Massey Products for Graph Homology

Ben Ward

Bowling Green State University

Warwick Geometry and Topology Online Seminar
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Recollection of A_∞-algebras

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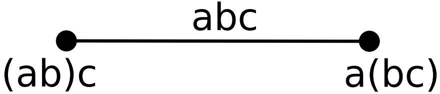

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Example of K_4.

String of 4 letters

Polytope

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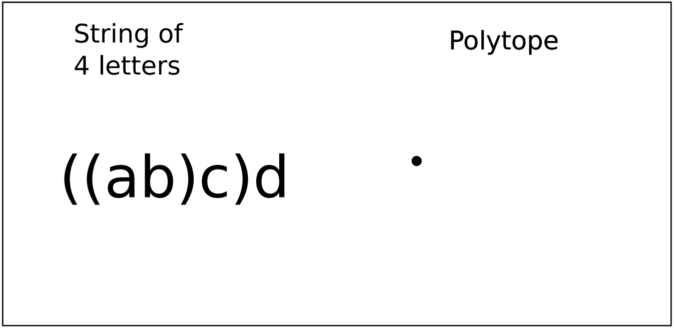

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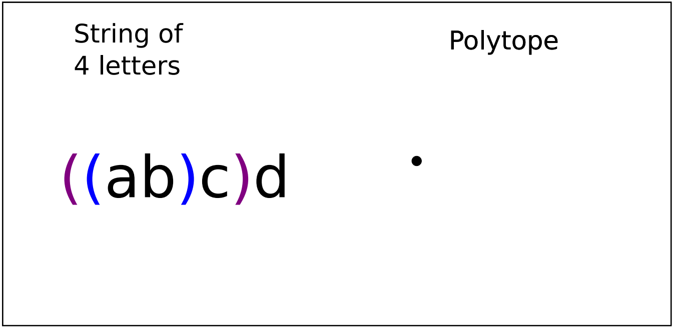

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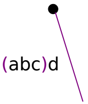

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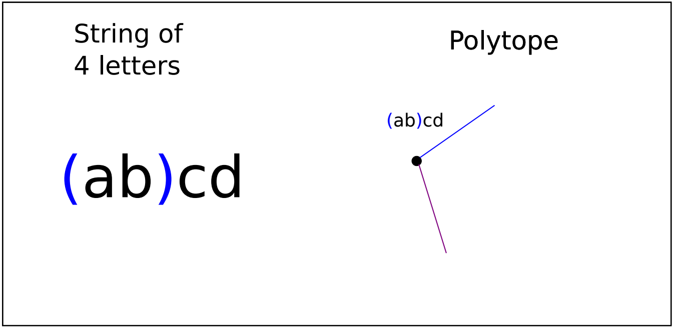

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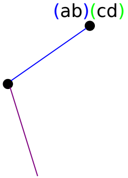

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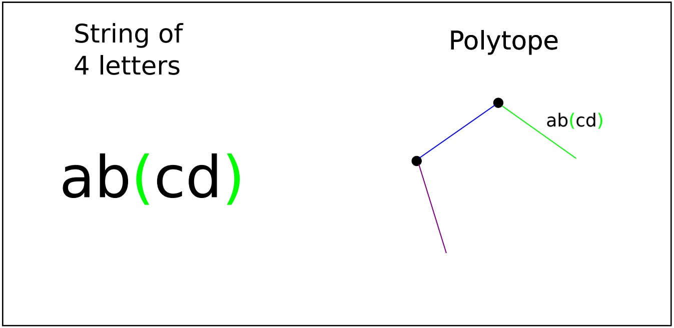

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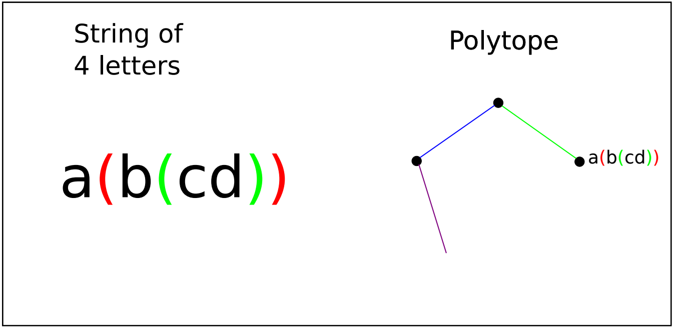

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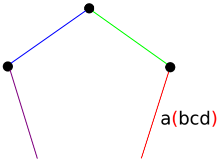

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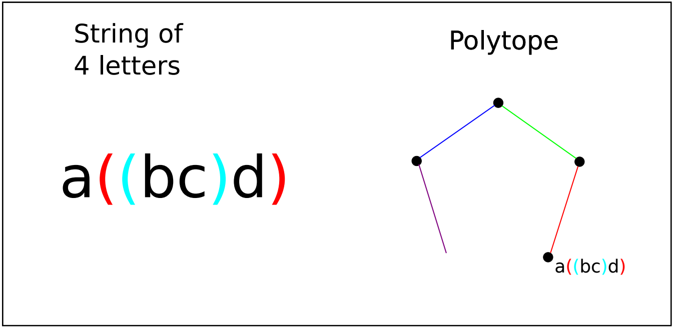

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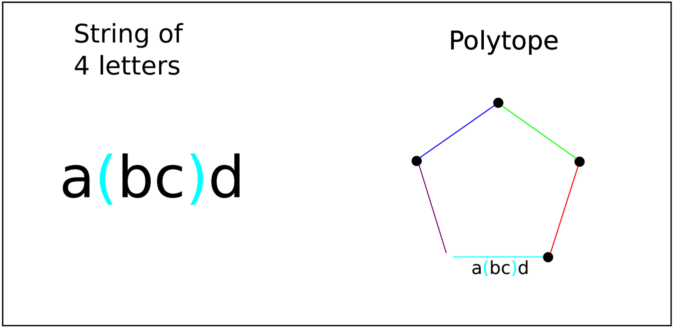

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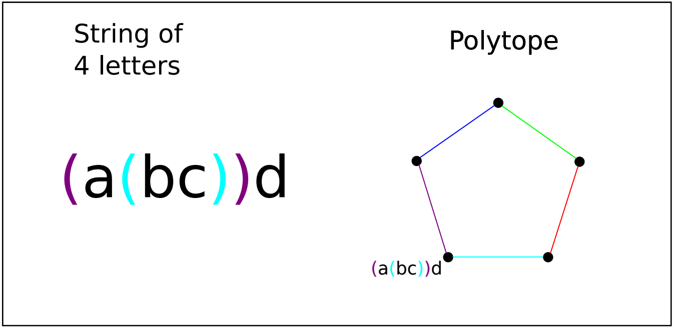

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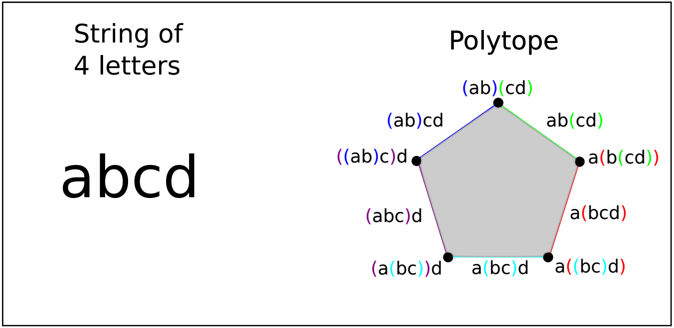
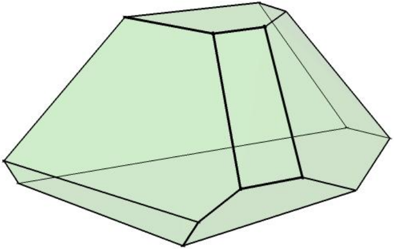


