BOUNDARIES

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Will mention collaborative work with:

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* Alessandra Iozzi, ETH.
* Jean Lécureux, Univ. of Orsay
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Outline

- Definitions & examples of CCC
- Walls & half-spaces
- Roller Boundary $\partial_R X$
- Simplicialized Roller $\mathcal{R}_\Delta X$, $\cancel{\partial_\Delta X}$
- Tits Boundary $\partial_T X$ & $\partial_R X$
- Maps $\mathcal{R}_\Delta^{(0)} X \rightleftarrows \partial_T X$
- Example $\mathcal{R}_\Delta X \rightleftarrows \partial_T X$ & General case
- Contact graph $\mathcal{C}X$ & $\partial_G \mathcal{C}X$

$\partial_G :=$ Gromov

$\partial_R :=$ Roller

$\partial_\triangle :=$ Simplicial

$\partial_T :=$ Tits

X : CAT(0) CC

$\mathscr{C}X$: Contact Graph

$$
\begin{tikzcd}
 & & \partial_G \mathscr{C} X & & \\
 & \partial_{R_{)(}} X \arrow[r, hook] \arrow[dl, <->] & \partial_{reg} X \arrow[u, "\cong"] \arrow[r, hook] & \partial_R X \arrow[dr] & \\
\partial_{T_{)(}} X \arrow[r, hook] & \partial_T X \arrow[rr, "\sim"] & & \mathfrak{R}_\triangle X & \mathfrak{R}^{(0)}_\triangle X \arrow[l, hook'] \\
 & & \partial_\triangle X \arrow[ul, "\sim"] \arrow[ur, "\sim"'] \arrow[u, phantom, "\circlearrowright \sim"] & &
\end{tikzcd}
$$

Focus

$\sim$: homotopy eq.

$\cong$: homeo

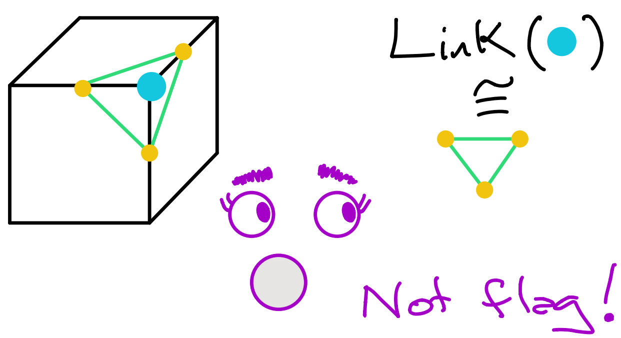
$\triangle$: simplicial/ized

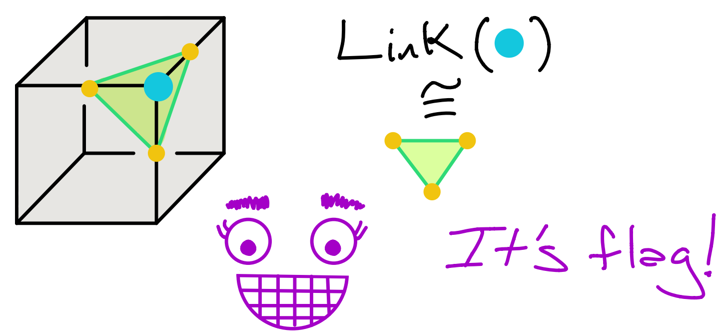
Def: X is CAT(0) cube complex if

- union of unit cubes glued isometrically along faces
- $\pi_1(X) = 1$
- No local positive curvature $\overset{\text{Gromov}}{\Longleftrightarrow}$

