Reconfiguration of square-tiled surfaces

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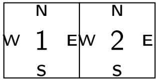
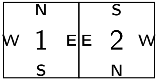
Joint work with Vincent Delecroix.

Square-tiled surfaces

Definition

- Square-tiled surface: gluing of N square tiles on their parralel sides $\rightsquigarrow$ closed orientable connected surface





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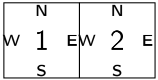
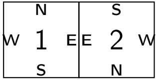
- Square-tiled surface: gluing of N square tiles on their parrallel sides $\rightsquigarrow$ closed orientable connected surface
- Quadratic: adjacencies = {NS,EW, NN, SS, EE, WW}
- Abelian: only {NS,EW}



Abelian



Quadratic

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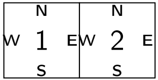

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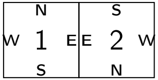

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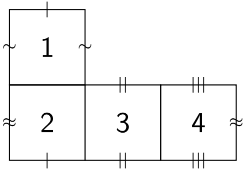
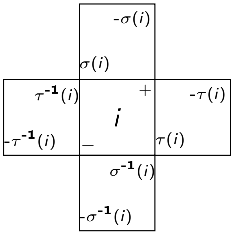


Quadratic encoding

Encoding

Triplet of involutions without fix-point $\rho, \sigma, \tau \in \mathfrak{S}_{2n}$ that generate a transitive subgroup of $\mathfrak{S}_{2n}$



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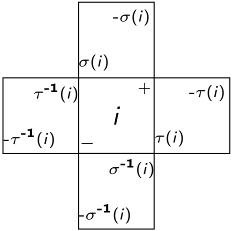

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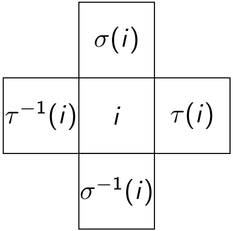
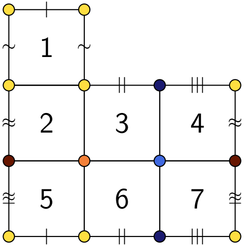


Stratum



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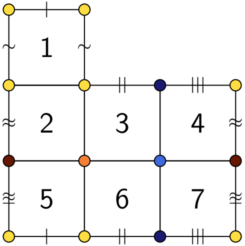
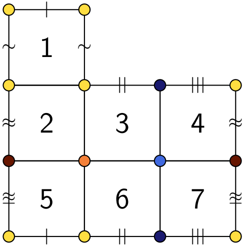


Euler's formula

- μ_i: # vertices of degree $i\pi$
- $\sum_i (i-2)\mu_i = 4g - 4$
- Stratum: $[1^{\mu_1}, 2^{\mu_2}, \ldots]$

Stratum



$[2^4, 6^1]$ so $g = 2$

Euler's formula

- μ_i: # vertices of degree $i\pi$
- $\sum_i (i-2)\mu_i = 4g - 4$
- Stratum: $[1^{\mu_1}, 2^{\mu_2}, \dots]$

Monte Carlo methods to sample efficiently

Reconfiguration

- Configuration space Ω
- Elementay operation $\leftrightarrow$
- Equivalent configurations: $\exists$ a sequence of operations leading from one to the other
- Reconfiguration graph: Vertices = configurations, edges = elementary operation

Monte Carlo methods to sample efficiently

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Usual questions

- Are any configurations equivalent ?
- How many reconfiguration steps separate any two configurations ?
- Application to sampling: Does the corresponding Markov chain mix well ?

Markov chains and mixing time

Random Walk P on the reconfiguration graph

- Irreducible: reconfiguration graph connected
- Aperiodic + Irreducible $\rightsquigarrow \exists!$ stationary distribution π
- + Symmetric $\rightsquigarrow \pi$ uniform

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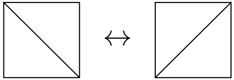
Mixing time

$$t_{mix}(\varepsilon) = \inf\{t : \max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{TV} \leq \varepsilon\}$$

where $\|\alpha - \beta\|_{TV} = \sup_{X \subset \Omega} |\alpha(X) - \beta(X)|$

Flips on triangulations

Elementary flip



Disarlo, Parlier 2014

Reconfiguration diameter of n-triangulations of genus g:

- Labeled vertices: $\Theta(g \log(g+1) + n \log(n))$
- Unlabeled vertices: $\Theta(g \log(g+1) + \log(n))$

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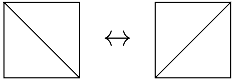

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Budzinski 2018

- For $g=0$, $t_{mix}=\Omega(n^{5/4})$
- t_{mix} polynomial in n ?