Associahedra

Example: K_5



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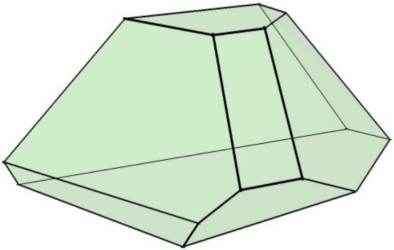

Example: $K_2 = \bullet$

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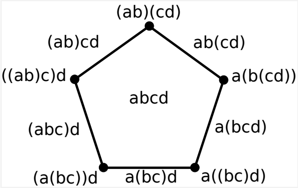
K_4		need $\mu_4 \colon A^{\otimes 4} \to A$ of degree 2

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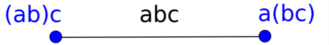
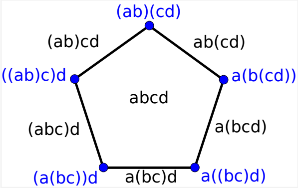
K_3		need $\mu_3 : A^{\otimes 3} \to A$ of degree 1
K_4		need $\mu_4 : A^{\otimes 4} \to A$ of degree 2

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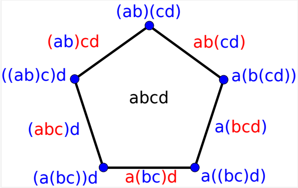
K_4		need $\mu_4 \colon A^{\otimes 4} \to A$ of degree 2

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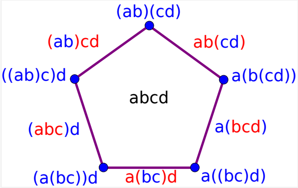
K_4		need $\mu_4 \colon A^{\otimes 4} \to A$ of degree 2

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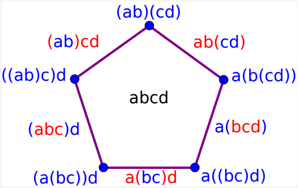
K_4		need $\mu_4 \colon A^{\otimes 4} \to A$ of degree 2

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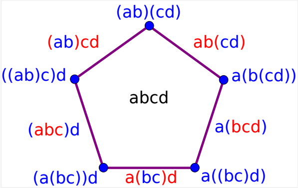
K_4		need μ_4: $A^{\otimes 4} \to A$ of degree 2

and so on... A_∞-algebra $= (A, \mu_2, \mu_3, \mu_4, \mu_5, \ldots)$

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Theorem (Kadeishvili)

Let A be a dg associative algebra over a field of characteristic zero. There exists an A_∞ structure on $H_*(A)$ such that $A \sim H_*(A)$ as A_∞-algebras.

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- We will call these higher operations "Massey products".

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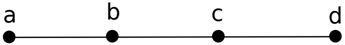

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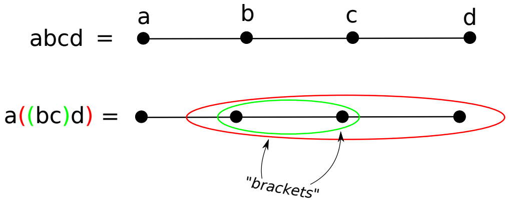

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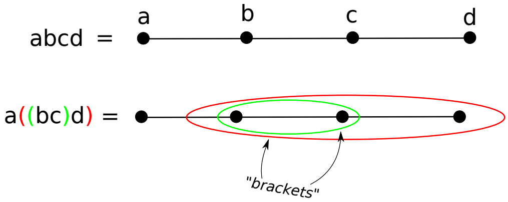

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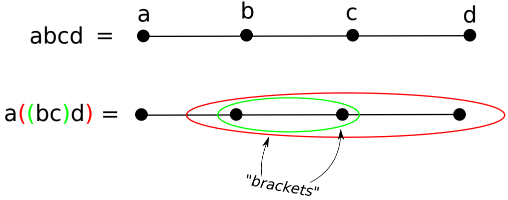

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Lemma (W.) The space of bracketings of *any graph* is contractible, in fact it is a polytope.

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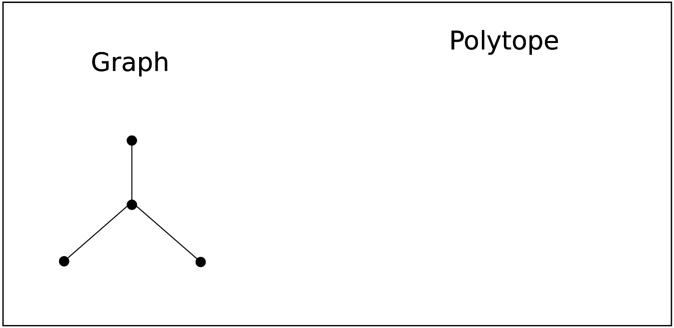

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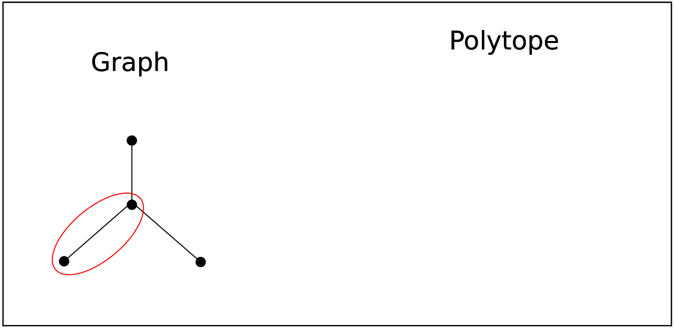

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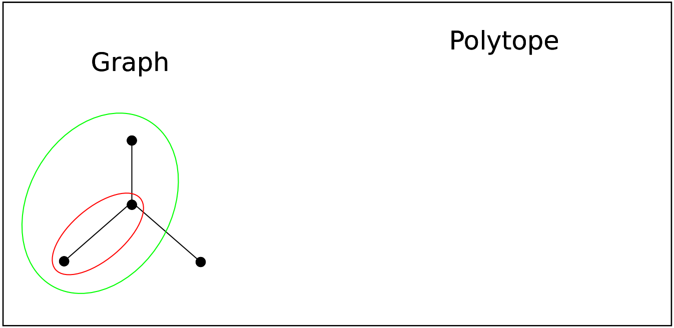

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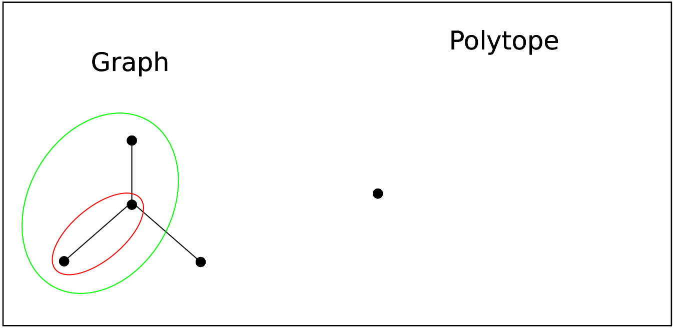

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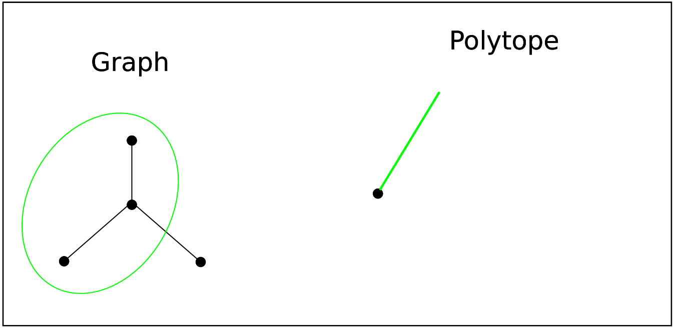

