The Fully Marked Surface Theorem

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<u>Observation</u> (Thurston 1976) If $S \subset M^3$ is a compact leaf of a Cod-1 foliation $\mathcal{F}$, then $|E_{\mathcal{F}}([S])| = |\chi(S)|$.

$E_{\mathcal{F}} :=$ Euler class of the tangent bundle to the leaves.

<u>Theorem</u> (Thurston 1976) If $\mathcal{F}$ also taut, M closed then S is Thurston Norm minimizing i.e. if T an incompressible surface and $[T] = [S] \in H_2(M)$, then $|\chi(S)| \leq \chi(T)$.

Define $\mathcal{F}$, taut, exmpls of T^2 non example

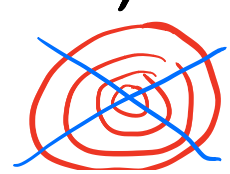
More general versions - $\exists$ Reebless M compact, M sutured manifold, $S \subset$ leaf ..., S coherent union of leaves

<u>Idea of Proof</u> - Special Case T connected

<u>case 1)</u> T is a leaf:

$|\chi(S)| = |E_{\mathcal{F}}([S])| = |E_{\mathcal{F}}([T])| = |\chi(T)|$

<u>Case 2)</u> Roussarie, Thurston $\Rightarrow$ T can be isotoped to be either a leaf or $\pitchfork \mathcal{F}$ except at finitely many saddle tangencies.



$$|E_{\mathcal{F}}([S])| = |E_{\mathcal{F}}([T])| = \Big|\sum_{x \text{ tangency}} \sigma(x)\Big| \leq \sum |\sigma(x)| = |\chi(T)|$$

$$\sigma(x) = \begin{cases} +1 & \text{if } + \text{normal to } \mathcal{F} \text{ at } x = + \text{ ...... } T \text{ ....} \\ -1 & \text{otherwise} \end{cases}$$

Recall $E_{\mathcal{F}}([T]) =$ obstruction to finding a section of $T(\mathcal{F})|T$.

This argument shows that if $T, \mathcal{F}$ are coherently oriented at all pts of tangency then T is norm minimizing.

We say such a T is fully marked
– allow T also to be a leaf –

Theorem (G–Yazdi, Acta to appear)
If S is a fully marked surface w.r.t. taut foliation $\mathcal{F}$ of the closed hyperbolic 3-manifold M, then $\exists$ taut $\mathcal{F}'$ with $T(\mathcal{F}) \simeq T(\mathcal{F}')$ and S' s.t. $[S'] = [S]$ and S' is a union of leaves of $\mathcal{F}'$

Need only M atoroidal.

Conjecture (G-Y) Theorem is an optimal converse to Thurston's theorem, ie. false in general if require $S' = S$.

Leaf $\xRightarrow{\text{Thurston}}$ Fully Marked

Fully marked $\xRightarrow{?}$ Leaf

Remark 1) Compact leaves of taut foliations of hyperbolic 3-manifolds can be eliminated by small perturbations of the foliation.

2) We think that the statement & proof should generalize to compact manifolds & sutured manifolds.

Idea of the Idea of the proof

A is the set of annuli $\partial N(L) - \mathring{N}(R)$. Similarly let $N_i = (S^3 - \mathring{N}(L_i)) - \mathring{N}(R_i)$ with $\partial N_i = R_i^+ \cup R_i^- \cup A_i$ $i = 1, 2$. Assume that R is oriented so that the $+$ side of D points into the ball containing R_1 where D is the disc along which R_1 and R_2 were summed. Let $E = (S^2 - D) \cap N$ where S^2 is the sphere separating R_1 and R_2. Let B_i be the ball bounded by S^2 containing R_i.

Key observation (Figure 1). If $P_i = (N - \mathring{N}(E)) \cap B_i$ then after a small isotopy

Sutured manifold example

$$\begin{aligned} N_1 &= P_1 \\ R_1^- &= R^- \cap P_1 \\ R_1^+ &= R^+ \cap P_1 \cup (N(E) \cap P_1 - \mathring{N}(E \cap R^-)) \end{aligned}$$

