THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2011

THREE-MANIFOLDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Define a topological *n–manifold*, M^n. **[4]**

b) Define ∂M, the *boundary* of M. Show that ∂M is a $(n-1)$–manifold and that $\partial\partial M = \emptyset$. **[6]**

c) Define S^3 as a subset of $\mathbb{R}^4$. Show that S^3 is a three-manifold. Prove that $\partial S^3 = \emptyset$. **[6]**

d) Define S^3 via a gluing of three-balls. **[3]**

e) Show that the two definitions of S^3 above are equivalent. **[6]**

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2. a) Define a *k–simplex*. **[3]**

b) Define a *triangulation* T and its *underlying topological space* $||T||$. **[4]**

c) Suppose T is a finite triangulation. Give a necessary and sufficient condition for $||T||$ to be a three-manifold. **[6]**

d) Consider the tetrahedron shown in Figure 1. Does the given triangulation determine a three-manifold? If it does not, give a reason. If it does, recognize the manifold. **[6]**

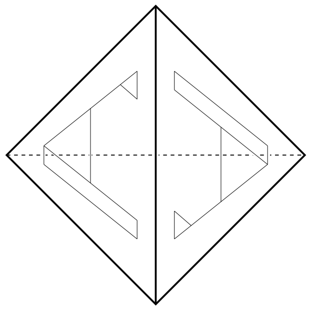

Figure 1: The front faces are glued to each other as shown. There are no other gluings.

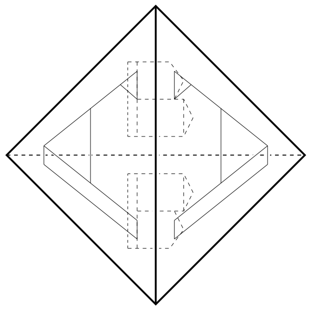

Figure 2: The front faces are glued by "closing the book", as are the back faces.

e) Consider the tetrahedron shown in Figure 2. Does the given triangulation determine a three-manifold? If it does not, give a reason. If it does, recognize the manifold. **[6]**

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3. a) Define a *bigon*. **[2]**

b) Suppose that $F^2 \subset M^3$ is a properly embedded surface. Define a *surgery disk* and *surgery bigon*. **[2]**

c) With F, M as above: define what it means for F to be *incompressible*, *boundary incompressible*, or *essential*. **[6]**

d) Define a *handlebody* V. **[2]**

e) Show that any surface F surface properly embedded in a handlebody V either is compressible, is boundary compressible, is a disk, or is a sphere. **[6]**

f) Let T^2 be the two-torus. Classify, up to proper isotopy, all connected essential surfaces in $T \times I$. **[7]**

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4. a) Define *fibered solid tori*. **[3]**

b) Define a *Seifert fibered space* $(M, \mathcal{F})$. Define the *singular* and *generic* fibers of $\mathcal{F}$, and their *multiplicities*. **[4]**

 Question 4 continued overleaf

c) Briefly describe the *base orbifold* B of a Seifert fibered space $(M, \mathcal{F})$. **[4]**

d) Giving the necessary definitions, state but do not prove the structure theorem for essential surfaces in Seifert fibered spaces. **[3]**

e) Let $X = K \tilde{\times} I$ be the orientation I–bundle over the Klein bottle K. Classify, up to isotopy, all Seifert fiberings of X. For each fibering, compute the base orbifold. **[7]**

f) Using the above or otherwise, classify up to isotopy the essential annuli in $X = K \tilde{\times} I$. **[4]**

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5. a) Suppose that $F \subset M$ is properly embedded. Define the notion of π_1–*injectivity.* **[3]**

b) Briefly define incompressibility. **[2]**

c) Show that if F is π_1–injective then F is incompressible. **[5]**

d) State Dehn's Lemma, the Loop Theorem and the Disk Theorem, clearly labelling each. **[6]**

e) Deduce the Disk Theorem from Dehn's Lemma and the Loop Theorem. **[3]**

f) Suppose that M is connected and irreducible. Suppose that $T \subset \partial M$ is a two-torus. Suppose that $\pi_1(M) \cong \mathbb{Z}$. Show that M is a solid torus. **[6]**

MATHEMATICS DEPARTMENT
FOURTH YEAR UNDERGRADUATE EXAMS

Course Title: THREE-MANIFOLDS

Model Solution No: 1

a) Define a topological n–*manifold*, M^n. **[4]**

(Book) An n–manifold M is a topological space which is Hausdorff and second countable and so that every point $x \in M$ has a neighborhood U that is homeomorphic to either $\mathbb{R}^n$ or to $\mathbb{R}^n_+$, the closed upper-half space.

b) Define ∂M, the *boundary* of M. Show that ∂M is a $(n-1)$–manifold and that $\partial\partial M = \emptyset$. **[6]**

(Book) ∂M is the set of points $x \in M$ which do not have neighborhoods homeomorpic to $\mathbb{R}^n$.

