n(S_1)$ and let C be any component of M_2.

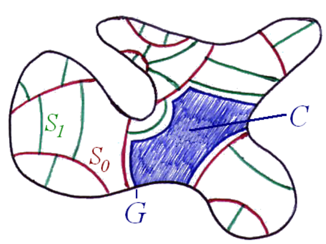

Figure 8.16: The component C contains an essential surface.

Proposition 8.3.1. *The component C is a compression body.*

Proof. Suppose that some component $G \subset \partial C$ is compressible into C. So let G_i, D_i be a sequence where $G_0 = G$ and D_i compresses G_i in the same direction as D_0, into C. Define $G_{i+1} = (G_i)_{D_i}$. So we get a sequence

$$G_0 \xrightarrow{D_0} G_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{n-1}} G_n.$$

See Figure 8.17.

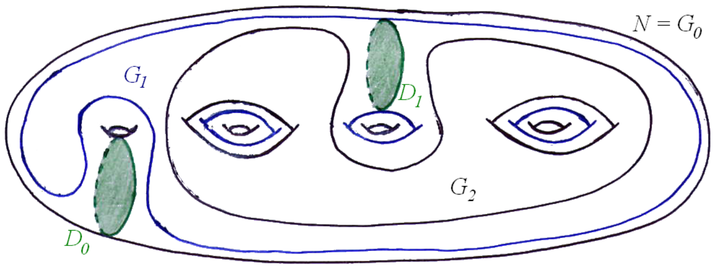

Figure 8.17: The first few terms in the sequence (G_i, D_i).

Note that G_{i+1} may be disconnected, as in Figure 8.18.

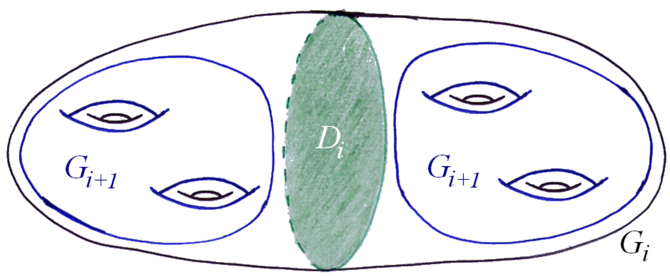

Figure 8.18: G_{i+1} may be disconnected.

Claim. If some component of G_n is a 2–sphere then it bounds a 3–ball in C.

Proof sketch. M is irreducible, thus C is irreducible as well. □

So cap off such 2–spheres, deleting them from G_n.

Claim. The closed surface G_n is incompressible in M.

Proof. As G_n is last in the sequence, G_n cannot compress into C. So suppose E is a surgery disk for G_n in the other direction. See Figure 8.19.

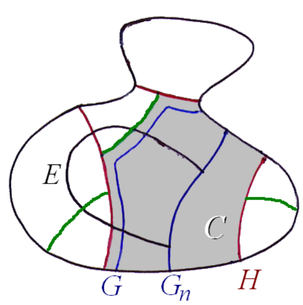

Figure 8.19: E is a compressing disk for G_n in the other direction.

Then we can do the following: Isotope E off of S_0, then off of S_1 and then off of $\{D_i\}$. It follows that E is a surgery disk for G_n in the compression body cobounded by G_0 and G_n. Thus G_n is the inner boundary of this compression body and so is essential. Thus E is trivial, as desired. □

To finish the proposition, deduce that the components of G_n are parallel to components of S_0 since G_n is essential, closed and disjoint from S_0 (as it lies in C).

Now let $S_2 \subset M_2$ be a collection of essential disks, cutting all compression bodies into products. Let $S_3 \subset M_3$ be a collection of vertical annuli (one per product). Finally $S_4 \subset M_4$ is a collection of disks cutting all handlebodies into 3–balls, as in Figure 8.20. This proves the existence of *short hierarchies.* □

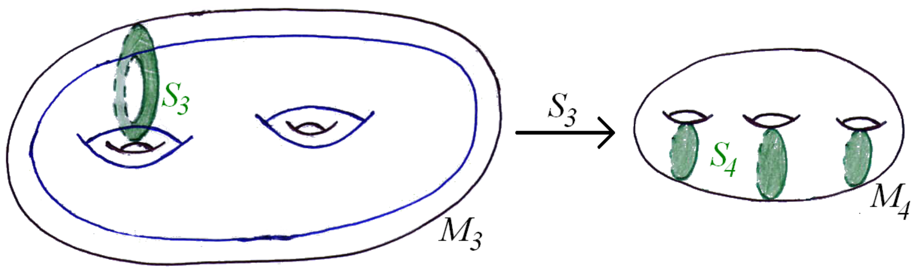

Figure 8.20: S_3 is a collection of vertical annuli; cut along these annuli to get a collection of handlebodies. Then cutting along S_4 gives a collection of 3-balls.