and

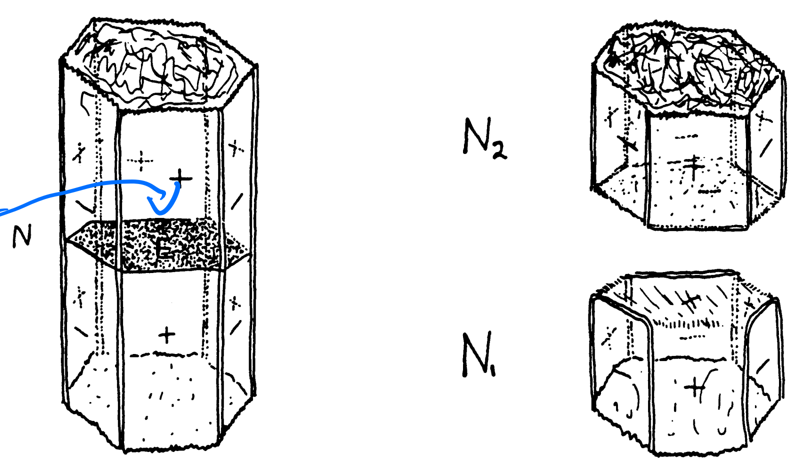
$$\begin{aligned} N_2 &= P_2 \\ R_2^+ &= R^+ \cap P_2 \\ R_2^- &= R^- \cap P_2 \cup (N(E) \cap P_2 - \mathring{N}(E \cap R^+)). \end{aligned}$$

Now $\mathcal{F}$ induces a codimension 1 transversely oriented foliation $\mathcal{G}$ on N so that $\mathcal{G}$ is transverse to A, tangent to $R^+ \cup R^-$, $\mathcal{G}$ and $\mathcal{G}|A$ have no Reeb components and $\mathcal{F}$ is obtained from $\mathcal{G}$ by gluing R^+ to R^-. By construction E is transverse to $\mathcal{G}$ in a neighborhood of ∂E. By Thurston [T-1] or Rousserie [R] and the Poincaré Hopf index formula one can isotope E so that E is transverse to $\mathcal{G}$ except along a finite number of saddle tangencies whose indices add up to $n-1$, if E is a $2n$ gon . Here we use the fact that $\mathcal{F}$ has no Reeb components. We will now construct the desired foliation on $S^3 - \mathring{N}(L_1)$. The other case is similar. Define $\mathcal{G}_1 = \mathcal{G}|N_1$. $\mathcal{G}_1$ is a singular foliation. Our goal is to heal the scar.

LEMMA. *If x is a point of E tangent to $\mathcal{G}_1$, then the normal to $\mathcal{G}_1$ at x points out of N_1.*

PROOF. It follows from the Poincaré Hopf formula that if M is a 3-manifold and $\vec{X}$ is a non-singular vector field pointing normal to ∂M, then $\chi(\partial_+) = \chi(\partial_-)$

a Fully marked disc



made optimally a leaf.

