GRAPHICALLY DISCRETE GROUPS & RIGIDITY

Joint work with

Alex Margolis,
Sam Shepherd,
Daniel Woodhouse.

AIM: Understand the relationship between the large-scale geometry of a group and its algebraic structure.

RIGIDITY PHENOMENA

"A group's geometry determines its algebra!"

<u>DEF</u>: Groups G and G' are <u>virtually isomorphic</u> if $\exists$ finite normal $K \trianglelefteq G$ and $K' \trianglelefteq G'$ so that

$$G/K \quad \text{and} \quad G'/K'$$

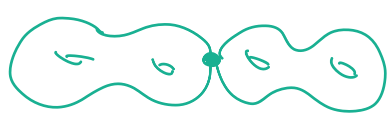
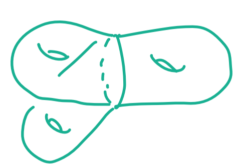
are abstractly commensurable: they have isomorphic finite-index subgroups.

<u>DEF</u>. A finitely generated group G is <u>quasi-isometrically rigid</u> if whenever G' is a group quasi-isometric to G, G and G' are virtually isomorphic.

EXAMPLES: QUASI-ISOMETRICALLY RIGID GROUPS.

- (Stallings–Dunwoody) Virtually free groups
- (Tukia; Gabai; Casson–Jungreis) Hyperbolic surface groups
- (Bourdon–Pajot; Xie; Haglund) Uniform lattices in isometry groups of Fuchsian buildings

NON-EXAMPLES: QUASI-ISOMETRICALLY RIGID GROUPS.

- $\pi_1\begin{pmatrix}\text{closed}\\ \text{hyperbolic}\\ n\text{-manifold}\end{pmatrix}$, $n \geq 3$
- Some free products  (Papasoglu – Whyte)
- Some amalgamated free products  (Malone; S.)

ACTION RIGIDITY.

DEF. A fg group G is <u>action rigid</u> if whenever G and G' act geometrically on the same proper geodesic metric space, G and G' are virtually isomorphic.

NOTE.

- If G is QI rigid, then G is action rigid.

  → Action rigidity can give evidence for & a step toward QI rigidity

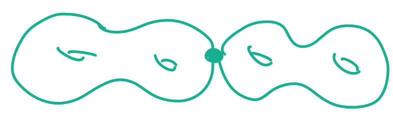
Ex. (MSSW) $\pi_1(N^3) *_{\mathbb{Z}} \pi_1(M^3)$ is action rigid

closed, hyperbolic 3-manifolds

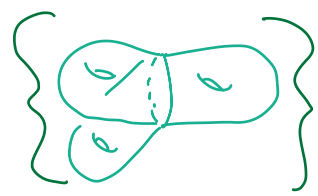
- Which quasi-isometric groups have a common model geometry?

EXAMPLES: ACTION RIGID GROUPS

- Any group that is QI rigid is action rigid
- (S.-Woodhouse) Free products of hyperbolic manifold groups*



- (MSSW) Fundamental groups of :



NON-EXAMPLES: ACTION RIGID GROUPS

- $\pi_1\begin{pmatrix}\text{closed} \\ \text{hyperbolic} \\ \text{n-manifold}\end{pmatrix}$, $n \geq 3$
- (Burger-Mozes; Wise) Groups acting on the product of trees $T_4 \times T_6$

OPEN QUESTION: If G is hyperbolic and $G \underset{\text{geom}}{\curvearrowright} \mathbb{H}^n_{\mathbb{F}}$, is G action rigid?

THEOREM. (Margolis – Shepherd – S. – Woodhouse)

If G is a free product of residually finite graphically discrete groups,

then G is action rigid within the family of residually finite groups.