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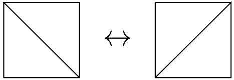

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Not on quadrangulations !

Flips on quadrangulation

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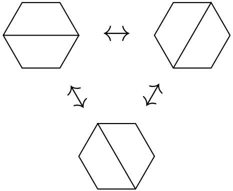
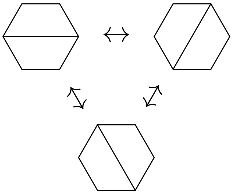


Flips on quadrangulation



Elementary flip

Caraceni, Stauffer 20

- For $g = 0$, $t_{mix} = \Omega(n^{5/4})$
- $t_{mix} = O(n^{13/2})$

Flips on quadrangulation

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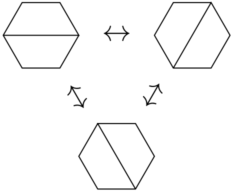

Caraceni, Stauffer 20

- For $g = 0$, $t_{mix} = \Omega(n^{5/4})$
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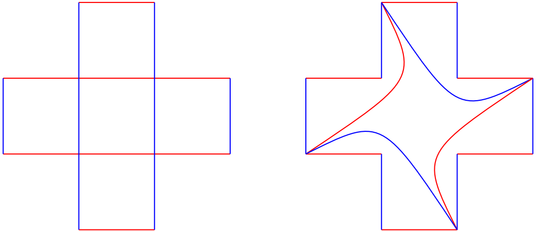
Preserves genus but not square-tiled surfaces !

Rotation

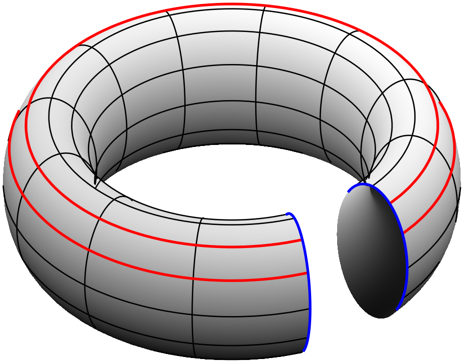
Elementary rotation



Preserves genus and square tiled-surface, but not Abelian/quadratic !

Playing Rubik's cube on a surface

Shearing move



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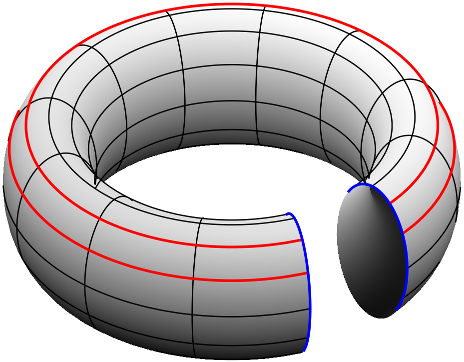
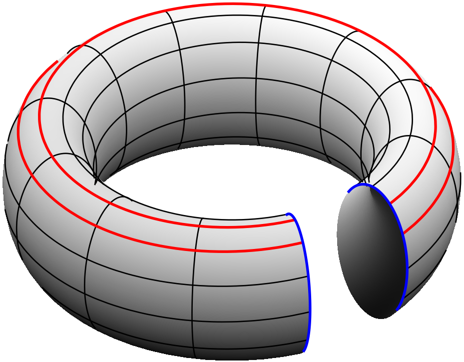
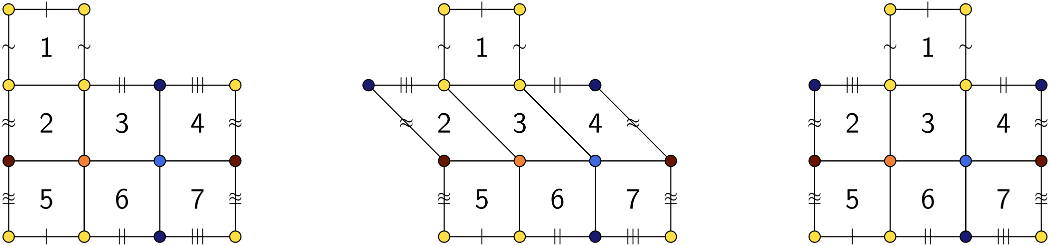


