Quasi-isometric rigidity of graphs of free groups with cyclic edge groups

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Def:
A (Q, A)-*quasi-isometry* is a function $f : (X, d_X) \to (Y, d_Y)$ such that

$$\frac{1}{Q} d_X(x_1, x_2) - A \leq d_Y(f(x_1), f(x_2)) \leq Q d_X(x_1, x_2) + A,$$

and for all $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leq A$.

Groups as metric objects:

Given a finitely generated group we can canonically associate it to the quasi-isometry class of its Cayley graph.

Question:

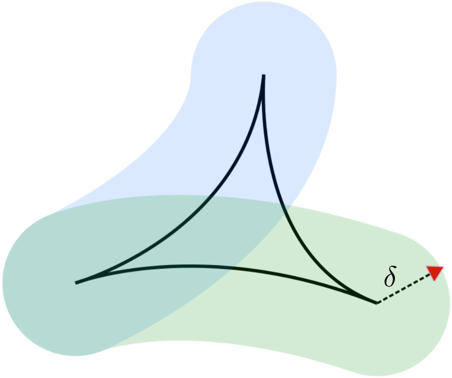
Can we classify groups and their properties up to quasi-isometry?

Rigidity:

A group G is *quasi-isometrically rigid* if $G \sim_{QI} G'$, then there are finite index subgroups $H \leqslant G$ and $H' \leqslant G'$ such that $H \cong H'$. (Or more generally *virtually isomorphic*).

Gromov hyperbolic groups

A group G is hyperbolic if all geodesic triangles in its Cayley graph are δ-slim.



Known Positive Results

The following hyperbolic groups are known to be quasi-isometrically rigid:

1. Finite groups
2. Two ended groups
3. Free groups [Stallings, 1968, Dunwoody, 1985, Karrass et al., 1973],
4. Cocompact Fuchsian groups [Tukia, 1988, Gabai, 1992, Casson and Jungreis, 1994],
5. Uniform Lattices in thick, right-angled Fuchsian (Bourdon)-buildings [Bourdon and Pajot, 2000, Haglund, 2006, Agol, 2013],
6. Certain graphs of groups with vertex groups that are cocompact Fuchsian, and infinite cyclic edge groups. – more later... [Taam and Touikan, 2019]

Guiding principal/conjecture:

When QI rigidity fails, it fails for a reason:

1. The group is quasi-isometric to a rank-1 symmetric space that has lots of incommensurable lattices,
2. The group splits as free product of one-ended groups [Whyte, 1999, Papasoglu and Whyte, 2002],
3. The group has *quadratically hanging* vertex groups in its JSJ decomposition [Malone, 2010, Stark, 2017, Dani et al., 2018].

Theorem (Shepherd-W.)

Let G be a cyclic amalgamations of finite rank free groups of the following form:

$$\mathbb{F}_n *_{\mathbb{Z}} \mathbb{F}_m = \langle \mathbb{F}_n, \mathbb{F}_m \mid w_1 = w_2 \rangle$$

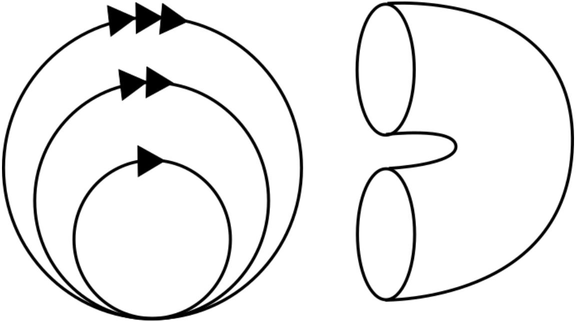
where $n, m \geq 2$ and $w_1 \in \mathbb{F}_n$ and $w_2 \in \mathbb{F}_m$ are suitably random/generic elements. Then G is quasi-isometrically rigid.

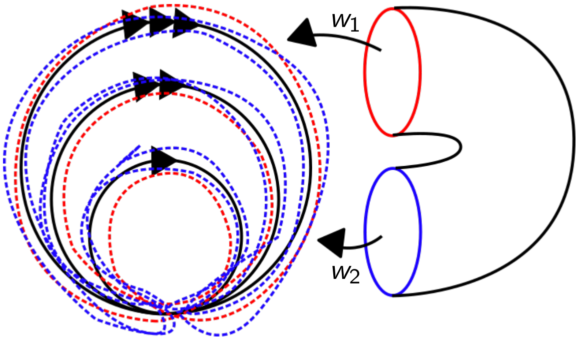
Theorem (Shepherd-W.)