EXAMPLES OF GRAPHICALLY DISCRETE GROUPS:

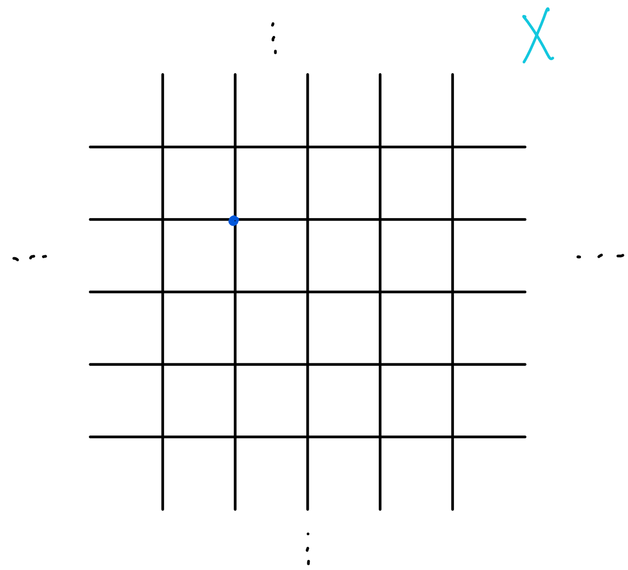
- (Bader – Furman – Sauer)
  - $\pi_1(M^n)$, M^n: finite-volume hyperbolic n-manifold, $n \geq 3$
  - irreducible lattices in connected center-free real semisimple Lie groups w/o compact factors (eg. $PSL_n\mathbb{R}$)
- (Trofimov) Virtually nilpotent groups
- (Kida) Mapping class groups
- (Guirardel – Horbez) $Out(F_n)$
- (Horbez – Huang) Various 2-dim'l ~~Artin~~ Artin groups
- (Farb – Mosher) Hyperbolic surface-by-free groups

GRAPHICALLY DISCRETE GROUPS

OBSERVATION:

Suppose X is a locally finite graph and $\mathrm{Aut}(X)$ is discrete.

If $\Gamma, \Gamma' \curvearrowright X$ geometrically, then Γ and Γ' are virtually isomorphic.



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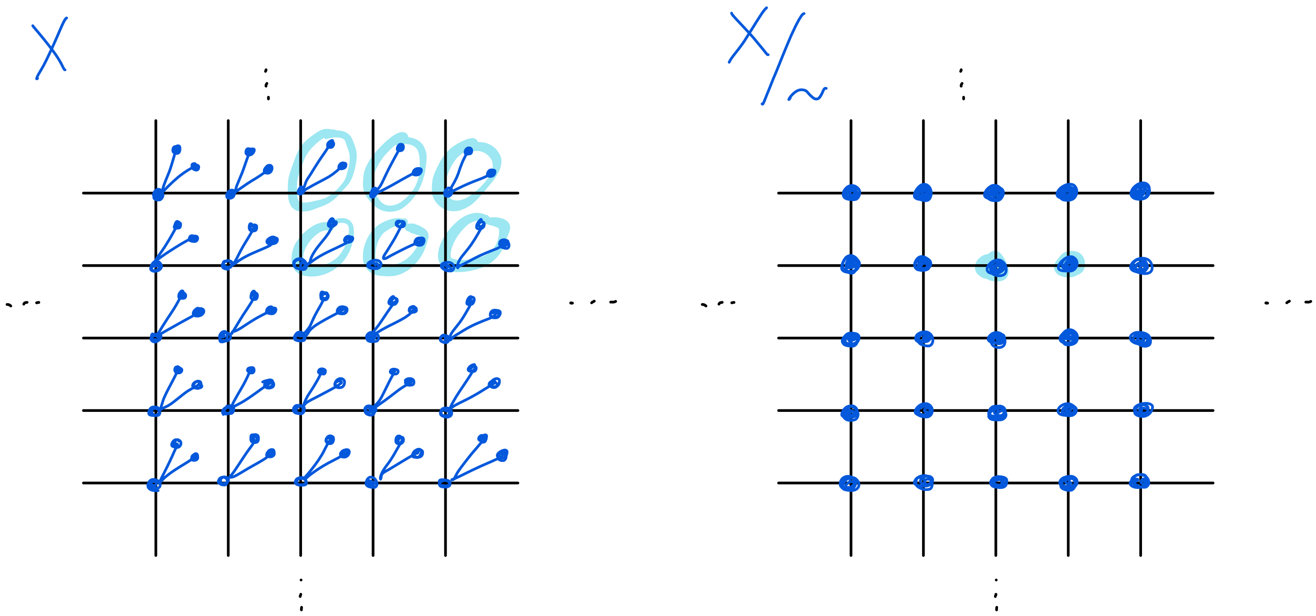
* It is too much to expect that whenever a group Γ acts geometrically on a locally finite graph X, $\mathrm{Aut}(X)$ is discrete.