8.4 Boundary Patterns

In this section, we again follow Lackenby.

Definition 8.4.1. A *boundary pattern* P for M^3 is a trivalent graph embedded in ∂M. We allow P to be the empty set, to be disconnected and to have simple closed curves as components.

Example 8.4.1. Trivalent graphs in $S^2 = \partial B^3$ are patterns for B^3. See Figure 8.21.

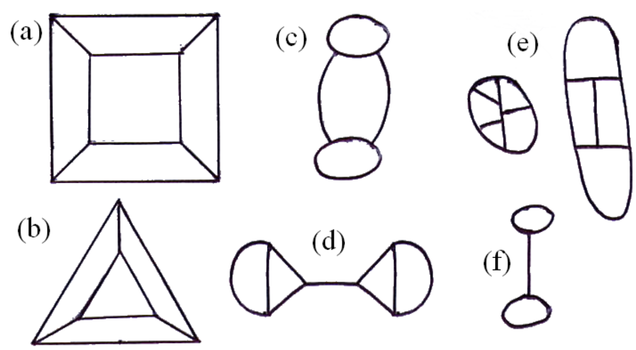

Figure 8.21: Six examples of trivalent graphs in S^2. Note that (e) is a disconnected pattern.

Suppose (M, P) is a manifold equipped with a boundary pattern. Suppose $S \subset M$ is properly embedded and ∂S is transverse to P. So ∂S misses the vertices of P and intersects the edges of P transversely. Let $N = M - n(S)$ and let

$$Q = (P - n(S)) \cup \partial S^+ \cup \partial S^-.$$

So Q is a pattern for N and we write $(M, P) \xrightarrow{S} (N, Q)$. See Figure 8.22.

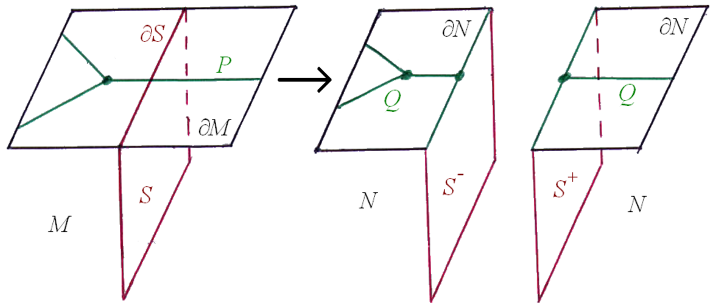

Figure 8.22: A picture of the cutting.

Definition 8.4.2. Let P be a boundary pattern for M. Then we call P *essential* if for any $(D, \partial D) \subset (M, \partial M)$ with ∂D transverse to P and $|\partial D \cap P| \leq 3$ we have

- a disk $E \subset \partial M$ such that $\partial E = \partial D$ and
- the intersection $E \cap P$ contains at most one vertex of P and contains no cycles of P.

Exercise 8.4.1. Verify that if P is essential then we get the implications shown in Figures 8.23 to 8.26:

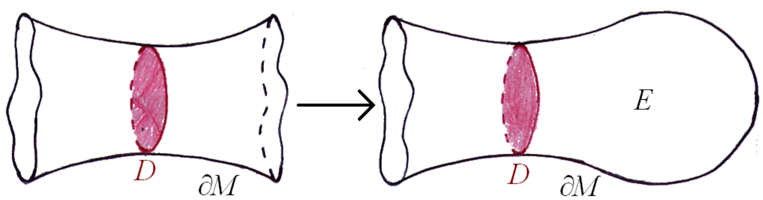

Figure 8.23: The case $\partial D \cap P = \varnothing$.