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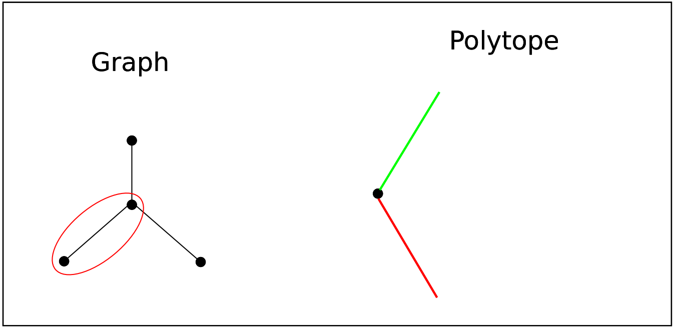

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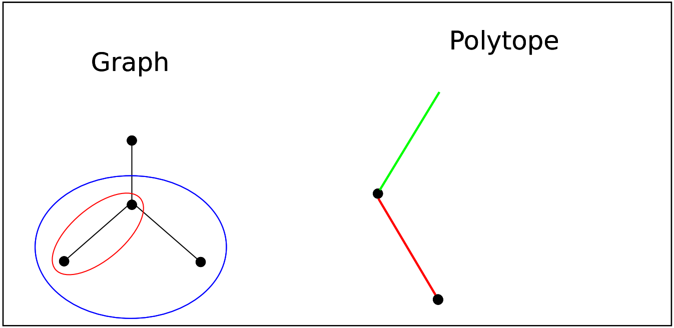

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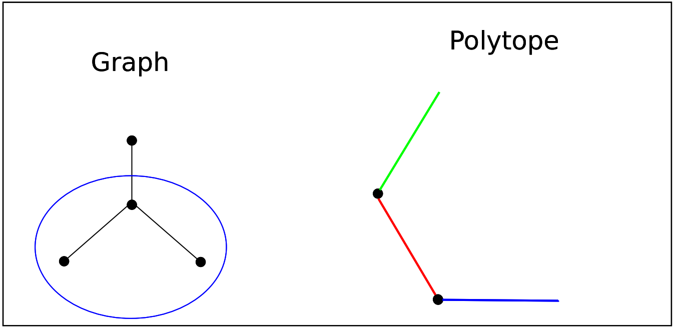

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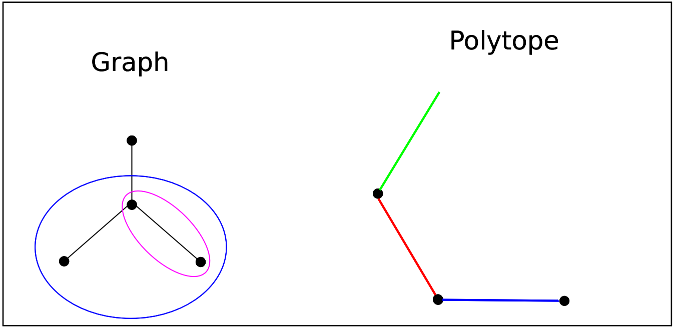

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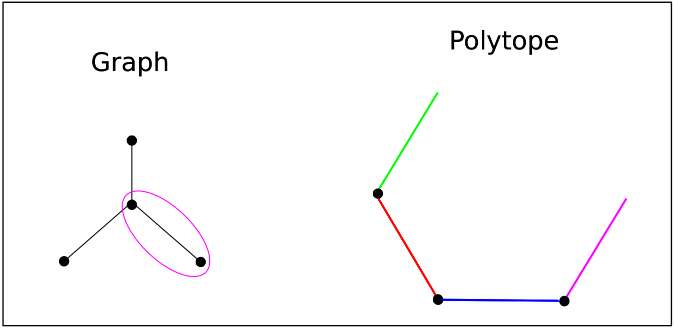

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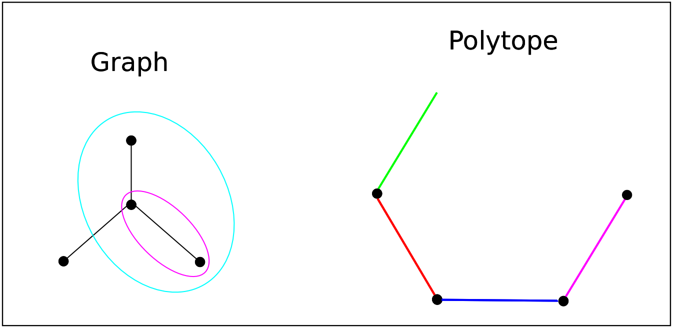

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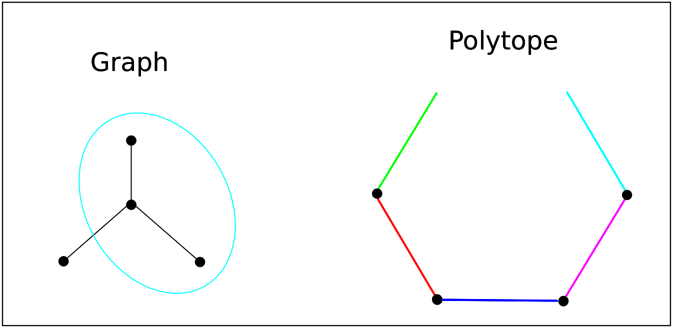

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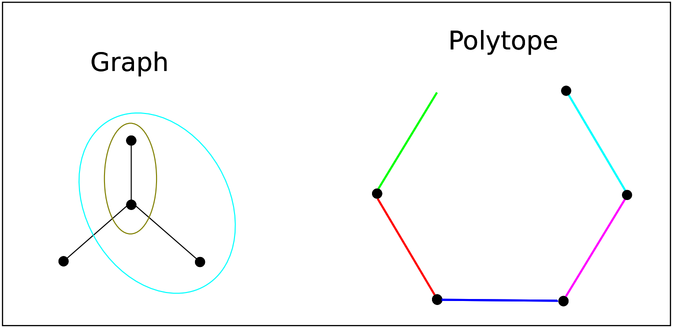

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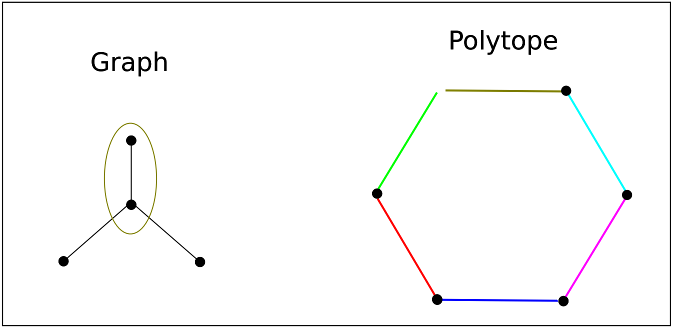

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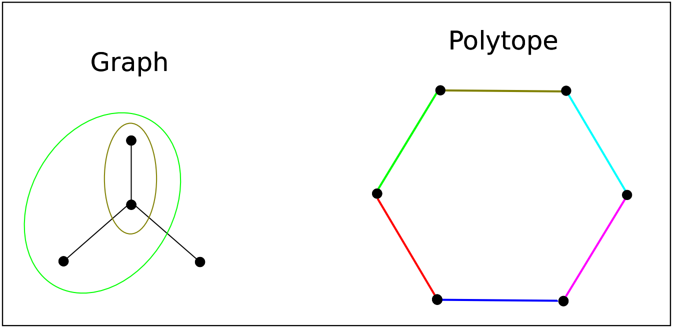

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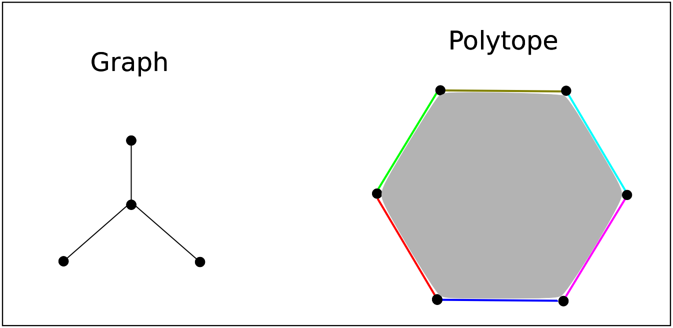

Bracketohedra???