Let G be a cyclic HNN extensions of the following form:

$$\mathbb{F}_n *_{\mathbb{Z}} = \langle \mathbb{F}_n \mid t w_1 t^{-1} = w_2 \rangle$$

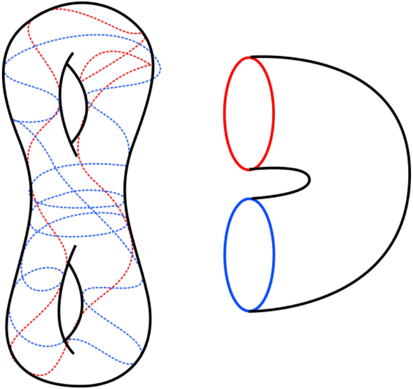
where $n \geq 2$ and $w_1, w_2 in \mathbb{F}_n$ are suitably random/generic elements. (We permit $w_1 = w_2$). Then G is quasi-isometrically rigid.





w_1
w_2

Taam and Touikan proved the corresponding result for a closed hyperbolic surface instead of a graph.



What kind of group is G?

1. G will be one-ended,
2. G will be hyperbolic,
3. G will be virtually special.

In short, quite a lot is already known about G.

The class of groups:

The $\mathcal{C}$ be the class of groups that split as finite graphs of groups with vertex groups isomorphic to $\mathbb{F}_n$ and edge groups isomorphic to $\mathbb{Z}$.

This is a wide class of groups:

The class $\mathcal{C}$ contains infinitely ended groups, Baumslag-Solitar groups, fundamental groups of surface amalgams – many groups we know are *not* quasi-isometrically rigid.

Theorem (Shepherd-W.)

Let G be a one-ended group, with JSJ decomposition over two-ended subgroups containing only virtually free rigid vertex groups and no quadratically hanging vertex groups. If G is hyperbolic (relative to virtually abelian subgroups) then G is quasi-isometrically rigid.

Note:

A virtually free vertex group could be two-ended.

NOTE:

There exist counterexamples if you allow hyperbolic relative to $\mathbb{F}_n \times \mathbb{Z}$ peripheral subgroups.

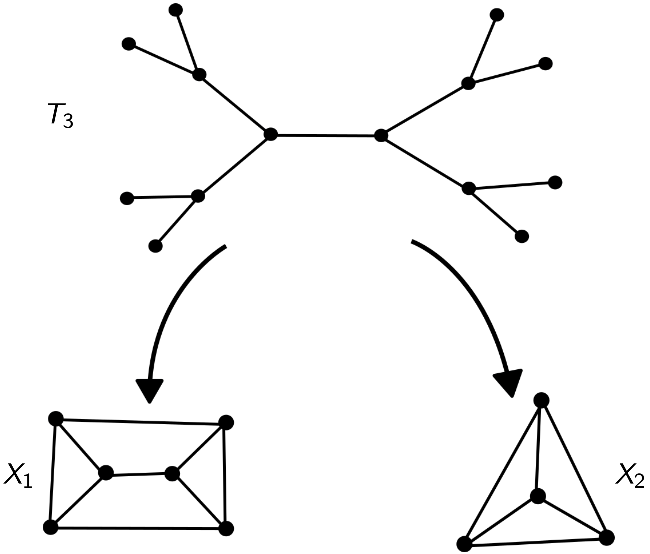
Prepared ingredients

1. JSJ theory for finitely presented groups[Guirardel and Levitt, 2017].
2. QI-invariance of the JSJ decomposition[Papasoglu, 2005].
3. Theory of rigid lines patterns in free groups [Cashen and Macura, 2011, Cashen, 2016].
4. QI-invariance of stretch ratio [Cashen and Martin, 2017].
5. Previous work studying when groups in $\mathcal{C}$ are separable and cubulated [Wise, 2000, Hsu and Wise, 2010].
6. Theory of cocompactly cubulated hyperbolic groups (Malnormal Special Quotient Theorem, cocompactly cubulated and hyperbolic $\implies$ virtually special). [Wise, 2009, Agol, 2013]

