MA3F2: Knot Theory

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Year 3/4 - Term 2

Contents

Chapter 1

Introduction

Definition 1. A knot is a loop in 3–space without self-intersections.

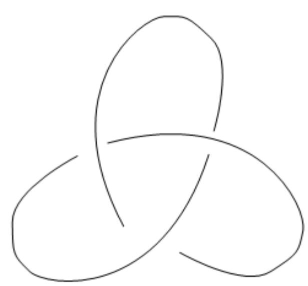
Considering particularly complicated knots in this form can be extremely difficult, so ideally we would like a lower dimensional form in which to discuss knots.

Definition 2. The shadow of a knot is its projection into $\mathbb{R}^2$.

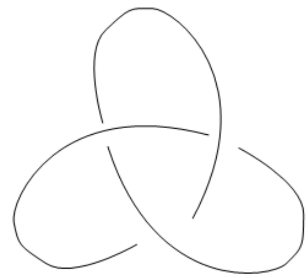
Definition 3. A diagram is a shadow with crossing information (indicating which strand passes over and which under) at the vertices.

Example 1. The following are knot diagrams:

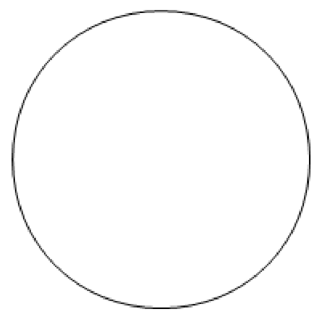
(i) The left trefoil:



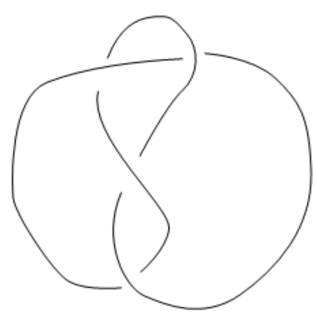
(ii) The right trefoil:



(iii) The unknot:



(iv) The figure eight:

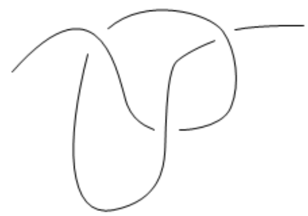


The definition of a diagram can be stated more precisely as:

Definition 4. A diagram is a smooth 4–valent graph in $\mathbb{R}^2$ that is the shadow of a knot, with all vertices transverse and with crossing information (distinguishing between over and under) at vertices.

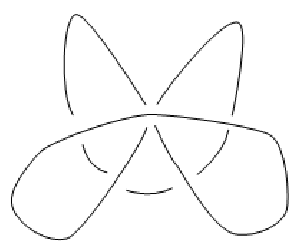
Example 2. The following are not diagrams:

(i) This is not a loop:

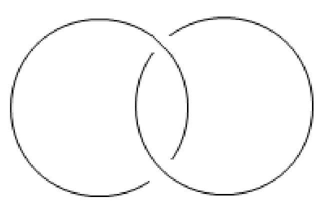


(ii) This has a cusp, a 2–valent vertex:

(iii) This has a 6–valent point, so we don't have accurate crossing information.



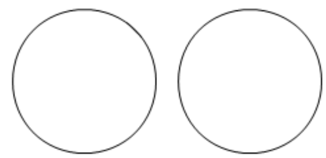
(iv) This is not the shadow of a knot.



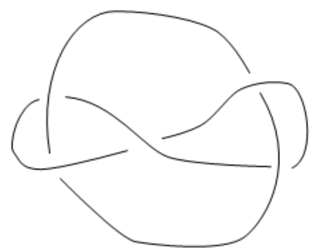
Definition 5. A link is the union of disjoint knots.

Example 3. The final object in the previous example is called the left Hopf link. Other links are:

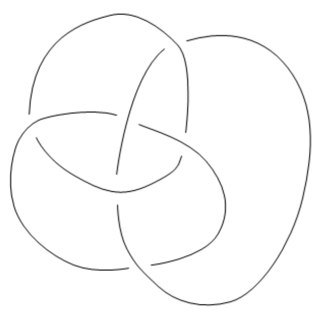
(i) The unlink:



(ii) The Whitehead link:



(iii) Borromean rings:

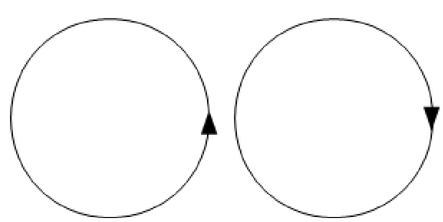


Definition 6. The knots comprising a link are called the components of the link.

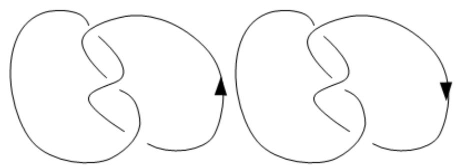
Example 4. A knot is a link with one component.

Definition 7. A knot has two possible orientations, clockwise or counter-clockwise.

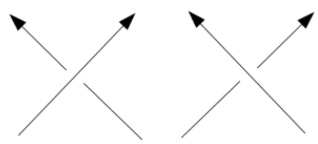
Example 5. Observe orientations on the unknot:



and on the right trefoil:



A link of n components has 2^n possible orientations, as we have 2 choices of orientation for each of the n knots. We adopt the convention that if a knot or link is oriented then every crossing of the diagram receives a handedness:



The first is right-handed and we say it has sign $+1$. The second is left-handed and we say it has sign -1.

Example 6. Every crossing in a diagram of DNA is right-handed.

Definition 8. Suppose that D is an oriented diagram. Define the writhe of D, $w(D)$, to be the sum of signs of crossings in D.

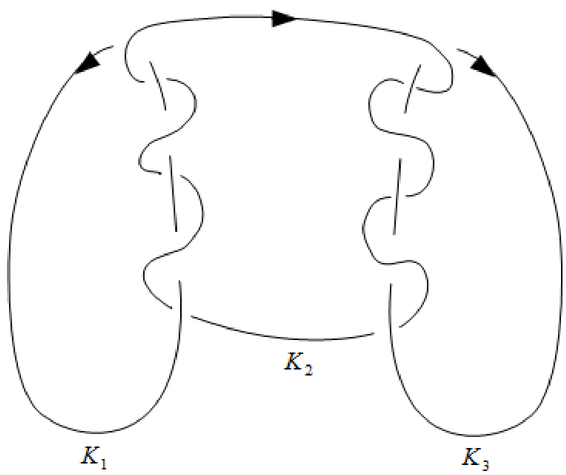
Example 7. The standard diagrams of the unknot and the figure eight shown above have writhe 0. The diagrams of the right-trefoil above have writhe -3 independent of the orientation. In fact, writhe is independent of the orientation in general.

Definition 9. Suppose K, K' are components of a link L. Define:

$$\mathrm{lk}(K, K') = \frac{1}{2} \sum \mathrm{sign}(c)$$

where c ranges over crossings between K and K'.

Example 8. Consider:



Then $\mathrm{lk}(K_1, K_3) = 0$, $\mathrm{lk}(K_1, K_2) = 3$ and $\mathrm{lk}(K_2, K_3) = -2$.

Remark. $\mathrm{lk}(K, K') \in \mathbb{Z}$ for any loop components.

Chapter 2

Reidemeister Moves

Definition 10. A knot K is isotopic to a knot K' (we write $K \backsimeq K'$) if there is a continuous motion of $\mathbb{R}^3$ mapping K onto K'.

Remark. This is an equivalence relation.

Example 9. Let $U = \{(x, y, 0)|x^2 + y^2 = 1\}$. This is isotopic to any simple closed curve in a plane embedded into $\mathbb{R}^3$.

Definition 11. The knot type of a knot K is:

$$[K] = \{K' \text{ a knot}|K' \backsimeq K\}$$

Both of the previous definitions apply to links. We now give some foundational results, mostly due to Reidemeister. We state the results for links, as knots are simply a special class of link.

Theorem 1 (Existence Theorem)**.** *For every link L there is an isotopic link L' such that $\pi_{xy}(L')$ is a diagram. $\pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2$ is projection to the xy–plane, adding crossing information.*

Lemma 2 (Existence of a Link)**.** *For every diagram D there is a link L so that $\pi_{xy}(L) = D$.*

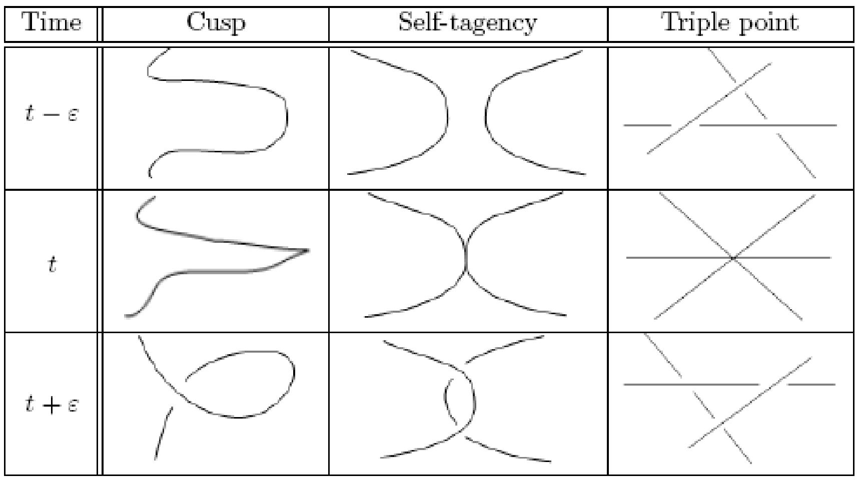
Proof. Given D a diagram in $\mathbb{R}^2$, include $\mathbb{R}^2$ into $\mathbb{R}^3$ as the xy–plane. We need to "fix" the crossings. Pick a disk $\mathbb{B}^2$ about a crossing. Let $\mathbb{B}^3$ be a ball with $\mathbb{B}^2$ as an equatorial disk. Label the arcs leaving the crossing as NE, NW, SE and SW. WE have crossing information, so assume without loss of generality that the SW to NE arc goes over. We change this arc so that it traverses over the surface of $\mathbb{B}^3$ and change the other arc so it traverses under the surface of $\mathbb{B}^3$, rounding the corners to avoid cusps. This gives the required link. $\square$

Lemma 3 (Uniqueness of a Link)**.** *If L, L' are links and $\pi_{xy}(L) = \pi_{xy}(L')$ as diagrams, then $L \backsimeq L'$.*

In order to simply study diagrams instead of loops, we must rephrase isotopy in this form. Suppose $\{K_t | t \in [0,1]\}$ is an isotopy between K and L, that is $K_0 = K$, $K_1 = L$ and $K_{t+\varepsilon}$ is a small deformation of K_t, giving $K \simeq L$. Let $D_t = \pi_{xy}(K_t)$. The problem is that K_t may not project to a diagram for some t.

Definition 12. Let Σ be the set of $t \in [0,1]$ such that $\pi_{xy}(K_t)$ is not a diagram.

Remark. Generically, Σ is finite, and the elements are of three types; in which the diagrams have cusps, self-tangencies or triple-points. We observe:



Definition 13. If D, D' are diagrams related as $t \pm \varepsilon$ in the above remark, then D, D' differ by a Reidemeister move, named R_1 for the cusp, R_2 for the self-tangency and R_3 for the triple point.

Definition 14. Say that D_t and $D_{t'}$ differ by R_0 if $[t, t'] \cap \Sigma = \varnothing$.

Example 10. Take a trefoil:

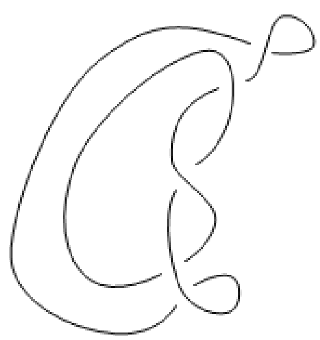
Apply R_0:

then R_2:

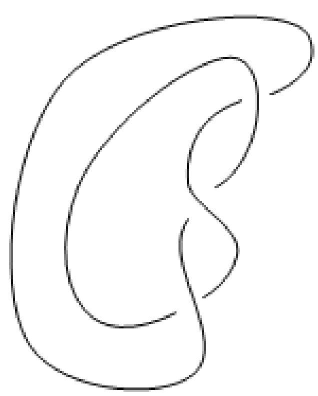
then R_3:

then R_1:

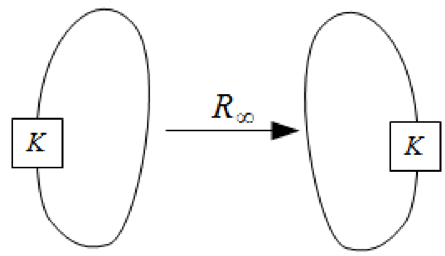
then R_2:



and finally R_1:



Remark. The operation above can be defined in general as R_∞ by the figure:



Every R_∞ is a sequence of R_0, R_1, R_2 and R_3 moves.

Theorem 4 (Reidemeister)**.** *Suppose K, K' have projections D, D'. Then $K \simeq K'$ iff D can be taken to D' by a sequence of Reidemeister moves.*

Definition 15. Any function $\phi : \mathcal{K} \to \mathbb{Z}$, where $\mathcal{K}$ is the set of knot types, is called a knot invariant.

Proposition 5. *Suppose we have a function on $\mathcal{D}$, the set of diagrams with one component, $\varphi : \mathcal{D} \to \mathbb{Z}$, and suppose that if $D \xrightarrow{R_i} D'$ then $\varphi(D) = \varphi(D')$. Then by the Reidemeister theorems, φ ascends to be a knot invariant.*

Remark. The idea here is that knots up to isotopy are in some sense the same as diagrams up to Reidemeister moves.

We wish to understand knots and links via their invariants.

Theorem 6. *If L, L' are link diagrams and $L \xrightarrow{R_i} L'$ then there is a bijection of components $C_i \to C_i'$ of L and L' such that $\mathrm{lk}(C_i, C_j) = \mathrm{lk}(C_i', C_j')$.*

Proof. It suffices to check the Reidemeister moves. R_1 does not introduce crossings between distinct components, so the linking number is unchanged. R_2 introduces a crossing of each handedness, so the linking number is unchanged. Observing the diagram of R_3, it is easy to construct a bijection between crossings that preserves the handedness and the components involved. $\square$

So linking number is a link invariant.

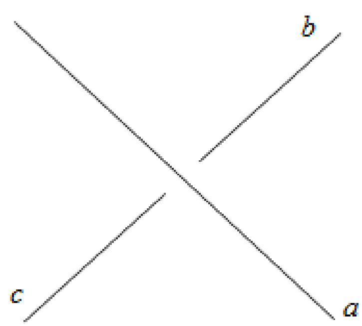
Example 11. The Hopf link has linking number ± 1 (depending on orientation) and the Whitehead link has linking number 0. So the Hopf link is not isotopic to the Whitehead link, and the Hopf links with different orientations are not isotopic.