Graph	Picture	Name

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	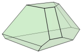	Associahedron

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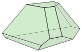
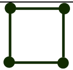
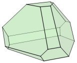
		Associahedron
		Cyclohedron

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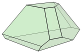
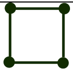
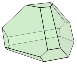
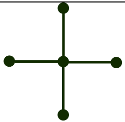
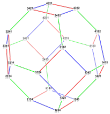
		Associahedron
		Cyclohedron
	 (thanks wikipedia)	Permutohedron

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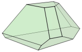
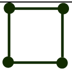
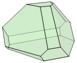
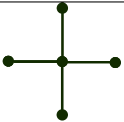
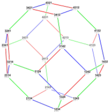
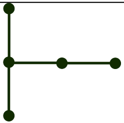
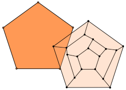
		Associahedron
		Cyclohedron
	 (thanks wikipedia)	Permutohedron
		????hedron

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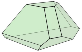
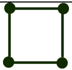
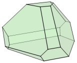
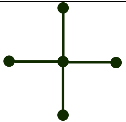
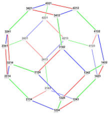
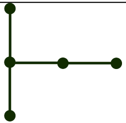
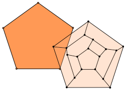
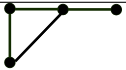
		Associahedron
		Cyclohedron
	 (thanks wikipedia)	Permutohedron
		????hedron
		????hedron

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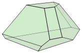
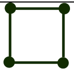
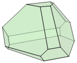
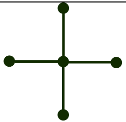
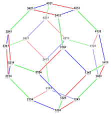
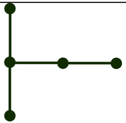
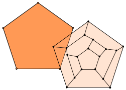
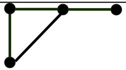
		Associahedron
		Cyclohedron
	 (thanks wikipedia)	Permutohedron
		????hedron
		????hedron

These are the only 3d Bracketohedra.

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Combinatorics	Multiply along a line
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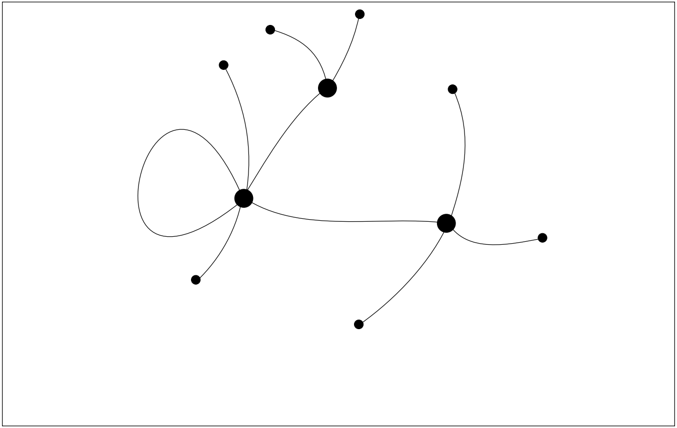

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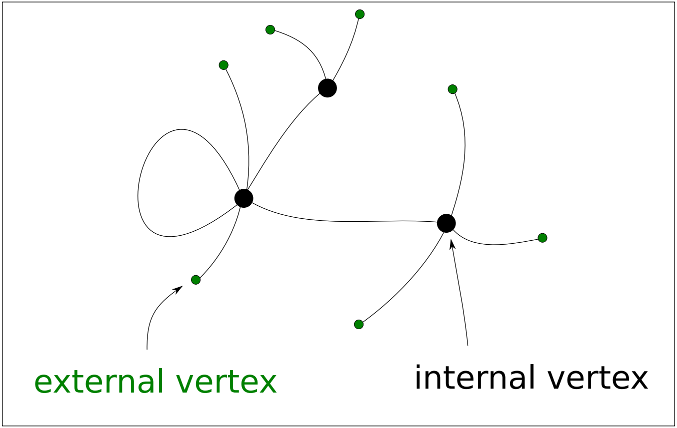

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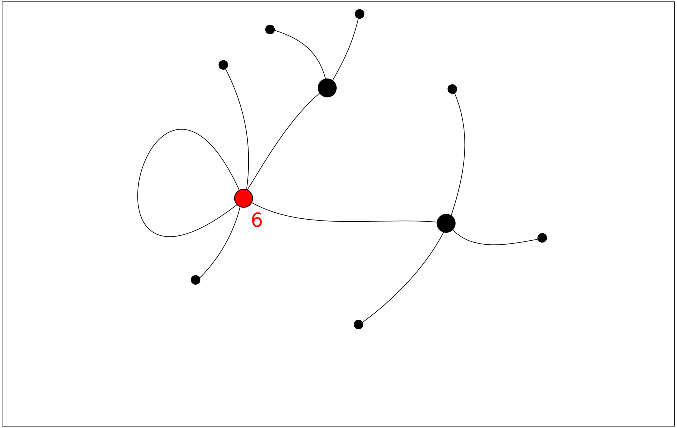

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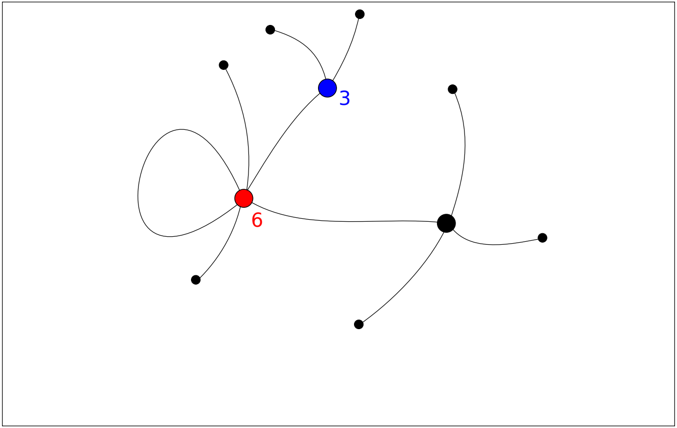

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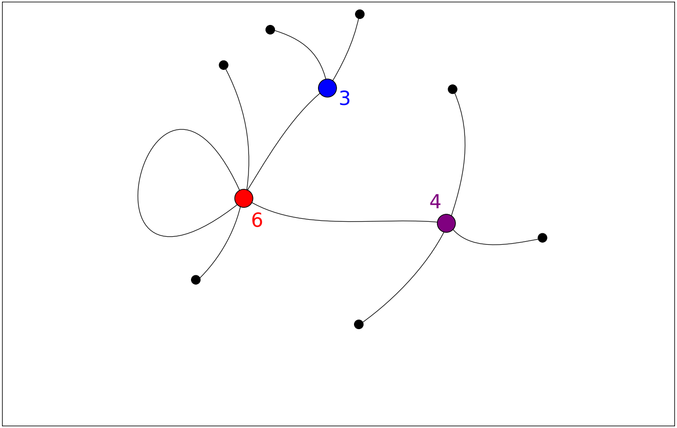

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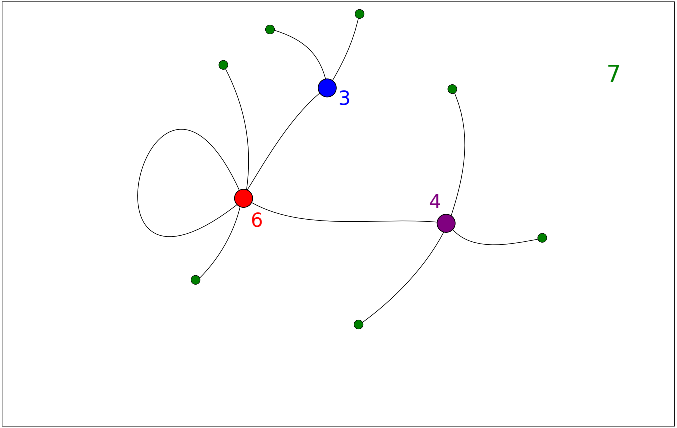