(Add leaves)

DEFINITION. A finitely generated group Γ is graphically discrete if whenever Γ acts geometrically on a locally finite graph X

there exists an $\mathrm{Aut}(X)$-invariant equivalence relation $\sim$ on VX so that

- equivalence classes are finite
- the image of $\mathrm{Aut}(X)$ in $\mathrm{Aut}(X/\sim)$ is discrete.



OBSERVATION: If Γ is graphically discrete and Γ & Γ' act geometrically on the same locally finite graph, then Γ & Γ' are virtually isomorphic.

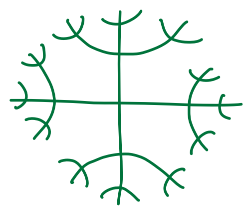
A BOUNDARY CRITERION

<u>Def.</u> Let X be a proper geodesic metric space. The <u>visual boundary</u> of X

$$\partial_\infty X := \left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{geodesic rays in } X \end{array} \right\}.$$

<u>Ex.</u>

$\partial_\infty \mathbb{H}^n \cong S^{n-1}$ sphere,



$\partial_\infty(\text{Tree}) \cong$ Cantor set

- (Gromov) If G is hyperbolic, then $\partial_\infty G$ is a quasi-isometry invariant.

- If $G \curvearrowright X$ by isometries, then $G \curvearrowright \partial_\infty X$ by homeomorphisms.

Theorem. (MSSW)

Let G be a hyperbolic group. Then G is graphically discrete

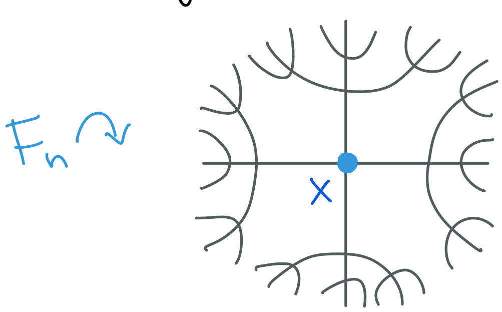
if and only if

for every geometric action of G on a locally-finite graph X and for all vertices $x \in X$, the image of the induced action

$$\mathrm{Stab}_{\mathrm{Aut}(X)}(x) \longrightarrow \mathrm{Homeo}(\partial_\infty X)$$

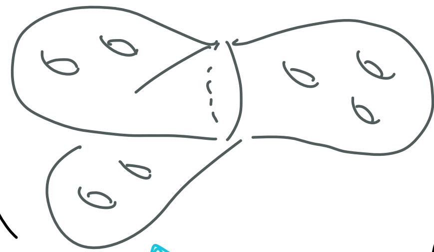
is finite.

EXAMPLE. The free group F_n, $n \geq 2$ is not graphically discrete.



EXAMPLE. Uniform lattices in the isometry group of a hyperbolic building are not graphically discrete

EXAMPLE.

π_1 () is not graphically discrete

SAME EULER CHARACTERISTIC

Theorem. (MSSW)
The fundamental group of a "simple surface amalgam" is graphically discrete if and only if all surfaces in the amalgam have distinct Euler characteristics.

Corollary. Graphical discreteness is not a quasi-isometry invariant.

Proof Outline:

Theorem (MSSW)

If G is a free product of residually finite graphically discrete groups, then G is action rigid within the family of residually finite groups.