Chapter 3

Link Colouring

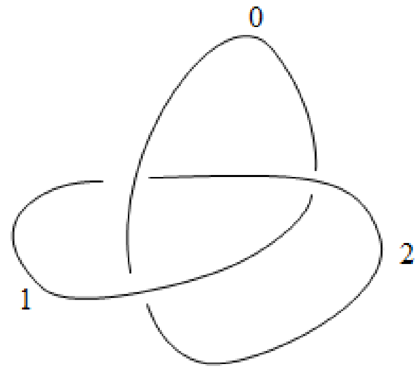
Definition 16. $a \subseteq D$ is an arc of D if it is a maximal path in the graph with all crossings over. $A(D)$ is the set of arcs in D.

Definition 17. A function $x : A(D) \to \mathbb{Z}/n\mathbb{Z}$ is a colouring (modulo n) if at every crossing:

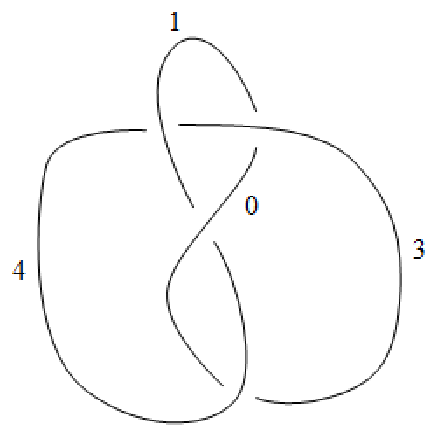


we have $2x(a) \equiv x(b) + x(c) \mod n$.

Example 12. The following is a colouring modulo 3:



and the following is a colouring modulo 5:



Definition 18. A colouring x is trivial if it is a constant function.

Remark. A constant function is always a colouring, as $2p \equiv p + p \mod n \,\forall\, p, n$.

Lemma 7. *The set of colourings under addition is a group.*

Lemma 8. *The set of trivial colourings under addition is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

Definition 19. We define $\mathrm{Col}(D, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Col}_n(D)$ to be the set of colourings quotiented by the trivial colourings.

Theorem 9. *If $D \xrightarrow{R_i} D'$ and $\mathrm{Col}_n(D) \neq \{0\}$ then $\mathrm{Col}_n(D') \neq \{0\}$.*

Proof. Let m be the colour of an arc. Performing R_1 on the arc gives us two arcs of colour m, so we do not change the trivialness or otherwise in either direction. Given two arcs of colours m and n respectively, when we perform R_2 we get a third arc which must be coloured $2n - m$. If $n = m$ then $2n - m = m$ also, so both colourings are trivial. With R_3 we only change the colouring of one arc, so both colourings are either trivial or both are non-trivial. $\square$

Corollary 10. *If D is a diagram of $K \in [U]$ (that is a diagram of the unknot) then* $\mathrm{Col}(D) = \mathbf{1}$.

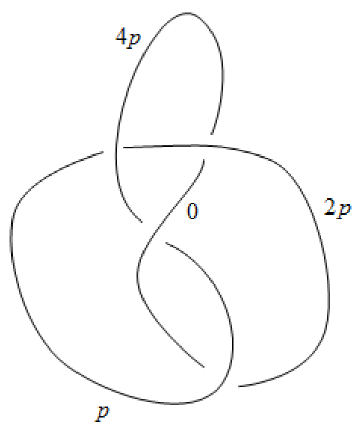
Corollary 11. *The unknot is not isotopic to the trefoil or to the figure-eight, as both have non-trivial colourings.*

Lemma 12. *If x is a colouring of D, and x is non-trivial, then for any arc $a \in A(D)$ there is a non-trivial colouring $y : A(D) \to \mathbb{Z}/n\mathbb{Z}$ so that $y(a) = 0$.*

Proof. Define $y(b) = x(b) - x(a)$ for any arc $b \in A(D)$. Clearly $y(a) = 0$. $\mathrm{Col}_n(D)$ is a group, so y is a colouring. If $x(c) \neq x(d)$ then:

$$y(c) = x(c) - x(a) \neq x(d) - x(a) = y(d) \qquad \square$$

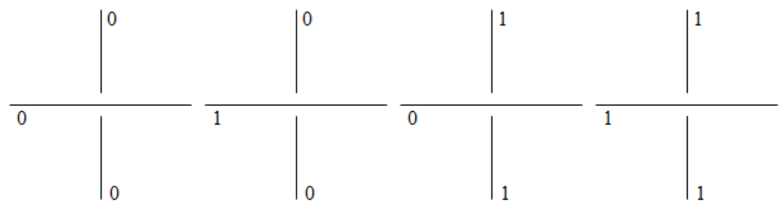
Example 13. Take the figure-eight knot, and colour two arcs that meet at a crossing 0 and p respectively, assuming $p \neq 0$. We must complete the colouring as below:



Considering crossings, we find $5p \equiv 0$. So for a non-trivial colouring in any modulus n, $5p \equiv 0 \mod n$. Hence $n \mid 5p$, and so either $5 \mid n$ or $n \mid p$. To see this, $5p \equiv 0 \mod n$ iff $\exists\, k$ such that $5p = nk$. Since 5 is prime, $5 \mid n$ or $5 \mid k$. If $5 \mid k$, then $p = \frac{nk}{5}$, and $\frac{k}{5} \in \mathbb{Z}$, so $n \mid p$. So any non-trivial colouring modulo n of the figure-eight must have $5 \mid n$ as if $n \mid p$ then $p \equiv 0 \mod n$ and the colouring was trivial. If D and D' differ by a Reidemeister move and D has a non-trivial 3–colouring then so does D'. We saw in an earlier example that a diagram of the trefoil has a non-trivial 3–colouring, and this diagram of the figure-eight cannot. So the trefoil is not isotopic to the figure-eight.

Proposition 13 (Colouring Links)**.** *Suppose L has $n \geq 2$ components. Then L has a non-trivial 2–colouring.*

Proof. In a 2–colouring we must colour the crossings in one of the following four ways:



Thus every component is monochromatic (note that a knot only has trivial 2–colourings). ☐

Definition 20. A link is split if L does not intersect the yz–plane and L meets the sets:

$$H^+ = \{(x, y, z)|x > 0\} \qquad H^- = \{(x, y, z)|x < 0\}$$

Definition 21. L is splittable if $L \backsimeq L'$ with L' split.

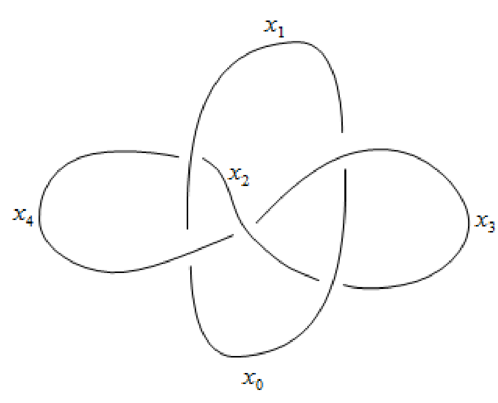
Example 14. The Hopf link is not splittable. If the Hopf link was isotopic to a split link it would have linking number 0, which it does not.

Proposition 14. *If L is splittable then L' has non-trivial colourings in every modulus.*

Proof. Split L. Colour each piece $L^+ = L \cap H^+$ and $L^- = L \cap H^-$ monochromatically, and then undo the Reidemeister moves, to rejoin the two links. ☐

Corollary 15. *The Borromean rings are not splittable as they do not have a non-trivial colouring in every modulues.*

Example 15. Consider the Whitehead link:



We find that:

$$\begin{aligned}
x_0 + x_1 &= 2x_3 \\
x_1 + x_2 &= 2x_4 \\
x_2 + x_3 &= 2x_0 \\
x_3 + x_4 &= 2x_2 \\
x_4 + x_2 &= 2x_1
\end{aligned}$$

Say $x_0 = 0, x_3 = p$. Then we get:

$$\begin{aligned}
x_1 &= 2p \\
x_2 &= -p \\
x_4 &= -3p \\
8p &= 0
\end{aligned}$$

Theorem 16. *Suppose that D is a diagram. Then the system of colouring equations has a dependency.*

Remark. This is similar to the earlier lemma, that for any arc a, if there exists a non-trivial colouring, then there is one with $x(a) = 0$.

Remark. In fact, we will show the stronger result that we can remove any one of the equations without losing information.

Recall that the shadow of a knot is the projection onto the xy–plane, forgetting the crossing information. The crossings are the vertices of the shadow, and the shadow minus the vertices are the edges.

Proposition 17. *The number of edges is twice the number of vertices whenever we have at least one vertex.*

Definition 22. If S is a shadow then the components of $\mathbb{R}^2 \setminus S$ are called faces, or regions.

Definition 23. Two regions are adjacent if their boundaries share an edge.

Proposition 18. *Every shadow has a checkerboard colouring, that is a 2–colouring of the regions so that adjacent regions have different colours.*

Proof. Define the parity of a region $e(R) \equiv |L \cap S| \mod 2$ where L is a ray based at a point $x \in R$ and L is transverse to S, so that there are no tangencies or triple points. We first show that parity is well-defined. Observe that if $x, y \in R$ then there is a polygonal path $P = \{x = x_0, x_1, x_2, ..., x_n = y\}$ in R connecting x and y. Let $e(x, L)$ be the parity of L based at x. Suppose L' is also based at x. We have $\{L_t | t \in [0,1]\}$, a family of rays such that $L_0 = L$, $L_1 = L'$, L_t is based at x for all t, and $L_{t+\varepsilon}$ is a small rotation of L_t. Let:

$$\Sigma = \{t \in [0,1] | L_t \text{ not transverse to } S\}$$

If at t, L_t is tangent to S, then at $t-\varepsilon$ there were no intersections with S local to the tangency, and at $t+\varepsilon$ there were two, or vice versa. If at t, L_t has a triple point, then there are two local intersections at both $t+\varepsilon$ and $t-\varepsilon$. So the parity does not change as t passes through Σ. Hence $e(x, L)$ is independent of the choice of L. Also if $x, y \in R$ then $e(x) = e(y)$; let $P = \{x = x_0, x_1, ..., x_n = y\}$ be a polygonal path. We make small permutations to x_i such that the ray L_i based at x_i and through x_{i+1} is transverse. Observe that L_i and $\overline{L_i \setminus [x_i, x_{i+1}]}$ give the same parity because $[x_i, x_{i+1}] \cap S = \varnothing$. So:

$$e(x_i, L_i) = e(x_{i+1}, \overline{L_i \setminus [x_i, x_{i+1}]}) = e(x_{i+1}, L_{i+1})$$

as above. So $e(x) = e(y)$. So $e(R)$ is well-defined. We now claim that if R, R' are adjacent then $e(R) \equiv e(R') + 1$. To see this, pick $x \in R$ and $y \in R'$ close to the shared edge, so that the ray L based at x and through Y is transverse. Since we may arrange that $[x, y]$ is a single point, the result is clear. $\square$

A shadow can be connected or disconnected, in the normal sense. A disconnected shadow implies that the link is splittable.

Definition 24. We say that a shadow D is reducible if there is a region R which is self-adjacent across a vertex.

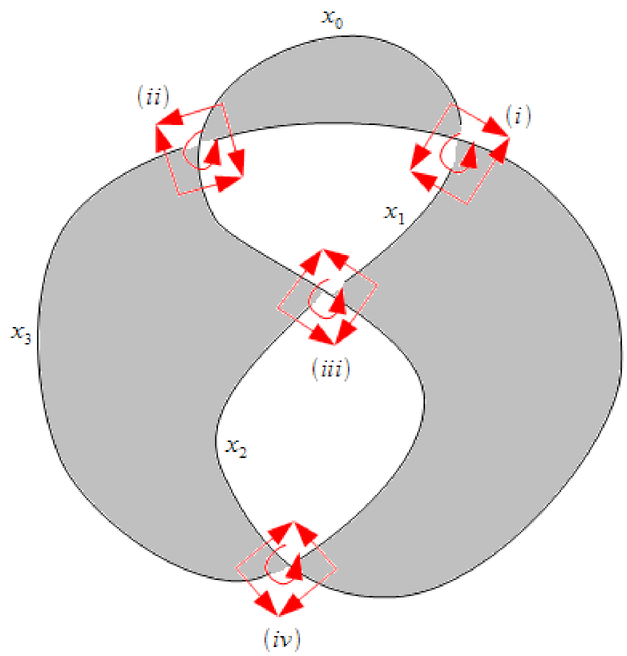
Theorem 19 (Jordan Curve Theorem)**.** *Any embedded loop in $\mathbb{R}^2$ is planar isotopic to a round circle.*

Proposition 20. *If a link L has diagram D then L is isotopic to a link L' with irreducible, or reduced, diagram D'.*

Proof. If D is reducible then there is a loop γ contained in some region R except for at a single crossing of the diagram. γ bounds a disk B. Let A be the cylinder $A = \pi_{xy}^{-1}(B)$. Inside of A, rotate L by π, perpendicular to the xy–plane. This reduces the number of crossings by 1. Repeat until D is reduced. $\square$

Around each crossing we may draw a small square that contains no other crossings. Orient these squares anti-clockwise. Pick a 2–colouring of the faces of D. Near the crossing, we have two choices for how to colour opposite pairs of faces. We now orient the edges of the square toward one colour, which we call light; the other colour will be called dark.

Example 16. Consider the following diagram:



We now consider the crossings. We add the arcs with positive sign if the orientation of the square agrees with the orientation of the edge passing through the arc, and with negative sign if they disagree. So we obtain:

(i) $-x_0 + x_2 - x_1 + x_2 = 0$

(ii) $+x_1 - x_0 + x_2 - x_0 = 0$

(iii) $-x_2 + x_0 - x_3 + x_0 = 0$

(iv) $+x_3 - x_2 + x_0 - x_2 = 0$

Or in matrix form:

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & -1 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Each row is a colouring equation, and adding them gives 0. So we have a dependency.

We now show this in general, proving the earlier theorem starting that there was a dependency in the colouring equations.

Proof. Consider an arc between two squares. On one side of the arc the region is coloured light and on the other side the region is coloured dark. So the edges at the end of the arc are oriented in the same direction, but are on opposite sides of their respective squares. So they contribute the value of the arc with positive and negative sign respectively. □

Definition 25. An overcrossing component $C \subseteq D$ is a component which only crosses over, so it consists of a single arc in a loop.

Proposition 21. *If we have an overcrossing component then C is splittable.*

Lemma 22. *If there are no overcrossing components then the number of arcs in D is equal to the number of crossings, so the colouring matrix is square.*

Proof. Orient D and assign every arc to the crossing at its head. This is a bijection. □

Definition 26. Let D be a diagram without overcrossing components. Let A_+ be the colouring matrix. Let A be the matrix obtained from A_+ by deleting any one column and any one row.

Example 17. For the trefoil we obtain the colouring equations:

(i) $x_0 + x_1 - 2x_2 = 0$

(ii) $x_1 + x_2 - 2x_0 = 0$

(iii) $x_2 + x_0 - 2x_1 = 0$

Then:

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

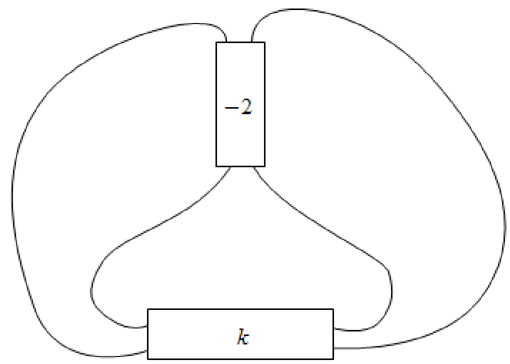
We may take:

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

which has determinant 3. We find that any choice of A gives a determinant of ± 3.