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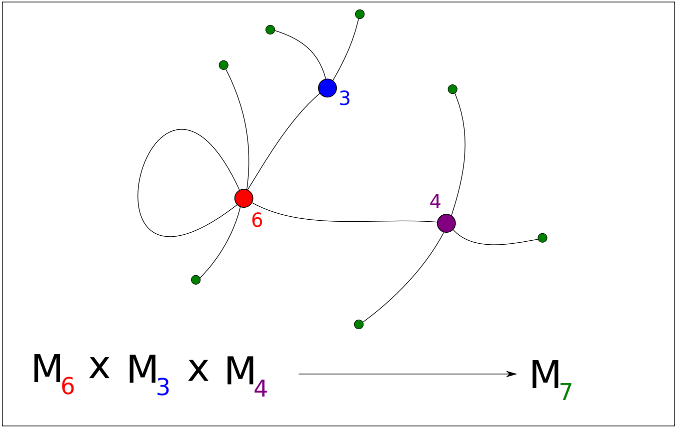

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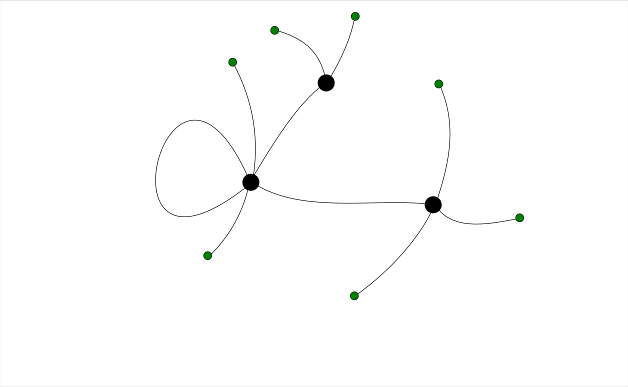

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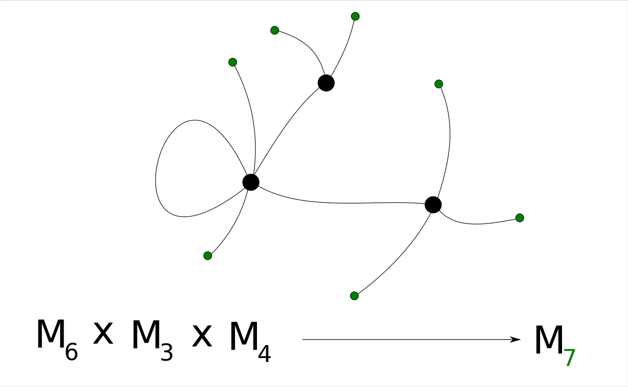

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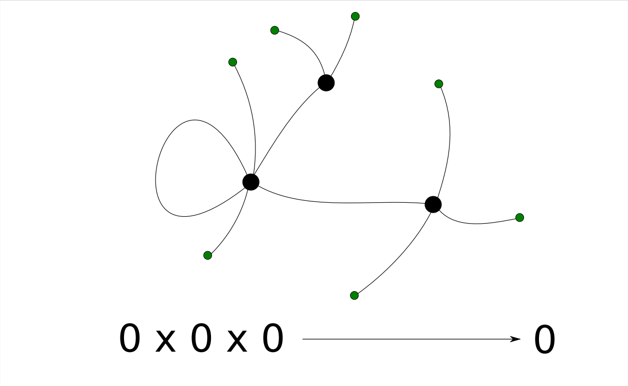

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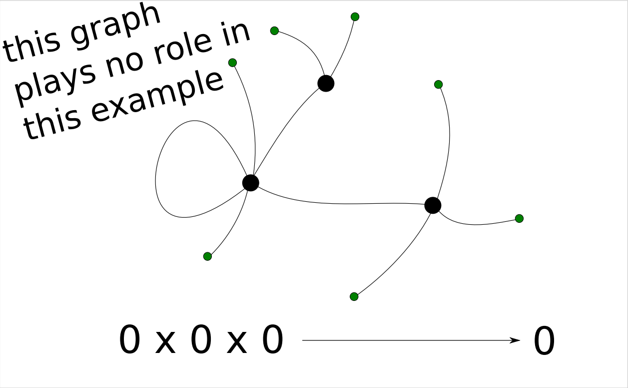

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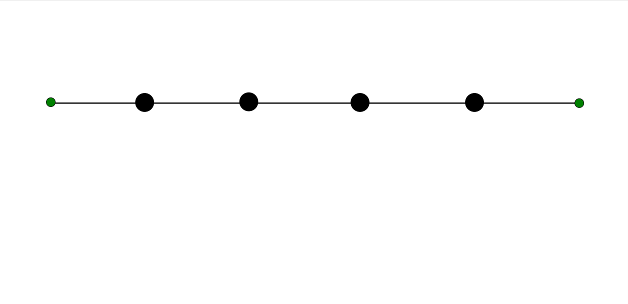

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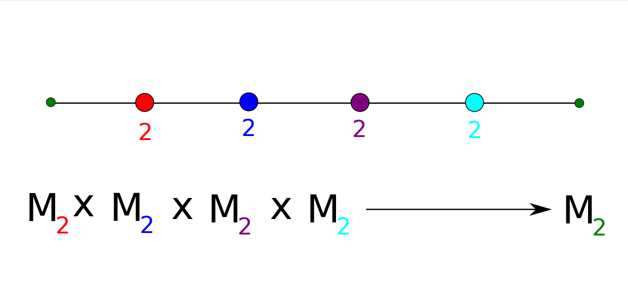

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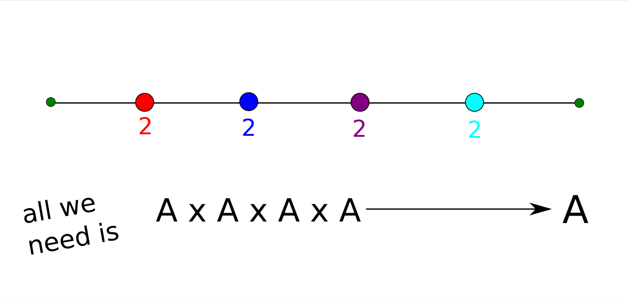

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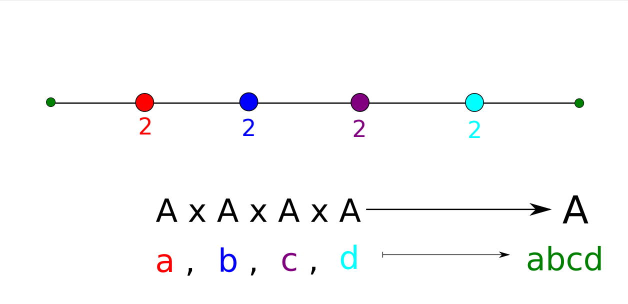

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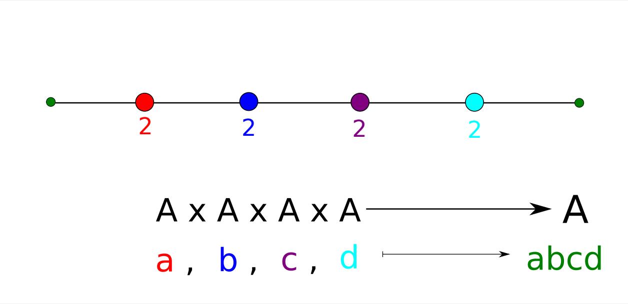

- Modular operads generalize associativity.

