Connected Floer homology of covering involutions

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Double branched covers

The map $f\colon X \to Y$ between n-dimensional manifolds is a *double branched cover* if away from a codimension-2 submanifold $B \subset Y$ it is a double cover, and along B it is $z \mapsto z^2$ (as a map $\mathbb{C} \to \mathbb{C}$) in the normal direction, multiplied with $\mathrm{id}\colon \mathbb{R}^{n-2} \to R^{n-2}$.

X is determined by (Y, B) together with a surjective map $\rho\colon \pi_1(Y \setminus B) \to \mathbb{Z}_2$, which is unique if $H_1(Y \setminus B; \mathbb{Z}) \cong \mathbb{Z}$

Example

$K \subset S^3$ defines a unique double branched cover $\Sigma(K)$.

Guiding principle

$\Sigma(K)$ carries imprtant information about K. Simple fact: For a knot K, $b_1(\Sigma(K)) = 0$ (i.e. it is a $\mathbb{Q}HS^3$). Indeed, $|H_1(\Sigma(K);\mathbb{Z})| = \det(K) = \Delta_K(-1)$, an odd integer.

Principle: If K is slice, then $\Sigma(K)$ bounds a rational homology disk (take the double branched cover of D^4 along the slice disk D).
Simple application:

Lemma

The torus knot $T_{3,5}$ is not slice.

Proof

Indeed, $\Sigma(T_{3,5}) = \Sigma(2,3,5)$ is the Poincaré homology sphere $(= \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0, \|(x,y,z)\| = 1\})$.

It bounds the (negative definite) E_8-plumbing, i.e. a smooth, simply connected 4-manifold with intersection form E_8. Therefore it cannot bound a homology disk (the union would have signature -8 and spin).

Indeed, more is true: suppose that $K_n = nT_{3,5}$ n-fold connected sum, then K_n is not slice. (Use Donaldson's diagonalizability thm.)

Recall that the concordance group $\mathcal{C}$ is the Abelian group of knots up to concordance. (Two knot are concordant is they cobounding an annulus in $S^3 \times [0,1]$); equivalently, slice knots are equivalent to the unknot.)

Corollary

$\mathcal{C}$ has a subgroup isomorphic to $\mathbb{Z}$.

More is true

Roughly the same argument gives the following:
Suppose that $G \subset \mathcal{C}$ is the subgroup spanned by 2-bridge knots (knots which admit diagrams with two maxima – and then two minima). The double branched cover of a 2-bridge knot has a genus-1 Heegaard splitting, hence a lens space, which bounds negative definite four-manifold with both orientation.

Theorem

$[T_{3,5}]$ *is of infinite order in* $\mathcal{C}/G$.

Detour

How much does the fact that the cover bounds a homology disk determines sliceness of K?

If K_1, K_2 differ by a Conway mutation (i.e. there is a small disk with bounday intersecting K_1 in four points, and K_2 is given by rotating that disk) then $\Sigma(K_1) \cong \Sigma(K_2)$. If one of them is slice, previous arguments will not obstruct the sliceness of the other. (Mutants also have the same HOMFLY, hence Jones and Alexander, but sometimes different genus, hence different knot Floer homology.)

The Kinoshita-Terasaka and Conway knots

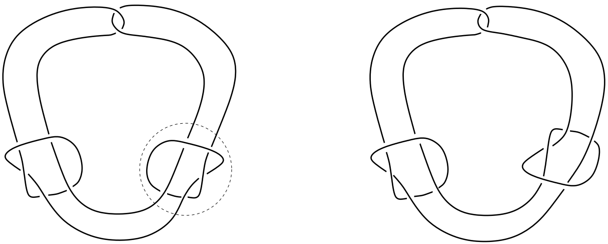

Figure: **The Kinoshita-Terasaka knot** KT **(on the left) and its Conway mutant, the Conway knot** C **(on the right).** The two knots are mutants of each other, as the dashed circle on the Kinoshita-Terasaka knot shows. KT is slice (easy), while C is not (recent result).