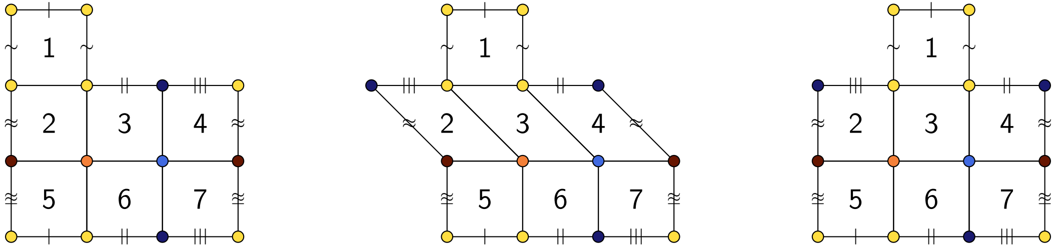


Shearing move



Shearing moves preserve the angle around the vertices !

Shearing move



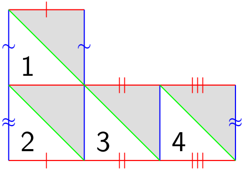
Shearing moves preserve the angle around the vertices !

Two settings

- Slow shears: One shear at a time
- Fast shears: Any number of shears on the same cylinder count as one

Hyperelliptic

- $\mu = [2^{mu_2}, 4g-2]$ or $[2^{\mu_2}, (2g)^2]$ $\rightsquigarrow$ always abelian
- Quadrangulation fixed under rotation of angle π
- Quotient gives a sphere



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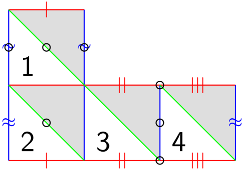

Hyperelliptic case

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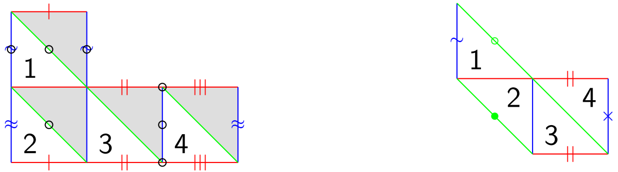

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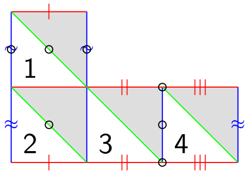
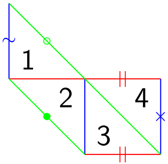
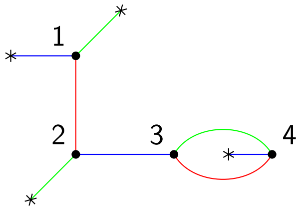
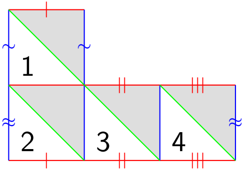
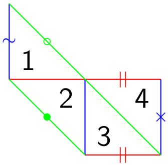