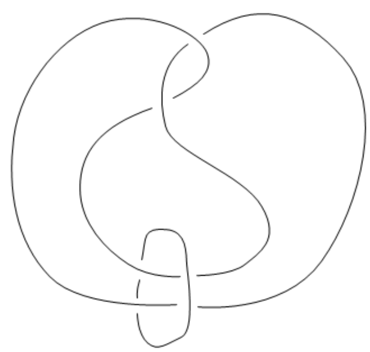
Definition 27. If L is a link with diagram, define $\det(L) = |\det(A)|$, where A is any reduced colouring matrix for D. This is the determinant of L.

Definition 28. The k–th twist knot K_n is:



where the a twist box marked n contains n right twists (a negative integer indicates left twists).

Example 18. Taking a Thorsten geometric limit of k–th twist knots, we get:



which is the Whitehead link. We can obtain the Borromean rings in a similar way by replacing the -2 twist box with an l twist box and taking both k and l to ∞.

Proposition 23. *D has a colouring mod n iff there are $\overline{x}, \overline{b} \in \mathbb{Z}^d$ solving $A\overline{x} = n\overline{b}$, where $d = |A(D)| - 1$.*

Now we have two cases. If $\det(A) = 0$, pick $\overline{y} \in \mathbb{Q}^d$ in the kernel, and let $y_i = \frac{p_i}{q_i}$ in lowest form. Let $Q = \mathrm{lcm}\{q_i\}$ and let $\overline{z} = Q\overline{y}$. Note that $\gcd\{z_i\} = 1$ as the y_is were in lowest form. As $A\overline{z} = 0$ and $\overline{z} \not\equiv \overline{0} \mod n$, D has non-trivial colourings in all moduli $n \geq 2$. If $\det(A) \neq 0$, fix any b, n. By Cramer's rule:

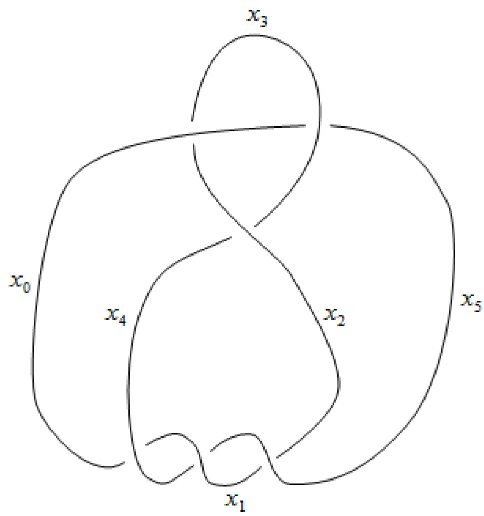
$$x_i = \frac{\begin{vmatrix} A_1 & A_2 & \cdots & nb & \cdots & A_d \end{vmatrix}}{\begin{vmatrix} A_1 & A_2 & \cdots & A_i & \cdots & A_d \end{vmatrix}}$$

gives $\overline{x}$ solving $A\overline{x} = n\overline{b}$. More simply:

$$\frac{n\begin{vmatrix} A_1 & \cdots & b & \cdots & A_d \end{vmatrix}}{\det(A)}$$

If $\det(A) = \pm 1$ then D only has trivial colourings. If $|\det(A)| \neq 1$ then let $n = |\det(A)|$. We may start guessing various $\bar{b}$s to find a colouring.

Example 19. Consider the 6_2–knot:



We obtain the colouring matrix:

$$A_+ = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

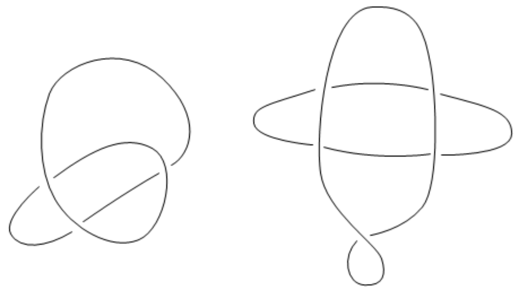
We may take the reduced matrix as:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

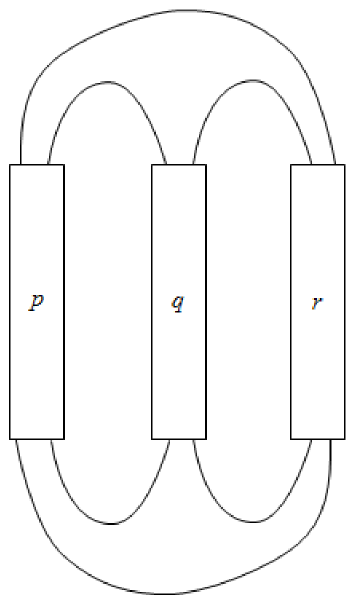
$\det(A) = 11$. We find that $x = (0, 3, 6, 5, 7, 10)$ is a colouring.

Definition 29. A diagram is alternating if every arc crosses over exactly one crossing.

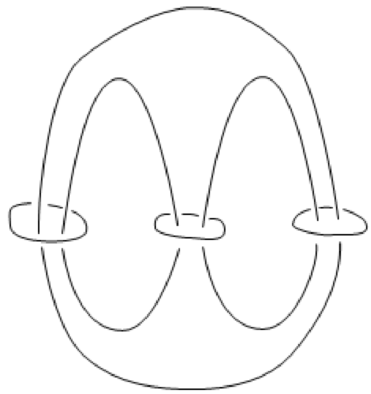
Example 20. The usual diagrams of the trefoil and the 6_2–knot are alternating. The following diagrams of the unknot and unlink are non-alternating:



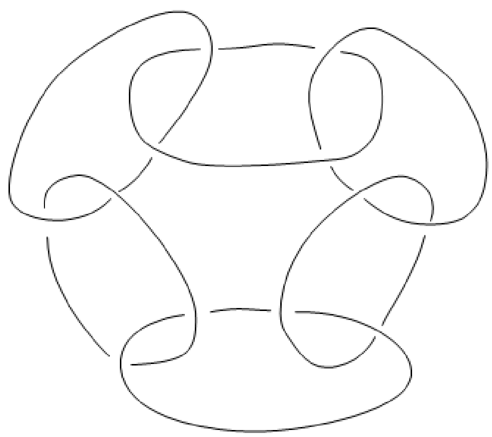
Definition 30. $P(p, q, r)$ is the (p, q, r)–pretzel knot:



with Thorsten geometric limit:



as $p, q, r \to \infty$. This can be drawn equivalently as:



3.1 Smith Normal Form

Recall that if $A \in M_d(\mathbb{Z})$ then there are $R, C, B \in M_d(\mathbb{Z})$ such that R and C are isomorphisms, B is diagonal and $B = RAC$.

Theorem 24. $\mathrm{Col}_n(D) \neq \{0\}$ *iff* $\gcd\{n, \det(D)\} > 1$.

Proof. Let A_+ and A be defined as above. Smith normal form gives R, C, B such that:

$$\begin{tikzcd} \mathbb{Z}^d \arrow[r, "A"] & \mathbb{Z}^d \arrow[d, "R"] \\ \mathbb{Z}^d \arrow[u, "C"] \arrow[r, "B"'] & \mathbb{Z}^d \end{tikzcd}$$

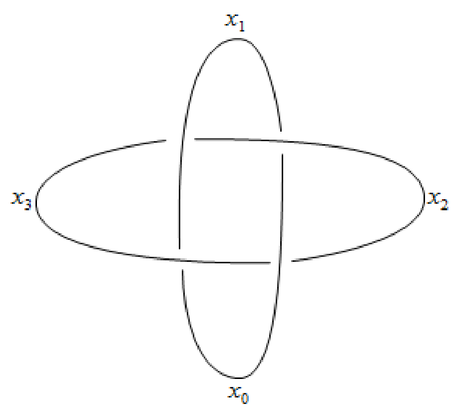
commutes, R and C are isomorphisms and B is diagonal. Reducing mod n we get:

$$\begin{array}{ccc} \mathbb{Z}_n^d & \xrightarrow{A_n} & \mathbb{Z}_n^d \\ C_n \uparrow & & \downarrow R_n \\ \mathbb{Z}_n^d & \xrightarrow[B_n]{} & \mathbb{Z}_n^d \end{array}$$

which must also commute. So C_n is an isomorphism and sends $\ker(B_n)$ to $\ker(A_n)$. Since B_n is diagonal, $\ker(B_n) \neq \{0\}$ iff some diagonal entry b_i in B_n has $\gcd\{b_i, n\} > 1$. Now $\det(A) = \det(RAC) = \det(B) = \prod_{j=1}^{d} b_j$. Thus $\exists$ a non-trivial colouring mod n iff $\exists b_i$ such that $\gcd\{b_i, n\} > 1$, iff $\gcd\{\det(A), n\} > 1$. $\square$

Thus we now have a method for computing colourings.

Example 21. Consider the following link:



Using our old methods we find that if x is a non-trivial colouring mod n then $2|n$. Using our new method we find the colouring matrix:

$$A_+ = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

We may take the reduced matrix as:

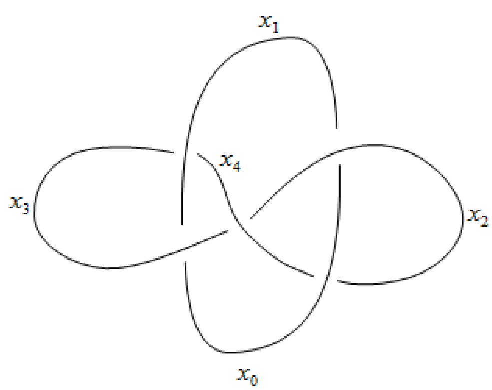
$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Using row and column operations:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

so $\det(A) = \pm 4$. Hence $\det(D) = |\det(A)| = 4$.

Example 22. Consider the Whitehead link:



with colouring matrix:

$$A_+ = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

We may reduce to:

$$A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Using row and column operations:

$$\begin{bmatrix} 2 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 2 & 0 & -1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

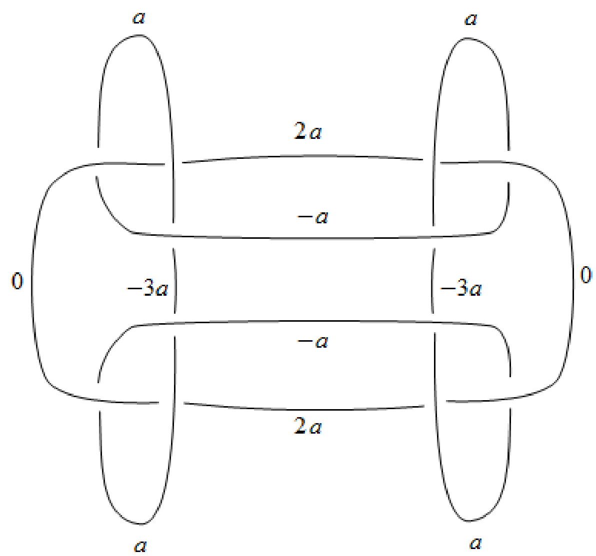
$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

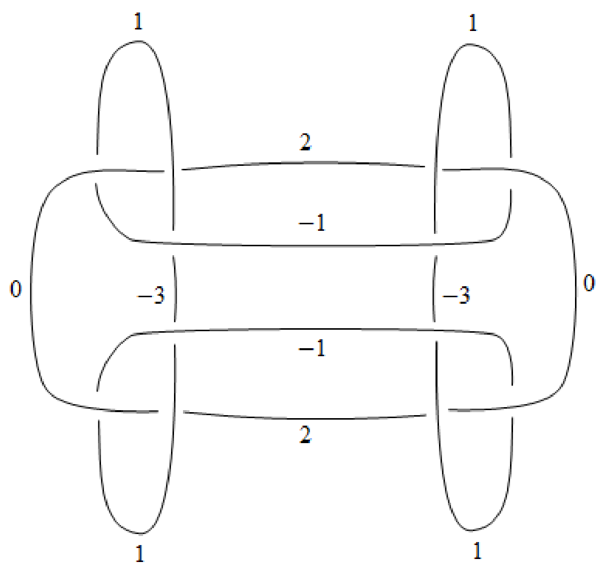
$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

so $\det(D) = 8$.

Example 23. Consider the boundary link L:



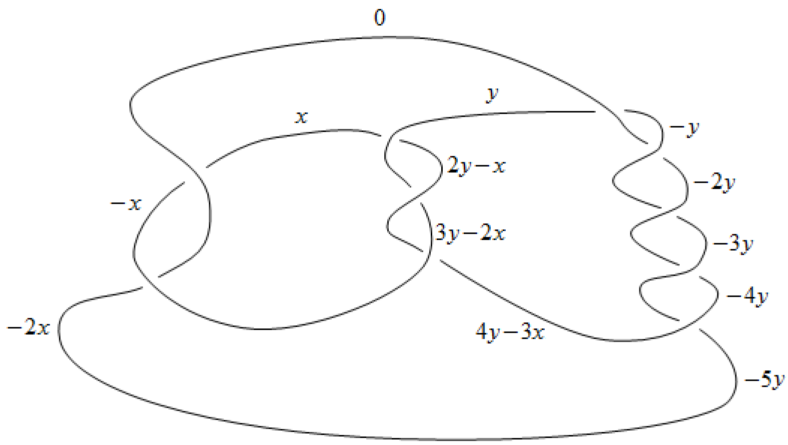
We use the symmetries to find the above colouring quickly. We have the colouring:



This is a colouring over $\mathbb{Z}$. So L has colourings in every modulus. Hence $\det(L) = 0$.

Remark. The unlink also has determinant 0, but clearly L above is not the unlink.

Example 24. Consider the pretzel knot $P(-2, 3, 5)$:



On three arcs we have two labels, so we find:

(i) $2x = 5y$

(ii) $3y = x$

(iii) $8y = 3x$

(iii) is dependent on (i) and (ii), as expected. We find:

$$5y = 6y$$

so $y = 0$. Then $x = 3 \cdot 0$, so $x = 0$. Thus $P(-2, 3, 5)$ only has trivial colourings. Hence $\det(P) = 1$.

Remark. The unknot U has deteminant 1, but clearly $P \not\simeq U$.

Suppose a diagram D has no overcrossing component. Thus $|D| = d + 1$ is the number of arcs, which is equal to the number of crossings.

Definition 31. The colouring group of D (and so of L) is $\mathrm{Col}(D) = \mathrm{Col}(L)$ given by:

$$\mathrm{Col}(D) := \langle x_1, ..., x_d | x_i + x_j = 2x_k \rangle$$

where x_i, x_j and x_k meet at a crossing with x_k the overcrossing arc.