Detour

With higher covers? Any prime power-fold cover has $b_1(Y) = 0$, so we get a map

$$\varphi\colon \mathcal{C} \to \oplus_{p^n} \Theta^3_{\mathbb{Q}}$$

sending $[K]$ to $([\Sigma^{p^n}(K)])_{p^n}$. (Here $\Theta^3_{\mathbb{Q}}$ is the homology cobordism group of rational homology three-spheres, where Y_1, Y_2 are equivalent if there is a cobordism between them with rational homology of the direct product.)

$\ker \varphi$ are the knots which are not slice but all covers bound rational homology disks.

Detour

Recent construction to show:

Theorem (Aceto - Meier - A. Miller - M. Miller - Park - S.)

$\mathbb{Z}_2^5 \subset \ker \varphi$. *(Take n-fold cover of S^3 along A and pull back B to get the candidate knots.)*

(Indeed, by other means: $\mathbb{Z}^\infty \subset \ker \varphi$; [Livingston].)

More refined ideas

Refine the tool: take the knot into accout (since for a slice knot we have both the 4-manifold and a slice disk in it).
Consider the pair $(\Sigma(K), \widetilde{K})$ and associate invariants to this (Alfieri-Celoria-Stipsicz)

Another refinement

Use the covering involution $\tau\colon \Sigma(K) \to \Sigma(K)$. (Recall, the branched cover is a map $\pi\colon \Sigma(K) \to S^3$ and τ interchanges points with equal π-image.)

τ induces involution on $\mathrm{HF}^-(\Sigma(K))$ — recall that this is the homology of a chain complex $\mathrm{CF}^-(\Sigma(K))$, which is defined from a Heegaard diagram of $\Sigma(K)$ and an almost complex structure (on the symmetric power $\mathrm{Sym}^g(\Sigma)$).

τ induces a map $\tau_\#$ on $\mathrm{CF}^-(\Sigma(K))$, which is a homotopy involution (since the diagram can be chosen to be τ-invariant, but the almost complex structure must be generic).

The group $\mathrm{CF}^-(\Sigma(K))$ (as any Heegaard Floer homology group) splits according to spin^c structures. $\Sigma(K)$ has a unique spin structure $\mathfrak{s}$ and $\tau_\#$ maps $\mathrm{CF}^-(\Sigma(K),\mathfrak{s})$ to itself. (Recall that $|H_1(\Sigma(K);\mathbb{Z})| = \det(K) = \Delta_K(-1)$ is odd.)

The map $\mathrm{id} + \tau_\# \colon \mathrm{CF}^-(\Sigma(K),\mathfrak{s}) \to \mathrm{CF}^-(\Sigma(K),\mathfrak{s})$ measures "how far is $\tau_\#$ from id".

Leads to isotopy invariant: the homology of the mapping cone of $\mathrm{id} + \tau_\#$, called branched Heegaard Floer homology. It distinguishes $T_{3,7}$ and $P(2,-3.-7)$, although $\Sigma(K)$'s are the same.

A concordance $C \subset S^3 \times [0,1]$ between K_1 and K_2 induces a four-manifold X_C (the double branched cover of $S^3 \times [0,1]$ along C) between $\Sigma(K_1)$ and $\Sigma(K_2)$. Since C is an annulus, X_C is a rational homology cobrodism. As such, X induces a map $F_{X_C}\colon \mathrm{CF}^-(\Sigma(K_1)) \to \mathrm{CF}^-(\Sigma(K_2))$.

Properies of F_{X_C}

- This map also splits accroding to spin^c structures;
- X_C admits a canonical spin structure $\mathfrak{s}_c$ extending the spin structures on the boundary components;
- The corresponding component $F_{X_C,\mathfrak{s}_c}$ homotopy commutes with the map induced by the covering involutions
- on the localized homology $H_*(\mathrm{CF}^- \otimes \mathbb{F}[U, U^{-1}]) \cong \mathbb{F}[U, U^{-1}]$ the map $F_{X_C,\mathfrak{s}_c}$ induces an isomorphism (here we use that C is an annulus, so X_C is a homology cobordism).