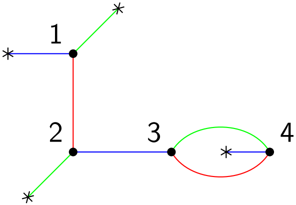




Strata of tricolored planar graphs







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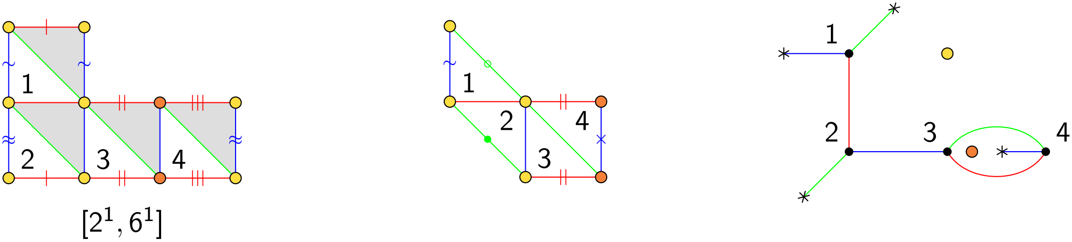
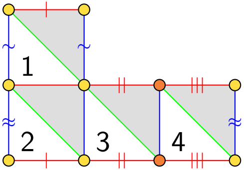
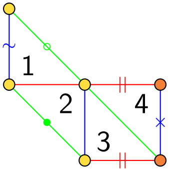
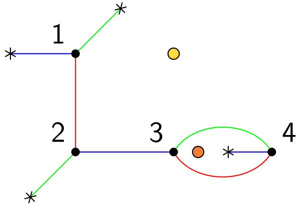




$[2^1, 6^1]$



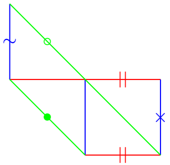


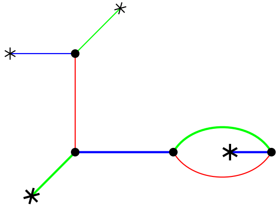
$([1^1, 3^1], 4)$

Stratum

- μ_i: number of faces of degree $3i$
- k: number of triangles
- Euler's formula : $(\sum_i (i-2)\mu_i) - k = 4g - 4 = -4$
- Hyperelliptic strata: $([1^{\mu_1}, 2^{\mu_2}, d^1], d + 2 - \mu_1)$

Shearing moves in tricolored planar graphs





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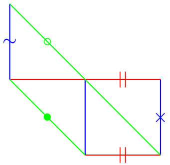
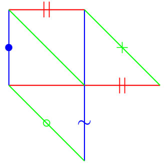
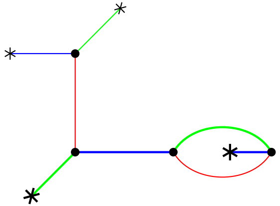
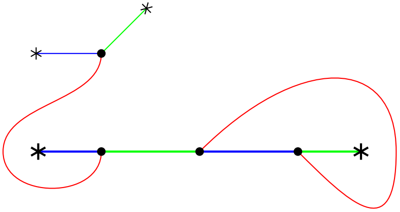
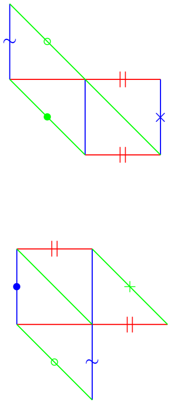
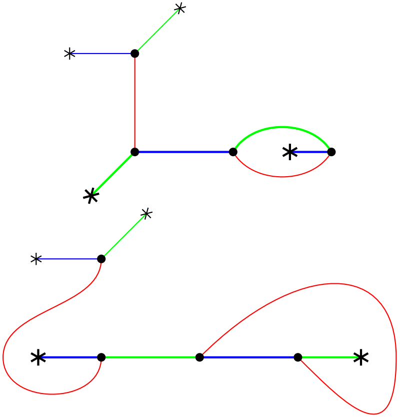












