Veering Polynomial II.
Flow graph & Thurston norm

Joint with M. Landry & Y. Minsky

Goals:

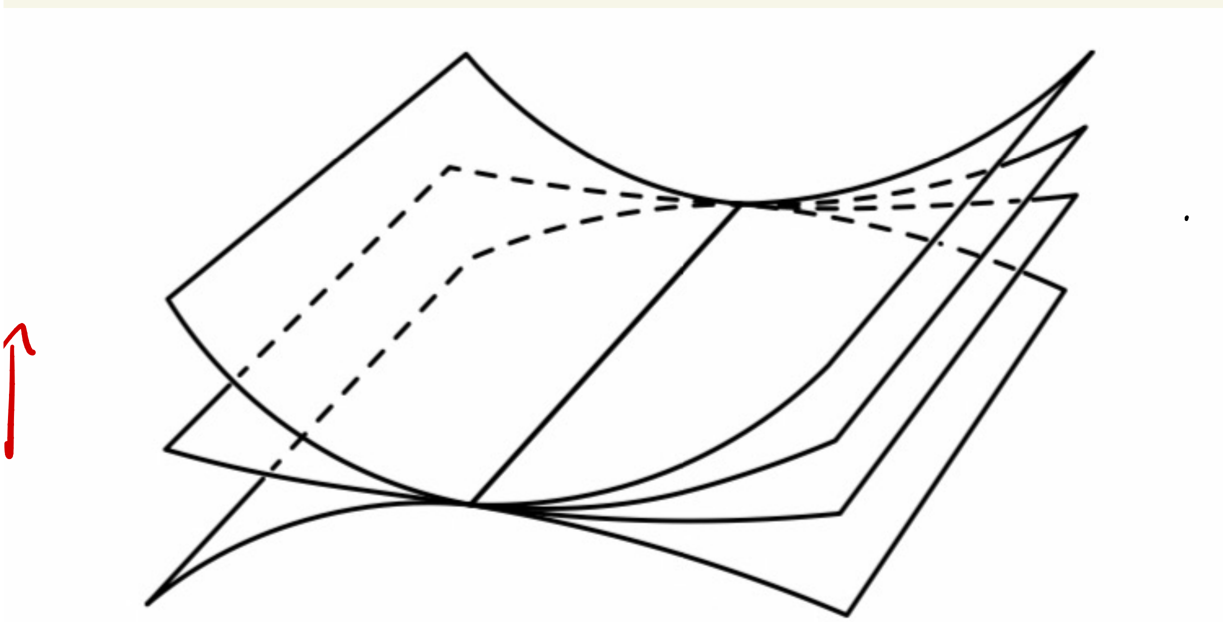
- understand the surfaces carried by τ and give the connection to the Thurston norm on $H_2(M, \partial M)$
- introduce the flow graph and its Perron polynomial & relate to V_τ
- explain the connection between the above & use to (not nec. fibered) faces of Thurston norm ball $B_x \subseteq H_2(M, \partial M)$