Example 25. For the trefoil, we get the group:

$$\begin{aligned} \langle x_1, x_2 | x_2 = 2x_1, x_1 = 2x_2, x_1 + x_2 = 0 \rangle &\cong \langle x_1 | -x_1 = 2x_2 \rangle \\ &\cong \langle a | 3a = 0 \rangle \\ &\cong \langle a | 3a \rangle \\ &\cong \mathbb{Z}/3\mathbb{Z} \end{aligned}$$

Proposition 25. $\mathrm{Col}(D)$ *is an isotopy invariant, or more precisely, the isomorphism type of* $\mathrm{Col}(D)$ *is an isotopy invariant.*

$\mathrm{Col}(L)$ is a cokernel:

$$\mathbb{Z}^d \xrightarrow{A^T} \mathbb{Z}^d \longrightarrow \mathrm{Col}(L) \longrightarrow 0$$

where the first $\mathbb{Z}^d$ refers to the crossings and the second to arcs. $B = RAC$ so $B^T = C^T A^T R^T$. So we have:

$$\begin{tikzcd}
\mathbb{Z}^d \arrow[r, "A^T"] & \mathbb{Z}^d \arrow[r] \arrow[d, "C^T"'] & \mathrm{Col}(L) \arrow[r] \arrow[d, "\text{isomorphism}"'] & 0 \arrow[d] \\
\mathbb{Z}^d \arrow[r, "B^T"] \arrow[u, "R^T"] & \mathbb{Z}^d \arrow[r] & \mathbb{Z}_{b_1} \times \mathbb{Z}_{b_2} \times \cdots \times \mathbb{Z}_{b_d} \arrow[r] & 0
\end{tikzcd}$$

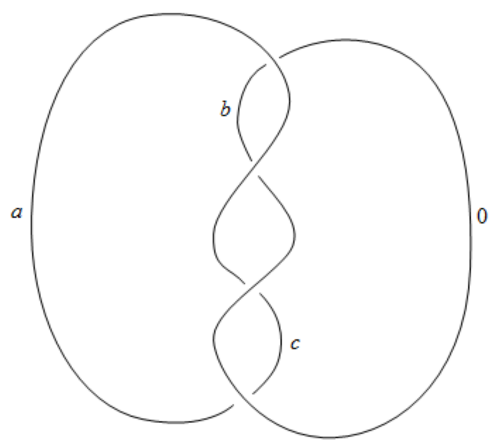
Here $\mathbb{Z}_b = \mathbb{Z}/b\mathbb{Z}$.

Corollary 26.

$$\det(L) = \begin{cases} |\,\mathrm{Col}(L)|, & |\,\mathrm{Col}(L)| < \infty \\ 0, & |\,\mathrm{Col}(L)| = \infty \end{cases}$$

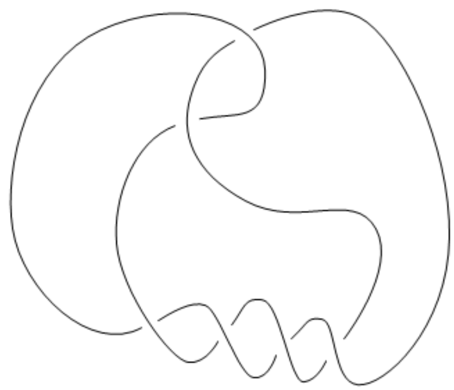
Corollary 27. *Since* $\mathrm{Col}(L)$ *is an isotopy invariant, so is* $\det(L)$*.*

Example 26. Consider the following diagram of the unlink:



We find the group $\langle a, b, c | a + c, b - 2a \rangle \cong \langle a, c | a + c \rangle \cong \langle a \rangle \cong \mathbb{Z}$. The boundary link from the earlier example has colouring group $\mathbb{Z} \times (\mathbb{Z}_3)^2$, so it is not the unlink, and hence not splittable, as each component is isotopic to the unknot.

Example 27. The 6_1–knot:



and the $9_{46} = P(-3,3,3)$ both have determinant 9, but $\mathrm{Col}(6_1) = \mathbb{Z}_9$ and $\mathrm{Col}(9_{46}) = (\mathbb{Z}_3)^2$.

Proposition 28. *If D is a diagram of L then the number of inequivalent colourings mod n is $\prod_{i=1}^{d} \gcd\{n, b_i\}$.*

Chapter 4

Mirrors and Invertability

4.1 Mirror Images

Definition 32. Suppose L is a link and $M \subseteq \mathbb{R}^3$ is an affine plane. Define $m(L)$ to be the reflection of L through the plane M.

Note. $m(m(L)) = L$.

Proposition 29. *The isotopy class* $[m(L)]$ *is independent of the choice of* M.

Proof. Suppose M, M' are any two planes. Let $m(L)$ and $m'(L)$ be the resulting links. Let φ be the result of reflecting in M and then in M'. We have two cases.

Case 1: M and M' are parallel. If M and M' are parallel then φ is a translation by distance $2d_{\mathbb{R}^3}(M, M')$ perpendicular to M in the direction from M to M'.

Case 2: $M \cap M'$ is a line. φ is a now a rotation about $M \cap M'$ through angle $2(\widehat{M, M'})$, the dihedral angle.

In either case, φ is a rigid motion of $\mathbb{R}^3$ so $m(L)$ is isotopic to $m'(L)$ and φ sends $m(L)$ to $m'(L)$. □

Proposition 30. *If* $L_0 \backsimeq L_1$ *then* $m(L_0) \backsimeq m(L_1)$.

Proof. If $\{L_t | t \in [0,1]\}$ is an isotopy between L_0 and L_1 then:

$$\{m(L_t) | t \in [0,1]\}$$

is the desired isotopy. □

Definition 33. Let $\overline{D}$ be the diagram D with all crossings reversed.

Proposition 31. *If* D *is a diagram of* L *then* $\overline{D}$ *is a diagram of* $m(L)$.*

*We sometimes write $\overline{D}$ as $m(D)$.

Proof. We have a canonical form for L given earlier, where the arcs lie in the xy–plane, except at crossings, where they traverse over and under a small ball around the crossing. Reflecting this in the xy–plane preserves the arcs and reverses the crossings. □

Definition 34. Say that L is achiral if $L \backsimeq m(L)$. Else L is chiral.

Example 28. The figure eight is achiral, as are the knots 6_3, 8_3, 8_9, 8_{12}, 8_{17} and 8_{18} from the standard knot tables.

Remark. Chirality dominates as the number of crossings goes to infinity.

Proposition 32. $\mathrm{Col}(L) \cong \mathrm{Col}(m(L))$.

Proof. Pick M to be a vertical plane disjoint from L. We have a bijection between crossings and arcs in D and $m(D)$, and so this induces an isomorphism on the colouring group. □

4.2 Invertability

Definition 35. Suppose L is an oriented link. Let $r(L) = -L$ be the same link with all orientations reversed.

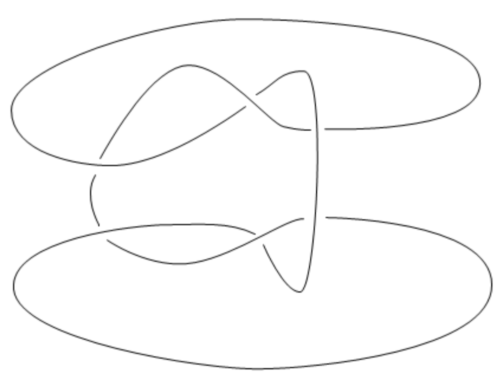
Definition 36. L is invertible if $L \backsimeq r(L) = -L$.

Proposition 33. *If $P = P(p_1, p_2, p_3, ..., p_n)$ is a pretzel knot and some p_i is even, then $P \backsimeq r(P)$.*

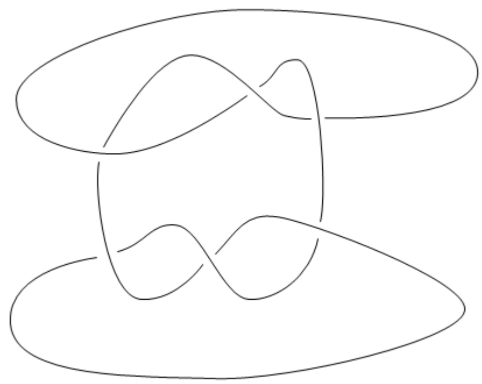
Example 29. Up to nine crossings, only 8_{17}, 9_{32} and 9_{33} are not invertible. However, as the number of crossings goes to infinity, non-invertability dominates.

Recall that D is alternating if every overcrossing arc crosses over exactly one crossing.

Example 30. The reef knot:



is non-alternating. However, the granny knot:



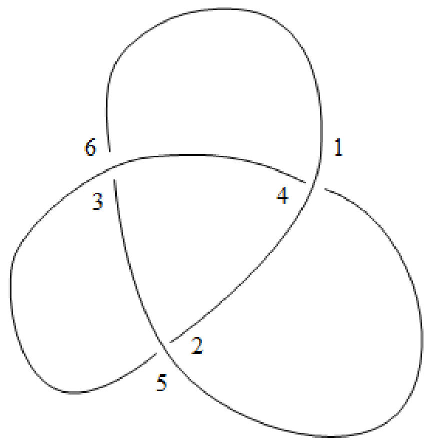
is alternating.

Proposition 34. *Every one-component shadow admits two realizations as an alternating diagram.*

Proof. We may label all crossings of D as follows. Pick a point and a direction. Number the crossings, each one twice, as you arrive at each. We claim that every crossing gets one even and one odd number. Recall that there is a checkerboard colouring of D. Orient dark regions anticlockwise and light regions clockwise. Label the boundary edges 0 and 1 at each end in the direction of the orientation, so the orientation points 0 to 1. At a crossing, we get eight labels, that agree across the edges. In this way, the edges across each crossing get two labels of opposite parity. Now we get an alternating diagram by insisting on odd edges passing over or under at crossings. □

Note. We find a function $f : \{1, 3, 5, ..., 2c - 1\} \to \{2, 4, ..., 2c\}$ where c is the number of crossings, called the code of the diagram.

Example 31. The diagram:



gives rise to the code $[4, 6, 2]$.

Note. We find that given a code, we can generate a diagram. In fact, every code, up to the condition that for every interval $[i, j] \cap \{1, 2, ..., 2c\}$ we have:

$$f(\{1, 3, ..., 2c - 1\} \cap [i, j]) \not\subseteq \{2, 4, ..., 2c\} \cap [i, j]$$

gives a diagram of a unique knot type.

Proposition 35. *Up to n crossings there are at most $n!2^n$ knot types.*

Chapter 5

The Alexander Polynomial

Recall that during the discussion of colourings we used the ring $\mathbb{Z}/n\mathbb{Z}$. Instead we may use the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$.

Definition 37. A Laurent polynomial is a finite sum $\Delta(t) = \sum_{k=-n}^{n} a_k t^k$ where $a_k \in \mathbb{Z}$.

Definition 38. We say $\Delta(t)$ is symmetric if $\forall\, n$, $a_n = a_{-n}$.

Example 32. $\Delta(t) = t - 3 + \frac{1}{t}$ is a symmetric Laurent polynomial.

We cannot usually use $\mathbb{Z}[t, t^{-1}]$ so we use $\mathbb{Z}[t, t^{-1}]/\Delta(t)$ for some $\Delta(t)$. The new colouring equation is:

$$(1-t)\alpha + t\beta - \gamma \equiv 0 \mod \Delta(t)$$

where α is the overcrossing arc, β is the left undercrossing arc (with respect to the orientation of α) and γ is the right undercrossing arc. We sometimes write:

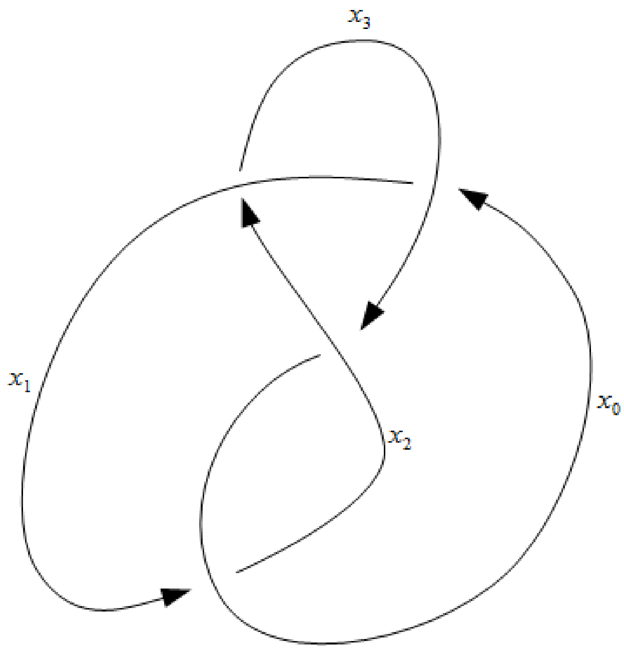
$$\mathrm{Key}(t) = \begin{bmatrix} \alpha & \beta & \gamma \\ 1-t & t & -1 \end{bmatrix}$$

Notice if we set $t = -1$ we get:

$$2\alpha(-1) - \beta(-1) - \gamma(-1) \equiv 0 \mod \Delta(-1)$$

which is the old colouring equation.

Example 33. Consider the figure-eight knot:



We get the colouring matrix:

$$A_+ = \begin{bmatrix} 1-t & -1 & t & 0 \\ 0 & 1-t & t & -1 \\ t & 0 & 1-t & -1 \\ t & -1 & 0 & 1-t \end{bmatrix}$$

Notice that $1 - t + t - 1 = 0$, so the sum of the columns is the zero vector, but the same is not true of the rows. Deleting a row and column:

$$A = \begin{bmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ t & -1 & 0 \end{bmatrix}$$

Then we get:

$$\begin{aligned} \det(A) &= 0 + t(1-t)^2 + (-t^2) - (0+0+0) \\ &= t^3 - 3t^2 + t \\ &= t^2\left(t - 3 + \frac{1}{t}\right) \end{aligned}$$

$\Delta_k(t) = t - 3 + \frac{1}{t}$ is the Alexander polynomial of the figure-eight knot.

Remark. This polynomial is a knot invariant. $\Delta_k(t)$ is only defined up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$.

Definition 39. For a ring R, $u \in R$ is a unit if $\exists\, w \in R$ such that $uw = 1$.

Note. The units of $\mathbb{Z}[t, t^{-1}]$ are $\pm t^k$ for $k \in \mathbb{Z}$.

Observe that for the figure-eight, the Alexander polynomial was symmetric.

5.1 The Winding Number

Suppose $x \in R$ is a point in a region R of D. Choose a generic ray L and orient L away from x. Compute $w(R)$, the winding number of D about R:

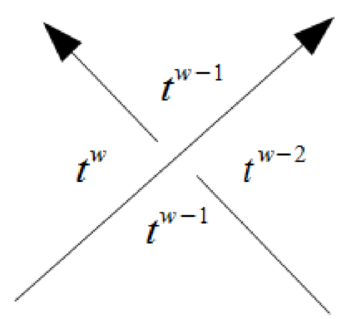
$$w(R) = \sum_{C \in L \cap D} \operatorname{sign}(c)$$

Proposition 36. *$w(R)$ is independent of the choice of L.*

Proposition 37. $w(R) \equiv e(R) \mod 2$.