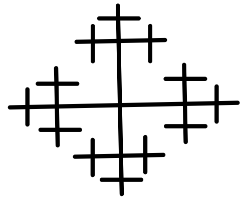
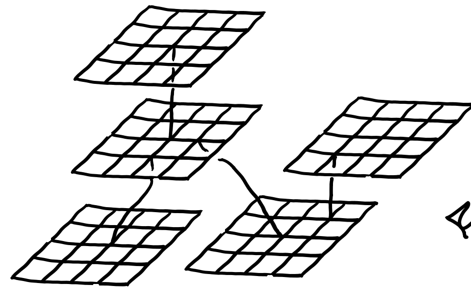
Link of every vertex is flag cpx:

3 squares vs $[0,1]^3$





Examples:

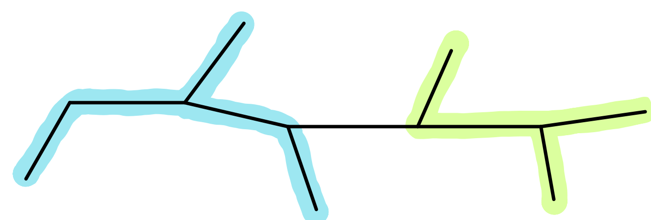
* Trees:  ,  $\curvearrowleft \mathbb{Z}^2 * \mathbb{Z}$
* RAAG $\rightsquigarrow$ Salvetti Cpx $\rightsquigarrow$ Univ. cover
* Small Cancellation $C'(1/6)$ (Wise)
* Coxeter groups (Niblo-Reeves)
* π_1(compact hyp 3-mnfld) (Bergeron-Wise, Agol)
* Closed under products
* Intimately related to Haagerup Property

Def: A <u>half-space</u> $h \subset X^{(0)}$ is subset s.t.

- h is edge convex
- $h^* = X^{(0)} \setminus h$ is edge convex
- h & h^* not empty

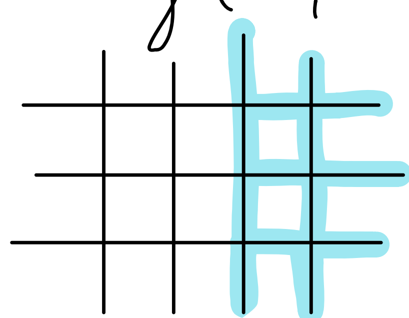
Collection of all half-spaces is $\mathcal{H}(X)$

<u>Examples:</u> Tree

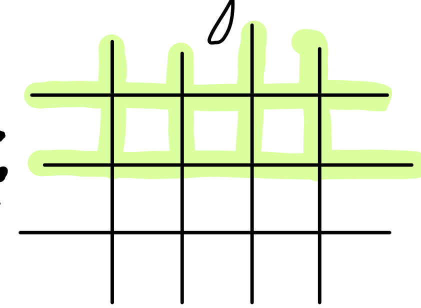


Products: $\mathcal{H}(X_1 \times X_2) \cong \mathcal{H}(X_1) \sqcup \mathcal{H}(X_2)$