Examples of modular operads

Let V be a vector space and $V \otimes V \xrightarrow{\langle -,- \rangle} \mathbb{Q}$ an inner product.
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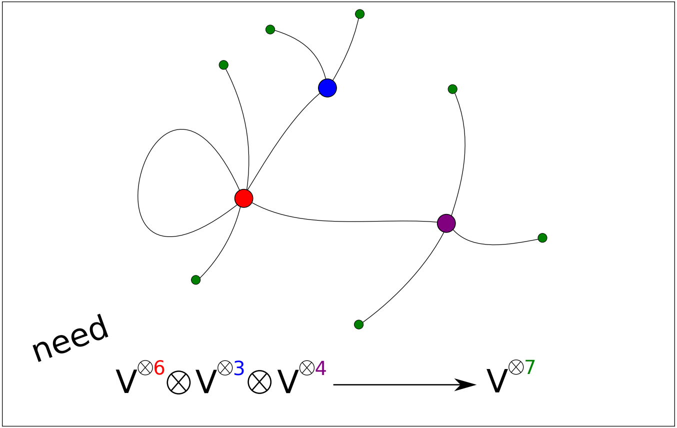

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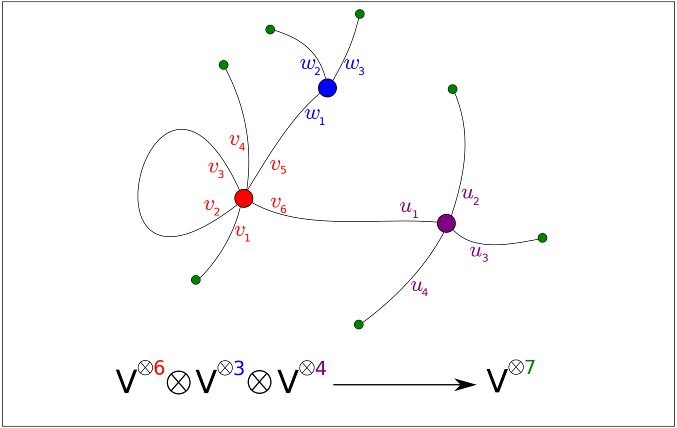

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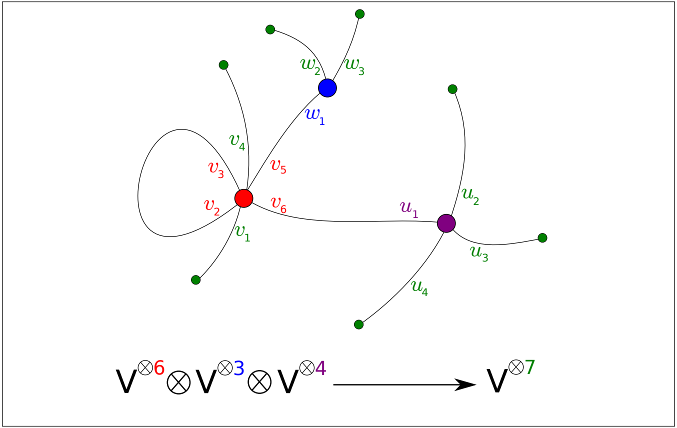

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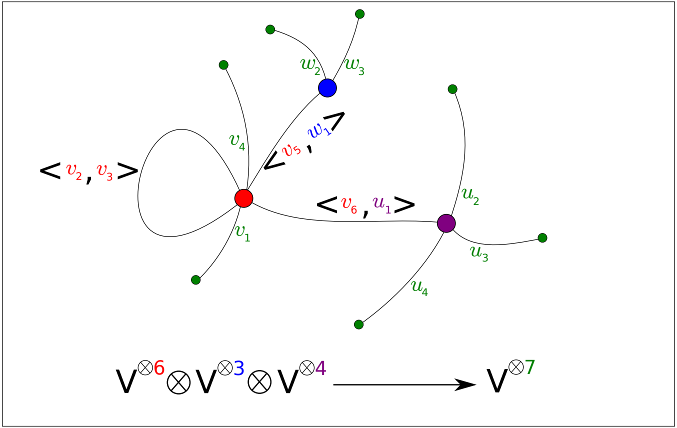

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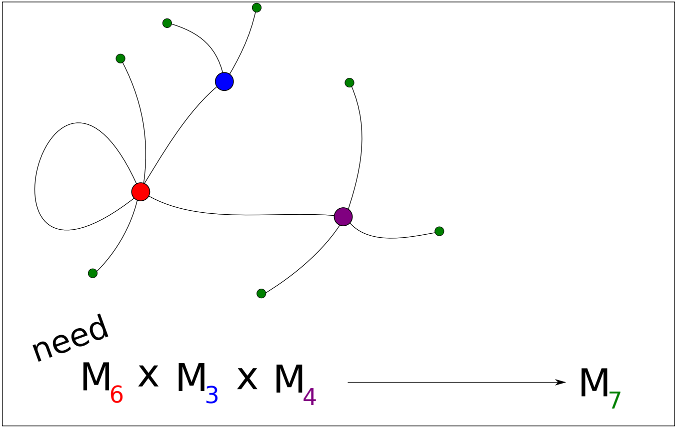

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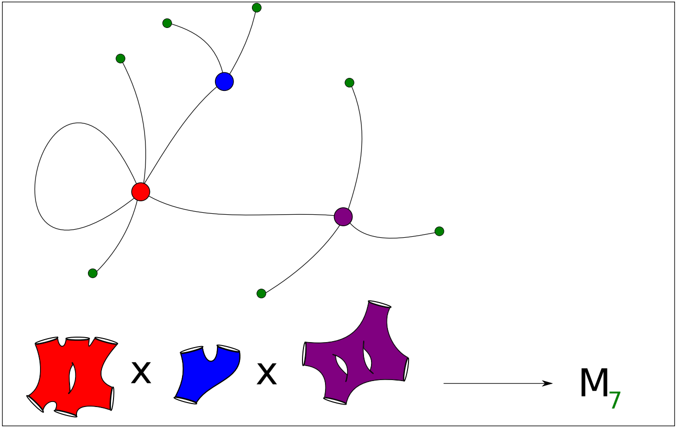

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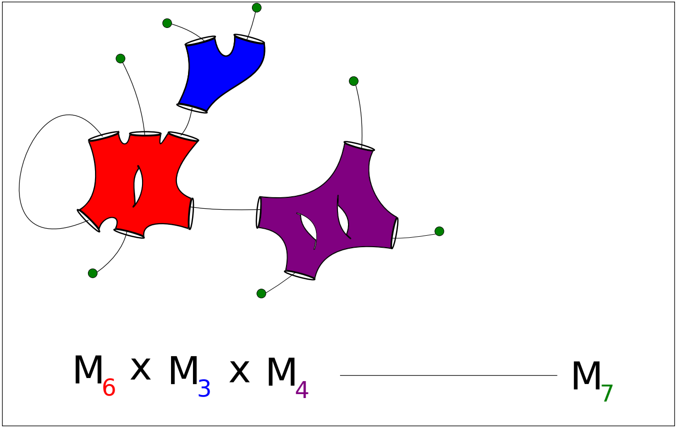

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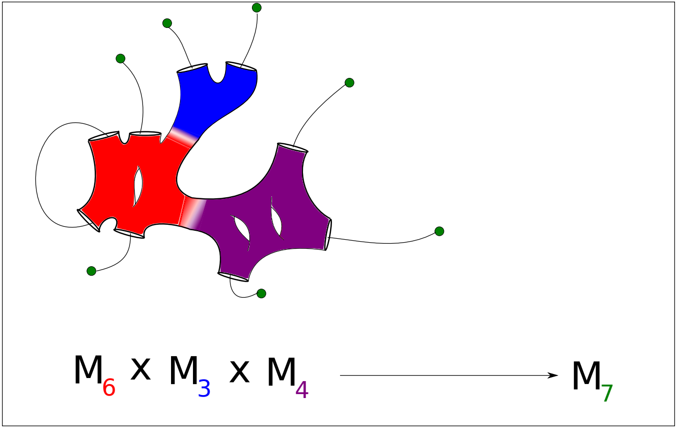