Shearing move

- swap colors + treadmill
- RG and GB in $O(1)$, RB in $O(n)$

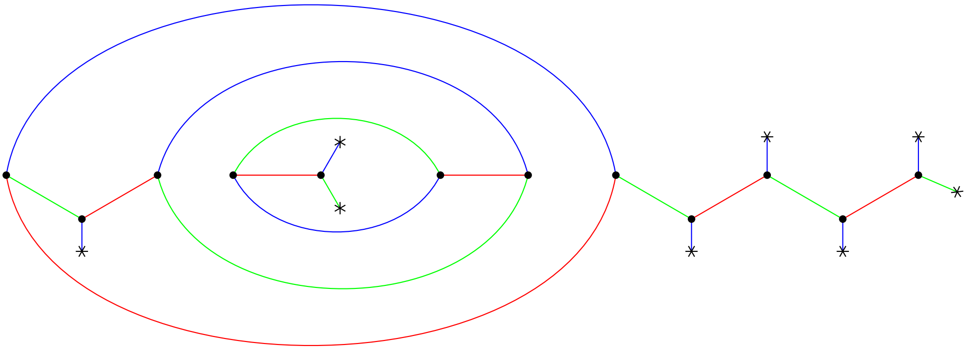
Contribution

Delecroix, L. 2023+

Reconfiguration diameter of unlabeled tricolored graphs:

- hyperelliptic strata: $O(kn)$ slow shears, $\Theta(k)$ fast shears
- $g = 0$ and $\mu_1 = 0$: $O(kn)$ slow shears, $\Theta(k)$ fast shears

Reach a "canonical" configuration



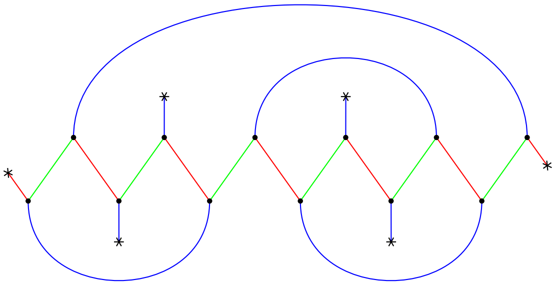
$([2^2, 3^1, 5^1], 8)$

Sketch of proof

1. Get to a path-like configuration: One RG cylinder finishing with halfedges
2. Reconfiguration within path-likes

1. Take a RG path
2. The RB path at the end of it is a fusion-path
3. Collapse the cylinders with a GB shear.

Blue dual tree



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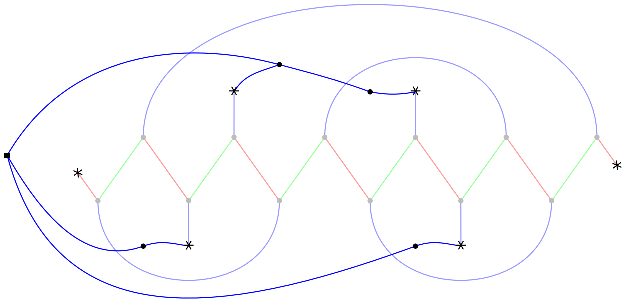
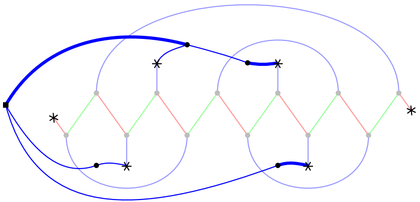


Proposition

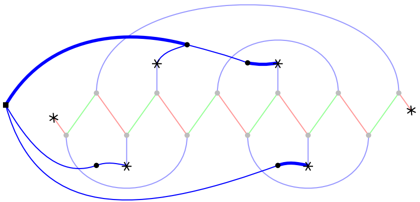
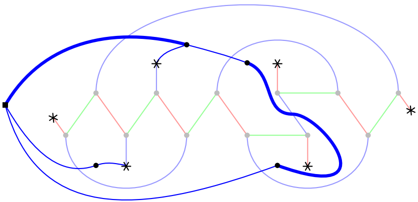
All path-like configurations corresponding to a blue dual tree are equivalent via $O(n)$ RG shears