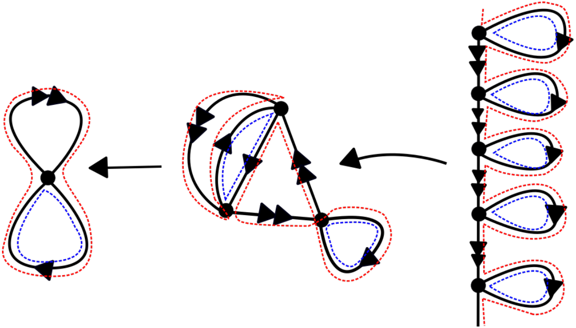
Our proof depends on a new proof and generalization the following theorem:

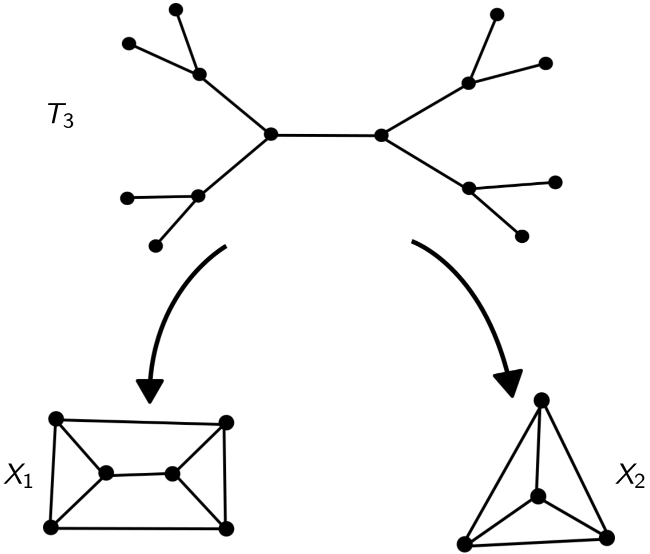
Theorem (Leighton)

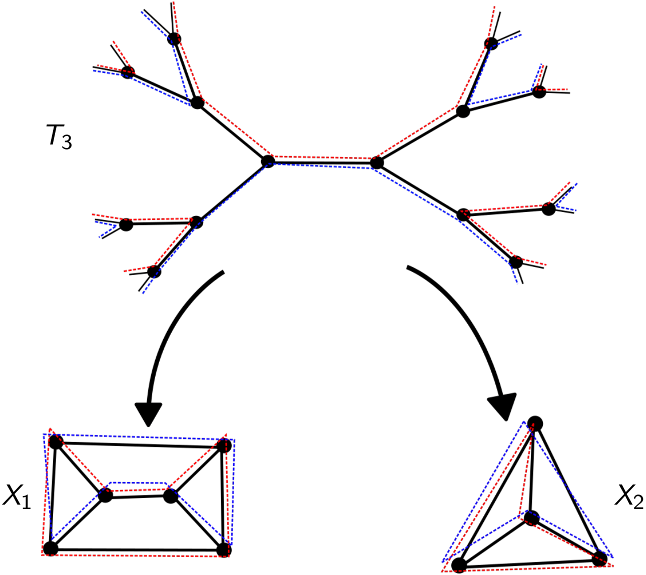
Let X_1 and X_2 be finite graphs with isomorphic universal covers. Then X_1 and X_2 have isomorphic finite covers (or a common finite cover).


T_3
X_1
X_2

A *graph with fins* is a graph X with a (finite) collection of geodesic, combinatorial loops X.




T_3
X_1
X_2


T_3
X_1
X_2

Theorem (W.)

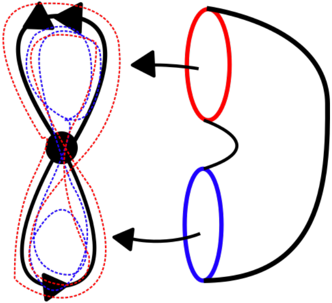
Let X_1 and X_2 be finite graphs with fins that have isomorphic universal covers. Then there exists a graph with fins $\widehat{X}$ that covers X_1 and X_2.

Further generalizations

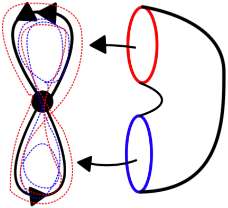
We actually show that $\widehat{X}$ also has some very nice symmetry properties that we use in the proof.
A more general statement has been proven [Gardam-W., Shepherd], which I am calling *Symmetry Restricted Leighton's Theorem*, but that is another story.