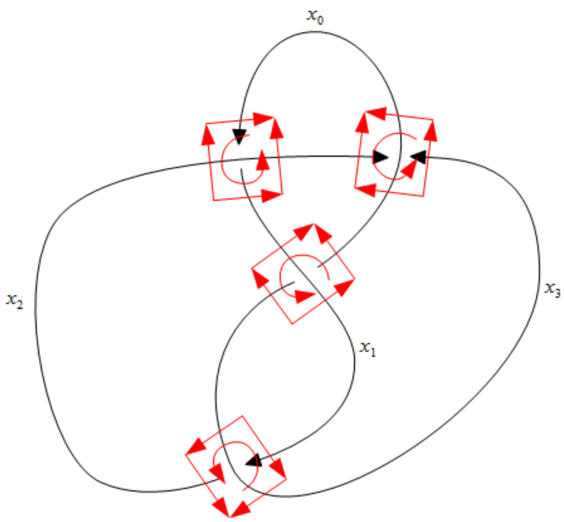
Proposition 38. *If R, R' are disjoint then $|w(R) - w(R')| = 1$.*

Now we label every region R with $t^{w(R)}$. So around each crossing we see:



We draw a small square around each crossing and orient edges from smaller to larger winding number.

Example 34. We do this for the figure eight:

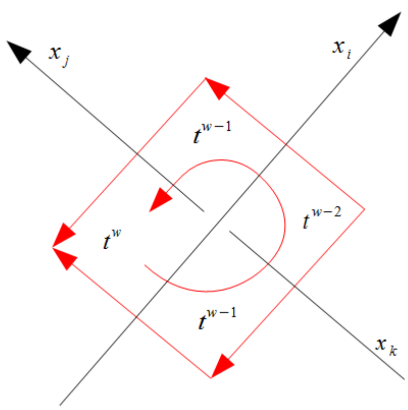


For each arc at every crossing we add the value that the edge points towards, multiplying by -1 when the orientation of the edge differs from the orientation of the square, to get:

$$\begin{bmatrix} t-1 & 0 & -t & 1 \\ 1 & t^{-1}-1 & 0 & -t^{-1} \\ -t & 1 & t-1 & 0 \\ 0 & -t^{-1} & 1 & t^{-1}-1 \end{bmatrix}$$

Now all rows and columns sum to zero.

At a general crossing we have:



We find that:

$$(-t^w + t^{w-1})x_i + t^w x_j - t^{w-1}(x_k) = 0 \iff t^{w-1}((1-t)x_i + tx_j - x_k) = 0$$

and so the coefficients in a row sum to zero. The columns sum to zero by the same reasoning as for the colouring matrix. Hence if $A_+(t)$ is the Alexander matrix and $A(t)$ is th result of deleting any one row and any one column then $\det(A(t)) \equiv \Delta_k(t)$ is independent of the choice of row or column. If we replace t by t^{-1} then:

$$\mathrm{Key}(t^{-1}) = \begin{bmatrix} \alpha & \beta & \gamma \\ 1-\frac{1}{t} & \frac{1}{t} & -1 \end{bmatrix}$$

Multiply by $-t$, as this does not change the determinant, and then:

$$-t\,\mathrm{Key}(t^{-1}) = \begin{bmatrix} \alpha & \beta & \gamma \\ 1-t & -1 & t \end{bmatrix}$$

So we get the mirror image of the usual configuration. So we find $\Delta_{m(K)}(t) \equiv \Delta_K\left(\frac{1}{t}\right)$ up to units. Reversing the orientation also gives:

$$\begin{bmatrix} \alpha & \beta & \gamma \\ 1-t & -1 & t \end{bmatrix}$$

So up to units, $\Delta_{r(K)}(t) \equiv \Delta_{m(K)}(t) \equiv \Delta_K\left(\frac{1}{t}\right)$.

Theorem 39. *$\Delta_K(t) \equiv \Delta_K\left(\frac{1}{t}\right)$ up to units.*

So $\Delta_K(t)$ is symmetric, and is equivalent (up to units) to the polynomial for the mirror image and for the reverse. Note that:

$$\mathrm{Key}(-1) = \begin{bmatrix} \alpha & \beta & \gamma \\ 2 & -1 & -1 \end{bmatrix}$$

which is the colouring equation. So $|\Delta_K(-1)| = \det(K)$.

Proposition 40. *Suppose L is split. Then $\Delta_L(t) \equiv 0$.*

Proof. We have:

$$A_+ = \left[\begin{array}{c|c} B_+ & 0 \\ \hline 0 & C_+ \end{array}\right]$$
$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C_+ \end{array}\right]$$

So $\Delta_L(t) = \det(A) = \det(B)\det(C_+) = 0$ as C_+ is unreduced. $\square$

Theorem 41. *Δ_K is a link invariant.*

Corollary 42. *If L is splittable then $\Delta_L(t) \equiv 0$.*

Chapter 6

Connect Sums

Definition 40. Suppose K, L are oriented knots. The connect sum $K\sharp L$ is defined by joining an arc of K and an arc of L by an arc α. Form $K\sharp L$ be doubling α, removing a neighbourhood of α in K and L, and joining the two arcs along the copies of α, preserving the orientation.

Theorem 43. *$[K\sharp L]$ depends only on $[K]$ and $[L]$.*

Sketch Proof. There are two steps. First we show that the connect sum is independent of the choice of α, and then that it is independent of the choice of $K' \in [K]$.

Step 1: Suppose α' is some other arc joining to a different point on K. We shrink L to be very small. We pull it along α' so it is very close to K. Then slide L along K to the end point of α. Then unreel along α and expand L again.

Step 2: If $K' \simeq K$ then $K'\sharp L \simeq K\sharp L$. Shrink L and pull it along α as above. Apply the isotopy taking K to K', carrying along L. Slide L to the correct position, unreel and expand. □

Proposition 44. *We have:*

(i) $K\sharp U \simeq K$.

(ii) $K\sharp L \simeq L\sharp K$.

(iii) $K\sharp(L\sharp M) \simeq (K\sharp L)\sharp M$.

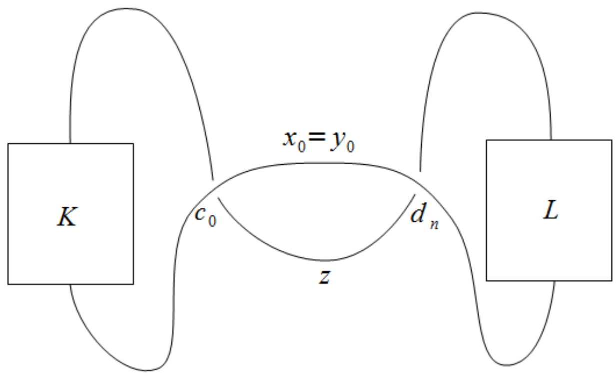
Remark. Inverses do not exist in general.

Definition 41. K is prime if $K \simeq L\sharp M$ implies $L \simeq U$ or $M \simeq U$.

Example 35. The reef and granny knots are not prime, but the trefoil and figure eight are.

Theorem 45. $\Delta_{K\sharp L} = \Delta_K \Delta_L$.

Proof. Recall that if A is the reduced matrix then $\det(A) = \Delta_K$ is the polynomial. We compute $K\sharp L$ with:



The key for d_n is given by:

$$\begin{bmatrix} x_0 & z & y_n \\ 1-t & t & -1 \end{bmatrix}$$

so A has block diagonal form (after deleting the row for c_0 and the column for x_0).

$$A = \left[\begin{array}{c|c|c} & & 0 \\ B & 0 & \vdots \\ & & 0 \\ \hline & & 0 \\ 0 & C & \vdots \\ & & 0 \\ \hline 0 \quad \cdots \quad 0 & 0 \quad \cdots \quad 0 \quad -1 & t \end{array}\right]$$

where B is the reduced matrix for K and C is the reduced matrix for L. Hence:

$$\Delta_{K\sharp L} = \det(A) = t\det(B)\det(C) = t\Delta_K\Delta_L \equiv \Delta_K\Delta_L \qquad \square$$

Corollary 46. $\det(K\sharp L) = \det(K)\det(L)$.

As a consequence, since $K\sharp U \backsimeq K$, $\Delta_K\Delta_U = \Delta_K$ so $\Delta_U = 1$. We find that the Alexander polynomials of the reef and granny knots agree, so the invariant cannot distinguish them.

Chapter 7

Bridge Number

Definition 42. Suppose that D is a diagram. Define the bridge number of D, $b(D)$, to be the number of overcrossing strands.

Remark. If D is alternating, $b(D)$ is the number of crossings of D.

Definition 43. The bridge number of K is:

$$b(K) = \min\{b(D)|D \text{ a diagram of } K' \backsimeq K\}$$

This, as with $c(K)$, is difficult to compute.

Note. Bridge presentation (that is any diagram realizing $b(K)$) may be enormously more complicated than a diagram relaizing the crossing number.

Definition 44. A $2n$–plat diagram in the xy–plane has n maxima (all at the same y–height), n minima (all at the same y–height) and no other maxima or minima with respect to the y co-ordinate.

Proposition 47. *For all knots K, $\exists\, n \in \mathbb{N}$ and a knot $K' \backsimeq K$ such that K' is in $2n$–plat position.*

Sketch Proof. Let D be any diagram for K. Pull all local maxima up and all local minima down using R_2 moves. □

Proposition 48. *K has a 2–plat diagram iff $K \backsimeq U$.*

Proof. The ($\Longleftarrow$) direction is trivial. Suppose K has a 2–plat diagram. Locate the highest crossing, and use an R_1 move to untwist it. The result follows by induction on the number of crossings. □

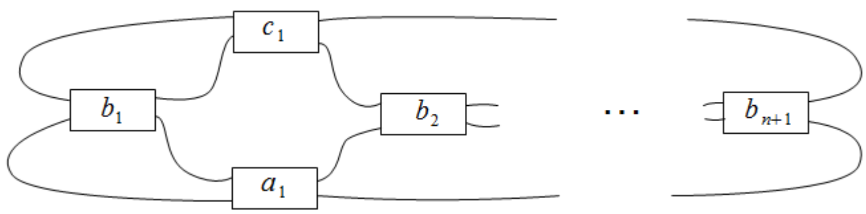
Proposition 49. *For any knot K, let n be half the minimal plat number. Then $n = b(K)$.*

Proof. We first show that $n \leq b(K)$. Let D be a minimal bridge diagram for K. Lift all bridges slightly out of the xy plane. This creates $b(K)$ maxima. Pull the other arcs down to create $b(K)$ minima. Hence $n \leq b(K)$. Now we show $b(K) \leq n$. Start with a minimal plat diagram for K, so it has $2n$ plats. Order the crossings by height. Drag the highest crossing so it goes under a small arc about one of the maxima. Do this for each crossing in turn. Thus $b(K) \leq n$. $\square$

Remark. There are only finitely many knots with crossing number less than or equal to n. There are infinitely many knots with bridge number at most n, if $n \geq 2$.

7.1 Flypes

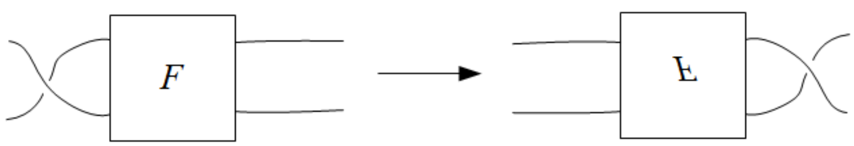
Theorem 50. *The general* 4*–plat diagram:*



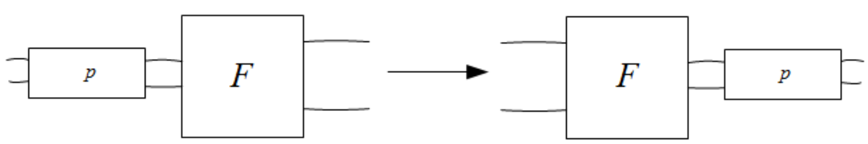
is isotopic to a knot with all $c_i = 0$.

To prove this, we need:

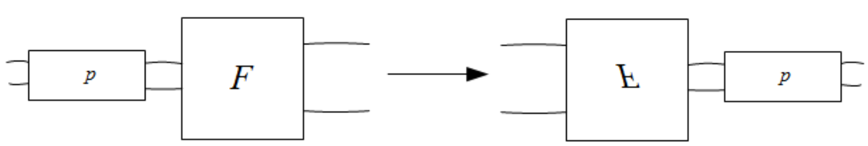
Definition 45. A flype is an isotopy of the form:



Remark. In general, we have that:



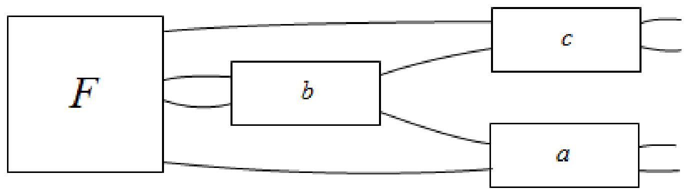
if p is even, and:



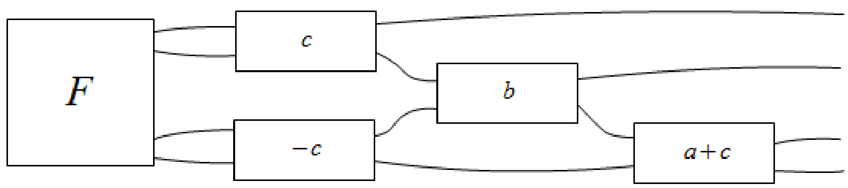
if p is odd.

We may also perform flypes when there are no crossings on either side, to create twists of the same number but opposite signs on each side.

Proposition 51. *We have an isotopy taking:*

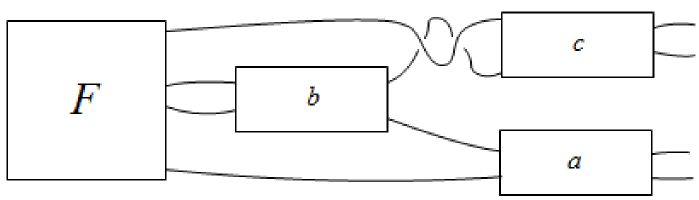


to:

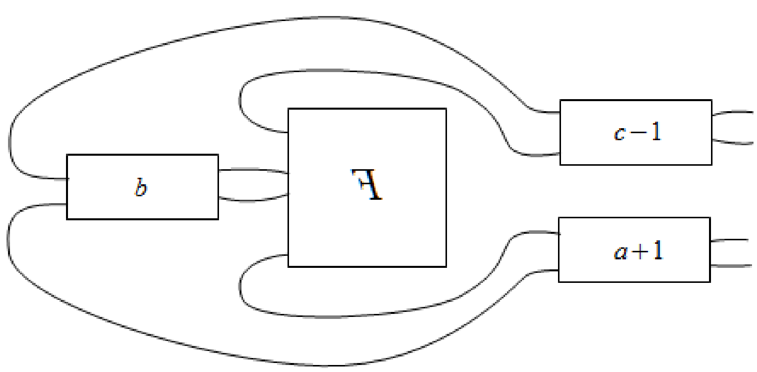


if bc is even, or the same with F upside-down and back-to-front if bc is odd.