(Book) Since ∂M is a subspace of M it is second countable and Hausdorff.

Suppose that $\phi\colon U \to \mathbb{R}^n_+$ is a homeomorphism. We claim that $\partial U = \phi^{-1}(\mathbb{R}^{n-1})$. The forward inclusion follows because points $p \in \mathbb{R}^n_{>0}$ have neighborhhoods homeomorphic to $\mathbb{R}^n$. The other inclusion is implied by the fact that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^n_+$.

Thus $U \cap \partial M = \partial U$. Let $\partial\phi = \phi|\partial U$. It follows that ∂M is an $n-1$–manifold.

(Book) By the above, every point $x \in \partial M$ has a neighborhood homeomorphic to $\mathbb{R}^{n-1}$: thus ∂M has no boundary points

c) Define S^3 as a subset of $\mathbb{R}^4$. Show that S^3 is a three-manifold. Prove that $\partial S^3 = \emptyset$. **[6]**

(Book and exercises) Since $S^3 \subset \mathbb{R}^4$ it is Hausdorff and 2nd countable. Define $S^3 = \{v \in \mathbb{R}^4 : |v| = 1\}$. Let N, S be the north and south poles $(0, 0, 0, \pm 1)$. Let $\rho\colon S^3 - \{S\} \to \mathbb{R}^3$ be sterographic projection, defined via

$$(x, y, z, w) \mapsto \frac{2}{w+1}(x, y, z).$$

This is a homeomorphism between a neighborhood of N and $\mathbb{R}^3$. Now, rotations of S^3 act via homeomorphisms, so every point has such a neighborhood. As in the discussion above, it follows that $\partial S^3 = \emptyset$.

d) Define S^3 via a gluing of three-balls. **[3]**

(Book) Let $B = \{v \in \mathbb{R}^3 : |v| \le 1\}$. Let $B_\pm$ be copies of B and let $\phi\colon \partial B_+ \to B_-$ be the induced identity map. Then $S^3 \cong B_+ \cup_\phi B_-$.

e) Show that the two definitions of S^3 above are equivalent. **[6]**

(Exercises) Define $S^3_\pm = S^3 \cap \mathbb{R}^4_\pm$ where $\mathbb{R}^4_\pm$ are the upper and lower half spaces. The projection $\rho\colon \mathbb{R}^4 \to \mathbb{R}^3$, sending (x, y, z, w) to (x, y, z), induces homeomorphisms from $S^3_\pm$ with the unit three-ball B. Finally, $S^3_+ \cap S^3_-$ is the common two-sphere boundary.

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MATHEMATICS DEPARTMENT
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Course Title: THREE-MANIFOLDS

Model Solution No: 2

a) Define a *k–simplex.* **[3]**

(Book) The set $\Delta^k = \{x \in \mathbb{R}^{k+1} \mid \sum x_i = 1, x_i \geq 0\}$ is the model k–simplex.

b) Define a *triangulation* T and its *underlying topological space* $||T||$. **[4]**

(Book) The I^{th} *facet* of Δ is the set $\{x \in \Delta \mid x_i = 0 \forall i \in I\}$. A *face* of a simplex is a codimension one facet.

A k–dimensional triangulation T is a collection of k–simplices and face pairings. A *face pairing* is an isometry of faces. The *underlying space* $||T||$ is the quotient of $\sqcup \Delta_j$ under the face pairings. We use $\pi \colon \sqcup \Delta_j \to ||T||$ to denote the quotient map.

c) Suppose T is a finite triangulation. Give a necessary and sufficient condition for $||T||$ to be a three-manifold. **[6]**

(Exercise) Since the triangulation is finite, $||T||$ is second countable. Additionally, since face pairings are isometries, we find $||T||$ is metrizable and so Hausdorff.

Thus, to be a three-manifold we need only check that every point of $||T||$ has a three-ball neighborhood. Fix a point $y \in ||T||$. Then there is a $\epsilon = \epsilon(y) > 0$ sufficiently small so that

- If x, x' are elements of $\pi^{-1}(y)$ then $N_\epsilon(x) \cap N_\epsilon(x') = \emptyset$.
- If $x \in \pi^{-1}(y)$ and x lies in Δ then $N_\epsilon(x) \subset \Delta$ meets only facets whose closure contains x.