Definition

This leads to

Definition

Take all module maps $f\colon \mathrm{CF}^-(\Sigma(K),\mathfrak{s}) \to \mathrm{CF}^-(\Sigma(K),\mathfrak{s})$ *which homotopy commute with* $\tau_\#$ *and induce isomorphism on the localized homology. Take* $f_{\max}$ *such map with maximal kernel. Then* $\mathrm{HFB}^-_{conn}(K) = H_*(\mathrm{Im}\ f_{\max}) \subset \mathrm{HF}^-(\Sigma(K),\mathfrak{s})$ *is the connected Floer homology of* K.

Theorem (Alfieri-Kang-S.)

- $\mathrm{HFB}^-_{conn}(K)$ *is a concordance invariant.*
- $\mathrm{HFB}^-_{conn}(K)$ *is a direct summand of* $\mathrm{HF}^-(\Sigma(K),\mathfrak{s}) = \mathbb{F}[U] \oplus A$ *(*A *is the* U*-torsion submodule)*
- $\mathrm{HFB}^-_{conn}(K) = \mathbb{F}[U] \oplus A_1$ *where* $A_1 \subset A$ *is a* U*-torsion submodule; we call it* $\mathrm{HFB}^-_{conn-red}(K)$.
- *for torus knots and alternating knots* $\mathrm{HFB}^-_{conn-red}(K) = 0$.

Simple examples

The vanishing of $\mathrm{HFB}^{-}_{conn-red}$ for alternating knots is standard (since $\mathrm{HF}^{-}(\Sigma(K), \mathfrak{s}) = \mathbb{F}[U]$ for those, so $A = 0$); for torus knots $T_{p,q}$ with pq odd is very simple:
The double cover is

$$\Sigma(2, p, q) = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^p + z^q = 0, \|(x, y, z)\| = 1\}$$

and the map τ sends (x, y, z) to $(-x, y, z)$. For $t \in S^1$ take the map

$$\varphi_t(x, y, z) = (t^{pq}x, t^{2q}y, t^{2p}z)$$

which acts on $\Sigma(2, p, q)$, it is the identity for $t = 1$ and is τ for $t = -1$, hence id and τ are isotopic.

Results

Computations (relying on lattice homolgy and other previous calculations) provide nonvanishing results for pretzel knots:

Theorem (Afieri-Kang-S.)

The connected reduced Floer homology of torus knots and alternating knots vanish. For the pretzel knot $P(2,-3,-q)$ *with* $q \geq 7$ *it is* $\mathbb{F}$*, and for* $P(-2n-1, 4n+1, 4m+3)$ *it is* $\mathbb{F}[U]/(U^n)$.

Theorem (Afieri-Kang-S.)

The quotien group $\mathcal{C}/(\mathcal{T}+\mathcal{A})$ *of the concordance group with the subgroups generated by torus knots* ($\mathcal{T}$) *and by alternating knots* ($\mathcal{A}$) *contains a direct summand isomorphic to* $\mathbb{Z}^{\infty}$.

Importance

Note that the assignment

$$[K] \mapsto \mathrm{HFB}^{-}_{conn-red}(K)$$

is not a homomorphism. It has the potential to capture

- torsion elements ($K \subset S^3$ with $[K] \neq 0$ but $[nK] = 0$ — no such is known except when $n = 2$ and K is concordant to an amphichiral knot)
- divisible elements ($[K] \neq 0$ with the property that for any $n \in \mathbb{N}$ there is L_n knot with $[nL_n] = [K] \in \mathcal{C}$ no candidates for such elements yet.)

Higher covers?

Thank you!