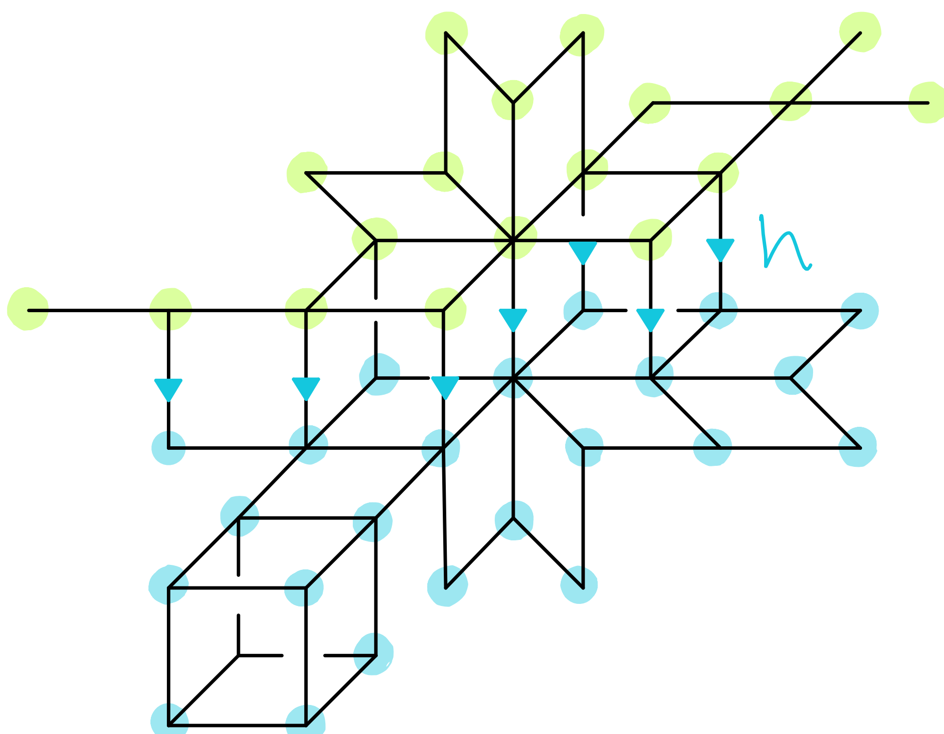
$h_1 \times X_2$:



; $X_1 \times h_2$:



<u>OR</u>: Remove "square-parallelism" class of (open) edge. This gives 2 components (Sageev), each is a half-space.



$\mathfrak{h} = \mathfrak{h}(X)$ denotes collection of half-spaces

Fact: $x, y \in X^{(0)}$: $x = y$ if & only if

$$\{h \in \mathfrak{h} : x \in h\} = \{h \in \mathfrak{h} : y \in h\}$$

TH: Combinatorial structure on X uniquely determines X as CCC. i.e.

$$X^{(0)} \rightsquigarrow \mathfrak{h}(X) \rightsquigarrow X(\mathfrak{h}(X))^{\circ} \cong X^{(0)}$$

(Sageev, Roller, Chatterji-Niblo, Nica)

So:

$i : X^{(0)} \hookrightarrow 2^{\mathfrak{H}}$; $x \mapsto U_x := \{h \in \mathfrak{H} : x \in h\}$

Roller Compactification & Boundary:

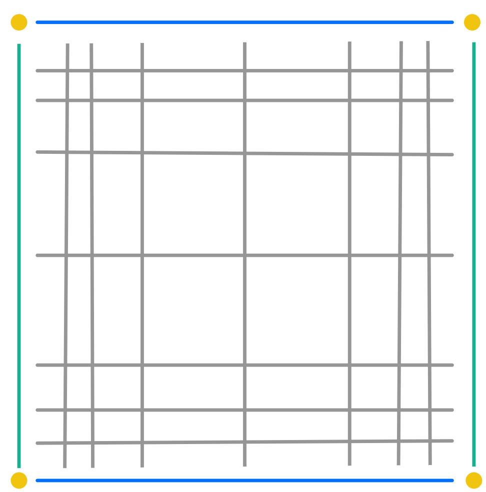
$$\overline{X} := \overline{i(X^{(0)})} < 2^{\mathfrak{H}} \quad \& \quad \partial_R X := \overline{X} \setminus i(X^{(0)})$$

- ☆ Totally disconnected
- ☆ Points $\rightsquigarrow$ subsets of $\mathfrak{H}$
- ☆ $\overline{X}$ has "extended edge metric"
- ☆ Well defined even if X not locally compact though $\partial_R X$ may not be closed.