The necessary and sufficient condition is that, for every $y \in ||T||$, the neighborhoods $N_\epsilon(x)$, for $x \in \pi^{-1}(y)$ glue together to give a ball with y at the center or on the boundary.

d) Consider the tetrahedron shown in Figure 3. Does the given triangulation determine a three-manifold? If it does not, give a reason. If it does, recognize the manifold. **[6]**

(Unseen, but similar to a given exercise) The front faces are glued via a rotation. This is not a three-manifold. Let x be the midpoint of the vertical edge and set $y = \pi(x)$. Then any sufficently small neighborhood of y in $||T||$ is a cone on the real projective plane, and so is not a three-ball.

e) Consider the tetrahedron shown in Figure 4. Does the given triangulation determine a three-manifold? If it does not, give a reason. If it does, recognize the manifold. **[6]**

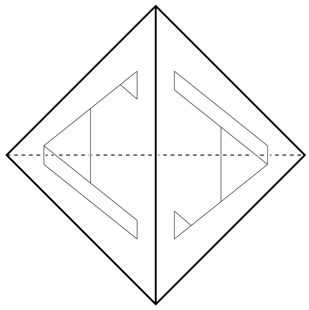

Figure 3: The front faces are glued to each other as shown. There are no other gluings.

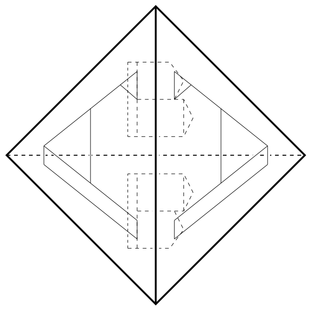

Figure 4: The front faces are glued by "closing the book", as are the back faces.

(Unseen, but easier than a given exercise) To see that $M = ||T||$ is a three-manifold note that distinct faces are glued in pairs. The three edges in M have degrees 1, 1, and 4 in the tetrahedron. Each point on an edge of degree one has a ball neighborhood in M because we closed the book. Any point on the edge of degree four has a neighborhood that is a union of four quarter-balls, each glued to the next by an orientation reversing map. Finally, there are two vertices. Each has link a two-sphere triangulated with two triangles.

We give two model proofs that M is a three-sphere.

First, consider the normal surface consisting of a single normal quad that meets the degree four edge in a single point. This gives a torus in M, cutting M into a pair of solid tori ($D \times S^1$). As their meridian disks meet in a single point we have the standard decomposition of S^3.

Second, compute $\pi_1(M)$ by taking the degree four edge as the spanning tree for $T^{(1)}$. There are two non-tree edges, and the two faces exactly kill these generators. Thus $\pi_1(M)$ is trivial. Since M is closed, the Poincaré conjecture implies M is a three-sphere.

MATHEMATICS DEPARTMENT
FOURTH YEAR UNDERGRADUATE EXAMS

Course Title: THREE-MANIFOLDS

Model Solution No: 3

a) Define a *bigon*. **[2]**

(Book) A *bigon* is a disk D with ∂D realized as a union of two arcs α, β, where $\alpha \cap \beta = \partial\alpha = \partial\beta$.

b) Suppose that $F^2 \subset M^3$ is a properly embedded surface. Define a *surgery disk* and *surgery bigon*. **[2]**

(Book) A *surgery disk* D for F is a disk embedded in M so that $D \cap F = \partial D$ and $D \cap \partial M = \emptyset$. A *surgery bigon* instead has $D \cap F = \alpha$ and $D \cap \partial M = \beta$.

c) With F, M as above: define what it means for F to be *incompressible*, *boundary incompressible*, or *essential*. **[6]**

(Book) The surface F is *incompressible* if for every surgery disk $(D, \partial D) \subset (M, F)$ we have that ∂D bounds a disk in F.

The surface F is *boundary incompressible* if for every surgery bigon $(D, \alpha, \beta) \subset (M, F, \partial M)$ we have that α cuts a bigon out of F.

To define *essential* there are three cases. If $F = S$ is a sphere then we require only that S does not bound a three-ball on either side. If $F = D$ is a disk then we only require that ∂D is essential in ∂M. If F is neither a sphere or disk then we require that F be incompressible and boundary incompressible in M.

d) Define a *handlebody* V. **[2]**

(Book) The three-ball $V_0 \cong B^3$ and the solid torus $V_1 \cong D \times S^1$ are handlebodies. In general, we obtain V_g as the boundary connect sum of g copies of V_1.

e) Show that any surface F surface properly embedded in a handlebody V either is compressible, is boundary compressible, is a disk, or is a sphere. **[6]**

(Exercise) Let $D \subset V$ be a collection of g disjoint essential disks so that $B = V - n(D)$ is a three-ball. Isotope F to be transverse to D and to minimize $|F \cap D|$. Suppose that α is an innermost simple closed curve of $F \cap D$. If the disk $E \subset D$ bounded by α is a compression disk then we are done. If it is a trivial surgery disk then we contradict the minimality of $|F \cap D|$.