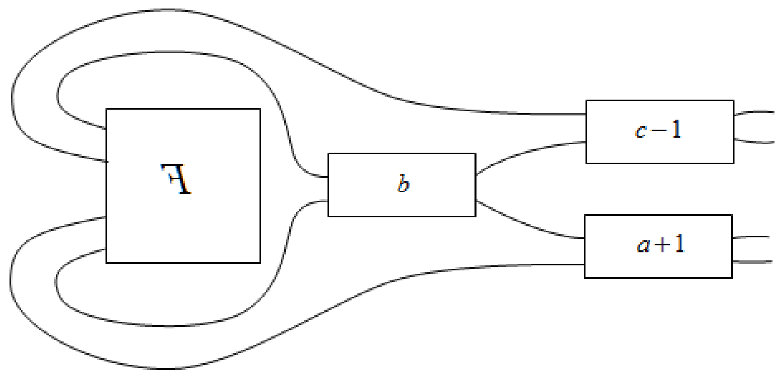
Proof. Using an R_2–move, we take the original arrangement to:



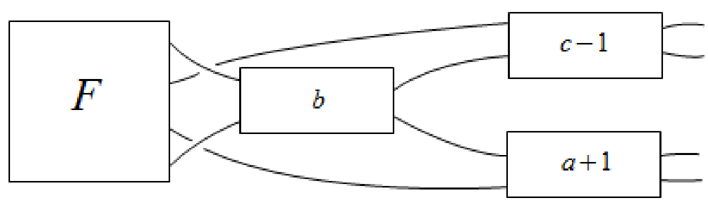
Now by a flype on F and the b twist box:



By a flype on F b times we get:



if b is even, or the same diagram with F upside down as well as backwards if b is odd. By R_2 and a flype, we get:



if b is even and the same diagram with F upside-down (but not backwards) if b is odd. Now if we do this $c-1$ more times we obtain the desired result. If $c < 0$ the result is analogous. $\square$

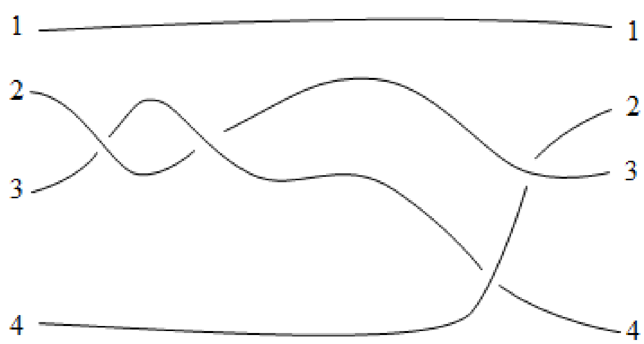
We may now prove the theorem by repeatedly applying the proposition, and then finishing with R_1 moves.

Chapter 8

Braids

If we cut off the minima and maxima from a plat, we find a braid.

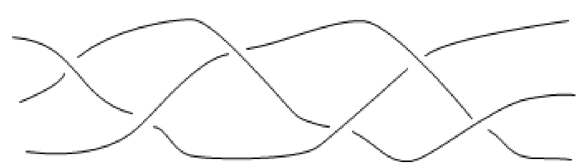
Example 36.



is a braid σ. It induces a permutation:

$$\pi_\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (2, 3, 4)$$

Example 37. The canonical braid is:



Formally:

Definition 46. A braid is a collection of strands σ connecting the points $(i, 0)$ to $(\pi_\sigma(i), 1)$ such that $\frac{d\sigma_y}{dt} > 0$.

Definition 47. $\sigma \backsimeq \tau$ if σ and τ are isotopic through braids.

Definition 48. If σ, τ are braids with the same number of strands, we may stack them by joining the end-points of σ to the start points of τ, to obtain $\sigma \cdot \tau$.

Note. $\sigma \cdot \tau \not\backsimeq \tau \cdot \sigma$ is general.

Definition 49. B_n is the n–strand braid group, the set of n–strand braids up to isotopy, together with the stacking operation above.

Example 38. B_1 is the trivial group. $B_2 \cong \mathbb{Z}$, the infinite cyclic group. B_3 can be understood, but B_4 and higher are extremely complicated.

We check the group axioms for B_n:

(i) There is an identity element, the trivial braid.

(ii) If σ, τ are braids then $\sigma \cdot \tau$ again has $\frac{d(\sigma \cdot \tau)_y}{dt} > 0$ so is a braid.

(iii) Associativity is obvious.

(iv) Inverses are given by mirror images.

We now give a second definition of the braid group.

Definition 50. Let σ_i be the braid of one crossing between strands i and $i+1$, with i on top. Then:

$$B_n \cong \left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i, & |i-j| \geq 2 \\ \sigma_i \cdot \sigma_j \cdot \sigma_i = \sigma_j \cdot \sigma_i \cdot \sigma_j, & |i-j| \leq 1 \end{array} \right\rangle$$

Note. The natural map $\pi : B_n \to \Sigma_n$, where Σ_n is the symmetric group on n elements, is an antihomomorphism, that is $\pi_{\sigma \cdot \tau} = \pi_\tau \pi_\sigma$.

Definition 51. Suppose $\sigma \in B_n$ is a braid. Let L_σ denote the braid closure of σ, formed by closing each strand without introducing additional crossings.

Definition 52. Suppose that L is an oriented link. Then the braid index of L is given by:

$$\mathrm{br}(L) = \min\{n | \exists\, \sigma \in B_n \text{ with } L_\sigma \backsimeq L\}$$

To show that $\mathrm{br}(L)$ is well-defined, we need:

Theorem 52 (Alexander)**.** *Every link is the closure of some braid.*

Example 39. $\mathrm{br}(L) = 1$ iff $L \backsimeq U$. $\mathrm{br}(L) = 2$ iff L is a $(2, p)$–torus link.

Remark. It is an open problem to find an algorithm that, given L, computes the braid index of L. Computing $b(L)$ is also open, but perhaps easier.

Proposition 53. *$\forall\, L$, $2\,\mathrm{br}(L) \geq b(L)$, but this bound is not sharp, that is $\forall\, n \in \mathbb{N}$, $\exists$ a link L such that $2\,\mathrm{br}(L) \geq b(L) + n$.*

Definition 53. For braids, we have the Markov moves:

M_I: We may replace $\sigma \cdot \tau$ by $\tau \cdot \sigma$.

M_{II}: If $\sigma \in B_n$, we may stabilize σ, by adding a new strand under σ and giving it a single positive or negative crossing with the lowest strand of σ. This crossing is added as the right-most crossing.

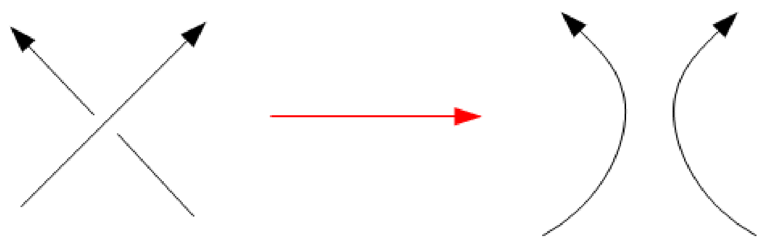
M_{II} in reverse is called destabilization.

Proposition 54. *If σ and τ differ by a Markov move then $L_\sigma \backsimeq L_\tau$.*

Theorem 55 (Markov)**.** *Suppose that $\sigma \in B_n$ and $\tau \in B_n$. Then $L_\sigma \backsimeq L_\tau$ iff there is a sequence of Markov moves between σ and τ.*

We will prove Alexander's theorem using Seifert circles and smoothing.

Definition 54. Suppose D is an oriented diagram and $c \in D$ is a crossing. We change D to D_c by removing the crossing as below:



Example 40. Smoothing a crossing in the standard diagram of the right trefoil gives the right Hopf link. Smoothing another crossing gives the unknot with a twist, which can be smoothed to give the unlink.

Definition 55. If D is an oriented diagram, smoothing all crossings gives the Seifert circles of D.

Remark. Clearly if D_c is a smoothing of D then D and D_c have the same Seifert circles.

Note. If D and D' differ by an R_1–move then D' has one more Seifert circle than D.

Definition 56. Circles $C_1 \neq C_2$ disagree if there is an arc α between C_1 and C_2 such that $\alpha \cap (C_1 \cup C_2) = \partial\alpha$ and the orientations of C_1 and C_2 disagree along α. Furthermore, α is an arc of conflict if, in addition to the above, $\alpha \cap D = \partial\alpha$.

Definition 57. Let $d(D)$ be the number of disagreements of the Seifert circles of D.

Consider the following algorithm. While $d(D) > 0$, find a conflict α and resolve the conflict. We can do this by performing an R_2–move along the arc of conflict. We now prove that this algorithm works.

Lemma 56. *Suppose that $d(D)$, the number of disagreements, is zero. Then D is isotopic to a braid closure L_σ.*

Proof. Let C be the disjoint union of the Seifert circles. Let R be a region of $\mathbb{R}^2 \setminus C$. R meets either one, two or at least three of the circles. If R meets C_i, C_j and C_k then two of these circles must disagree, giving a contradiction. It follows that every region meets at most two of the Seifert circles. So we have at most two sets of nested Seifert circles. By unsmoothing it is clear that D is isotopic to a braid closure. $\square$

Lemma 57. *If $d(D) > 0$ then there is a conflict.*

Proof. We use induction on $c(D)$, the number of crossings of D. Note that if D_c is D smoothed at c, then $d(D_c) = d(D)$. Suppose $c(D) = 0$. Since there is a disagreement, by the combinatorial intermediate value theorem there is a conflict between two arcs of the diagram somewhere between the arcs of disagreement. Suppose that $c(D) = n$ and the lemma holds for D_c where c is some crossing. $d(D) = d(D_c) > 0$, so D_c has a disagreement. Hence D_c has an arc of conflict, α. Let β be an arc between the two strands where c was removed. If $\alpha \cap \beta = \varnothing$ then we are done, as α is an arc of conflict for D. If $\alpha \cap \beta \neq \varnothing$, we may assume that $|\alpha \cap \beta| = 1$. Let β be between arcs c_i and c_j, and α between c_k and c_l. Note that if $c_i \neq c_l$ then α' is an arc of conflict for D, where α' is formed by travelling along α from c_l until reaching β, and then travelling along β to c_i. if $c_i \neq c_k$ then α'' is an arc of conflict for D, where α'' is formed by travelling along α from c_k until reaching β, and then travelling along β to c_i. If $c_i = c_l$ and $c_i = c_k$ then α was not an arc of conflict for D_c. $\square$

Lemma 58. *If α is an arc of conflict for D, then resolving α with an R_2–move reduces $d(D)$ by exactly one.*

Proof. Let α be an arc of conflict between circles C_1 and C_2, and let C_i be another circle. Resolving and smoothing gives circles C_1' and C_2', with C_2' inside C_1' and orientations agreeing. Note that C_i disagrees with C_1' iff it disagrees with C_1, and similarly for C_2. $\square$

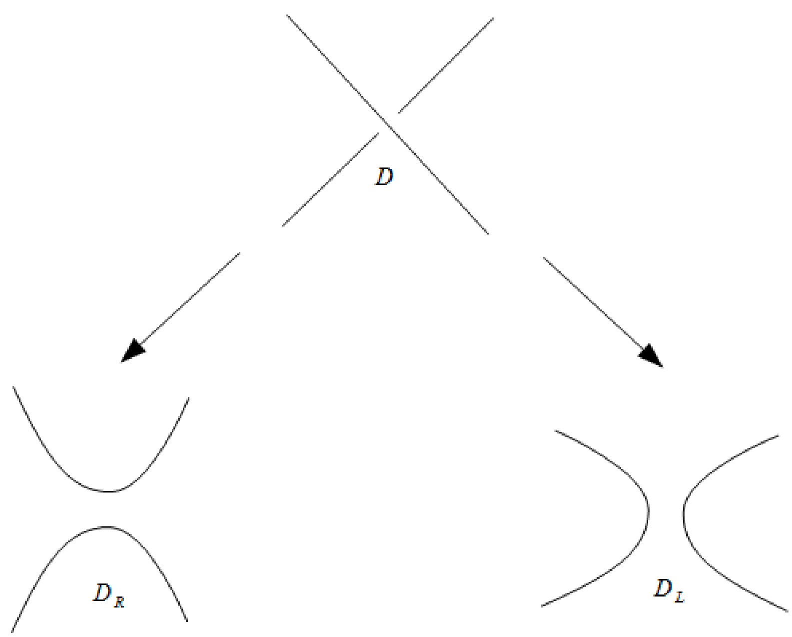
We can now prove correctness of the algorithm to give Alexander's theorem.

Proof. If $d(D) > 0$ then $\exists$ an arc of conflict. Resolving this conflict decreases $d(D)$ by exactly one. If $d(D) = 0$ then D is a braid closure. $\square$

Chapter 9

The Kauffman Bracket Polynomial

Definition 58. Suppose D is an unoriented diagram and $c \in D$ is a crossing. Then we may smooth left or right as follows:



where D_R is the right smoothing and D_L is the left smoothing. We define a bracket such that:

$$\langle D \rangle = A \langle D_R \rangle + B \langle D_L \rangle$$

for variables A and B. We allow A and B to commute, and make the bracket insensitive to R_0–moves. We have axioms:

(i) $\langle D\rangle = A\langle D_R\rangle + B\langle D_L\rangle$, where D_R is D with a crossing c right-smoothed, and D_L is D with c left-smoothed. This is called the Skein relation.

(ii) $\langle D\rangle = 1$ if D has no crossings.

(iii) $\langle U \coprod D\rangle = CD$ where U has no crossings and C is a variable.*

Example 41.

(i) The bracket of the standard diagram of the Hopf link is:

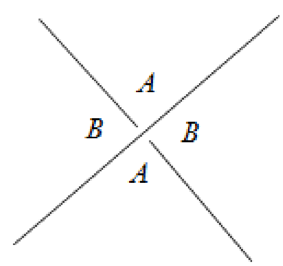
$$A^2C + 2AB + B^2C$$

(ii) The bracket of the standard diagram of the trefoil is:

$$A^3C + 3A^2B + 3AB^2C + B^3C^2$$

9.1 Kauffman States

Kauffman states provide a method for organizing the bracket computations. Label all crossings as below:



If there are $c(D)$ crossings then there are $2^{c(D)}$ possible complete smoothings. On each crossing we can consider the A (right) or B (left) smoothing. Each of these is a Kauffman state. We find that:

$$\langle D\rangle = \sum_{s\in S} A^a B^b C^{c-1}$$

where a is the number of A smoothings in s, b is the number of B smoothings in s, c is the number of circles, and S is the set of Kauffman states.

Proposition 59.

(i) $\langle D \coprod E\rangle = \langle D\rangle \langle E\rangle C$

*Here $U \coprod D$ denotes $U \cup D$ with $U \cap D = \varnothing$, so that $U \coprod D$ is still the diagram of a link.