Example: $X = \text{tree} \Rightarrow \partial_R X \cong \partial_G X$

↳ Gromov

Products: $\overline{X_1 \times X_2} = \overline{X_1} \times \overline{X_2}$



$$\left.\begin{array}{l} \partial_R X_1 \times X_2 \\ X_1 \times \partial_R X_2 \\ \partial_R X_1 \times \partial_R X_2 \end{array}\right\} = \partial_R (X_1 \times X_2)$$

⚠ Totally disconnected

So: $\partial_R X$ is a union of CAT(0) CC's,

$$d(y,y') = \tfrac{1}{2}\#(U_y \,\triangle\, U_{y'}) \in \mathbb{N} \cup \{\infty\}$$

Def: $y \sim y'$ if $d(y,y') < \infty$

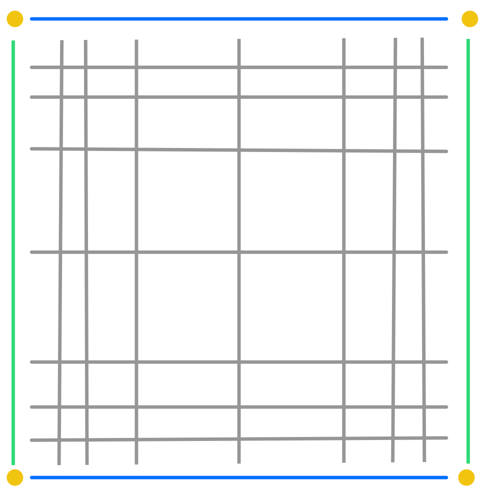
Class is a "largest" CAT(0) CC $\subseteq \partial_R X$

$$\boxed{\mathcal{R}_\Delta^{(0)} X := \{[y] : y \in \partial_R X\}}$$

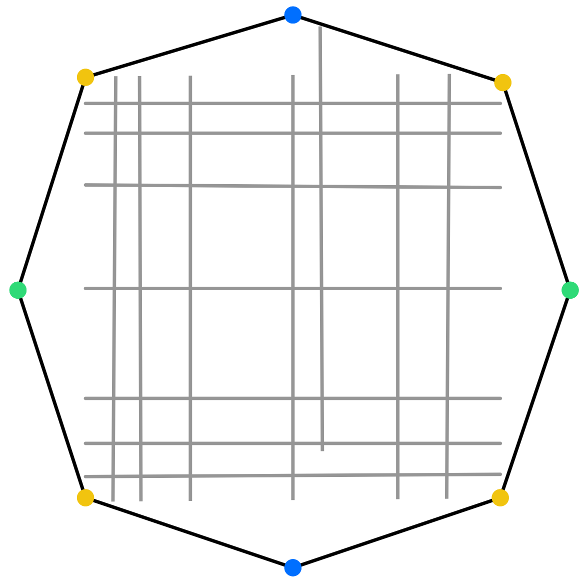
$$[y] \leq [z] \;:\; [z] \subseteq \overline{[y]}$$

$\mathcal{R}_\Delta X$ = Geom. Real. of n-simplex: $[y_1] \leq \cdots \leq [y_{n+1}]$.

Example: $\mathbb{Z}^2$:



$\partial_R \mathbb{Z}^2$



$\mathcal{R}_\Delta \mathbb{Z}^2$

Simplicialized Roller Boundary

$\partial_T X = \{[g] : g \text{ CAT(0) geodesic ray}\}$

$g \sim g' \Leftrightarrow$ Bounded Hausdorff distance

Tits Metric makes $\partial_T X$ CAT(1)

Ex: $\partial_T \mathbb{Z}^2 \cong S^1$

$\partial_T F_2 \cong$ uncountable discrete

Define Map : $\partial_R X \xrightarrow{\varphi} \partial_T X$ (FLM1)