FIGURE 1

Normals at tangencies point up

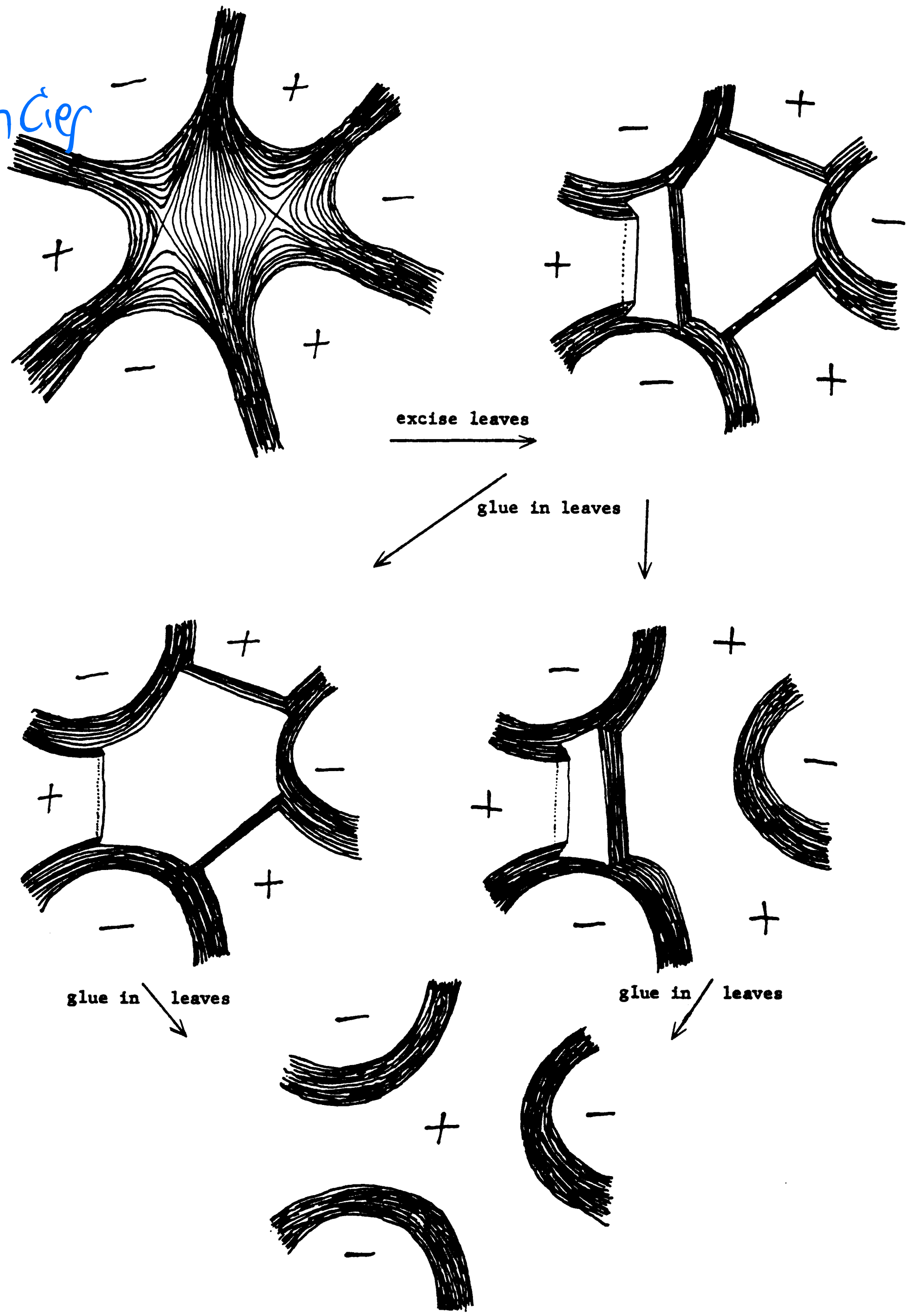


FIGURE 3

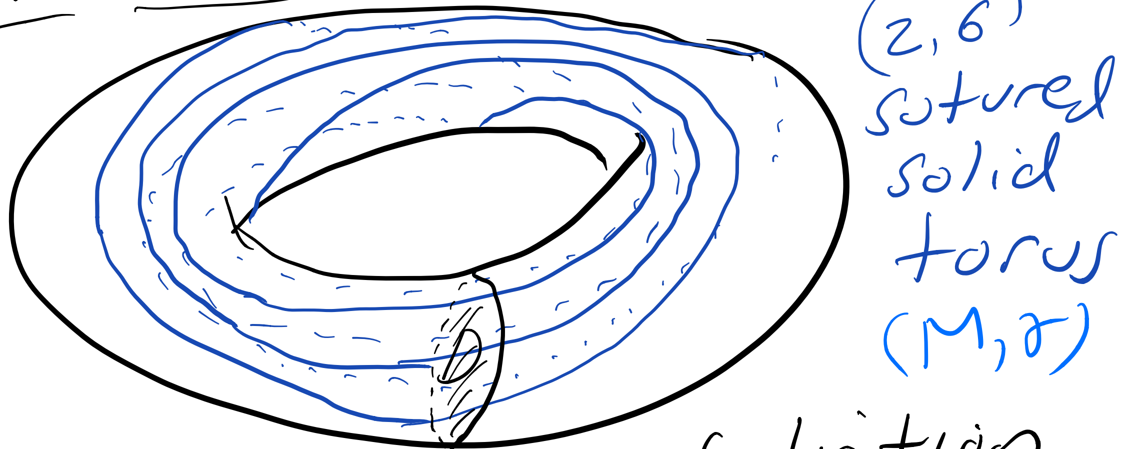
Application The fully marked surface theorem enables "Hierarchical decomposition" of a taut foliation to smaller pieces via sutured manifold decomposition.

Eg. (1985) Murasugi Sum II

S

If $K = \partial S$ is a fibered link with fiber S then for $i = 1, 2$ $K_i = \partial S_i$ is a fibered link with fiber S_i. The monodromy f of S $=$ "$f_2 \circ f_1$" where f_i is the monodromy of S_i.

Brilliant observation of Mehdi Yazdi (Acta Math to appear)



If $\mathcal{F}$ is a taut foliation on (M, γ) then D is fully marked (up to isotopy)

Conjecture (Thurston 1976) If M^3 is atoroidal $a \in H^2(M, \mathbb{Z})$, $x^*(a) = 1$ then there is some [taut] Foliation $\mathcal{F}$ such that $E_{\mathcal{F}} = a$

Theorem (G-Yazdi) There $\exists$ ∞ly many closed hyperbolic 3-manifolds M with $z \in H^2(M, \mathbb{Z})$ $z \in$ dual thurston unit ball s.t. z satisfies parity condition but $z \neq E_{\mathcal{F}}$ for any taut $\mathcal{F}$.