(ii) $\langle D\sharp E\rangle = \langle D\rangle \langle E\rangle$

Definition 59. We say that D, D' are regularly isotopic if they differ by R_0, R_2 and R_3–moves only.

Proposition 60. *The bracket is an invariant of regular isotopy if* $A = B^{-1}$ *and* $C = -(A^2 + A^{-2})$.

Proof. The required identities can be shown by algebra on the bracket. □

Remark. The bracket is not invariant under R_1–moves.

Remark. The bracket becomes a knot invariant if we set $-A^3 = 1$, that is $A = \sqrt[6]{1} \in \mathbb{C}$, but then $\langle D\rangle = 1 \,\forall\, D$.

Proposition 61. *The writhe* $w(D)$ *is a regular isotopy invariant.*

Definition 60. The Kauffman polynomial is given by:

$$X_K(A) = (-A^{-3})^{w(D)} \langle D\rangle$$

where K is a knot or link, and D is any diagram of K.

Theorem 62. X_K *is an isotopy invariant.*

Proof. It suffices to check invariance under R_1–moves as $w(D)$ and $\langle D\rangle$ are regular isotopy invariants. Let D' be a link K containing a right-handed twist (a right crossing that can be removed by and R_1–move), and let D be the diagram after the R_1–move. We denote by X_D the Kauffman polynomial calculated with diagram D.

$$\begin{aligned} X_{D'}(A) &= (-A^{-3})^{w(D')} \langle D'\rangle \\ &= (-A^{-3})^{w(D)+1}(-A^3) \langle D\rangle \\ &= (-A^{-3})^{w(D)} \langle D\rangle \\ &= X_D(A) \end{aligned}$$

The result is similar for a left-handed twist. □

Proposition 63.

(i) $\langle m(D)\rangle = \langle D\rangle_{|_{A=A^{-1}}}$.

(ii) $X_{m(D)}(A) = X_D(A^{-1})$.

(iii) $X_{r(D)}(A) = X_D(A)$.

Sketch Proof.

(i) Mirroring switches left and right crossings, so switches A and $B = A^{-1}$.

(ii) $w(m(D)) = -w(D)$.

(iii) Reversing orientations leaves left/right crossings, and hence writhe, unchanged. □

Example 42. If D has no crossings then $\langle D \rangle = 1$, $w(D) = 0$, so $X_U(A) = (-A^{-3})^0 1 = 1$. If K is the right Hopf link, then:

$$\begin{aligned} X_K(A) &= (-A^{-3})^2(-A^4 - A^{-4}) \\ &= -A^{-2} - A^{-10} \end{aligned}$$

If K is the right trefoil, then:

$$\begin{aligned} X_K(A) &= (-A^{-3})(-A^5 - A^{-3} + A^{-7}) \\ &= A^{-4} + A^{-12} - A^{-16} \end{aligned}$$

Theorem 64 (Dehn)**.** *The right trefoil RT is not isotopic to the left trefoil LT.*

Proof. We find:

$$\begin{aligned} X_{RT} &= A^{-4} + A^{-12} - A^{-16} \\ X_{LT} &= A^4 + A^{12} - A^{16} \end{aligned}$$

so the knots have different Kauffman polynomials. □

Definition 61. Define the Jones polynomial:

$$V_K(t) = X_K\left(t^{-\frac{1}{4}}\right)$$

Example 43. $V_{RT}(t) = -t^4 + t^3 + t$.

It remains an open question as to whether $V_K = 1$ iff $K \simeq U$. This is true up to 17 crossings.

Proposition 65.

(i) $V_U = 1$.

(ii) Let D_+ be a diagram with a right crossing. Let D_- be the same diagram with the right crossing replaced by a left crossing, and let D_0 be D_+ after the right crossing has been left smoothed. Then:

$$\frac{1}{t}\langle D_+ \rangle - t \langle D_- \rangle = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \langle D_0 \rangle$$

Proof of (ii). This can be shown with simple algebra on the bracket. Let D' be the D_+ with the right crossing right smoothed. Then:

$$\begin{aligned} \langle D_- \rangle &= A \langle D' \rangle + A^{-1} \langle D_0 \rangle \\ \langle D_+ \rangle &= A \langle D_0 \rangle + A^{-1} \langle D' \rangle \end{aligned}$$

So then:

$$A\langle D_+\rangle - A^{-1}\langle D_-\rangle = (A^2 - A^{-2})\langle D_0\rangle$$

Recall that $X_K(A) = (-A^{-3})^{w(D)}\langle D\rangle$ where D is any diagram of K. Equivalently:

$$\langle D\rangle = -A^{3w(D)}X_K(A)$$

IF $n = w(D_0)$ then $n+1 = w(D_+)$ and $n-1 = w(D_-)$. So:

$$\begin{aligned}
A(-A^3)^{n+1}X_{D_+}(A) - A^{-1}(-A^3)^{n-1}X_{D_-}(A) &= (A^2 - A^{-2})(-A^3)^n X_{D_0}(A) \\
&= -A^4 X_{D_+} + A^{-4}X_{D_-} \\
&= (A^2 - A^{-2})X_{D_0}
\end{aligned}$$

So then:

$$\frac{1}{t}V_{D_+} - tV_{D_-} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{D_0}$$ □

Proposition 66. *If K, L are oriented links, then:*

(i) $V_{K\sharp L} = V_K V_L$.

(ii) $V_{K\coprod L} = CV_K V_L = (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_K V_L$.

Example 44. Let K be a non-splittable link with one component a trefoil and the other an unknot, and let L be the right trefoil, oriented in the same direction of the trefoil component of K. We may make two connect sums $K\sharp L$ depending on whether we join L to the trefoil component or the unknot component of K. By the proposition, they have the same Jones polynomial, but they are clearly not isotopic.

Theorem 67. *$\exists$ prime knots K and L such that $V_K = V_L$ but $K \backsimeq L$.*

Theorem 68 (Thistlethwaite)**.** *$\exists$ a link L such that $L \not\backsimeq U$ but $V_L = 1$.*

Example 45. Take the standard diagram of the right trefoil, D_+. Reverse one crossing (D_-) and left smooth it (D_0). Note that $D_- \backsimeq U$ and $D_0 \backsimeq H$, the Hopf link.

$$\begin{aligned}
\frac{1}{t}V_{D_+} - tV_{D_-} &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{D_0} \\
V_{D_+} &= t(tV_{D_-} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{D_0}) \\
&= t^2 + (t^{\frac{3}{2}} - t^{\frac{1}{2}})V_H
\end{aligned}$$

Let H_+ be the standard diagram of H. Reverse one crossing (H_-, the unlink) and left smooth ($H_0 \backsimeq U$). So:

$$\begin{aligned}
V_{H_-} &= -t^{\frac{1}{2}} - t^{-\frac{1}{2}} \\
V_{H_0} &= 1
\end{aligned}$$

Hence:

$$\begin{aligned} V_{H_+} &= -t^{\frac{5}{2}} - t^{\frac{3}{2}} + t^{\frac{3}{2}} - t^{-\frac{1}{2}} \\ &= -t^{\frac{5}{2}} - t^{\frac{1}{2}} \end{aligned}$$

Hence:

$$\begin{aligned} V_T = V_{D_+} &= t^2 - (t^{\frac{3}{2}} - t^{\frac{1}{2}})(t^{\frac{5}{2}} + t^{\frac{1}{2}}) \\ &= t^2 - t(t-1)(t^2+1) \\ &= t^2 - t^4 - t^2 + t^3 + t \\ &= -t^4 + t^3 + t \end{aligned}$$

Proposition 69. *Any diagram can be converted to a diagram of the unknot via crossing changes.*

We may now give an algorithm to compute V_D. Suppose D requires m crossing changes:

$$D = D_m, D_{m-1}, ..., D_1, D_0 \backsimeq U$$

As before, we may write:

$$V_{D_K} = t^2 V_{D_{k-1}} + t(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_{D_k^0}$$

assuming the crossing was positive. Now apply this recursively. This algorithm does not complete in polynomial time with respect to the number of crossings. It is an open problem to find such an algorithm. Now recall that a diagram is alternating if every arc crosses over exactly one crossing. If D is alternating then the A, B labelling of crossings gives a two-colouring of the regions of D, as every region contains only A labels or only B labels.

Definition 62. Suppose that D is alternating. Write $n = c(D)$, the crossing number, X for the number of A regions and Y for the number of B regions. Finally, let w be the writhe.

Proposition 70. *Suppose that D is alternating and reduced. Then the top (highest power) term of $\langle D \rangle$ is given by $A^n C^{Y-1}$ and the bottom (lowest power) term of $\langle D \rangle$ is given by $A^{-n} C^{X-1}$. Substituting for C, we find the highest term is $(-1)^{Y-1} A^{n+2Y-2}$ and the lowest term is $(-1)^{X-1} A^{-n-2X+2}$.*

Proof. Perform only A smoothings, and then the B regions either side of each crossing are distinct, because the diagram is reduced. So we get Y loops, and hence making all A smoothings gives $A^n C^{Y-1}$. Similarly making all B smoothings gives $A^{-n} C^{X-1}$. We claim that any other smoothing has a power of A strictly between $n + 2Y - 2$ and $-n - 2X + 2$, so is neither highest or lowest. To show this, suppose that s is a state other than the state with all A smoothings. Say s has $b \geq 1$ B smoothings. We need to show that $A^a B^b C^{c-1}$ has a lower degree than A^{n+2Y-2}. We do this by induction on b. In the base case $b = 1$, $A^{n-1} B C^{Y-2}$ is the term given. That is, switching any one smoothing in the all A state to a B smoothing reduces the number of loops by exactly one. In general, if $b \geq 2$, we may change a previous state $A^{n-b+1}B^{b-1}C^R$ to at most $A^{n-b}B^bC^{R+1}$, so the total degree in A cannot go up. We obtain the bound for the lowest term similarly. □

Proposition 71. *If D is alternating and connected then $X + Y = n + 2$.*

Proof. We use the Euler number of D. Let $V = n$, the number of vertices, $E = 2n$, the number of edges, and $F = X + Y$, the number of faces. As the Euler number of a sphere is 2, $n - 2n + X + Y = 2$, so $X + Y = n + 2$. □

Definition 63. If $p(t) \in \mathbb{Z}[t, t^{-1}]$, define $\operatorname{span}(p)$ to be the highest degree minus the lowest degree.

Example 46.

(i) $\operatorname{span}(t^n) = 0$.

(ii) $\operatorname{span}(t^n + 1) = n$ if $n \geq 1$.

(iii) $\operatorname{span}(t^n + t^{-n}) = 2n$ if $n \geq 0$.

Theorem 72. *If D is alternating, reduced and connected, then* $\operatorname{span}(\langle D\rangle) = 4n$.

Proof. We have:

$$\begin{aligned}
\operatorname{span}(\langle D\rangle) &= n + 2Y - 2 - (-n - 2X + 2) \\
&= 2n + 2X + 2Y - 4 \\
&= 2n + 2(X + Y) - 4 \\
&= 2n + 2(n + 2) - 4 \\
&= 4n
\end{aligned}$$

□

Corollary 73. *If D is alternating, reduced and connected, then* $\operatorname{span}(V_D) = n$. *So the Jones polynomial detects the crossing number of knots with an alternating presentation.*

Chapter 10

Tangles

Definition 64. A tangle is a link with two arcs and n loops inside of a ball with endpoints at the north-east, south-east, south-west and north-west corners.

Definition 65. We say tangles S, T are equivalent, or isotopic, if there is an isotopy of the ball, fixed on the boundary, sending S to T.

Definition 66. We define the following canonical tangles:

(i) The 0 tangle has arcs joining north-west to north-east, and south-west to south-east, without crossings.

(ii) The ∞ tangle joins north-west to south-west and north-east to south-east without crossings.

(iii) The 1 tangle has arcs joining opposite corners with the north-east to south-west arc passing over in a single crossing.

(iv) The -1 tangle has arcs joining opposite corners with north-east to south-west arc passing under in a single crossing.

Remark. The above tangles are pairwise inequivalent.

Definition 67. The closure (or numerator) of T, denoted $N(T)$, is the link obtained by connecting the north pair and the south pair, outside the ball.

Definition 68. If S, T are tangles then $S + T$ is the tangle formed by joining the north-east endpoint of S to the north-west endpoint of T, and similarly for the south endpoints.

Remark. Addition of tangles is not commutative, but it is associative. We have $N(S + T) \backsimeq N(T + S)$. 0 is the additive identity.

Definition 69. Write $n = 1 + \cdots + 1$ for the $-n$ twist box. Similarly we may write $-n$ for the sum of n -1 tangles.

Definition 70. The reciprocal of of a tangle F is the reflection through the north-west to south-east line, and is denoted by $\frac{1}{F}$.

Definition 71. The dot of tangles $S \cdot T$ is the tangle $\frac{1}{S} + T$.

Example 47. $N(3 \cdot 2)$ is the 5_2–knot.

Definition 72. $a_1 a_2 a_3 \cdots a_n$ is the tangle:

$$(\cdots(((a_1 \cdot a_2) \cdot a_3) \cdot a_3) \cdot \cdots \cdot a_n)$$

These are Conway's rational tangles.

Example 48. $N((a_1 \cdot a_2) \cdot a_3)$ gives a 4–plat.

Proposition 74. *We have:*

$$N(a_1 a_2 \cdots a_n) \simeq \begin{cases} N(a_n a_{n-1} \cdots a_1), & n \text{ odd} \\ N((-a_n)(-a_{n-1}) \cdots (-a_1)), & n \text{ even} \end{cases}$$

Recall that rational numbers may be expressed as continued fractions.

Example 49.

(i) As a continued fraction:

$$\begin{aligned} \frac{23}{10} &= 2 + \frac{3}{10} \\ &= 2 + \frac{1}{\frac{10}{3}} \\ &= 2 + \frac{1}{3 + \frac{1}{3}} \\ &= [3, 3, 2] \end{aligned}$$

(ii) As a continued fraction:

$$\begin{aligned} \frac{17}{10} &= 1 + \frac{7}{10} \\ &= 1 + \frac{1}{\frac{10}{7}} \\ &= 1 + \frac{1}{1 + \frac{3}{7}} \\ &= 1 + \frac{1}{1 + \frac{1}{\frac{3}{7}}} \\ &= 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3}}} \\ &= [3, 2, 1, 1] \end{aligned}$$

(iii) As a continued fraction:

$$\begin{aligned}
\frac{17}{12} &= 1 + \frac{5}{12} \\
&= 1 + \frac{1}{\frac{12}{5}} \\
&= 1 + \frac{1}{2 + \frac{2}{5}} \\
&= 1 + \frac{1}{2 + \frac{1}{\frac{5}{2}}} \\
&= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} \\
&= [2, 2, 2, 1]
\end{aligned}$$

Theorem 75 (Conway)**.** *The map $a_1 a_2 a_3 \cdots a_n \mapsto [a_1, a_2, ..., a_n]$ from rational tangles to $\mathbb{Q} \cup \{\infty\}$ is a bijection.*

Remark. Neither rational tangles nor rational numbers have unique expressions as above. For example:

$$2 = 1 + \frac{1}{1}$$

So by the theorem:

$$[2] = [1\ 1]$$

Example 50. The rational tangles $T_{\frac{17}{10}}$ and $T_{\frac{17}{12}}$ are not isotopic, but:

$$N(T_{\frac{17}{10}}) \backsimeq N(T_{\frac{17}{12}})$$

By $T_{\frac{17}{10}}$ we mean the rational tangle corresponding to $\frac{17}{10} \in \mathbb{Q}$ by Conway's bijection.