Let $y \in \partial_R X$, $U_y = \{h : y \in h\}$

$\forall h_1, \dots, h_n \in U_y$; $\bigcap_{i=1}^{n} h_i \neq \emptyset$

$\Longrightarrow$ $\bigcap_{h \in U_y} \partial_T h \neq \emptyset$

Caprace Lytchak & Balser Lytchak

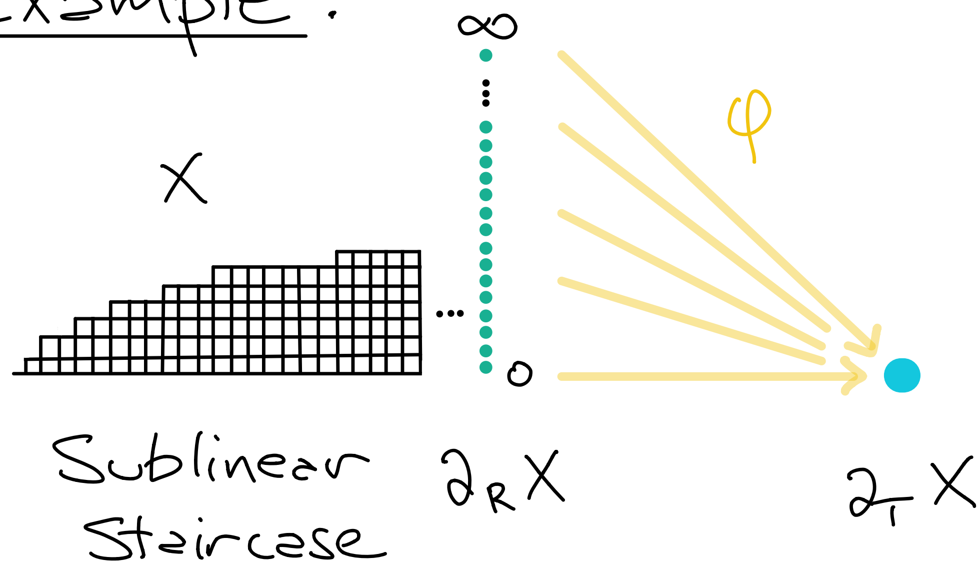
&

intrinsic radius $\leq \pi/2$

&

has unique circum center. $= \varphi(y)$

Example:



Why?

$\partial_T h = \{0\}$

$\forall h \in \mathfrak{h}(X)$

In fact: $\varphi(y) = \varphi(y')$ if $y \sim y'$

(FFH)

$$\Longrightarrow \exists\, \varphi' : \mathcal{R}^{(0)}_{\Delta} X \longrightarrow \partial_T X$$

Important property of $\mathcal{H}$: Helly

$h_1, \dots, h_n \in \mathcal{H}$ s.t. $\underbrace{h_i \cap h_j}_{\text{pairs} = 2} \neq \emptyset$

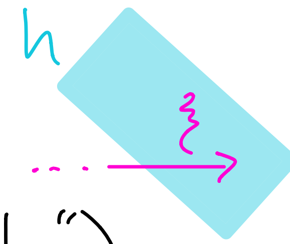
$\Rightarrow \bigcap_{i=1}^{n} h_i \neq \emptyset$

* The same is true in $\mathbb{R}^d$ w/ $d+1$ & CAT(0) convex sets.

* Helly dimension of CCC is 1.

* pair-wise intersect $\Rightarrow$ FIP $\Rightarrow$ total intersection in $\partial_R X$ nonempty

Can go backwards: $\partial_T X \xrightarrow{\psi} \mathcal{R}_\Delta^{(0)} X$

Let $[\xi] \in \partial_T X$, Let $T_\xi = \{ h \in \mathfrak{H} :$  $\}$

(ξ is "deep in h")

$\Rightarrow$ T_ξ has nonempty finite intersect.

$\Rightarrow$ $\bigcap_{h \in T_\xi} h \subset \partial_R X$ nonempty (CFI)