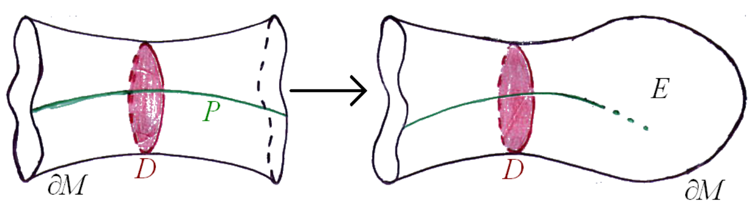

Figure 8.24: The case $|\partial D \cap P| = 1$ is not possible.

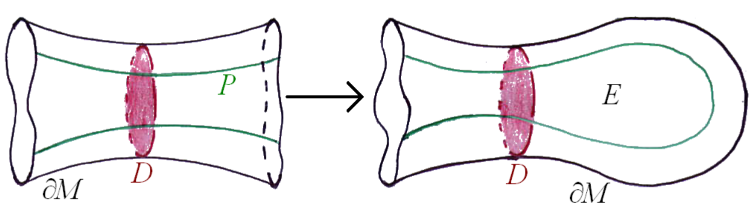

Figure 8.25: The case $|\partial D \cap P| = 2$.

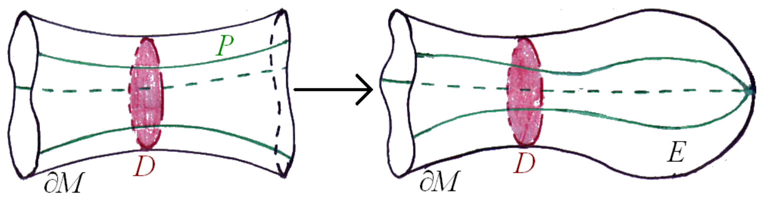

Figure 8.26: The case $|\partial D \cap P| = 3$.

Exercise 8.4.2. Analyse the examples of (B^3, P) given above. Which are, and which are not, essential?

Exercise 8.4.3. Give necessary and sufficient conditions for P to be an essential pattern for B^3.

Example 8.4.2. If $M_0 = \mathbb{T}^3 = I^3/\sim$ then $S_0 = \{z = 0\}$ is an essential torus, $S_1 = \{x = 0\} \subset M_1$ is an essential annulus and $S_2 = \{y = 0\} \subset M_2$ is an essential disk. We can see this in Figure 8.27.

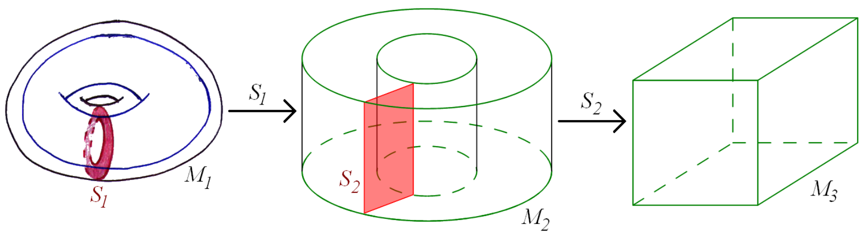

Figure 8.27: Pictures of these cuttings with boundary patterns. For M_3, P_3 is the 1–skeleton of the cube.

Definition 8.4.3. Let $P \subset \partial M$ be a pattern. We say P is *homotopically essential* if the following condition hold. For any map $f\colon (D, \partial D) \longrightarrow (M, \partial M)$ (which need not be an embedding) transverse to P, we define $Z = Z_f = \partial D \cap f^{-1}(P)$. If $|Z| \leq 3$ then there is a homotopy $H\colon D \times I \longrightarrow M$ such that

- for all t: $H_t|Z = f|Z$,
- $H_0 = f$,
- $H_1(D) \subset \partial M$ and finally
- $H_1(D)$ contains at most one vertex of P and contains no cycles of P.

Exercise 8.4.4. If P is homotopically essential, then P is essential.

Theorem 8.4.1 (9.1 in Lackenby)**.** *If P is essential, then it is homotopically essential.*

We will indicate a proof, using *special hierarchies*, in the next lecture.

Exercise 8.4.5. Theorem 8.4.1 implies the Disk Theorem. As a hint, recall that we allow $P = \varnothing$.

We pause to give another example of a hierarchy.

Example 8.4.3. Consider the knot $K \subset S^3$ shown in Figure 8.28: the $(1, 1, -3)$–pretzel knot. The surface shown is a spanning surface for K. This is one of the two so-called *checkerboard surfaces* for this diagram of K.