Proof sketch:

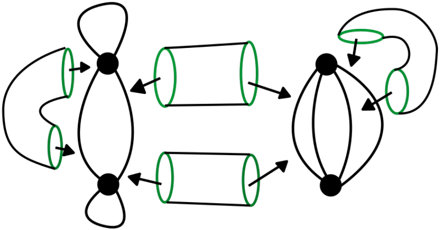
Suppose that we have our graph of spaces X such that $G = \pi_1 X$ (constructed in some canonical way):



Then the group G' also has a decomposition as a graph of groups X' in a similarly canonical way:

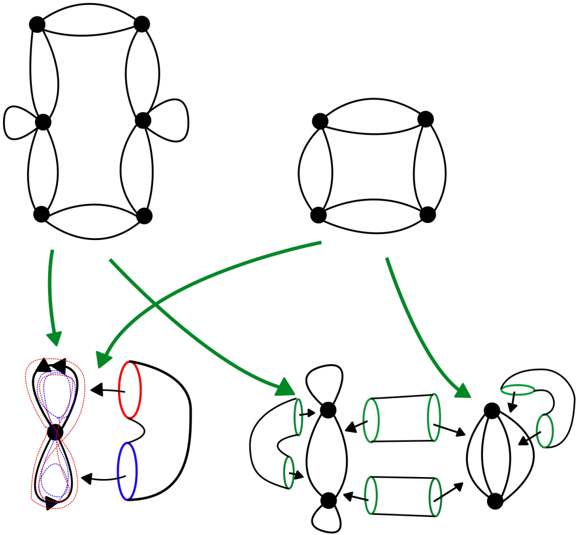


X

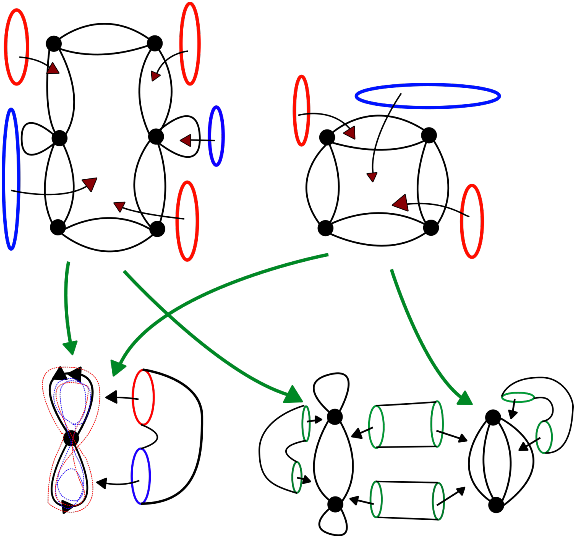


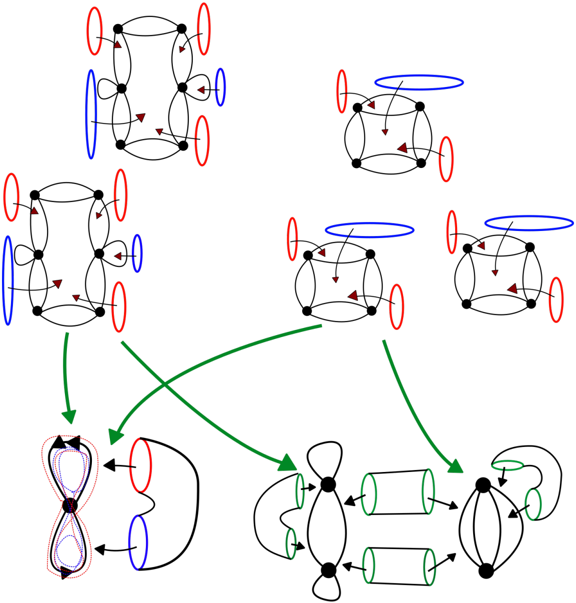
X'

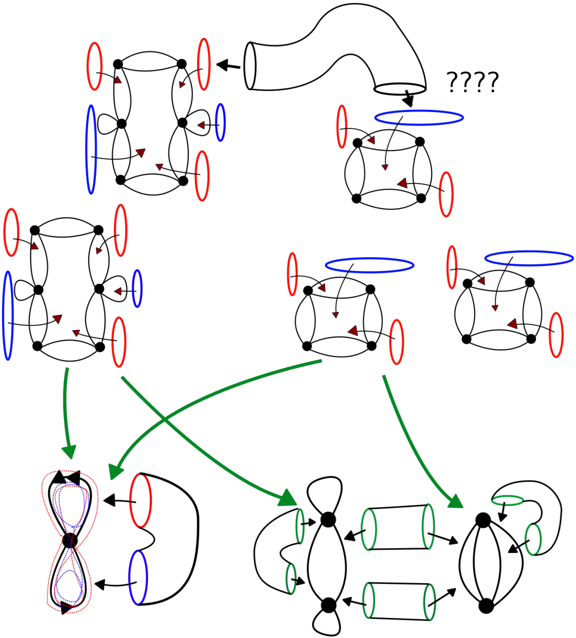
Take common covers of the vertex spaces:

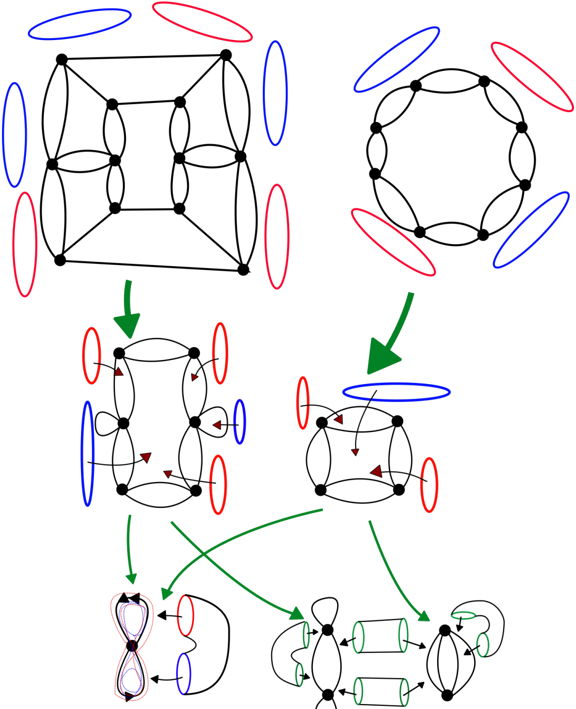


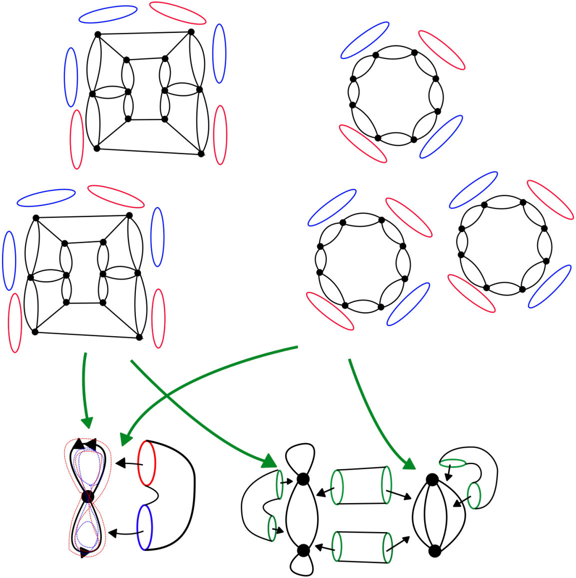
Common covers as graphs with fins:

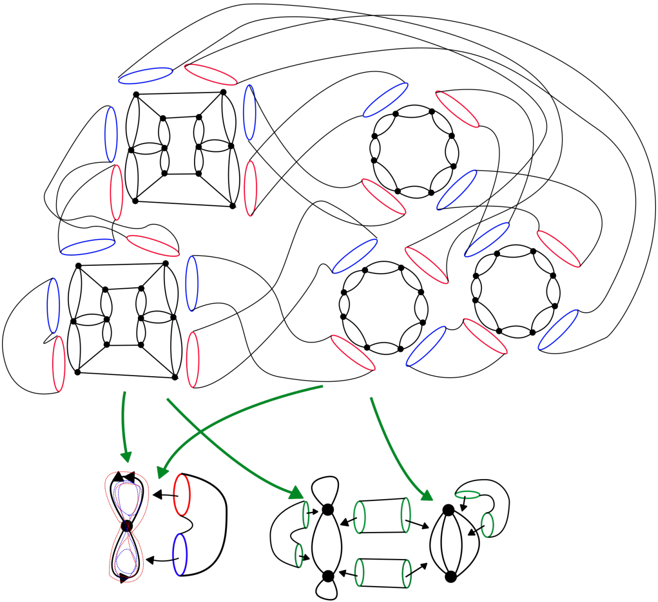












Conjecture (Haglund)

Let X_1 and X_2 be compact special cube complexes with isomorphic universal covers. Then X_1 and X_2 have isomorphic finite covers (or a common finite cover).

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