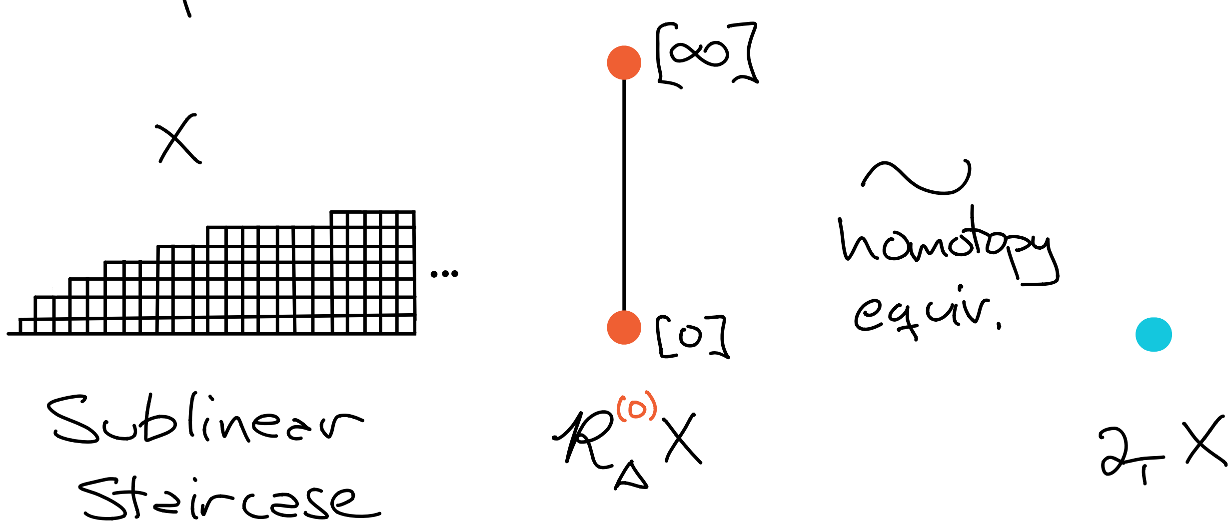
$\Rightarrow$ $\exists\, y \in \partial_R X$ s.t. $\overline{[y]} = \bigcap_{h \in T_\xi} h$ (FFH)

& $\psi[\xi] = [y]$ is well defined.

$\boxed{\text{TH:}}$ (FFH) $\exists$ $\mathrm{Aut}\,X$-equivariant homotopy equivalence:

$$\partial_T X \longrightarrow \mathcal{R}_\Delta X.$$

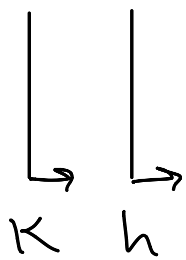
<u>Example</u>:



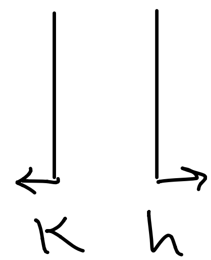
Now: Switch gears: $\partial_R X$ & Contact Graph.

