Isotopy and Equivalence of Knots in 3-manifolds

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A **Knot** is a (tame) embedding $K : S^1 \hookrightarrow M^3$

Two Knots K, J in M are

1. **Equivalent** : If $\exists$ o.p. homeo $h : M \longrightarrow M$ s.t. $h(K) = J$
2. **Isotopic** : If $\exists$ 1-parameter family of embeddings $f_t : S^1 \times [0,1] \to M$ s.t. $f_0 = K$ and $f_1 = J$
3. **Ambient Isotopic** : $\exists$ 1-parameter family of homeos $h_t : M \times [0,1] \longrightarrow M$ s.t. $h_0 = Id$ and $h_1(K) = J$.

(3) $\Rightarrow$ (2), (1)

(2) $\Rightarrow$ (3) (Isotopy extension)

(1) $\Rightarrow$ (3) ?

Thm (Fisher, '60) $Homeo^+(S^3)$ is path-connected

(1), (2), (3) equivalent for S^3

Def: $Mod^+(M) = \pi_0(Homeo^+(M)) = Homeo^+(M) / \text{isotopy}$

Mapping class group of M

Q: Does equivalence imply isotopy in general M?

A: No!

$$\left\{\begin{array}{c}\text{Free homotopy} \\ \text{classes of loops}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Conjugacy classes} \\ \text{in } \pi_1(M)\end{array}\right\}$$

$\Rightarrow$ If $h: M \to M$ acts nontrivially on conjugacy classes (e.g. on H_1), then K and $h(K)$ may not be ambient isotopic.

<u>Thm 1</u> (ABDPR) Suppose M is a prime, orientable, closed 3-mfd and $h \in \mathrm{Homeo}^+(M)$ fixes every isotopy class of knot in M. Then h is isotopic to the identity.

$\mathrm{Homeo}^+(M)$ "sees" knot isotopy classes.

<u>Corollary</u>: M prime, orientable, closed. Then Isotopy $\Leftrightarrow$ Equivalence iff $\mathrm{Mod}^+(M) = \{1\}$

M prime $\nearrow$ M irreducible, "every isotopy" $\longleftrightarrow$ "every homotopy"

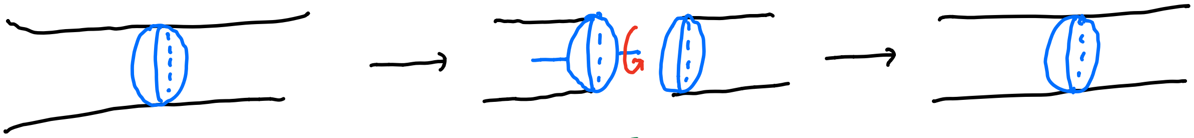
M prime $\searrow$ $M = S^1 \times S^2$, need "every isotopy"

$\boxed{M = S^1 \times S^2}$

$$1 \to \mathbb{Z}/2 \to \underset{\substack{\cong \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2}}{\mathrm{Mod}^+(S^1 \times S^2)} \to \mathbb{Z}/2 \to 1$$

Gluck twist $\mathcal{G}$

reverse or. on both factors
$1 \mapsto -1 \in \pi_1(S^1 \times S^2) = \mathbb{Z}$.



Gluck twist $\quad \pi_1(SO(3)) = \mathbb{Z}/2$

Thm 2 (ABDPR) For every winding number $w \neq 0 \in \mathbb{Z}$, $\exists$ K with winding number w s.t. $\mathcal{G}(K) \not\simeq K$. If w is odd, $\mathcal{G}(K) \simeq K \Rightarrow K \simeq S^1 \times \{*\}$

$\Rightarrow$ Thm 1 when $M = S^1 \times S^2$

$\boxed{M \text{ irreducible}}$ $\quad \mathrm{Mod}^+(M) \hookrightarrow \mathrm{Out}(\pi_1(M)) = \dfrac{\mathrm{Aut}(\pi_1)}{\mathrm{Inn}}$ (Many people ...)

We show if $h: M \to M$ acts trivially on conjugacy classes then $h_* = 1 \in \mathrm{Out}(\pi_1) \Rightarrow h \simeq \mathrm{Id}$.

Def (Grossman '75) A group G has Property A if every conjugacy class preserving automorphism is inner.

e.g. Abelian groups have Prop. A.

Thm 3 (ABDPR) Every orientable 3-mfd group has Property A.

$\Rightarrow$ Thm 1 when M irreducible

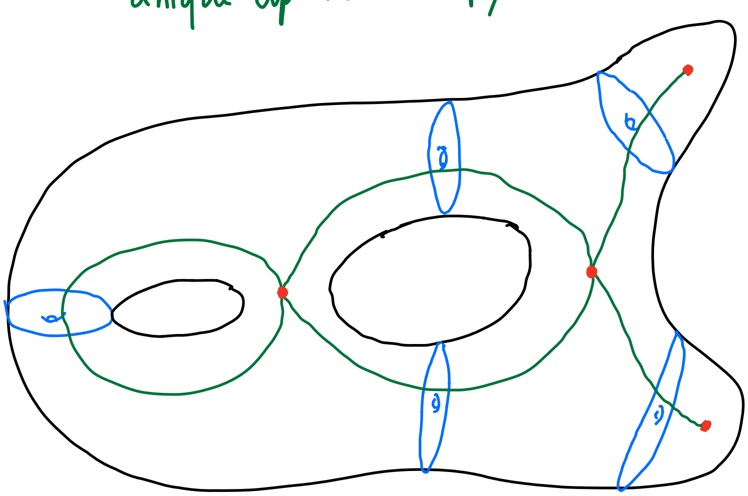
Idea of Proof : Use the Prime and JSJ-decomposition

1. Free products (Neshchadim '96)
   $\Rightarrow$ Non-prime 3-manifolds have Prop. A

2. Hyperbolic + relatively hyperbolic (Minasyan-Osin '10)
   $\Rightarrow$ hyperbolic JSJ-components have Prop. A.

3. Most Seifert fibered 3-mfd groups with or without bdy (Allenby-Kim-Tang, '03, '09) All except $S^2(p,q,r)$ and $T^2(p)$.
   We finished remaining cases.

- If M non-prime or has trivial JSJ-decomposition ✓
- Else : M is Haken.
  Waldhausen $\Rightarrow$ Any $\varphi \in \mathrm{Out}(\pi_1)$ repped by homeo $h: M \to M$ unique up to isotopy.



M

h class-preserving

- h preserves JSJ-decomposition
- Restriction to each piece class-preserving

Grossman ('75) introduced Property A and showed:

- Prop A + Conjugacy separable $\Rightarrow$ Out(G) is residually finite.

- Surface groups, free groups have Prop. A.
  $\Rightarrow$ Mod(S_g), Out(F_n) are residually finite.
  unknown   not linear, $n \geq 3$

Q: Are all 3-mfd groups conjugacy separable?