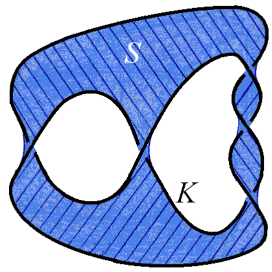

Figure 8.28: A diagram of the $(1, 1, -3)$–pretzel and S, one of its two checkerboard surfaces.

Near a twist we see a half-twisted band, as in Figure 8.29.

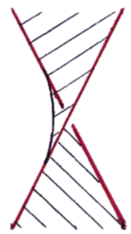

Figure 8.29: A half twisted band.

Let $N = N(K)$ be a regular neighbourhood and write $X = X_K = S^3 - n(K)$. See Figure 8.30. Let S_0 be the remains of the spanning surface in X.

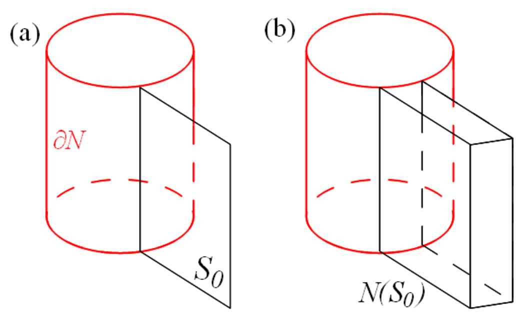

Figure 8.30: (a) A picture of $N(K)$, S_0 and (b) $N(S_0)$.

Let $M_0 = X$ and cut M_0 along S_0 to get M_1. Thus, as M_1 is a genus two handlebody, we find that $\partial S_0^{\pm}$ gives a pattern to ∂M_1, shown in Figure 8.31.

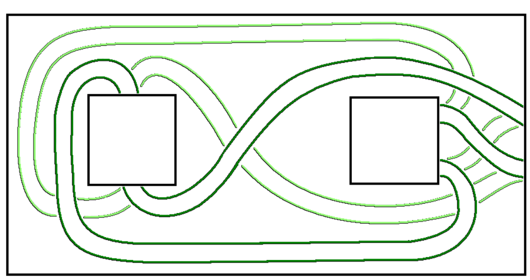

Figure 8.31: A pattern to ∂M given by $\partial S^{\pm}$. Note that M_1 is the handlebody on the outside.

The two components of P in ∂M_1 cobound an annulus, the remains of ∂N. We take S_1 to be the union of a pair of disks as in Figure 8.32.

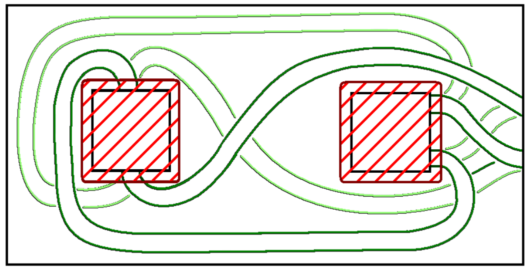

Figure 8.32: The essential surface S_1 in M_1, consisting of two disks which meet ∂M_1 in two loops around the holes.

Now cut along S_1 to get $M_2 \cong B^3$.

Exercise 8.4.6. Show that (M_2, P_2) is homeomorphic to the pattern shown in Figure 8.33.

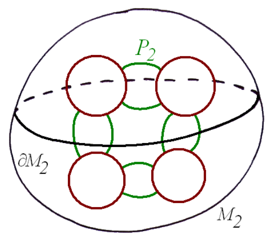

Figure 8.33: A 3–ball with a pattern.

Exercise 8.4.7. Show that $P_2 \subset \partial M_2$ is essential. Figure 8.34 may be helpful.

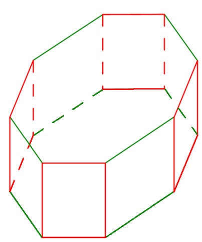

Figure 8.34: $(M_2, P_2) \cong \text{Oct} \times I$ where Oct denotes an octagon.

Claim. The surface $S_0 \subset X$ is essential.

Proof. Suppose $(D, \partial D) \subset (X, S_0)$ is a surgery disk. So consider $D \cap S_1 \subset D$. This is a collection of simple loops and arcs.