Fact: $h, K \in \mathfrak{H}$ can have 5-relationships

$h \subset K$



$h \cap K = \emptyset$

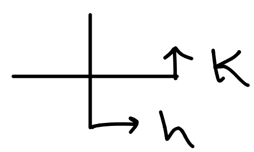


None: $h \pitchfork K$

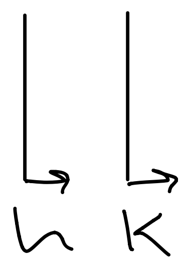
$h \cap K,\ h^* \cap K,$

$h \cap K^*,\ h^* \cap K^*$

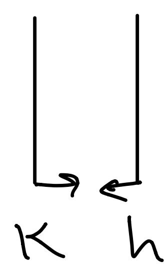
$\neq \emptyset$



$K \subset h$

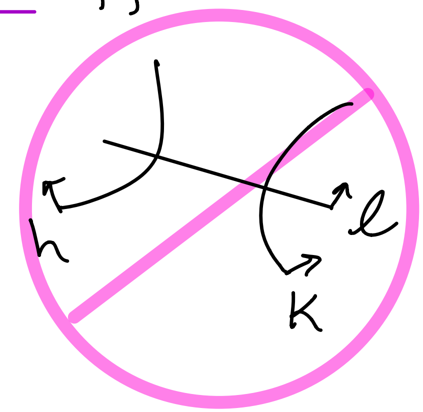


$h^* \subset K$



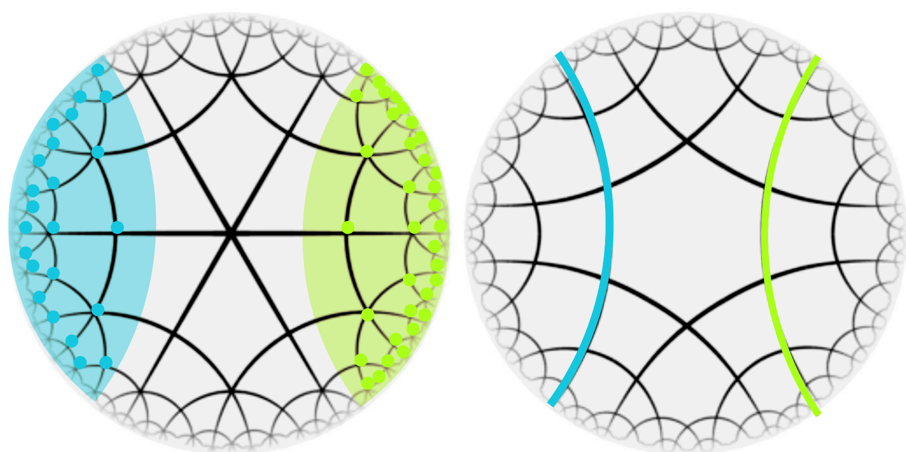
★ **Important Property of pairs of half-spaces**

$h, k \in \mathcal{H}$ are **strongly separated** if parallel & $\nexists\, \ell \in \mathcal{H}$ s.t. $h \pitchfork \ell \pitchfork k$



Example:

X square complex



Walls of X

Def: X is irreducible if $X \neq X_1 \times X_2$

$\Leftrightarrow \mathcal{H}(X) \neq \mathcal{H}_1 \sqcup \mathcal{H}_2$

s.t. $h_1 \pitchfork h_2 \ \forall h_i \in \mathcal{H}_i$

$\boxed{\text{TH:}}$ (Caprace Sageev) X nonelementary (& essential) is irreducible iff $\exists h, k \in \mathcal{H}$ strongly separated.

Def: For X irreducible, $\xi \in \partial X$ is regular
if $\exists\, h_{n+1} \subset h_n$ p.w. strongly separated
s.t. $\{\xi\} = \bigcap_n h_n$

- Can define for X reducible, collection is $\partial_{reg} X$

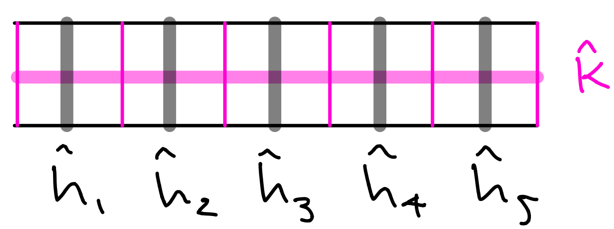
- Intervals between regular points are "visible": $\xi, \eta \in \partial_{reg} X$ distinct
$\Rightarrow \mathcal{I}(\xi,\eta) \setminus \{\xi,\eta\} \subset X$
i.e $m(\xi,\eta,\bullet) : \overline{X} \setminus \{\xi,\eta\} \to X$ (Fernós)

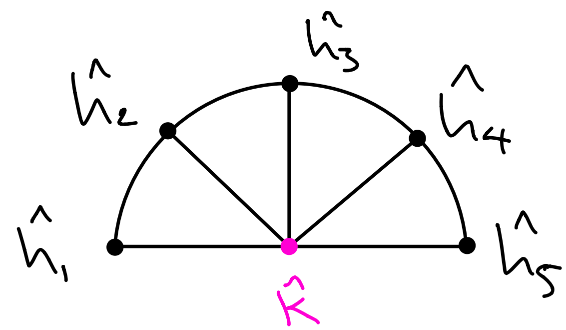
The Contact Graph $\mathcal{C}X$ & its Boundary.