So we may suppose that there are no such simple closed curves. Suppose that α is an outermost arc of $F \cap D$. If the bigon $E \subset D$ is a boundary compression for F we are done. If E is trivial then again we contradict the minimality of $|F \cap D|$. So we may we suppose that $F \cap D = \emptyset$.

So F is contained in B. If F has genus then, following the proof of Alexander's theorem, F compresses. Thus F is a disk or sphere and we are done.

f) Let T^2 be the two-torus. Classify, up to proper isotopy, all connected essential surfaces in $T \times I$. **[7]**

(Unseen, but similar to a given exercise) By an exercise in class, all essential surfaces in I-bundles may be isotoped to be *vertical*, a union of fibers, or *horizontal*, transverse to all fibers. Thus all vertical surfaces are annuli. These are determined, up to isotopy by their projection into T. The projection is a simple closed curve and thus is determined up to isotopy by its *slope*, an element of $\mathbb{Q} \cup \{\infty\}$.

Suppose now that $F \subset T \times I$ is horizontal. Let d be the intersection number between F and any generic fiber. Let A, B be vertical annuli meeting in a single fiber and cutting $T \times I$ into a ball. Thus $F \cap A$ is a collection of d loops and the same holds for B. It follows, as F is connected, that $d = 1$ and F is isotopic to $T \times \{1/2\}$.

MATHEMATICS DEPARTMENT
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Course Title: THREE-MANIFOLDS

Model Solution No: 4

a) Define *fibered solid tori.* **[3]**

(Book) Suppose that $0 \leq q < p$ are integers, with $\gcd(p, q) = 1$. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Starting with $D \times I$ we glue $D \times \{1\}$ to $D \times \{0\}$ via the map $\phi(z) = z \cdot \exp(2\pi i \cdot q/p)$ to obtain a solid torus $V_{p,q}$. Since $D \times I$ is fibered by vertical intervals (the second coordinate) the quotient V is fibered by circles. Letting $D = D \times \{0\} \subset V$, we note that the central circle meets D once while all other circles meet D exactly p times.

b) Define a *Seifert fibered space* $(M, \mathcal{F})$. Define the *singular* and *generic* fibers of $\mathcal{F}$, and their *multiplicities.* **[4]**

(Book) A partition $\mathcal{F}$ of a three-manifold M, into circles, is a *Seifert fibering* if for every circle $\alpha \in \mathcal{F}$ there is a pair (p, q), as above, and a system of regular neighborhoods of α that are all fiber homeomorphic to $V_{p,q}$.

If α has invariant (p, q) then p is the *multiplicity* of α. Furthermore, α is *singular* if $p > 1$ and *generic* if $p = 1$.

c) Briefly describe the *base orbifold* B of a Seifert fibered space $(M, \mathcal{F})$. **[4]**

(Book) We define $B = M/S^1$; that is, we form the quotient space where points are the circles of $\mathcal{F}$. The fibered solid tori in M provide charts homeomorphic to the unit disk $D^2 \subset \mathbb{C}$, except at the singular fibres, where B is modelled on D modulo the action of the p^{th} roots of unity.

d) Giving the necessary definitions, state but do not prove the structure theorem for essential surfaces in Seifert fibered spaces. **[3]**

(Book) Supose that $(M, \mathcal{H})$ is a Seifert fibered space. A surface F in $(M, \mathcal{H})$ is *vertical* if F is a union of fibers. On the other hand F is *horizontal* if the intersection of F with any fiber is transverse. All essential surfaces in M may be properly isotoped to be either vertical or horizontal. Each vertical surface is an S^1–bundle over a 1-orbifold (T, K, A, M). Each horizontal surface is an orbifold cover of B.

e) Let $X = K \tilde{\times} I$ be the orientation I–bundle over the Klein bottle K. Classify, up to isotopy, all Seifert fiberings of X. For each fibering, compute the base orbifold. **[7]**

(Unseen, but easier than a given exercise) We construct X out of the unit cube I^3 by gluing the right and left faces by translation and by gluing the front and back faces by translation followed by 180 degree rotation. Thus the intervals parallel to the z–axis give the second coordinate of $K \tilde{\times} I$. See Figure 5.