Step One: Promote the common model geometry

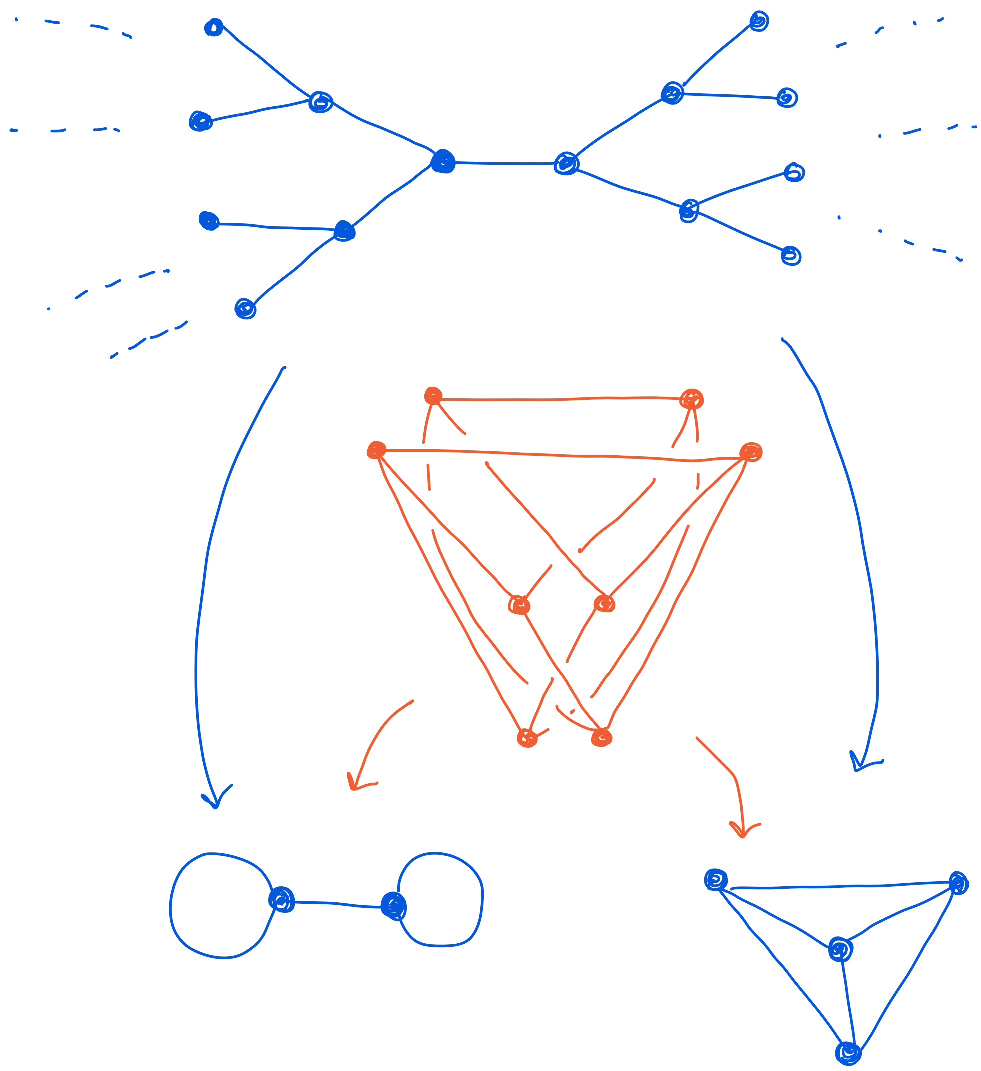
- Show if $G, G' \curvearrowright X$ geometrically, proper, geodesic metric space

then $G, G' \curvearrowright Y$ geometrically, locally finite graph with a tree of spaces decomposition preserved by G & G'.

→ Graphical discreteness of vertex groups implies pairs G_v, G'_v are commensurable inside $\mathrm{Aut}(X_v)$

PROOF, STEP TWO : APPLY A GENERALIZATION OF LEIGHTON'S GRAPH COVERING THEOREM.

Theorem (Leighton, 1982) If two finite graphs have isomorphic universal covers, then the graphs have isomorphic finite covers.



THANK YOU!