Idea Step 1 (Yazdi) Construct candidate M, $z \in H^2(M, \mathbb{Z})$

Step 2 Use Fully marked surface theorem to reduce $\mathcal{F}$ to a taut $\mathcal{F}_1$ on $(6,2)$-solid torus s.t. D not fully marked.

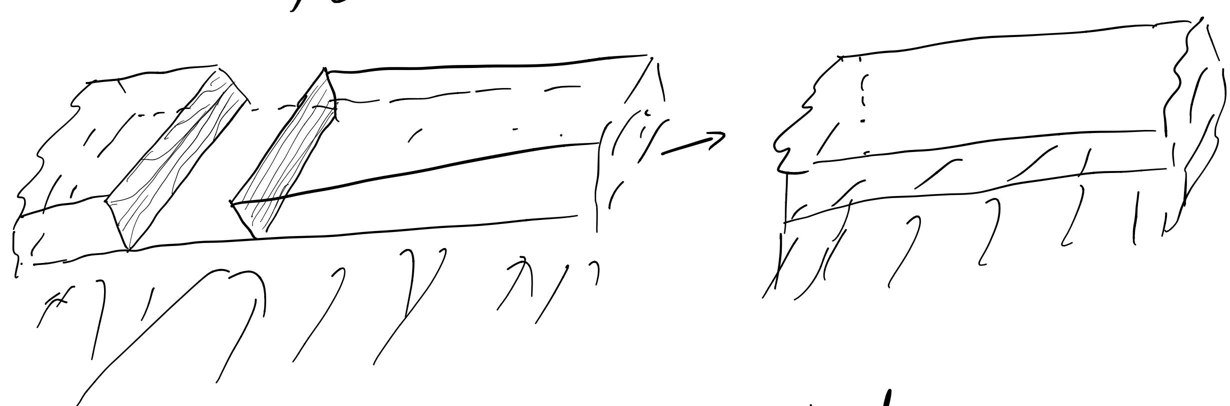
Step 3 (Yazdi) $\mathcal{F}_1$ does not exist.

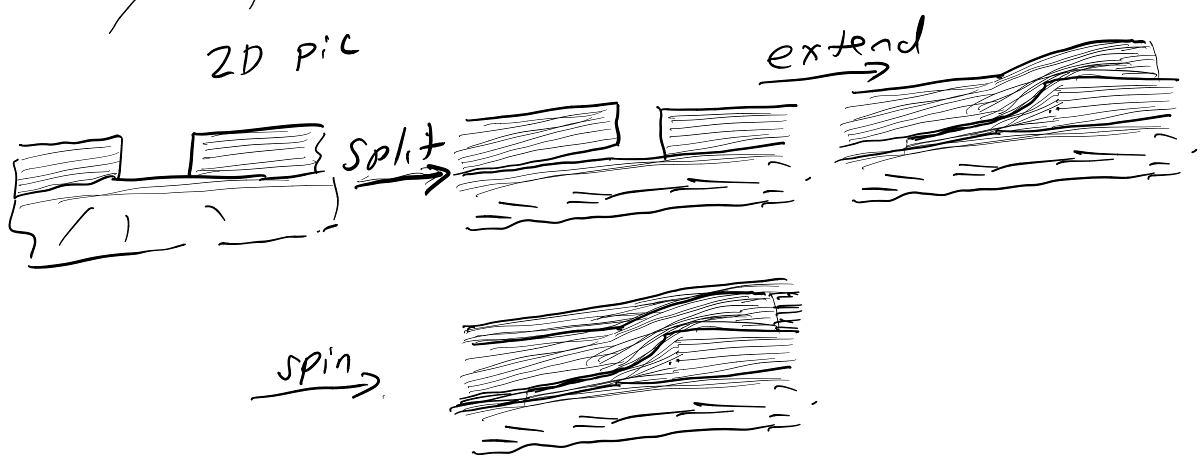
Idea of Proof of FMST

A) Some technique

i) After splitting along $\mathring{N}(S)$ only add bits of leaves to make $\exists$ locally horizontal and vertical