1. Suppose α is an innermost loop. Then α bounds E in D. So $(E, \alpha) \subset (M_2, \partial M_2)$ and $\alpha \cap P_2 = \varnothing$ which implies that we may isotope E past S_1, reducing $|S_1 \cap D|$. See Figure 8.35.

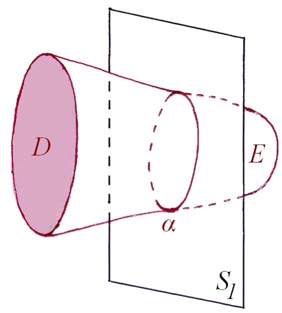

Figure 8.35: We may isotope E past S_1, reducing $|S_1 \cap D|$.

2. Suppose $\alpha \subset D$ is an outermost arc of $S_1 \cap D$. So α cuts off a bigon E. So $(E, \partial E) \subset (M_2, \partial M_2)$ is a bigon and $\partial E \cap P_2$ is exactly two points. But (M_2, P_2) is essential and we continue as usual.

So we may assume that $D \cap S_1 = \varnothing$. So $(D, \partial D)$ embeds in $(M_2, \partial M_2)$ with $\partial D \cap P_2 = \varnothing$. Since M_2 is a ball we find that D is parallel to a disk $D' \subset S_0$. So S_0 is incompressible. Now by Lemma 20.2 (1.10 in Hatcher) S_0 is boundary incompressible. It is also possible to directly prove that by repeating the proof using bigons. See Figure 8.36. $\square$

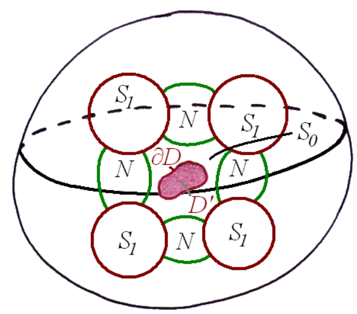

Figure 8.36: D is parallel to a disk $D' \subset S_0$.

We now give the ideas necessary to prove Theorem 8.4.1. We need a few more definitions.

Definition 8.4.4. Suppose $S \subset (M, P)$ is properly embedded and suppose $P \subset \partial M$ is an essential pattern. A surgery bigon D for S is a *pattern surgery* if $|\beta \cap P| \leq 1$ where $\partial D = \alpha \cup \beta$ and $\alpha = \partial D \cap S$. Say D is *trivial* if α cuts a bigon E out of S with $\partial E = \alpha \cup \gamma$ and $|\gamma \cap P| \leq 1$. Otherwise call D a *pattern compression*.

Definition 8.4.5. If S is essential and all pattern surgeries are trivial, we call S *pattern essential*. See Figure 8.37.

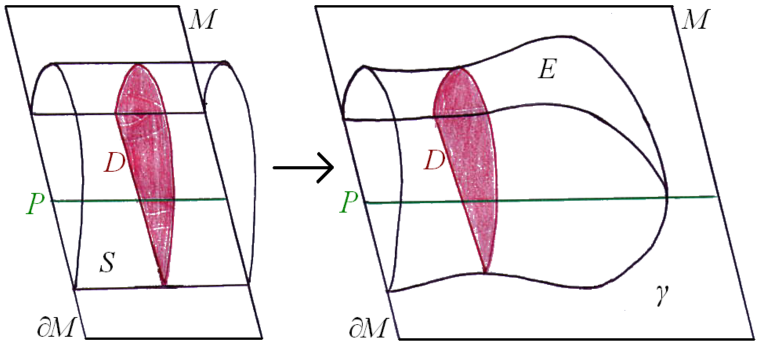

Figure 8.37: A picture of what it means to be pattern essential.

Definition 8.4.6. A *special hierarchy* is a sequence $(M_i, P_i) \xrightarrow{S_i} (M_{i+1}, P_{i+1})$ where all P_i are essential and all S_i are pattern essential. We do not allow S_i to be a sphere.

Proposition 8.4.1. *If $S \subset (M, P)$ is essential we may isotope S to be pattern essential.*

Proof. Exercise. □

Using the above one can show the following two propositions which imply Theorem 8.4.1.

Proposition 8.4.2. *If P is a pattern for $M \cong B^3$ and is essential, then P is homotopically essential.*

Proposition 8.4.3. *If $(M, P) \xrightarrow{S} (N, Q)$ are all essential and $Q \subset \partial N$ is homotopically essential, then P is homotopically essential in M.*