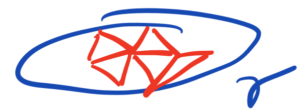
Coarse embeddings and homological filling functions

Rob Kropholler

Jt with Mark Pengitore

Def (Filling function)

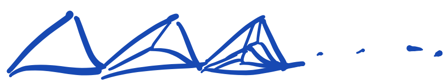


Let X be a simply connected simp. cx.

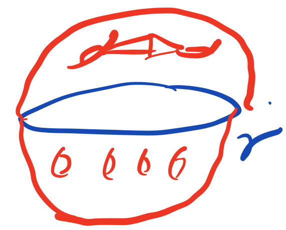
Let $\gamma: S^1 \underset{\text{simp. map}}{\to} X^{(1)}$. $\exists$ a simp. map $f: D^2 \to X$ s.t. $f|_{\partial D^2} = \gamma$. Define $|f| =$ # 2-cells in the triangulation

Define $\text{Area}(\gamma) = \min\{|f| \mid f: D^2 \to X,\ f|_{\partial D^2} = \gamma\}$

$\text{Fill}_X : \mathbb{N} \to \mathbb{N}$ $\quad \text{Fill}_X(n) = \sup\{\text{Area}(\gamma) \mid \ell(\gamma) \leq n\}$



Def (Homological filling function)



Let X be a simp cx with $H_1(X)=0$.

Let $\gamma: S^1 \to X^{(1)}$. Then $\exists$ a simplicial map of a surface Σ with 1 ∂ comp to X $f:\Sigma \to X$ s.t $f|_{\partial\Sigma} = \gamma$ $|f|$ = # 2-cells in triangulation

$\mathrm{HArea}(\gamma) = \min \{ f : \Sigma \to X$ as above $\}$.

homotopy type is not fixed

$\mathrm{HFill}_X(n) = \sup \{ \mathrm{HArea}(\gamma) \mid \ell(\gamma) \leq n \}$

Rmk $\mathrm{HFill}_X \leq \mathrm{Fill}_X$

Relation to gps and equivalence

Suppose G acts properly cocompactly on a simply connected space X

Define $\mathrm{Fill}_G := \mathrm{Fill}_X$

$$\mathrm{HFill}_G := \mathrm{HFill}_X$$

This is well defined up to equivalence given by the partial ordering $f \preceq g$ $f, g : \mathbb{N} \to \mathbb{N}$
if $\exists\, C > 0$ s.t $f(n) \leq C g(Cn + C) + Cn + C$
$\forall n \in \mathbb{N}$

Ex. $a_k x^k + \dots + a_1 x + a_0 \sim x^k$

$$a^n \sim b^n \quad \forall a, b \in \mathbb{N}$$

Thm (Gersten, Bowditch)
A finitely presented gp G is hyperbolic iff $HFill_G \sim n$ iff $Fill_G \sim n$

Thm (Abrams-Brady-Dani-Young)
$\exists$ gps G with $\underset{\text{exp}}{Fill_G} \not\sim \underset{\text{polynomial}}{HFill_G}$

Thm (Brady-K-Soroko)
$\exists$ gps with unsolvable word problem
but $HFill \sim n^4$

Passing to subgps. Given $H \leq G$ can one describe

Ex. $HFill_H$ in terms of $HFill_G$

$Fill_H$ $Fill_G$

CAT(0) $Fill \sim n^2$

Subgps of RAAGs $\subset$ or products of hyp gps

$Fill \sim n^2$

with exp. filling function.

Def G has geometric dimension (gd) $\leq n$ if

$G = \pi_1(X)$ X is an aspherical n-dim ~~cell~~ simp. cx.

$\tilde{X} \simeq *$

Thm. (Gersten)

Let G be a fin. pres. gp $H < G$ fin pres.

Suppose gd $G \leq 2$ Then

$$HFill_H \precsim HFill_G$$

Cor. If H is a fin. pres. subgp of a $C'(\frac{1}{6})$ gp then H is hyp.

Idea of pf.