Corollary 76. *Any rational tangle $a_1 a_2 \cdots a_n$ is isotopic to a tangle that is either 0, ± 1, ∞ or such that all the a_i have the same sign, and $|a_i| \geq 2$. Here we may have $a_n = 0$.*

The proof of this corollary uses Lagrange's formula:

$$a + \frac{1}{-b} = (a - 1) + \frac{1}{1 + \frac{1}{b-1}}$$

To see this:

$$
\begin{aligned}
(a-1) + \frac{1}{1+\frac{1}{b-1}} &= a + \frac{1}{1+\frac{1}{b-1}} - \frac{1+\frac{1}{b-1}}{1+\frac{1}{b-1}} \\
&= a + \frac{-\frac{1}{b-1}}{1+\frac{1}{b-1}} \\
&= a + \frac{-1}{b-1+1} \\
&= a + \frac{1}{-b}
\end{aligned}
$$

By the bijection, this means that we can find an isotopy $[-b, a] \mapsto [b-1, 1, a-1]$.

Chapter 11

Surfaces

We can classify up to homeomorphism surfaces which are compact, orientable and without boundary by using the genus, thought of as the number of "handles". Denote by S_g the compact orientable surface without boundary of genus g. To obtain surfaes with boundary, we cut n disks out of S_g.

Definition 73. $S_{g,n}$ is the orientable, compact, connected surface with genus g and n boundary components.

11.1 The Euler Characteristic

If S is a surface decomposed into 2–cells, then the Euler characteristic is:

$$\chi(S) = V - E + F$$

where V is the number of 0–cells (or vertices), E is the number of 1–cells (or edges), and F is the number of 2–cells (or faces).

Theorem 77. $\chi(S) = 2 - 2g - n$.

Definition 74. At every point x in a surface S there is a notion of right or left handedness. We say S is orientable if there is a consistent choice of right handedness in S.

Example 51. The annulus is orientable, but the Möbius band is not.

Definition 75. A surface $S \subseteq \mathbb{R}^3$ is a spanning surface for $K \subseteq \mathbb{R}^3$ if:

(i) $S \cap K = \partial S = K$.

(ii) S is orientable, compact and connected.

Example 52. U is spanned by a round disk. U is also spanned by surfaces of higher genus.

Theorem 78 (Classification)**.** *All compact connected surfaces are determined (up to homeomorphism) by their:*

(i) Orientability.

(ii) Number of boundary components.

(iii) Euler characteristic.

Remark. Not all surfaces can be embedded in $\mathbb{R}^3$. For example, $M_0 \cup_\partial M_1$, where M_0 and M_1 are Möbius bands, is the Klein bottle, which cannot be embedded in $\mathbb{R}^3$. However, all orientable surfaces can be embedded in $\mathbb{R}^3$.

Proposition 79. *Every oriented knot has a spanning surface.*

Proof. Let D be a diagram of K. Let $\{C_i\}$ be the set of Seifert circles of D. Attach a disk to each circle. These disks are nested as the circles are. Now unsmooth and attach half-twisted bands at each crossing. We claim that this canonical surface is orientable by the Seifert circles. □

Note. Different diagrams may lead to different spanning surfaces.

Proposition 80. *If K is a knot with diagram D, the canonical surface for D has one boundary component, is orientable and has genus $g = \frac{1}{2}(c - s + 1)$, where c is the number of crossings and s is the number of Seifert circles.*

Proof. $V - E + F = \chi(S) = 2 - 2g - 1$. The number of faces is $s + c$, the number of edges is $6c$ and the number of vertices is $4c$. So:

$$\begin{aligned} V - E + F &= 4c - 6c + s + c \\ &= s - c \\ &= 1 - 2g \end{aligned}$$

So $c - s = 2g - 1$, and hence:

$$\frac{(c - s + 1)}{2} = g$$ □

Definition 76. The Seifert genus of K is given by:

$$g(K) = \min\{g(S) | S \text{ spans } K\}$$

Corollary 81. *If D is a diagram of K then $g(K) \leq \frac{1}{2}(c - s + 1)$.*

Proposition 82. $g(K) = 0$ *iff* $K \simeq U$.

Proof. By the above corollary, $g(U) \leq \frac{1}{2}(0 - 1 + 1) = 0$, so $g(U) = 0$. If K is spanned by a disk, then the disk gives an isotopy of K to a small round circle. □

Example 53. The trefoil, figure eight, 6_1 and 9_{46} knots all have genus 1.

Theorem 83. *Suppose K, L are knots. Then $g(K\sharp L) = g(K) + g(L)$.*

Proof. We first show that $g(K\sharp L) \leq g(K) + g(L)$. It suffices to find a spanning surface S for $K\sharp L$ with genus at most $g(K) + g(L)$. Pick spanning surfaces S_K and S_L for K and L such that S_K and S_L have minimal genus. Connect S_K and S_L along a band attached to ∂S_K and ∂S_L to get a spanning surface for $K\sharp L$ with genus $g(K) + g(L)$. Now we show that $g(K\sharp L) \geq g(K) + g(L)$. Fix a minimal spanning surface S for $K\sharp L$. Let P be a large 2–sphere that splits K from L. Consider the intersection of S with P. Note that $(K\sharp L) \cap P$ is exactly two points. By moving S slightly we find that $S \cap P$ is a collection of arcs and circles. Since only two endpoints are available, there is exactly one arc and a collection of circles. We say one such circle c is innermost if one component D of $S \setminus c$ meets S only along c. By the Jordan curve theorem, if $S \cap P$ has a circle, then there is an innermost circle. If there are no circles in $S \cap P$ then $S \cap P$ is just an arc, which we may use to cut S. Cutting S along the arc gives spanning surfaces S'_K and S'_L of K and L respectively. If circles of $S \cap P$ exist, then there is an innermost circle c, which is spanned by a disk D, that is $\partial D = c$ and $D \cap S = c$. We may cut S along c and glue in two copies of D (a process called compression). If c separates S then after compression we ignore the component not meeting the knot. So after compression we find a new spanning surface S' that meets P in fewer circles. $\square$

Corollary 84. *If $K\sharp L \simeq U$ then both K and L are isotopic to U.*

Proof. If $K\sharp L \simeq U$ then $g(K\sharp L) = 0$, so $g(K) = g(L) = 0$. Hence $K \simeq L \simeq U$. $\square$

Definition 77. Let $\mathbb{T}^2$ be the 2–torus in $\mathbb{R}^2$ in standard position. If a knot K is embedded in $\mathbb{T}^2$, then call K a torus knot.

Definition 78. We label a simple loop through the hole of the torus by $\frac{1}{0}$ and a simple loop around the hole by $\frac{0}{1}$, in reference to the vertical and horizontal slope. We can define more curves in this way; for example $\frac{3}{1}$ moves vertically three times as fast as horizontally, so it meets $\frac{1}{0}$ once, and $\frac{0}{1}$ three times. The general $\frac{m}{n}$ curve meets $\frac{1}{0}$ n times and $\frac{0}{1}$ m times.

Proposition 85.

(i) The $(1, n)$ and $(n, 1)$ torus knots are isotopic to U.

(ii) The figure eight is not a torus knot.

(iii) The (p, q) torus knot is isotopic to the (q, p) torus knot.

(iv) If $\gcd\{p, q\} = n$ then the (p, q) torus link has n components.

Note. The (p, q) torus knot is the curve of slope $\frac{p}{q}$ on $\mathbb{T}^2$.

Definition 79. Let D be an oriented link diagram. Define the Conway polynomial ∇_D to satisfy:

(i) $\nabla_U = 1$.

(ii) $\nabla_{D_+} - \nabla_{D_-} = z\nabla_{D_0}$, where D_+ has a right crossing, D_- has this crossing replaced by a left crossing and D_0 has this crossing smoothed.

Theorem 86. ∇_K *(calculated for any diagram of K) is an isotopy invariant of links.*

Remark. Just as with V_K there is an algorithm to compute ∇_K. We use a double recursion on the number of crossing changes required to get U and on the number of crossings.

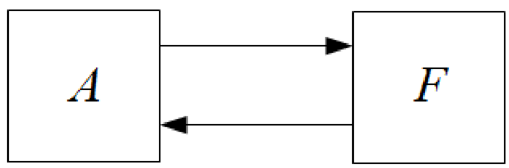
Example 54. Let H be the Hopf link, S the split link, and U the unknot. Then:

$$\nabla_H - \nabla_S = z\nabla_U = z$$

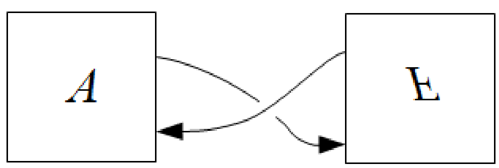
So we may calculate ∇_H from ∇_S.

Proposition 87. *Let S be the split link. Then $\nabla_S = 0$.*

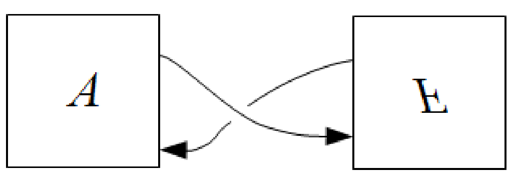
Proof. Consider D:



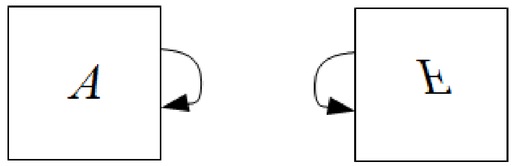
This is isotopic to both D_+:



and D_-:



Smooth to find D_0:



So:

$$\nabla_{D_+} - \nabla_{D_-} = z\nabla_{D_0}$$

Since $D_+ \backsimeq D_- \backsimeq D$ (meaning the knots represented are isotopic), we find $z\nabla_{D_0} = 0$, so $\nabla_{D_0} = 0$. $\square$

Example 55. Using this information in the previous example, we find $\nabla_H = z$.

Example 56. Let T be the right trefoil. Then:

$$\begin{aligned}\nabla_T - \nabla_U &= z\nabla_H \\ \nabla_T &= z^2 + 1\end{aligned}$$

Proposition 88.

(i) $\nabla_{r(D)}(z) = \nabla_D(z)$.

(ii) $\nabla_{m(D)}(-z) = \nabla_{m(D)}(z)$.

(iii) If K is a knot, then ∇_K is a polynomial in z^2.

Proof of (ii). Fix D_+ with positive crossing. D_- has a negative crossing. So $m(D_+)$ has a negative crossing and $m(D_-)$ has a positive crossing.

$$\nabla_{m(D_-)}(z) - \nabla_{m(D_+)}(z) = z\nabla_{m(D_0)}(z)$$

Substitute $z = -w$.

$$\begin{aligned}\nabla_{m(D_-)}(-w) - \nabla_{m(D_+)}(-w) &= -w\nabla_{m(D_0)}(-w) \\ \nabla_{m(D_+)}(-w) - \nabla_{m(D_-)}(-w) &= w\nabla_{m(D_0)}(-w)\end{aligned}$$

Compare to:

$$\nabla_{D_+}(z) - \nabla_{D_-}(z) = z\nabla_{D_0}(z)$$

By double induction:

$$\begin{aligned}\nabla_{D_-}(z) &= \nabla_{m(D_-)}(-z) \\ \nabla_{D_0}(z) &= \nabla_{m(D_0)}(-z)\end{aligned}$$

So $\nabla_{m(D_+)}(-z) = \nabla_{D_+}(z)$ as desired. $\square$

Proposition 89.

(i) If K is a knot, $\nabla_K(0) = 1$.

(ii) If K is a link of two components then the linear term of ∇_K is the linking number.

Proof of (ii). Let K be represented by D_+. Let c be a positive crossing between distinct components. This crossing adds $\frac{1}{2}$ to the linking number in D_+, $-\frac{1}{2}$ to the linking number of D_-, and D_0 is a knot. So:

$$\begin{aligned} \nabla_{D_+} - \nabla_{D_-} &= z\nabla_{D_0} \\ &= z(1 + a_2 z^2 + a_4 z^4 + \cdots) \end{aligned}$$

by (i) and the previous proposition. So the linear terms of ∇_{D_+} and ∇_{D_-} differ by 1. So ∇_{D_+} is one more than the linear term of ∇_{D_-}, which is the linking number of D_- by induction. □

Theorem 90. *If L is a link, than $\nabla_L(x - x^{-1}) = \Delta_L(x^2)$.*

Corollary 91. $\nabla_L(2i) = \pm\det(L)$.

Proof. $i - \frac{1}{i} = 2i$, so $\nabla_L(2i) = \Delta_L(-1) = \pm\det(L)$. □

Definition 80. The HOMFLY (Hoste, Ocheau, Millet, Freyd, Lickorish, Yetter) polynomial P_K satisfies:

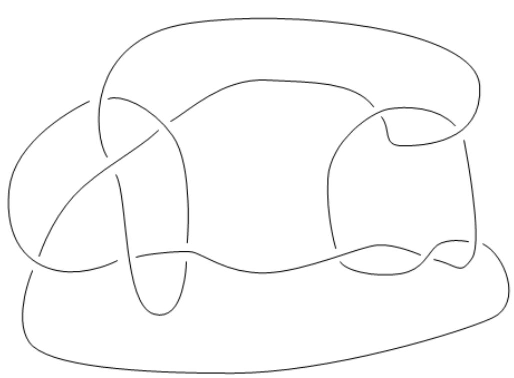
(i) $P_U = 1$.

(ii) $\alpha P_{D_+} - \alpha^{-1}P_{D_-} = zP_{D_0}$ where D_+ has a positive crossing, D_- replaces this by a negative crossing, and D_0 has this crossing smoothed.

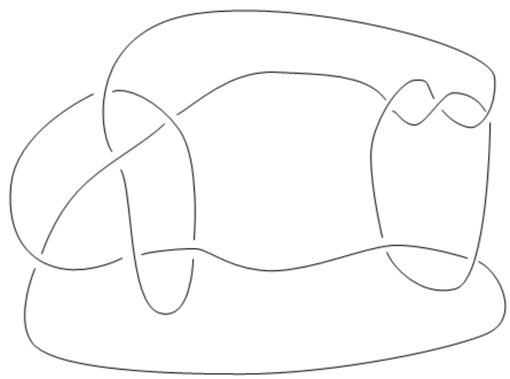
Remark. P_K is a polynomial in z and a Laurant polynomial in α. At $\alpha = 1$ we find $P_K(1, z) = \nabla_K(z)$. At $\alpha = t^{-1}$, $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ we find $P_K(t^{-1}, t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = V_K(t)$.

Theorem 92. *P_K is an isotopy invariant.*

Example 57. Consider the Kinoshita-Terasaka knot:



and the Conway knot:



These two knots are not isotopic, but they have the same HOMFLY polynomial, and both have Conway polynomial 1.