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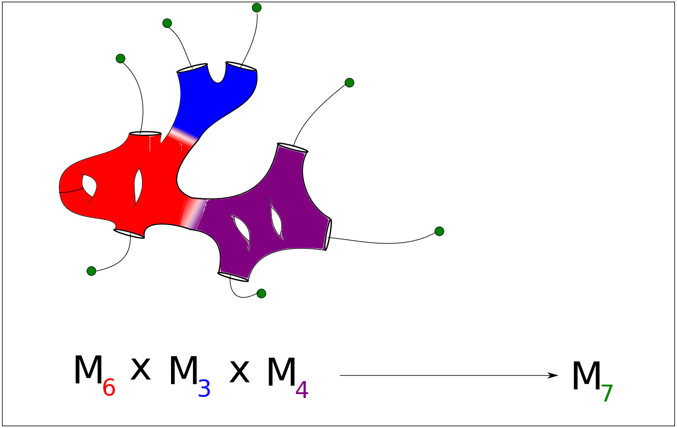

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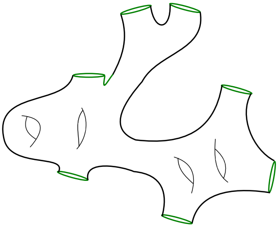

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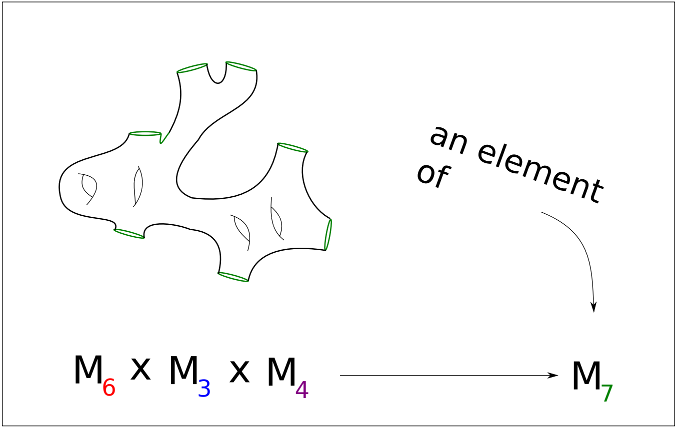

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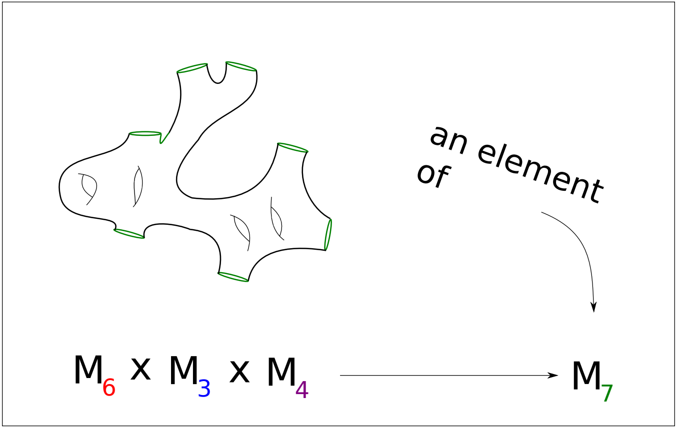

- Surfaces form a modular operad by gluing.

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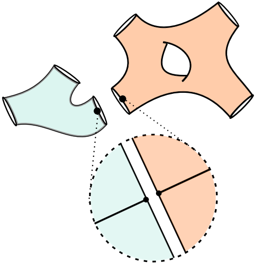

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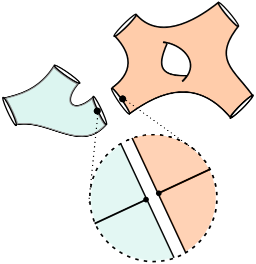

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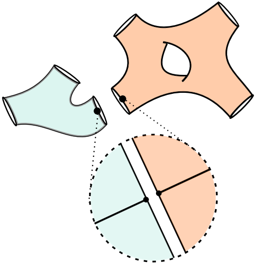

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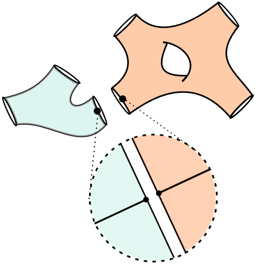

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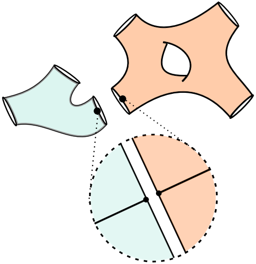

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It's preferable to separate out the genus: $\mathcal{M} = \{\mathcal{M}_{g,n}\}$.

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Algebraic structure	Associativity	Modular Operad
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3. Generalize classical Koszul duality theory from operads to groupoid colored operads.

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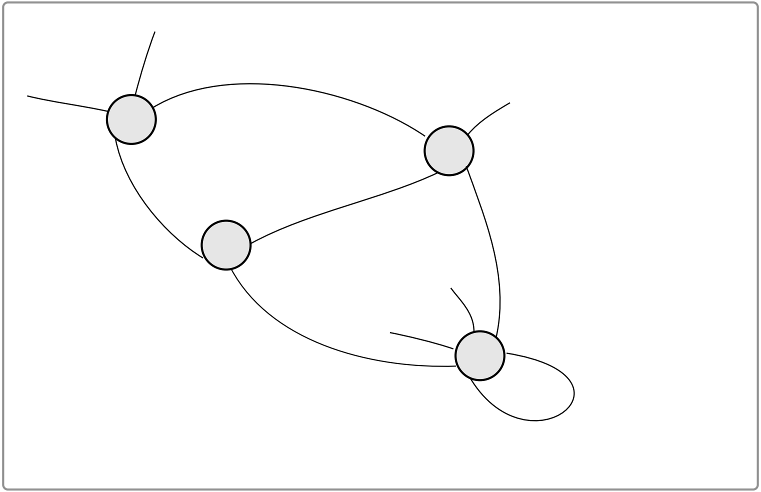

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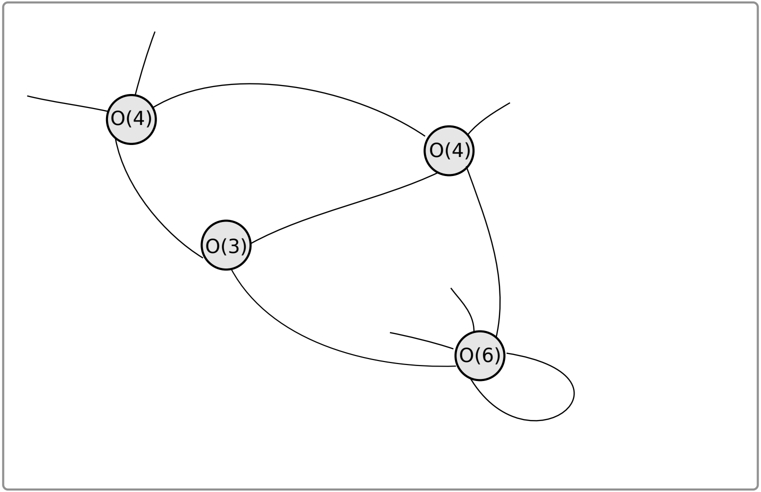