$h \in \mathfrak{H}$ the associated wall is $\{h, h^*\} =: \hat{h} = \hat{h}^*$

$\mathcal{C}X$ is the graph with vertex set $\{\hat{h} : h \in \mathfrak{H}\}$
& edges connecting $\hat{h}$ & $\hat{k}$ if either

$\hat{h} \pitchfork \hat{k}$ OR $\hat{h}$ & $\hat{k}$ adjacent in X

Ex: $X =$ 



$\mathcal{C}X$

Facts about $\mathcal{C}X$: (Hagen)

- $\mathcal{C}X$ is hyperbolic (in fact quasi-tree)
- For $x \in X$ let $\pi(x) = \{\hat{h} : \hat{h} \text{ adjacent to } x\}$.
  $\rightarrow$ $\pi(x)$ is a clique in $\mathcal{C}X$ (may be ∞)
  $\pi : X \to \{\text{cliques in } \mathcal{C}X\}$
  is $\mathrm{Aut}(X)$-equivariant

- Define:

$$d_{\mathcal{C}X}(\pi(x), \pi(y)) = \min\{d_{\mathcal{C}X}(\hat{h}, \hat{k}) : \hat{h} \in \pi(x), \hat{k} \in \pi(y)\}$$

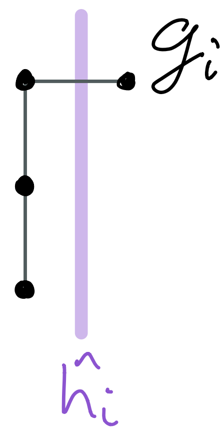
$$\Longrightarrow d_{\mathcal{C}X}(\pi(x), \pi(y)) \le d(x, y).$$

Hierarchy paths à la Masur-Minsky:

(Behrstock-Hagen-Sisto): $x, y \in X$

$\exists$ edge-geodesic $g : x \to y$ obtained by

concatenation $g = g_1 \cdots g_n$ s.t. $\exists\, \hat{h}_1, \dots, \hat{h}_n$

with 

& $(\hat{h}_1, \dots, \hat{h}_n)$ is geodesic in $\mathcal{C}X$.

$\boxed{\text{TH:}}$ (FLM)

$\exists$ Aut(X) equivariant Homeomorphism

$$\partial_r X \longrightarrow \partial \mathcal{C}X$$

Observation: $d_{\mathcal{C}X}(\hat{h}, \hat{k}) \geq 3 \Rightarrow \hat{h}, \hat{k}$ strong sep.

Proof: If $\hat{h}, \hat{k}$ not strongly separated then either $\hat{h} = \hat{k}$, or $\hat{h} \pitchfork \hat{k}$ $(\Rightarrow d_{\mathcal{C}X}(\hat{h}, \hat{k}) \leq 1)$

OR: $\hat{h} /\!/ \hat{k}$ & $\exists\, \hat{\ell} \pitchfork \hat{k}$ & $\hat{\ell} \pitchfork \hat{h}$

$$\Rightarrow d_{\mathcal{C}X}(\hat{h}, \hat{k}) = 2. \quad \blacksquare$$

$\boxed{\text{TH:}}$ (FLM)

$\exists$ Aut(X) equivariant Homeomorphism

$$\partial_r X \longrightarrow \partial CX$$

"Idea": Let $h_{n+1} \subset h_n$ be ∞ descending

chain of p.w strongly separated half spaces.

$$\Rightarrow \langle \hat{h}_n, \hat{h}_m \rangle_{\hat{h}_0} \xrightarrow{n,m\to\infty} \infty \quad \text{so} \quad \hat{h}_n \longrightarrow \eta \in \partial_G CX.$$

This can be reversed to produce $\xi \in \partial_{reg} X$ ∎.

Thank you!!