I. Cones in (co)homology.

Identify: $H^1(M) = H_2(M, \partial M)$

$\tau \rightsquigarrow \mathrm{cone}_2(\tau) \subseteq H_2(M, \partial M)$

cone spanned by surfaces (nonneg.) carried by $\tau^{(2)}$

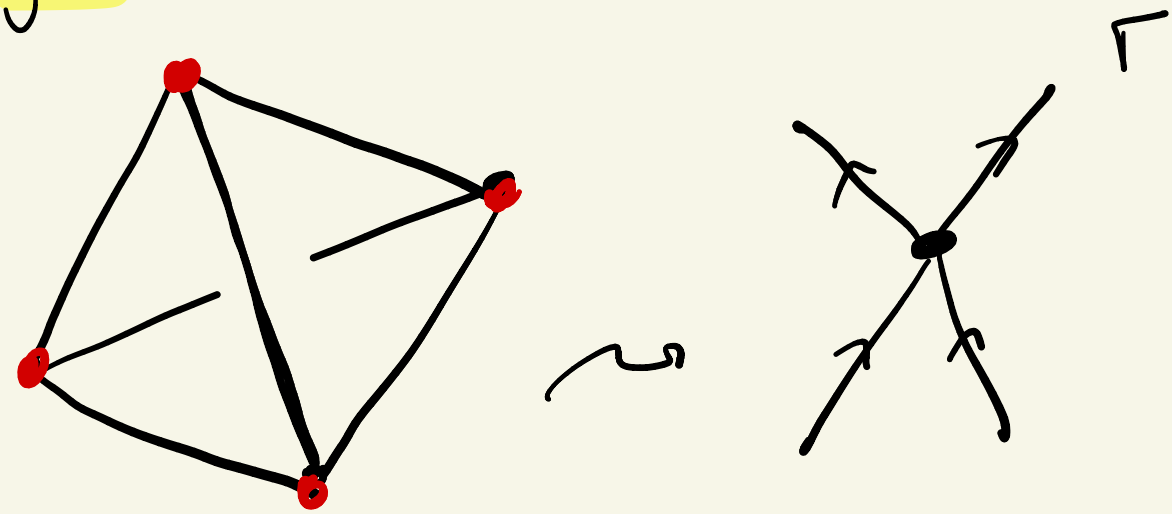


Alt: classes rep. by nonneg. 2-cycles on $\tau^{(2)}$.

For any $\alpha \in \mathrm{Cone}_2(\tau)$ & any (pos. closed) transverse curve γ to $\tau^{(2)}$,

$$(\gamma, \alpha) \geq 0.$$

Let Γ be the directed dual graph to τ.



$\mathrm{Cone}_1(\Gamma) \subseteq H_1(M;\mathbb{R})$ generated by directed dual cycles.

<u>Theorem</u> (Duality)

$$\mathrm{Cone}_2(\tau) = \mathrm{Cone}_1^{\vee}(\Gamma) = \left\{ \alpha \;\middle|\; \begin{array}{l} (\gamma, \alpha) \geq 0 \\ \forall \text{ transverse curves } \gamma \end{array} \right\}$$

II. Thurston norm

Note: $[\Gamma] \in H_1(M)$ &

if $S < \tau^{(2)}$ then

(Γ, S) = # of triangles in ideal triangulation of S

Define

$$c_\tau = -1/2\,(\Gamma, \cdot) \in H^2(M, \partial M)$$

so if $S < \tau^{(2)}$ then

$$-c_\tau([S]) = -\chi(S)$$
$$= \chi([S])$$

$\Rightarrow$ On $\mathrm{cone}_2(\tau)$,

$-c_s$ = Thurston norm χ.