ii) Extending across annular ditches





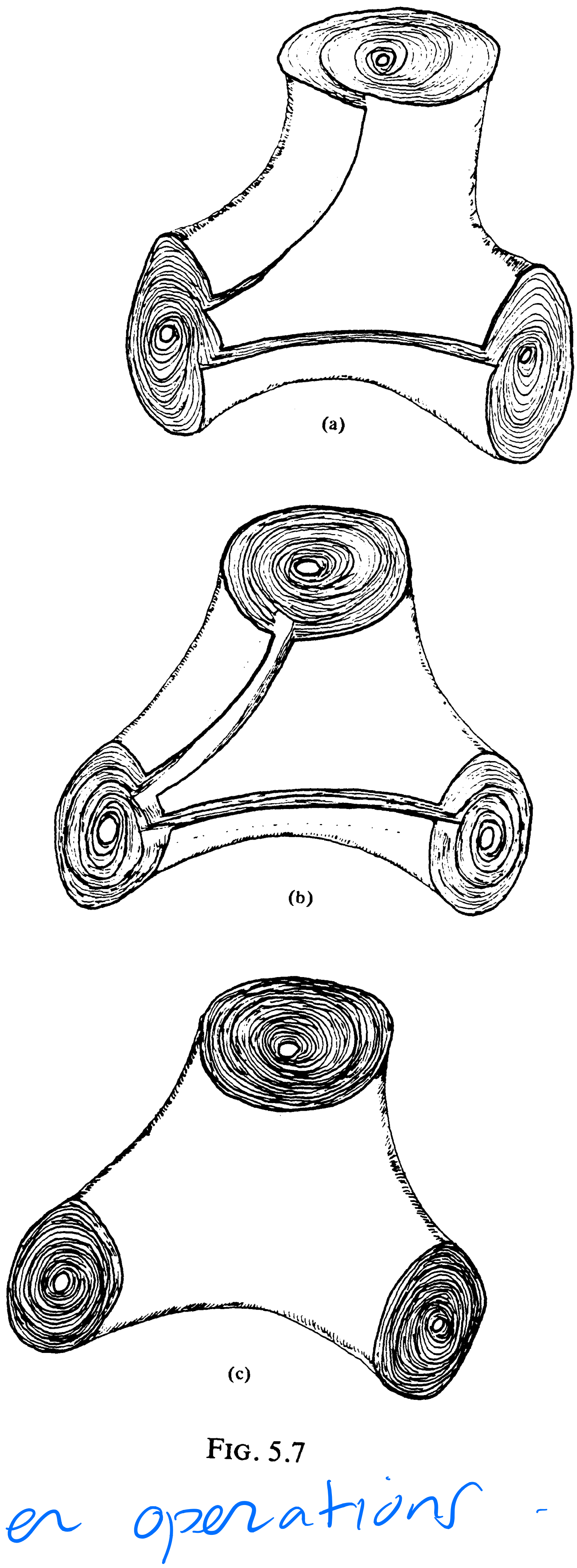


FIG. 5.7

+ other operations - - - -

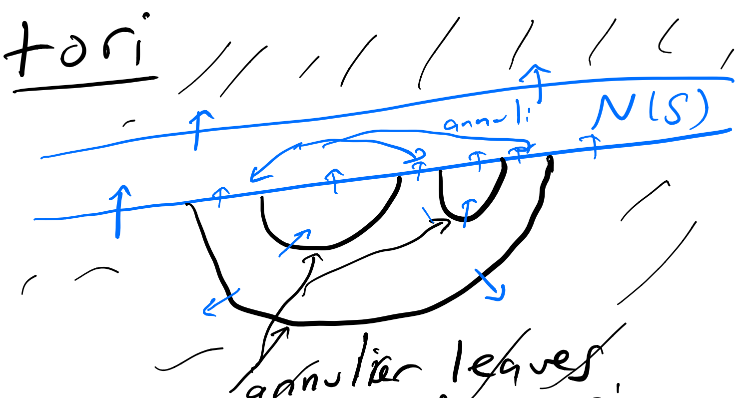
B) Keeping track of the
i) homotopy class and ii) tautness

i) Find a vector field $\vec{X} \pitchfork \mathcal{F}$ and $\pitchfork S$ <u>coherently.</u>

ii) Find $\gamma_1, \dots, \gamma_n$ simple closed curves in M s.t. γ_i <u>coherently</u> $\pitchfork \mathcal{F}$ and S and every leaf of $\mathcal{F}|M - \mathring{N}(S)$ hit by some γ_i.

- All operations add stuff to $\mathcal{F}|M-\mathring{N}(S)$ transverse to $\vec{X}$ and $\pitchfork$ to the γ_i's.

C) <u>Proposition</u> Coherent transversals exist $\Longleftrightarrow$ there are no <u>bad</u> <u>solid</u> <u>tori</u>



D) Eliminate bad solid tori by replacing S by S' where $[S]=[S']$