Def Coarse embedding $(H \hookrightarrow_{c.e.} G)$

$\varphi : H \to G$ is a Coarse embedding if $\exists \rho_-, \rho_+ : \mathbb{R}_+ \to \mathbb{R}_+$

s.t. $\lim_{t \to \infty} \rho_-(t) = \lim_{t \to \infty} \rho_+(t) = \infty$ and

$$\rho_-(d(h_1, h_2)) \leq d(\varphi(h_1), \varphi(h_2)) \leq \rho_+(d(h_1, h_2))$$

Ex. $H < G$ then the inclusion is a coarse embedding

Coarse embedding is much wilder than subgps.

Ex.

$\mathbb{Z} \hookrightarrow_{c.e.}$ Any infinite gp

$\mathbb{Z}^n \hookrightarrow_{c.e.} \pi_1(M^{n+1})$ M^{n+1} closed hyp

n+1 manifold

Thm (K-Pengitore)

Let G, H be fin. pres. gps suppose. $\operatorname{gd} G \leq 2$ and $H \hookrightarrow_{c.e.} G$ then

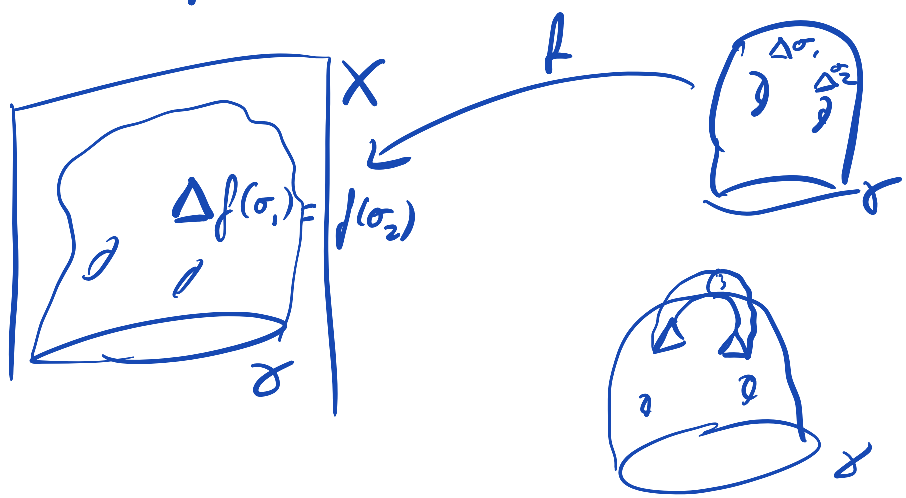
$HFill_H \preceq HFill_G$

Cor.

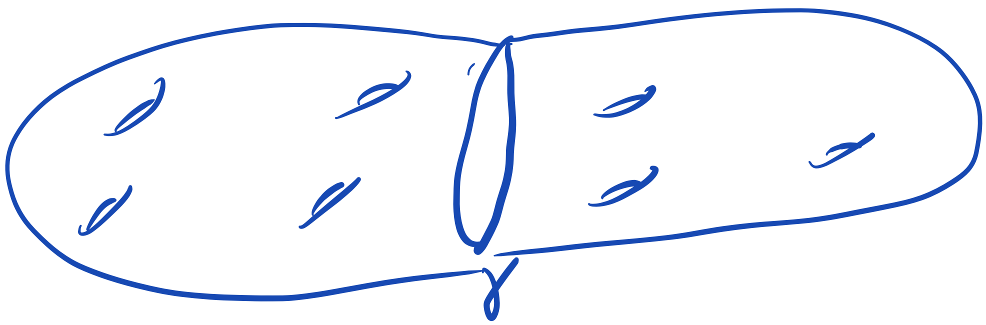
If G is an inf pres. $C'(\frac{1}{6})$ gp H is a fin pres. subgp then H is hyp.

Ideas from the proof.

1. Given a surface filling can replace with a surface filling with no cancellations



② In the case that G has gd 2 surface fillings with no cancellations are unique.



③ adjust the space X for G s.t.

we get a subspace Y which is q.i. to H

Think of mapping cylinder.