Reconfiguration of blue dual trees



new Glue-cut operation preserving path-likes

Reconfiguration of blue dual trees

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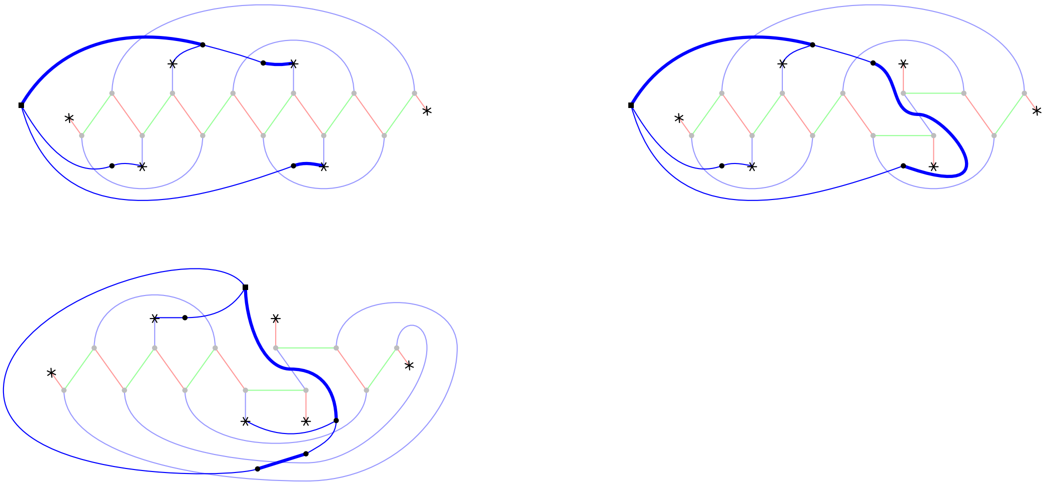

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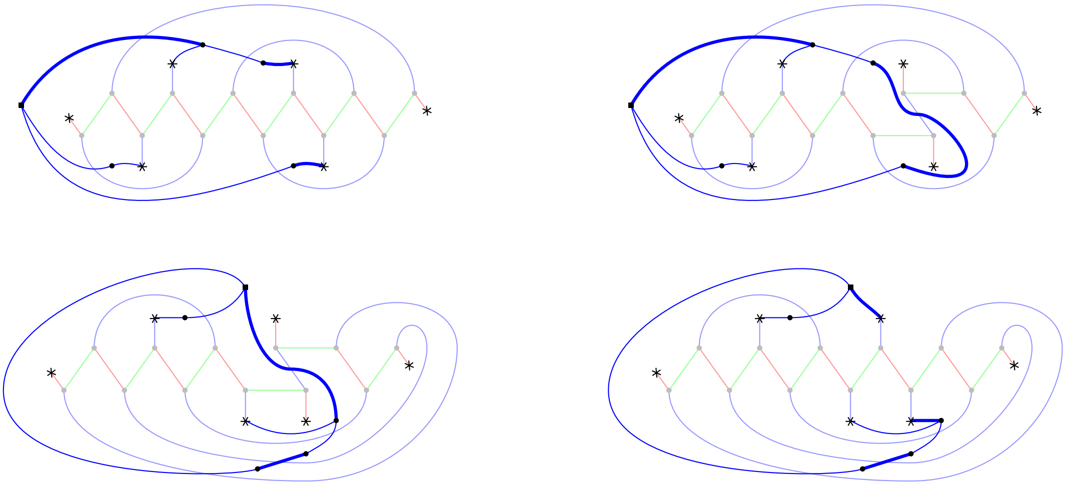
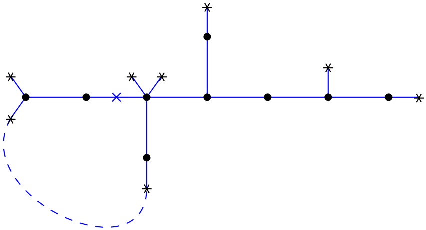


new Glue-cut operation preserving path-likes

1. Blue dual tree → Blue dual path
2. Sort the vertices on the path



Future work

Rapid mixing in hyperelleptic case ?

- Among path-like configurations with the glue-cut operation ?
- In general

Connectivity in the general case

- Non planar $\Rightarrow$ no dual tricolored planar graph
- Hyperelleptic case negligible, not in all strata

Future work

Rapid mixing in hyperelleptic case ?

- Among path-like configurations with the glue-cut operation ?
- In general

Connectivity in the general case

- Non planar $\Rightarrow$ no dual tricolored planar graph
- Hyperelleptic case negligible, not in all strata

Thanks !