<u>Theorem</u>

τ determines a face F_τ of $B_\chi(M)$ such that

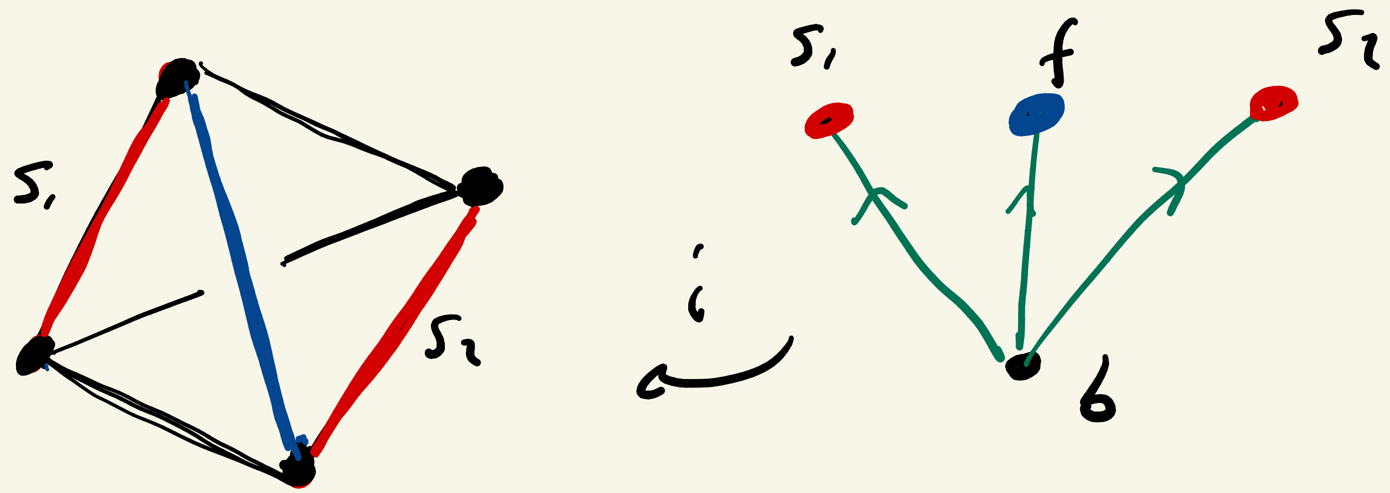
$\mathrm{Cone}_2(\tau) = \mathbb{R}_+ F_\tau$.

This cone is subspace of $H_2(M, \partial M)$ on which

$$-e_\tau = \chi.$$

Moreover, if $\alpha \in \mathbb{R}_+ F_\tau$ and S is genus minimizing among taut rep.s of α, then $S < \tau$.

III. The flow graph & the veering polynomial.



Define $\Phi = \Phi_\tau$ (flow graph)

vertices $\rightsquigarrow$ edges of τ

edges $\rightsquigarrow$ 3 outgoing Φ-edges at each vertex

Rmk: Φ comes with an embedding $i: \Phi \to M$ & pushing it slightly upward makes it positive transverse to $\tau^{(2)}$.

$$\Rightarrow \quad \text{cone}_1(\Phi) \subseteq \text{cone}_1(\Gamma)$$

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$\Phi \rightsquigarrow P_\Phi$ (Perron Polynomial)

Let A_Φ be its adjacency matrix

$$(A_\Phi)_{ab} = \sum_{e:\, a \to b} e \in \mathbb{Z}[C_1(\Phi)]$$

then

$$P_\Phi = \det(I - A_\Phi)$$

$$= 1 + \sum_c (-1)^{|c|} c \in \mathbb{Z}[H_1(\Phi)]$$

where $\{c\} = \{\text{multicycles of } \Phi\}$

$$:= \operatorname{supp}(P_\Phi)$$

why care?

- $i(\operatorname{supp} P_\Phi) \subseteq H_1(M;\mathbb{R})$ generates $\operatorname{cone}_1(\Gamma)$
- P_Φ determines V_τ

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Theorem In $H_1(M;\mathbb{R})$,

$$\operatorname{cone}_1(\Gamma) = \operatorname{cone}_1(\Phi)$$

$$= \left\{ \sum t\, i(c) : \begin{array}{l} c \in \operatorname{supp}(\Phi) \\ t \geq 0 \end{array} \right.$$

The inclusion $i : \Phi \to M$
induces $i_* : \mathbb{Z}[H_1(\Phi)] \to \mathbb{Z}[G]$

Theorem

$$V_\tau = i_*(P_\Phi)$$

veering polynomial

Fiber detection Theorem

Let τ be a veering triangulation and F_τ the corresponding face. TFAE

1. $\mathrm{supp}(P_\Phi) \subseteq H_1(M;\mathbb{R})$ lies in an open halfspace

3. $\exists\, \alpha \in H^1(M)$ s.t. $(\gamma, \alpha) > 0$ $\forall$ closed transverse curves γ

3. τ is layered

4. F_τ is fibered.

Example in layered case:

Uses computation of Anna Parlak

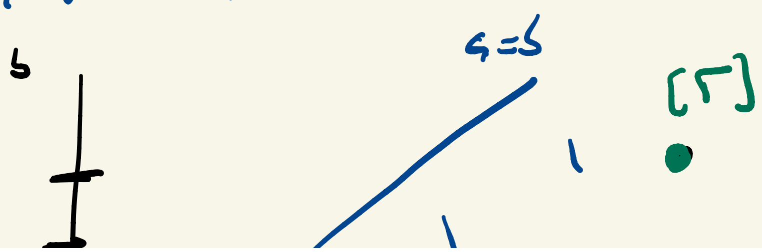
$$M = S^3 \setminus 9^2_{52}$$

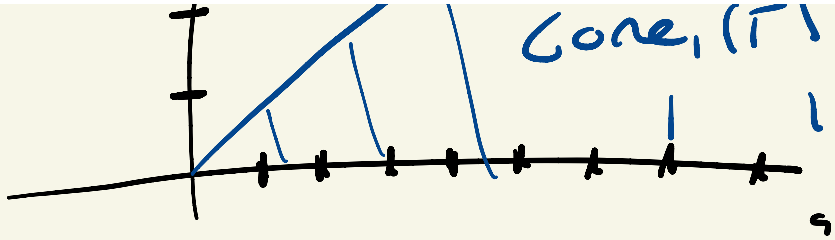
pulled from veering census of Giannopolous-Schleimer-Segerman

$$V_\tau = \Theta_\tau \cdot V^{AB}$$

$$= (a^4 b - a^2 b - ab - a^3 - a^2 + 1) \cdot (1 + a^5 b^3)(1 + a^3)$$

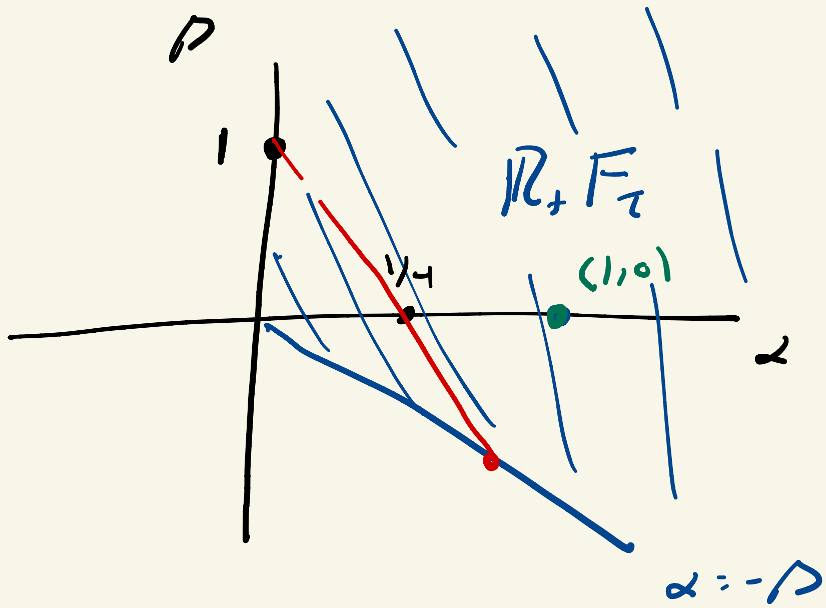
$$H_1(M; \mathbb{R}) = \mathbb{R}_a \oplus \mathbb{R}_b$$





$[\Gamma] = a^8 b^4$

Dual cone in $H_1(M) = \mathbb{R}\alpha \oplus \mathbb{R}\beta$



In (α, β) coordinates,

$$e_\tau(\alpha,\beta) = -(4\alpha + \beta).$$

So for example, if $\Sigma = (1,0)$

then $|\chi(\Sigma)| = \chi((1,0))$

$= 4$ and

$f: \Sigma \to \Sigma$ has stretch factor equal to largest root of (McMullen)

$$\Theta_\tau^{(1,0)} = t^4 - t^3 - 2t^2 - t + 1$$

$$\sim 2.081$$