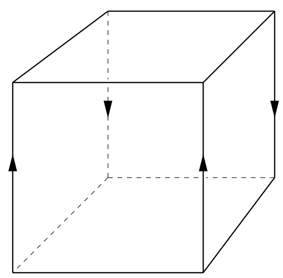

Figure 5: The orientation I–bundle over K.

Let K be the zero-section of the I–bundle structure on X. Note that K is essential and without boundary. Let $\mathcal{F}$ be any fibering of X. Thus K may, after isotopy, be made vertical with respect to $\mathcal{F}$. Note that $\mathcal{F}|K$ determines $\mathcal{F}|N(K)$. Thus, by a theorem from class, $\mathcal{F}|K$ determines $\mathcal{F}$. Finally, since there are only two isotopy classes of two-sided curves in K, there are exactly two Seifert fiberings of X; we call them $\mathcal{F}$ and $\mathcal{G}$.

The intervals parallel to the x–axis glue to give $\mathcal{F}$ while those parallel to the y–axis give $\mathcal{G}$. To compute the base orbifolds, note that the yz–plane meets every circle of $\mathcal{F}$ once while the zx–plane meets every generic fiber of $\mathcal{G}$ twice. Taking quotients, the base orbifold for $\mathcal{F}$ is a copy of the Mobius band while the base for $\mathcal{G}$ is a copy of $D(2, 2)$.

f) Using the above or otherwise, classify up to isotopy the essential annuli in $X = K \tilde{\times} I$. **[4]**

(Unseen, but easier than a given exercise) Suppose that A is an essential annulus in X. By a theorem from class, every essential annulus in X may be made vertical with respect to $\mathcal{F}$ or $\mathcal{G}$. If with respect to $\mathcal{F}$ then A covers the unique essential arc in the Mobius band. If with respect to $\mathcal{G}$ then A covers the unique essential separating arc in $D(2, 2)$.

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MATHEMATICS DEPARTMENT
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Course Title: THREE-MANIFOLDS

Model Solution No: 5

a) Suppose that $F \subset M$ is properly embedded. Define the notion of π_1–*injectivity*. **[3]**

(Book) Pick a point $z \in F$. We say F is π_1–injective if the inclusion map $i\colon F \to M$ induces an injection $i_*\colon \pi_1(F, z) \to \pi_1(M, z)$.

b) Briefly define incompressibility. **[2]**

(Book) A properly embedded surface $F \subset M$ is incompressible if every surgery disk $(D, \partial D) \subset (M, F)$ is trivial. That is, if $D \cap F = \partial D$ then ∂D bounds a disk in F.

c) Show that if F is π_1–injective then F is incompressible. **[5]**

(Book) Suppose that D is a surgery disk for F. Let $\alpha = \partial D$. It follows that $[\alpha] = 1 \in \pi_1(M, z)$. As F is π_1–injective, we find that $[\alpha] = 1 \in \pi_1(F, z)$, as well. By an exercise from class, any simple close curve in F that is trivial in $\pi_1(F, z)$ bounds a disk in F.

d) State Dehn's Lemma, the Loop Theorem and the Disk Theorem, clearly labelling each. **[6]**

(Book) Dehn's Lemma - Suppose that $\alpha \subset \partial M$ is a simple closed loop. If α bounds a singular disk in M, then it bounds an embedded disk.

Loop Theorem - Suppose that $F \subset \partial M$ is a component. If F is not π_1–injective then there is an essential simple closed curve in the kernel.

Disk Theorem - Suppose that $F \subset \partial M$ is a component. If F is not π_1–injective then F is compressible.

e) Deduce the Disk Theorem from Dehn's Lemma and the Loop Theorem. **[3]**

(Exercise) Suppose that F is not π_1–injective. Let α be a nontrivial element of the kernel. By the Loop Theorem there is an essential simple closed loop β that also lies in the kernel. By Dehn's lemma β bounds a disk in M. This gives a compressing disk for F.

f) Suppose that M is connected and irreducible. Suppose that $T \subset \partial M$ is a two-torus. Suppose that $\pi_1(M) \cong \mathbb{Z}$. Show that M is a solid torus. **[6]**

(Exercise) Note that $\pi_1(T) \cong \mathbb{Z}^2$. Since $\mathbb{Z}^2$ does not inject into $\mathbb{Z}$ the map i_* has kernel. Choose an essential simple closed curve μ in the kernal. By Dehn's Lemma the loop μ bounds a disk D in M. Let T_D be the result of compressing T along D. So T_D is a two-sphere. As M is irreducible, T_D bounds a three-ball B. Note that $D \cap B = \emptyset$, as otherwise B contains T, a boundary component. Thus, we have that $B \cup N(D) \cong D \times S^1$, as desired.