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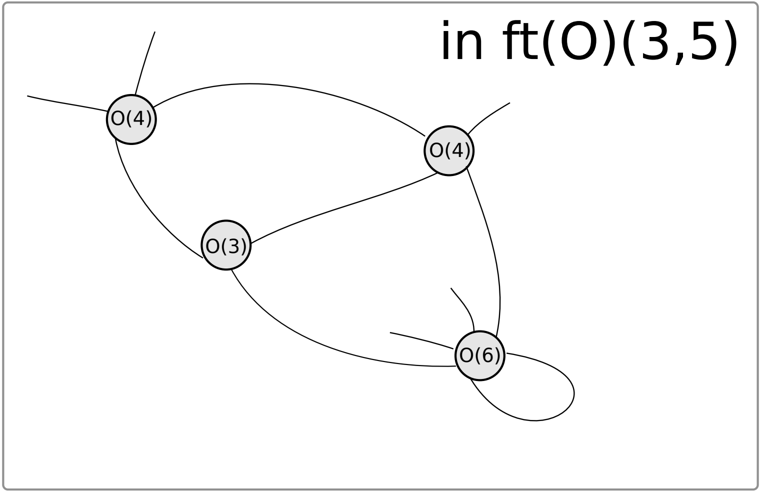

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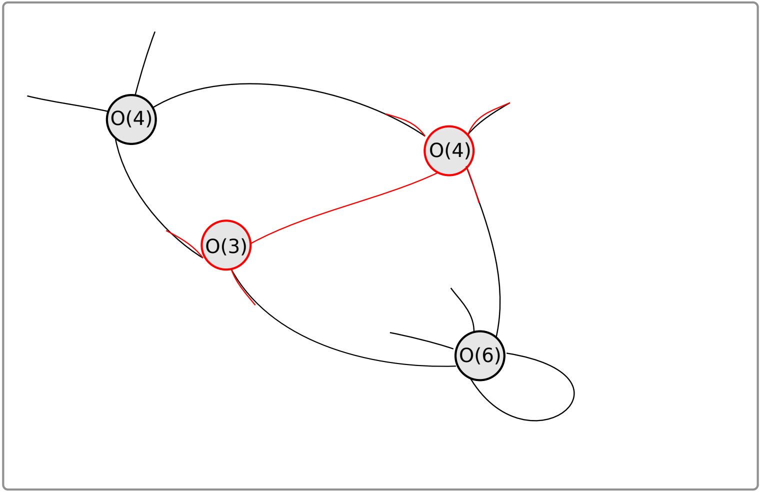

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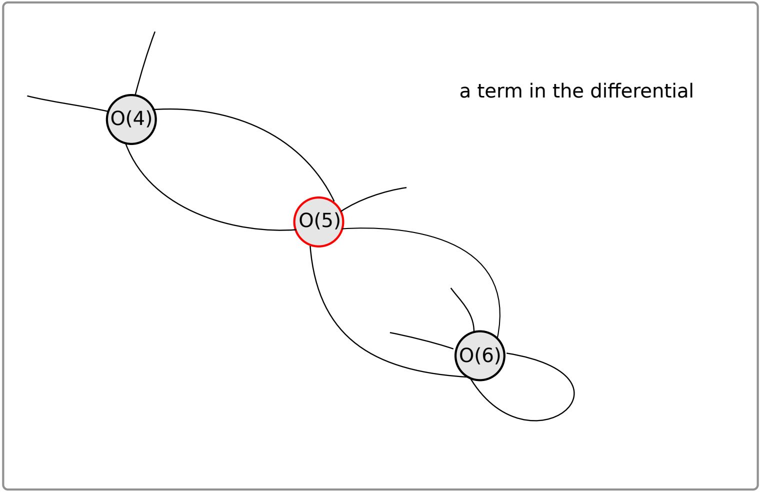

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Thus every cycle in $\mathrm{ft}(\mathrm{Com})(g, n)$ is a boundary in $\mathrm{ft}(\mathcal{H}_{Lie})(g, n)$, i.e. every commutative graph homology class is represented, via Massey products, by a graph labeled by Lie graph homology classes.

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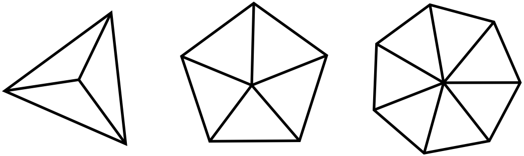

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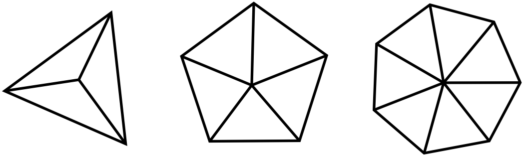

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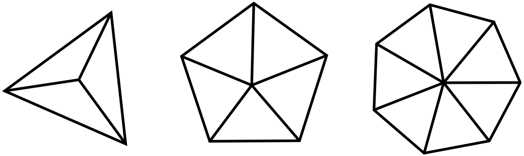

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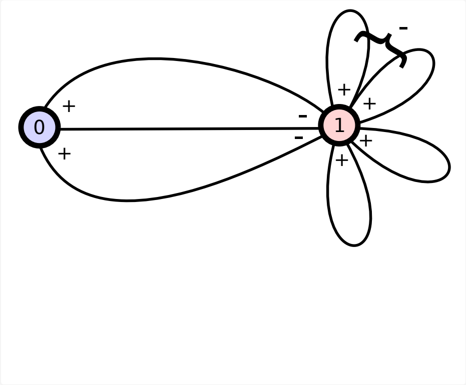

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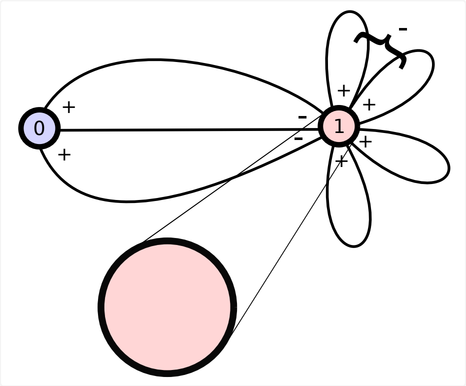

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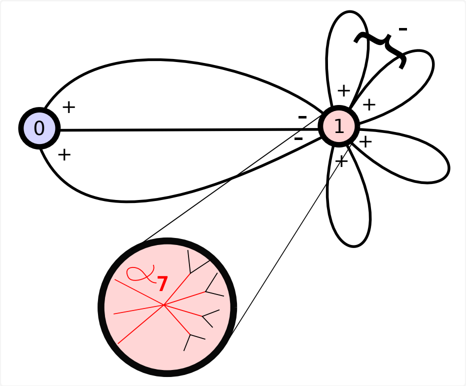

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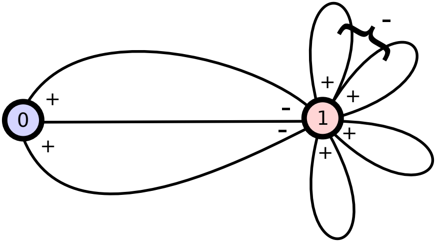

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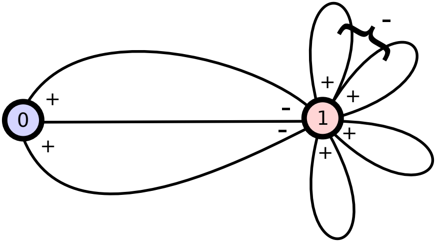

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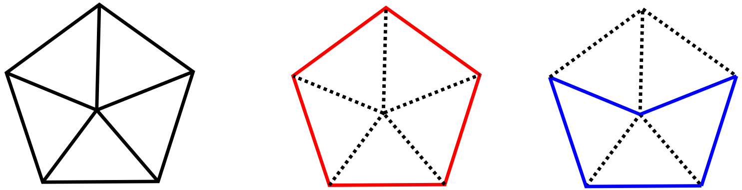

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Bibliography

- Massey Products for graph homology. arXiv:1903.12055; to appear in Int. Math. Res. Not.
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